The periods of certain automorphic forms of arithmetic type

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To the memory of Takuro Shintani

The automorphic forms to be considered in this paper are the holomorphic ones on the upper half plane $\mathfrak{H}=\{z\in \boldsymbol{C}\mid \mathrm{Im}(z)>0\}$ with respect to a congruence subgroup of the multiplicative group $B^{\times}$ of an indefinite quaternion algebra B over $\boldsymbol{Q}$. As shown in our previous papers, we have the notion of rationality of such forms over the algebraic closure $\overline{\boldsymbol{Q}}$ of $\boldsymbol{Q}$. Let f be such a form of weight $2k$ with $0<k\in\boldsymbol{Z}$ which is a $\overline{\boldsymbol{Q}}$-rational eigenform of Hecke operators. Then we shall define two "fundamental periods" $u_{+}(f)$ and $u_{-}(f)$ which are nonzero complex numbers determined up to algebraic factors with the property that all the periods of f belong to $\overline{\boldsymbol{Q}}u_{+}(f)+\overline{\boldsymbol{Q}}u_{-}(f)$. It can be shown that $\pi\langle f, f\rangle/[u_{+}(f)\overline{u_{-}(f)}]$ is an algebraic number, where $\langle f, f\rangle$ is the normalized Petersson inner product (Theorem 4.4). Now it is well known that there is an elliptic cusp form g belonging to the same eigenvalues for Hecke operators as f. The main purpose of the present paper is to show that under certain assumptions on f, both $u_{+}(f)/u_{+}(g)$ and $u_{-}(f)/u_{-}(g)$ are algebraic numbers (Theorem 4.7). We obtain this result by first generalizing the integrals considered by Shintani in [16] and applying our results of [14] and [15] to the generalized integrals. It should be noted that the algebraicity of $\langle f, f\rangle/\langle g, g\rangle$ was proved in [15, II] in a more general case. Let us now give a summary of the contents.

We start with a somewhat more general setting with a totally real algebraic number field F of degree n as a basic field. We take a quaternion algebra B over F which is unramified at r archimedean primes of F with $0<r\leqq n$. For a cusp form f of "even weight" with respect to a congruence subgroup Γ of $B^{\times}$, we consider an integral of the form

$$M(z, f)=\int_{D}\overline{\theta(z, w)}f(w)\,\mathrm{Im}(w)^{2k}d\mu(w) \qquad (D=\Gamma\backslash\mathfrak{H}^{r})$$

with a certain theta function $\theta(z, w)$ on $\mathfrak{H}^{n}\times\mathfrak{H}^{r}$. We shall show that $M(z, f)$ is a Hilbert cusp form of half-integral weight and its Fourier coefficients are given by the "periods" of f (Theorems 2.2 and 3.1). Specializing this to the case $F=\boldsymbol{Q}$, we shall show that

$$M(z, f\mid T(p))=M(z, f)\mid T(p^{2})$$

for every odd prime p, where $T(p)$ and $T(p^2)$ are Hecke operators (Theorem 3.2). These generalize the results of Shintani [16] which deal with the case $B=M_2(\boldsymbol{Q})$. Our methods of proof are simpler than those of [16]. The quantities $u_+(f)$ and $u_-(f)$ will be defined in Section 4 by means of the cohomology groups attached to Γ. We shall show that the Fourier coefficients of $M(z, f)$ are $u_+(f)$ times algebraic numbers if f is "primitive". This fact combined with several results of arithmeticity proved in [15, II] and [14] concerning inner products will yield the algebraicity of $u_+(f)/u_+(g)$ and $u_-(f)/u_-(g)$ under the assumption that $L(z, f)\neq 0$.

1. Automorphic forms on $\mathfrak{H}^r$.

The symbols $\mathfrak{H}$, $\overline{\boldsymbol{Q}}$, F, and n will have the same meaning as in the introduction throughout the paper. We denote by $\tau_1, \cdots, \tau_n$ the embeddings of F into $\boldsymbol{R}$, and by I_F the free $\boldsymbol{Z}$-module generated by $\tau_1, \cdots, \tau_n$; we put $\boldsymbol{C}I_F=I_F\otimes_{\boldsymbol{Z}}\boldsymbol{C}=\sum_{\nu=1}^n \boldsymbol{C}\tau_\nu$. We embed F into $\boldsymbol{R}^n$ by the map $a\mapsto(a^{\tau_1}, \cdots, a^{\tau_n})$ and identify $F\otimes_{\boldsymbol{Q}}\boldsymbol{R}$ and $F\otimes_{\boldsymbol{Q}}\boldsymbol{C}$ with $\boldsymbol{R}^n$ and $\boldsymbol{C}^n$ through the map. If $p=\sum_{\nu=1}^n p_\nu\tau_\nu\in I_F$ and $z=(z_1, \cdots, z_n)\in\boldsymbol{C}^n$, we put $z^p=\prod_{\nu=1}^n z_\nu^{p_\nu}$; this is meaningful for $p\in\boldsymbol{C}I_F$ if $0<z_\nu\in\boldsymbol{R}$ for all ν. We use the letter F also for the element $\sum_{\nu=1}^n\tau_\nu$ of I_F; thus $z^F=z_1\cdots z_n$; in particular, $a^F=N_{F/\boldsymbol{Q}}(a)$ for $a\in F$. We put also

$$e(x)=\exp(2\pi ix) \qquad \text{for} \quad x\in\boldsymbol{C}, \tag{1.1}$$

$$e_F(z)=e(\textstyle\sum_{\nu=1}^n z_\nu) \qquad \text{for} \quad z=(z_1, \cdots, z_n)\in\boldsymbol{C}^n. \tag{1.2}$$

We write $x\gg 0$ and also $0\ll x$ for an element $x=(x_1, \cdots, x_n)\in\boldsymbol{R}^n$ if $x_\nu>0$ for all ν.

Let B be a quaternion algebra over F unramified at $\tau_1, \cdots, \tau_r$ and ramified at $\tau_{r+1}, \cdots, \tau_n$; we assume $0<r\leqq n$; $M_2(F)$ is included as a special case. Put $B_{\boldsymbol{R}}=B\otimes_{\boldsymbol{Q}}\boldsymbol{R}$. Then there is an isomorphism

$$B_{\boldsymbol{R}}\longrightarrow M_2(\boldsymbol{R})^r\times\boldsymbol{H}^{n-r}, \tag{1.3}$$

which sends an element x of F to $(x^{\tau_1}, \cdots, x^{\tau_n})$, where $\boldsymbol{H}$ denotes the Hamilton quaternions. For every $\alpha\in B_{\boldsymbol{R}}$, we denote by α_ν its projection to the ν-th factor of $M_2(\boldsymbol{R})^r\times\boldsymbol{H}^{n-r}$. Thus α_ν belongs to $M_2(\boldsymbol{R})$ or to $\boldsymbol{H}$ according as $\nu\leqq r$ or $\nu>r$. In particular, $x_\nu=x^{\tau_\nu}$ for $x\in F$.

For every integer $m\geqq 0$, we obtain an $\boldsymbol{R}$-rational polynomial representation $\sigma_m:\boldsymbol{H}^\times\to GL_{m+1}(\boldsymbol{C})$ by composing an injection of $\boldsymbol{H}^\times$ into $GL_2(\boldsymbol{C})$ with the representation of GL_2 by symmetric tensors of degree m; we understand that σ_0 is the trivial representation. Changing it for an equivalent representation, and choosing a suitable isomorphism (1.3), we may assume:

(1.4) *α_ν has algebraic entries for every $\alpha\in B$ and every $\nu\leqq r$;*

(1.5) $\sigma_m(\alpha_\nu)$ *has algebraic entries for every* $\alpha\in B^\times$, *every* $\nu>r$, *and every* $m\geqq 0$;

(1.6) $\sigma_m(x^\iota)={}^t\overline{\sigma_m(x)}$ *for all* $x\in \boldsymbol{H}^\times$.

Here and throughout the paper, ι denotes the main involution of any quaternion algebra (B, $M_2(\boldsymbol{R})$, $\boldsymbol{H}$, etc.). For an element α of $B_{\boldsymbol{R}}$ or $M_2(\boldsymbol{R})$ or $\boldsymbol{H}$, we put $N(\alpha)=\alpha\alpha^\iota$ and $\mathrm{Tr}(\alpha)=\alpha+\alpha^\iota$. We let an element α of $B_{\boldsymbol{R}}^\times$ act on $(\boldsymbol{C}\cup\{\infty\})^r$ by the rule

$$\alpha(z_1,\cdots,z_r)=(\alpha_1 z_1,\cdots,\alpha_r z_r),$$

$$\alpha_\nu z_\nu=(a_\nu z_\nu+b_\nu)(c_\nu z_\nu+d_\nu)^{-1}\qquad\text{for}\quad \alpha_\nu=\begin{pmatrix} a_\nu & b_\nu\\ c_\nu & d_\nu\end{pmatrix}.$$

We denote by $B_{\boldsymbol{R}+}^\times$ the set of all α of $B_{\boldsymbol{R}}$ such that $N(\alpha)\gg 0$, and put $B_+^\times=B\cap B_{\boldsymbol{R}+}^\times$.

Let $k=\sum_{\nu=1}^r k_\nu\tau_\nu\in I_F$ and $\kappa=\sum_{\nu=r+1}^n \kappa_\nu\tau_\nu\in I_F$ with $\kappa_\nu\geqq 0$. We define a factor of automorphy $J(\alpha,z)^k$ and a representation $\sigma_\kappa: B_{\boldsymbol{R}}^\times\to GL_d(\boldsymbol{C})$ with $d=\prod_{\nu>r}(\kappa_\nu+1)$ by

$$J(\alpha,z)^k=\prod_{\nu=1}^r|N(\alpha_\nu)|^{-k_\nu/2}(c_\nu z_\nu+d_\nu)^{k_\nu}\quad\left(\alpha\in B_{\boldsymbol{R}}^\times,\ z\in\boldsymbol{C}^r,\ \alpha_\nu=\begin{pmatrix} * & *\\ c_\nu & d_\nu\end{pmatrix}\right),\tag{1.7}$$

$$\sigma_\kappa(\alpha)=\sigma_{\kappa_{r+1}}(\alpha_{r+1})\otimes\cdots\otimes\sigma_{\kappa_n}(\alpha_n).\tag{1.8}$$

For a $\boldsymbol{C}^d$-valued function f on $\mathfrak{H}^r$ and an element α of $B_{\boldsymbol{R}+}^\times$, we define a $\boldsymbol{C}^d$-valued function $f\|_{k,\kappa}\alpha$ on $\mathfrak{H}^r$ by

$$(f\|_{k,\kappa}\alpha)(z)=\{\textstyle\prod_{\nu>r}N(\alpha_\nu)^{\kappa_\nu/2}\}\cdot J(\alpha,z)^{-k}\sigma_\kappa(\alpha)^{-1}f(\alpha z).$$

Let $\mathfrak{r}$ denote the maximal order of F and $\mathfrak{o}$ a maximal order of B. For every positive integer N, put

$$\Gamma_N=\{\gamma\in\mathfrak{o}\,|\,N(\gamma)=1,\ \gamma-1\in N\mathfrak{o}\}.$$

By a congruence subgroup of B, we understand a subgroup Γ of $B_+^\times$ such that $\Gamma_N\subset\Gamma$ and $[\Gamma\mathfrak{r}^\times:\Gamma_N\mathfrak{r}^\times]<\infty$ for some N. We then denote by $\mathcal{M}_{k,\kappa}(\Gamma)$ the set of all $\boldsymbol{C}^d$-valued holomorphic functions f on $\mathfrak{H}^r$ which are holomorphic at every cusp (if any), and which satisfy $f\|_{k,\kappa}\gamma=f$ for all $\gamma\in\Gamma$. (The cusp condition is necessary only when $B=M_2(\boldsymbol{Q})$.) The subspace of $\mathcal{M}_{k,\kappa}(\Gamma)$ consisting of the cusp forms (that is, the elements vanishing at all cusps) is denoted by $\mathcal{S}_{k,\kappa}(\Gamma)$; naturally $\mathcal{S}_{k,\kappa}=\mathcal{M}_{k,\kappa}$ if B is a division algebra. We write simply $\mathcal{M}_k$, $\mathcal{S}_k$, and $f\|_k\gamma$ for $\mathcal{M}_{k,0}$, $\mathcal{S}_{k,0}$ and $f\|_{k,0}\gamma$. We define a measure μ on $\mathfrak{H}^r$ by

$$d\mu(z)=\prod_{\nu=1}^r y_\nu^{-2}dx_\nu dy_\nu\qquad(z_\nu=x_\nu+iy_\nu)\tag{1.9}$$

and the inner product $\langle f,g\rangle$ of two elements f and g of $\mathcal{S}_{k,\kappa}(\Gamma)$ by

$$\langle f,g\rangle=\mu(D)^{-1}\int_D {}^t\overline{f(z)}g(z)\,\mathrm{Im}(z)^k d\mu(z)\qquad(D=\Gamma\backslash\mathfrak{H}^r).\tag{1.10}$$

If $B=M_2(F)$, we can introduce the notion of (Hilbert) modular forms of half-integral weight as follows. Put

$$\theta_F(z)=\sum_{x\in\mathfrak{r}} e(\sum_{\nu=1}^n x_\nu^2 z_\nu) \qquad (z=(z_1, \cdots, z_n)\in\mathfrak{H}^n),$$

(1.11)
$$\Gamma_\theta=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\mathfrak{r}) \,\middle|\, c\in 4\mathfrak{d}\right\},$$

where $\mathfrak{d}$ denotes the different of F. Then we have, as shown in [13, p. 342],

(1.12)
$$\theta_F(\gamma z)=j(\gamma, z)\theta_F(z), \quad j(\gamma, z)=\zeta_\gamma(cz+d)^{F/2} \qquad (\gamma\in\Gamma_\theta)$$

with a root of unity ζ_γ, and moreover

$$j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right)=\left(\frac{c}{d\mathfrak{r}}\right)(cz+d)^{F/2} \quad \text{if} \quad 0\ll d\equiv 1 \qquad (\mathrm{mod}\ 4\mathfrak{d}),$$

where $(-)$ denotes the quadratic residue symbol in F. Now let $h=\sum_{\nu=1}^n h_\nu\tau_\nu\in (1/2)I_F$ with h_ν satisfying $2h_1\equiv \cdots \equiv 2h_n\equiv 1$ (mod 2), and let Δ be a congruence subgroup of Γ_θ. Then we denote by $\mathcal{M}_h(\Delta)$ the set of all holomorphic functions f on $\mathfrak{H}^n$ holomorphic also at the cusps such that

$$f(\gamma z)=j(\gamma, z)(cz+d)^{h-(F/2)}f(z) \qquad \text{for every } \gamma=\begin{pmatrix} * & * \\ c & d \end{pmatrix}\in\Delta.$$

The subspace of $\mathcal{M}_h(\Delta)$ consisting of the cusp forms is denoted by $\mathcal{S}_h(\Delta)$.

We now define a 3-dimensional subspace V of B over F by

(1.13)
$$V=\{\alpha\in B \mid \mathrm{Tr}(\alpha)=0\}$$

and an F-valued F-bilinear symmetric form S on V by

(1.14)
$$S(\alpha, \beta)=\mathrm{Tr}(\alpha\beta^\iota)=-\alpha\beta-\beta\alpha \qquad (\alpha, \beta\in V).$$

We are going to apply our results of [13] and [15] to this form S. As in [13, § 3], we denote by V_ν the completion of V at τ_ν, and by S_ν the $\boldsymbol{C}$-bilinear extension of S to $V_\nu\otimes_{\boldsymbol{R}}\boldsymbol{C}$. The space $\mathfrak{Z}_m$ of [13, (2.6)] in the present case is isomorphic to $\mathfrak{H}$. We define a map $p: \boldsymbol{C}\to M_2(\boldsymbol{C})$ by

(1.15)
$$p(w)=\begin{pmatrix} w & -w^2 \\ 1 & -w \end{pmatrix} \qquad (w\in\boldsymbol{C}).$$

The restriction of p to $\mathfrak{H}$ is essentially a special case of [13, (2.7)]. If $\mathrm{Im}(w)\neq 0$, we have

(1.16)
$$\gamma p(w)\gamma^\iota=(cw+d)^2 p(\gamma w) \qquad \text{for} \quad \gamma=\begin{pmatrix} * & * \\ c & d \end{pmatrix}\in GL_2(\boldsymbol{R}).$$

We also put

(1.17)
$$[\alpha, w]=(w \quad 1)\varepsilon\alpha\begin{pmatrix} w \\ 1 \end{pmatrix}, \quad \varepsilon=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad (w\in\boldsymbol{C},\ \alpha\in V_\nu\otimes_{\boldsymbol{R}}\boldsymbol{C}).$$

Then $S_\nu(\alpha, p(w))=[\alpha, w]$ for $\nu\leqq r$. For $\alpha\in V$, $w=(w_1, \cdots, w_r)\in \boldsymbol{C}^r$, and $k\in I_F$ as above, we put

$$[\alpha, w]^k=\prod_{\nu=1}^r[\alpha_\nu, w_\nu]^{k_\nu}. \tag{1.18}$$

If $\mathrm{Im}(w_\nu)\neq 0$ for all ν, we have

$$[\beta^\iota\alpha\beta, w]^k=|N(\beta)|^k J(\beta, w)^{2k}[\alpha, \beta w]^k \qquad (\alpha\in V,\ \beta\in B^\times), \tag{1.19}$$

$$[\beta\alpha\beta^\iota, \beta w]^k=|N(\beta)|^k J(\beta, w)^{-2k}[\alpha, w]^k \qquad (\alpha\in V,\ \beta\in B^\times). \tag{1.20}$$

For $\lambda=\sum_{\nu>r}\lambda_\nu\tau_\nu\in I_F$ with $\lambda_\nu\geqq 0$, let $\mathcal{P}_\lambda$ denote the vector space over $\boldsymbol{C}$ of all polynomial functions on $V_{r+1}\times\cdots\times V_n$ which are S_ν-harmonic (in the sense of [13, p. 322]) and homogeneous of degree λ_ν on V_ν. Define the action of $B^\times$ on $\mathcal{P}_\lambda$ by

$$t^\beta(\alpha)=t((\beta_\nu\alpha_\nu\beta_\nu^\iota)_{\nu>r}) \qquad (\beta\in B^\times,\ \alpha\in\prod_{\nu>r}V_\nu,\ t\in\mathcal{P}_\lambda). \tag{1.21}$$

LEMMA 1.1. *The above representation of $B^\times$ on $\mathcal{P}_\lambda$ is equivalent to $\sigma_{2\lambda}$.*

PROOF. It is sufficient to prove the corresponding assertion for $GL_2(\boldsymbol{C})$ and its representation on the space $\mathcal{P}'_\lambda$ of homogeneous harmonic functions of degree λ on the 3-dimensional space $\{x\in M_2(\boldsymbol{C})\,|\,\mathrm{Tr}(x)=0\}$. To $p\in\mathcal{P}'_\lambda$, assign a function p^* on $\boldsymbol{C}^2$ by $p^*(x)=p(\varepsilon^{-1}x\cdot{}^t x)$ for $x\in\boldsymbol{C}^2$. It can easily be shown that $p\mapsto p^*$ gives an isomorphism of $\mathcal{P}'_\lambda$ onto the space of all homogeneous polynomial functions on $\boldsymbol{C}^2$ of degree 2λ, which commutes with the action of $GL_2(\boldsymbol{C})$.

Thus we can find a $\boldsymbol{C}^d$-valued function u on $\prod_{\nu>r}V_\nu$ whose components form $\overline{\boldsymbol{Q}}$-rational basis of $\mathcal{P}_\lambda$ over $\boldsymbol{C}$ such that

$$u(\beta\alpha\beta^\iota)=\sigma_{2\lambda}(\beta)u(\alpha) \qquad (\alpha\in V,\ \beta\in B^\times). \tag{1.22}$$

If $\lambda=0$, we understand that u is the constant 1.

2. A generalization of the Shintani integral.

In order to state our first main theorem, it is necessary to normalize Haar measures of certain subgroups of $SL_2(\boldsymbol{R})$. Put

$$a_t=\begin{pmatrix} t & 0\\ 0 & t^{-1}\end{pmatrix},\quad n_s=\begin{pmatrix} 1 & s\\ 0 & 1\end{pmatrix},\quad k_\theta=\begin{pmatrix} \cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$$

$$(0\neq t\in\boldsymbol{R},\ s\in\boldsymbol{R},\ \theta\in\boldsymbol{R}),$$

$$A=\{a_t\,|\,0<t\in\boldsymbol{R}\},\qquad A'=\{a_t\,|\,0\neq t\in\boldsymbol{R}\},$$

$$N=\{n_s\,|\,s\in\boldsymbol{R}\},\qquad K=\{k_\theta\,|\,\theta\in\boldsymbol{R}\}.$$

Let H be a subgroup of $SL_2(\boldsymbol{R})$ such that $\beta^{-1}H\beta$ for some $\beta\in SL_2(\boldsymbol{R})$ coincides with A, A', N, $N\{\pm 1\}$, or K. Then we define an invariant measure dh on H so that: the whole measure of H is 1 if $\beta^{-1}H\beta=K$;

$$\int_H \varphi(h)dh=\begin{cases}\int_0^\infty \varphi(\beta a_t\beta^{-1})t^{-1}dt & \text{if } \beta^{-1}H\beta=A\,,\\ \int_{-\infty}^\infty \varphi(\beta a_t\beta^{-1})|t^{-1}|dt & \text{if } \beta^{-1}H\beta=A'\,.\end{cases} \tag{2.1}$$

Notice that this is independent of the choice of β. If $\beta^{-1}H\beta=N$ or $N\{\pm 1\}$, we fix an arbitrary invariant measure on H. In particular, if $H=N$, we take $d(n_s)=ds$. Since $SL_2(\boldsymbol{R})=ANK$, we define an invariant measure dg on $SL_2(\boldsymbol{R})$ as usual by $d(ank)=dadndk$ ($a\in A$, $n\in N$, $k\in K$). This can also be characterized by

$$\int_{SL_2(\boldsymbol{R})}\phi(g(i))dg=(1/2)\int_{\mathfrak{H}}\phi(z)d\mu(z) \tag{2.2}$$

for an integrable function ϕ on $\mathfrak{H}$. We shall consider the product $SL_2(\boldsymbol{R})^r$ and its subgroups H such that $\beta^{-1}H\beta$ for some $\beta\in SL_2(\boldsymbol{R})^r$ is a product of copies of A, A', N, $N\{\pm 1\}$, or K. We normalize Haar measures on $G=SL_2(\boldsymbol{R})^r$ and such H by taking the product of the above measures. If C is a discrete subgroup of H, then we normalize a measure $d(Ch)$ on $C\backslash H$ so that

$$\int_H \varphi(h)dh=\int_{C\backslash H}\{\textstyle\sum_{c\in C}\varphi(ch)\}\,d(Ch).$$

A measure on $H\backslash G$ can be normalized in a similar way. We shall simply write dg for $d(Hg)$ if there is no fear of confusion.

We now take a congruence subgroup Γ of $B^\times$ such that $N(\gamma)=1$ for all $\gamma\in\Gamma$, and let Γ act on V by $\alpha\mapsto\gamma\alpha\gamma^{-1}$ for $\alpha\in V$ and $\gamma\in\Gamma$. Put

$$V^*=\{\alpha\in V\,|\,N(\alpha_\nu)<0 \quad \text{for all } \nu\leqq r\}. \tag{2.3}$$

This is stable under Γ. Identify B with a subset of $M_2(\boldsymbol{R})^r$ through the map $\alpha\mapsto(\alpha_1,\cdots,\alpha_r)$, and put, for each $\alpha\in V$, $\neq 0$,

$$H_\alpha=\{g\in SL_2(\boldsymbol{R})^r\,|\,g\alpha=\alpha g\}\,,\qquad \Gamma_\alpha=H_\alpha\cap\Gamma. \tag{2.4}$$

Given $f\in S_{2k,2\lambda}(\Gamma)$ and $0\neq\alpha\in V$, we put

$$P(f,\alpha,\Gamma)=\int_{\Gamma_\alpha\backslash H_\alpha}[\alpha,\,hw]^k\cdot{}^t\overline{u(\alpha)}f(hw)d(\Gamma_\alpha h) \qquad (w\in\mathfrak{H}^r), \tag{2.5}$$

where u is the function of (1.22). Notice that the integrand is in fact Γ_α-invariant. Obviously H_α is a subgroup of $SL_2(\boldsymbol{R})^r$ of the above type, and therefore the measures of H_α and $\Gamma_\alpha\backslash H_\alpha$ can be normalized in the above described manner.

LEMMA 2.1. *Integral* (2.5) *is convergent and independent of* w. *Moreover, it is* 0 *unless* $\alpha \in V^*$.

PROOF. Postponing the proof of convergence, let us prove here only the independence from w and the vanishing. Call the right-hand side of (2.5) $\varphi(w)$. Then φ is holomorphic in w, and $\varphi(hw)=\varphi(w)$ for all $h \in H_\alpha$, and hence φ must be a constant. Now (1.19) shows that

$$\varphi=[\alpha,\, w]^k \int_{\Gamma_\alpha \backslash H_\alpha} J(h,\, w)^{-2k} \cdot {}^t\overline{u(\alpha)} f(hw) d(\Gamma_\alpha h)\,.$$

Suppose $N(\alpha_\nu)>0$ for some $\nu \leqq r$. Since $\mathrm{Tr}(\alpha_\nu)=0$, we have $\alpha_\nu(w_\nu)=w_\nu$ for some $w_\nu \in \mathfrak{H}$. With this choice of w_ν, we have $[\alpha,\, w]^k=0$, and hence $\varphi=0$. Next suppose $N(\alpha)=0$. This happens only when $B=M_2(F)$. We can find an element δ of $SL_2(F)$ such that $\delta^{-1}\alpha\delta=\begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix}$ with $0 \neq p \in F$. Put $H'=\delta^{-1}H_\alpha\delta$, $\Gamma'=\delta^{-1}\Gamma_\alpha\delta$, and $f'=f\|_{2k}\delta$. Then

$$\begin{aligned}(-p)^{-k}\varphi(\delta w) &= \int_{\Gamma_\alpha \backslash H_\alpha} J(h\delta,\, w)^{-2k} f(h\delta w) d(\Gamma_\alpha h) \\ &= \int_{\Gamma' \backslash H'} J(\delta h',\, w)^{-2k} f(\delta h' w) d(\Gamma' h') \\ &= \int_{\Gamma' \backslash H'} f'(h' w) d(\Gamma' h')\,.\end{aligned}$$

Observe that $H'=(\{\pm 1\}N)^n$. Since f' is a cusp form, the last integral is 0. This completes the proof.

We note here a few easy relations:

$$P(f,\, \beta\alpha\beta^{-1},\, \Gamma)=P(f\|_{2k,2\lambda}\beta,\, \alpha,\, \beta^{-1}\Gamma\beta) \qquad (\beta \in B_+^\times)\,, \tag{2.6}$$

$$P(f,\, \alpha,\, \Gamma')=[\Gamma_\alpha : \Gamma'_\alpha]P(f,\, \alpha,\, \Gamma) \quad \text{if} \quad [\Gamma : \Gamma']<\infty\,, \tag{2.7}$$

$$P(f,\, c\alpha,\, \Gamma)=c^{k+\lambda}P(f,\, \alpha,\, \Gamma) \qquad \text{if} \quad c \in (\boldsymbol{R}^\times)^n\,. \tag{2.8}$$

A $\boldsymbol{C}$-valued function η on V is called *locally constant* if there are two $\boldsymbol{Z}$-lattices L and M in V such that $\eta(v)=0$ for $v \notin L$ and $\eta(v)=\eta(v')$ for $v-v' \in M$; further we say that η is *Γ-invariant* if $\eta(v)=\eta(\gamma v \gamma^{-1})$ for all $\gamma \in \Gamma$.

THEOREM 2.2. *Let* $k=\sum_{\nu=1}^r k_\nu \tau_\nu$ *and* $\lambda=\sum_{\nu=r+1}^n \lambda_\nu \tau_\nu$ *be elements of* I_F *with* $k_\nu>0$ *and* $\lambda_\nu \geqq 0$. *Further let* Γ *be a congruence subgroup of* $B^\times$ *such that* $N(\gamma)=1$ *for all* $\gamma \in \Gamma$, f *an element of* $\mathcal{S}_{2k,2\lambda}(\Gamma)$, η *a* Γ*-invariant locally constant function on* V, *and* q *an element of* F *such that* $q^{\tau_\nu}>0$ *for* $\nu \leqq r$ *and* $q^{\tau_\nu}<0$ *for* $\nu>r$. *Let* $R(\Gamma)$ *be a complete set of representatives of* V^* *modulo the action* $\alpha \mapsto \gamma\alpha\gamma^{-1}$ *for all* $\gamma \in \Gamma$. *Then an infinite series*

(2.9) $$\sum_{\alpha\in R(\Gamma)}\eta(\alpha)|N(\alpha)|^{-\xi/2}P(f,\alpha,\Gamma)e_F(-qN(\alpha)z)$$

is convergent and defines a Hilbert cusp form of weight h (that is, an element of $\mathcal{S}_h(\Delta)$ with a congruence subgroup Δ of $SL_2(F)$), where $\xi=\sum_{\nu=1}^r\tau_\nu$, and

(2.10) $$h=\sum_{\nu=1}^n h_\nu\tau_\nu,\quad h_\nu=\begin{cases} k_\nu+(1/2) & \text{for } \nu\leqq r,\\ \lambda_\nu+(3/2) & \text{for } \nu>r.\end{cases}$$

REMARK. That each term of (2.9) depends only on the class of α modulo Γ follows from (2.6). Series (2.9) is only superficially dependent on the choice of Γ. In fact, $\mu(\Gamma\backslash\mathfrak{H}^r)^{-1}$ times (2.9) depends only on f, η, and q, and is independent of Γ. This will follow from our proof of the theorem; it can easily be derived also from (2.7).

Our proof requires a theta function on $\mathfrak{H}^n\times\mathfrak{H}^r$ we introduced in [13] and [15], whose explicit form is

(2.11) $$\begin{aligned}\theta(z,w,t)&=\theta(z,w,t;\eta,q)\\ &=y^{\xi/2}\,\mathrm{Im}(w)^{-2k}\sum_{\alpha\in V}\overline{\eta(\alpha)}t(\alpha)[\alpha,\bar{w}]^k e\big(\textstyle\sum_{\nu=1}^n q_\nu R_\nu[\alpha,z,w]\big)\\ &\qquad (z\in\mathfrak{H}^n,\ w\in\mathfrak{H}^r,\ t\in\mathcal{P}_\lambda),\end{aligned}$$

(2.12) $$R_\nu[\alpha,z,w]=\begin{cases} N(\alpha_\nu)z_\nu+(i/2)y_\nu\,\mathrm{Im}(w_\nu)^{-2}|[\alpha_\nu,w_\nu]|^2 & (\nu\leqq r),\\ N(\alpha_\nu)\bar{z}_\nu & (\nu>r).\end{cases}$$

Here $y_\nu=\mathrm{Im}(z_\nu)$; k, η, ξ, and q are the same as in the theorem. This is essentially the same as [15, I, (6.6)] specialized to the case $m=1$. Notice that the present R_ν is 1/2 times R_ν of [15, I, (6.7)] with $-\bar{z}$ in place of z. Now [15, I, (6.10)] in the present case becomes

(2.13) $$\theta(z,\beta w,t;\eta,q)=N(\beta)^kJ(\beta,w)^{2k}\theta(z,w,t^\beta;\eta^\beta,N(\beta)^2q)\qquad(\beta\in B_+^\times),$$

where $\eta^\beta(\alpha)=\eta(\beta\alpha\beta^\iota)$. Our θ is also a special case of [13, (7.6)], and therefore, by [13, Proposition 7.1], it satisfies

(2.14) $$\theta(\gamma z,w,t)=\overline{j(\gamma,z)}(c\bar{z}+d)^{h-(F/2)}\theta(z,w,t)\qquad \text{for every } \gamma=\begin{pmatrix} * & * \\ c & d\end{pmatrix}\in\Delta,$$

where Δ is a congruence subgroup of $SL_2(F)$, $j(\gamma,z)$ is defined by (1.12), and h is defined by (2.10) (cf. [15, I, (6.11)]). We now define $\Theta(z,w;\eta,q)$, or simply $\Theta(z,w)$, to be a column vector whose components are $\theta(z,w,t_i;\eta,q)$ for $i=1,\cdots,d$, with components $t_1,\cdots,t_d$ of the column vector u of (1.22). From (2.13) we obtain

(2.15) $$\Theta(z,\beta w;\eta,q)=N(\beta)^kJ(\beta,w)^{2k}\sigma_{2\lambda}(\beta)\Theta(z,w;\eta^\beta,N(\beta)^2q)\qquad(\beta\in B_+^\times).$$

Given Γ, f, and η as in the theorem, we consider an integral

$$\int_D {}^t\overline{\Theta(z,w)}f(w)\,\mathrm{Im}(w)^{2k}d\mu(w) \qquad (D=\Gamma\backslash\mathfrak{H}^r). \tag{2.16}$$

This can be written $\mu(D)\langle\Theta(z,w), f(w)\rangle$. Since f is a cusp form, the convergence of (2.16) is straightforward. Now our theorem will follow from

PROPOSITION 2.3. *Series* (2.9) *is equal to* $2^r q^{\xi/2}$ *times* (2.16).

PROOF. Let R' be a complete set of representatives of $V-\{0\}$ modulo Γ. Since

$$V-\{0\}=\bigcup_{\alpha\in R'}\{\gamma^{-1}\alpha\gamma\,|\,\gamma\in\Gamma_\alpha\backslash\Gamma\},$$

integral (2.16) times $y^{-\xi/2}$ is equal (at least formally) to

$$\sum_{\alpha\in R'}\eta(\alpha)\sum_{\gamma\in\Gamma_\alpha\backslash\Gamma}\int_{\gamma D}[\alpha,w]^k\cdot{}^t\overline{u(\alpha)}f(w)\boldsymbol{e}(-\textstyle\sum_{\nu=1}^n q_\nu\overline{R_\nu[\alpha,z,w]})d\mu(w)$$

$$=\sum_{\alpha\in R'}\eta(\alpha)\int_{D_\alpha}[\alpha,w]^k f_\alpha(w)\boldsymbol{e}(-\textstyle\sum_{\nu=1}^n q_\nu\overline{R_\nu[\alpha,z,w]})d\mu(w),$$

where $D_\alpha=\Gamma_\alpha\backslash\mathfrak{H}^r$ and $f_\alpha(w)={}^t\overline{u(\alpha)}f(w)$. This termwise integration will be justified later. For simplicity, put $G=SL_2(\boldsymbol{R})^r$ and $\boldsymbol{i}=(i,\cdots,i)$ $(\in\mathfrak{H}^r)$. By (2.2), the integral over D_α can be written in the form

$$2^r\int_{\Gamma_\alpha\backslash G}[\alpha,g\boldsymbol{i}]^k f_\alpha(g\boldsymbol{i})\boldsymbol{e}(-\textstyle\sum_{\nu=1}^n q_\nu\overline{R_\nu[\alpha,z,g\boldsymbol{i}]})dg$$

$$=2^r\int_{H_\alpha\backslash G}\boldsymbol{e}(-\textstyle\sum_{\nu=1}^n q_\nu\overline{R_\nu[\alpha,z,g\boldsymbol{i}]})\left\{\int_{\Gamma_\alpha\backslash H_\alpha}[\alpha,hg\boldsymbol{i}]^k f_\alpha(hg\boldsymbol{i})dh\right\}dg$$

$$=2^rP(f,\alpha,\Gamma)\int_{H_\alpha\backslash G}\boldsymbol{e}(-\textstyle\sum_{\nu=1}^n q_\nu\overline{R_\nu[\alpha,z,g\boldsymbol{i}]})dg.$$

By Lemma 2.1, we may assume that $\alpha\in V^*$. The last integral over $H_\alpha\backslash G$ is equal to

$$\begin{aligned}&\boldsymbol{e}(-\textstyle\sum_{\nu\leq r}q_\nu N(\alpha_\nu)\bar{z}_\nu-\sum_{\nu>r}q_\nu N(\alpha_\nu)z_\nu)\\ &\times\int_{H_\alpha\backslash G}\exp(-\pi\textstyle\sum_{\nu=1}^r q_\nu y_\nu\,\mathrm{Im}(g_\nu i)^{-2}|[\alpha_\nu,g_\nu i]|^2)dg.\end{aligned} \tag{2.17}$$

Take $\beta\in G$ so that $(\beta\alpha\beta^{-1})_\nu$ is diagonal for every $\nu\leq r$, and put $H=\beta H_\alpha\beta^{-1}$. Then $H=A'^r$. Changing g for $\beta^{-1}g$, we find that the last integral over $H_\alpha\backslash G$ is equal to

$$\int_{H\backslash G}\exp(-4\pi\textstyle\sum_{\nu=1}^r q_\nu|N(\alpha_\nu)|\,y_\nu\,\mathrm{Im}(g_\nu i)^{-2}|g_\nu i|^2)dg. \tag{2.17$'$}$$

Observe that if ϕ is an H-invariant function on $\mathfrak{H}^r$, then

$$2^r\int_{H\backslash G}\phi(g\boldsymbol{i})dg=\int_{R^r}\phi(\boldsymbol{i}+u)du\,.$$

Applying this to (2.17′), we find that (2.17) is equal to

$$2^{-2r}|qN(\alpha)y|^{-\xi/2}\boldsymbol{e}_F(-qN(\alpha)z)\,.$$

Thus (2.16) is equal to

$$2^{-r}q^{-\xi/2}\sum_{\alpha\in R(\Gamma)}\eta(\alpha)|N(\alpha)|^{-\xi/2}P(f,\,\alpha,\,\Gamma)\boldsymbol{e}_F(-qN(\alpha)z)\,,$$

and hence we obtain our proposition.

Now (2.14) shows that (2.16) as a function of z behaves like a Hilbert modular form of weight h under Δ. It remains to show that the transform of (2.16) under every element of $SL_2(F)$ has a Fourier expansion with constant term 0. Take an arbitrary $\alpha=\begin{pmatrix} * & * \\ c & d\end{pmatrix}\in SL_2(F)$. By [13, Prop. 7.1], $(c\bar{z}+d)^{-h}\theta(\alpha z,\,w)$ is a finite linear combination of functions of the form $\theta(bz,\,w,\,t\,;\,\eta',\,q)$ with $0\ll b\in F$ and locally constant functions η' on V. As can be seen from [13, p. 340], this expression is independent of t. Therefore (2.16) transformed by α is a finite linear combination of integrals of the above type, and hence has a Fourier expansion with constant term 0. This completes the proof of Theorem 2.2.

Let us now justify the termwise integration in the above proof and prove the convergence of (2.5) and (2.9). We consider here only the case $B=M_2(F)$, since the case of division algebra can be treated in a similar and easier way. Write the variable w on $\mathfrak{H}^n$ as $w=u+iv$, and recall that D $(=\Gamma\backslash\mathfrak{H}^n)$ is contained in the union of a compact subset K of $\mathfrak{H}^n$ and $\bigcup_{\beta\in T}\beta(W)$, where T is a finite subset of $SL_2(F)$, and

$$W=\{u+iv\in\mathfrak{H}^n\,\big|\,|u_\nu|\leqq a,\ v_\nu>1\quad\text{for all }\nu\}$$

with a positive constant a. Observe that w_ν/v_ν belongs to a compact subset of $\mathfrak{H}$ if $w\in W$. Fix one $\beta\in T$. Then we see that

$$\mathrm{Im}(R_\nu[\alpha,\,z,\,\beta w])=\mathrm{Im}(R_\nu[\beta^{-1}\alpha\beta,\,z,\,w])\geqq v_\nu^{-2}y_\nu P_\nu[\alpha]\qquad\text{for all}\quad w\in W$$

with a positive definite quadratic form P_ν independent of w. Similarly

$$|\eta(\alpha)[\alpha,\,\beta w]^k|=|\eta(\alpha)[\beta^{-1}\alpha\beta,\,w]^kJ(\beta,\,w)^{-2k}|\leqq v^{2k}g(\alpha)|J(\beta,\,w)|^{-2k}$$

for all $w\in W$ with a polynomial function g on V. Thus $\theta(z,\,\beta w,\,t)$ has a majorant

$$y^{F/2}\sum_{\alpha\in L}g(\alpha)\exp(-2\pi\sum_{\nu=1}^{n}q_\nu y_\nu v_\nu^{-2}P_\nu[\alpha])|J(\beta,\,w)|^{2k}$$

on W, where L is a lattice of V. Multiply each term by $(f(w)v^{2k})\circ\beta$ and integrate over W. Since $|(f\|_{2k}\beta)(w)|\leqq M\cdot\exp(-\lambda v^{F/n})$ on W with positive constants M and λ, each integral is majorized by

$$(2.18) \qquad My^{F/2}g(\alpha)\int_{W}\exp(-\lambda v^{F/n}-2\pi\sum_{\nu=1}^{n}q_{\nu}y_{\nu}v_{\nu}^{-2}P_{\nu}[\alpha])v^{2k-2F}du\,dv\,.$$

We now need an elementary

LEMMA 2.4. *Let $0<a\in\boldsymbol{R}$, $0<b\in\boldsymbol{R}$, and $r\in\boldsymbol{R}$. Then for every $q\in\boldsymbol{R}$, >0, there exists a constant C depending only on a, q, r such that*

$$\int_{1}^{\infty}\exp(-ax^{1/n}-bx^{-2})x^{r}dx\leqq Cb^{-q}\,.$$

PROOF. Put $p=r+2q+(1/2)$. Then the Schwarz inequality shows that the square of the integral in question is majorized by

$$\int_{1}^{\infty}\exp(-2ax^{1/n})x^{2p}dx\int_{1}^{\infty}\exp(-2bx^{-2})x^{-4q-1}dx\,.$$

Putting $x^{-2}=t$, we see that the last integral is smaller than

$$(1/2)\int_{0}^{\infty}e^{-2bt}t^{2q-1}dt=(1/2)\Gamma(2q)(2b)^{-2q}\,,$$

which proves our assertion.

This lemma shows that the integrals of (2.18) as well as their sum over all $\alpha\in L$ are convergent. The same can be shown for the integrals over K in a similar and simpler way. This justifies the termwise integration in the proof of Proposition 2.3, and at the same time proves the convergence of (2.9), provided $P(f,\alpha,\Gamma)$ is convergent. As for $P(f,\alpha,\Gamma)$, if $F[\alpha]$ is a field or $N(\alpha)=0$, we see easily that $\Gamma_{\alpha}\backslash H_{\alpha}$ is compact, so that there is no problem. If $N(\alpha)\neq 0$ and $F[\alpha]$ is not a field, then $N(\alpha)\ll 0$ and $F[\alpha]$ is isomorphic to $F\oplus F$. Before treating this case, let us first express $P(f,\alpha,\Gamma)$ for each $\alpha\in V^{*}$ as an integral of a holomorphic r-form over a cycle.

Given $\alpha\in V^{*}$, take $\beta\in SL_{2}(\boldsymbol{R})^{r}$ so that $(\beta\alpha\beta^{-1})_{\nu}=\begin{pmatrix}c_{\nu} & 0\\ 0 & -c_{\nu}\end{pmatrix}$ for $\nu=1,\cdots,r$ with $0<c_{\nu}\in\boldsymbol{R}$. Then $\beta H_{\alpha}\beta^{-1}=A^{r}$. Now $F[\alpha]$ is either a quadratic extension of F or $F\oplus F$. If $F[\alpha]$ is a field, it has exactly $2r$ real archimedean primes, and hence its unit group has rank $n+r-1$; therefore $\{\pm 1\}\Gamma_{\alpha}/\{\pm 1\}$ is isomorphic to $\boldsymbol{Z}^{r}$. Define an isomorphism ω of $(\boldsymbol{R}^{\times})^{r}$ onto H_{α} by

$$(2.19) \qquad \omega(s)=\left(\beta_{1}^{-1}\begin{pmatrix}s_{1} & 0\\ 0 & s_{1}^{-1}\end{pmatrix}\beta_{1},\ \cdots,\ \beta_{r}^{-1}\begin{pmatrix}s_{r} & 0\\ 0 & s_{r}^{-1}\end{pmatrix}\beta_{r}\right)$$

for $s=(s_{1},\cdots,s_{r})\in(\boldsymbol{R}^{\times})^{r}$, and put $v(s)=\omega(s)v^{0}$ with a point v^{0} of $\mathfrak{H}^{r}$. Let X be a fundamental domain of $(\boldsymbol{R}^{\times})^{r}/\omega^{-1}(\Gamma_{\alpha})$. Then, with $f_{\alpha}={}^{t}\overline{u(\alpha)}f$, we have

$$(2.20) \qquad P(f,\alpha,\Gamma)=\int_{\Gamma_{\alpha}\backslash H_{\alpha}}[\alpha,\,hv^{0}]^{k}f_{\alpha}(hv^{0})dh$$

$$=\int_X [\alpha, v(s)]^k f_\alpha(v(s))|s_1 \cdots s_r|^{-1} ds_1 \cdots ds_r .$$

We have $\beta_\nu v(s)_\nu = s_\nu^2 (\beta v^0)_\nu$, so that

$$J(\beta, v(s))_\nu^{-2} dv(s)_\nu = 2 s_\nu (\beta v^0)_\nu ds_\nu . \tag{2.21}$$

On the other hand, by (1.20),

$$[\alpha, v(s)]_\nu = J(\beta, v(s))_\nu^2 [\beta\alpha\beta^{-1}, \beta v(s)]_\nu = -2c_\nu s_\nu^2 (\beta v^0)_\nu J(\beta, v(s))_\nu^2 .$$

This combined with (2.21) yields

$$[\alpha, v(s)]_\nu s_\nu^{-1} ds_\nu = -c_\nu dv(s)_\nu , \tag{2.22}$$

and hence

$$P(f, \alpha, \Gamma) = (-1)^r |N(\alpha)|^{\xi/2} \int_{v(X)} [\alpha, w]^{k-\xi} \cdot {}^t\overline{u(\alpha)} f(w) dw_1 \wedge \cdots \wedge dw_r \tag{2.23}$$

with a suitable orientation of $v(X)$.

Next let us assume that $F[\alpha] = F \oplus F$. This happens only when $B = M_2(F)$. Then we may assume that $\beta \in SL_2(F)$ and $\beta\alpha\beta^{-1} = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}$ with $c \in F$. Observe that

$$\beta\Gamma_\alpha\beta^{-1} = \left\{ \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \middle| s \in U \right\}$$

with a subgroup U of $\mathfrak{r}^\times$ of finite index. Defining again $\omega(s)$, $v(s)$, and X in the same fashion as above, we see that (2.23) holds also in the present case. Now the convergence of $P(f, \alpha, \Gamma)$ can be shown by decomposing X into two parts corresponding to $|s_1 \cdots s_n| \geqq 1$ and $|s_1 \cdots s_n| \leqq 1$. The integral over the first part is convergent, since f is rapidly decreasing at the cusp $i\infty$. The convergence of the other part can be seen by the transformation $w \mapsto \varepsilon(w)$ with $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

We now state special cases of some results of [15] concerning Θ with later applications in view.

PROPOSITION 2.5. *Let Δ be a congruence subgroup of $SL_2(F)$ for which* (2.14) *holds. For $g \in \mathcal{M}_h(\Delta)$, put*

$$f(w) = \int_\Phi g(z) \Theta(z, w; \eta, q) \operatorname{Im}(z)^h d\mu(z) \qquad (\Phi = \Delta \backslash \mathfrak{H}^n) .$$

Then $f \in \mathcal{M}_{2k, 2\lambda}(\Gamma)$ with a congruence subgroup Γ of $B^\times$.

PROOF. The function $\theta(-\bar{z}, w, t; \eta, q)$ is a finite linear combination of functions of [15, I, (6.6)]. Therefore, changing z for $-\bar{z}$, we obtain our assertion immediately from [15, I, Theorem 6.2].

Let us now assume

(2.24) *F has a subfield E such that $[E:\boldsymbol{Q}]=r$ and the restrictions of $\tau_1, \cdots, \tau_r$ to E are all different.*

Let $\tau_1', \cdots, \tau_r'$ be the restrictions of $\tau_1, \cdots, \tau_r$ to E. Define a function $\Theta^*(z, w)$ on $\mathfrak{H}^r \times \mathfrak{H}^r$ by

$$\Theta^*(z, w)=y^{\sigma}\Theta(z^*, w; \eta, q),$$

$$z_\nu^*=\begin{cases} q^{-\tau_\nu}z_\nu & \text{if } \nu \leqq r, \\ q^{-\tau_\nu}\bar{z}_\lambda & \text{if } \nu>r \text{ and } \tau_\nu=\tau_\lambda' \text{ on } E, \end{cases}$$

where σ is an element of $(1/2)I_E$ defined by

$$\sigma=\mathrm{Res}_{F/E}(\lambda)+(3/2)([F:E]-1)\textstyle\sum_{\nu=1}^r \tau_\nu'.$$

Define also an element μ of $(1/2)I_E$ by

$$\mu=\textstyle\sum_{\nu=1}^r (k_\nu+2)\tau_\nu'-\mathrm{Res}_{F/E}(\lambda)-(3/2)[F:E]\sum_{\nu=1}^r \tau_\nu'.$$

It can easily be seen that

$$\Theta^*(\gamma z, w)=\overline{j(\gamma, z)}(c\bar{z}+d)^{\mu-(E/2)}\Theta^*(z, w) \quad \text{for every } \gamma=\begin{pmatrix} * & * \\ c & d \end{pmatrix}\in \Delta'$$

with a congruence subgroup Δ' of $SL_2(E)$.

PROPOSITION 2.6. *For $g\in \mathcal{M}_\mu(\Delta')$ with such a Δ', put*

$$f(w)=\int_{\Phi'} g(z)\Theta^*(z, w)\,\mathrm{Im}(z)^\mu d\mu(z) \qquad (\Phi'=\Delta'\backslash\mathfrak{H}^r).$$

Then $f\in \mathcal{M}_{2k,2\lambda}(\Gamma)$ with a congruence subgroup Γ of $B^\times$. Moreover, if g has algebraic Fourier coefficients, then f is arithmetic in the sense of [15, II, § 2].

PROOF. Our assertions follow immediately from [15, I, Theorems 6.3 and 6.4]. It should be noted that the $\overline{\boldsymbol{Q}}$-rationality of f defined on [13, p. 329] is consistent with that of [15, II, § 2] as can easily be shown.

We conclude this section by giving transformation formula (2.14) in a more explicit form, which will be needed in the next section. The symmetric form S being as in (1.14), choose a basis $\{\beta_1, \beta_2, \beta_3\}$ of V over F so that $qS(\beta_i, \beta_j)\in \mathfrak{r}$ for all i and j with an element q of F as in Theorem 2.2; put $L=\sum_{i=1}^3 \mathfrak{r}\beta_i$ and use the same letter S to denote the matrix $(S(\beta_i, \beta_j))$ of size 3. For $u, v\in V$, put

(2.25) $f(z, w; u, v)$

$$=y^{\xi/2}\,\mathrm{Im}(w)^{-2k}\textstyle\sum_{\alpha-u\in L} \boldsymbol{e}_F(qS(\alpha, v))t(\alpha)[\alpha, \bar{w}]^k \boldsymbol{e}(\sum_{\nu=1}^n q_\nu R_\nu[\alpha, z, w]).$$

This is a special case of (2.11) and of [13, (7.6)]. Let $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\mathfrak{r})$; suppose $bq\{S\}\equiv 0\ (2\mathfrak{d}^{-1})$, $cq^{-1}\{S^{-1}\}\equiv 0\ (2\mathfrak{d})$, $cq^{-1}S^{-1}\equiv 0\ (\mathfrak{d})$, and $0\ll d\equiv 1\ (4\mathfrak{d})$, where $\{S\}$ denotes the vector whose components are the diagonal elements of S. Then

$$f(\gamma z,\ w\,;u,\ v)=\left(\frac{2c\cdot\det(qS)}{d\mathfrak{r}}\right)(c\bar{z}+d)^h e(X)f(z,\ w\,;u',\ v') \tag{2.26}$$

with $u'=au+cv$, $v'=bu+dv$, and $X=(1/2)\mathrm{Tr}_{F/Q}(qS(u,\ v)-qS(u',\ v'))$. This follows from [13, Lemma 7.2].

3. The commutativity with Hecke operators.

Let us now assume $F=\boldsymbol{Q}$. We are going to specialize Theorem 2.2 to this case and then study the behavior of (2.9) under Hecke operators. For each prime p, let $\boldsymbol{Z}_p$ and $\boldsymbol{Q}_p$ be the p-completions of $\boldsymbol{Z}$ and $\boldsymbol{Q}$. We fix a maximal order $\mathfrak{o}$ in B, and put $B_p=B\otimes_{\boldsymbol{Q}}\boldsymbol{Q}_p$, $V_p=V\otimes_{\boldsymbol{Q}}\boldsymbol{Q}_p$. For every lattice $\mathfrak{a}$ in B, we denote by $\mathfrak{a}_p$ its p-closure in B_p. Let e be the discriminant of B, that is, the product of all primes p ramified in B. If $p\nmid e$, we can find an isomorphism μ_p of B_p onto $M_2(\boldsymbol{Q}_p)$ such that $\mu_p(\mathfrak{o}_p)=M_2(\boldsymbol{Z}_p)$. Then

$$\mu_p(V_p)=\left\{\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\in M_2(\boldsymbol{Q}_p)\right\}.$$

Let us fix a positive integer m prime to e and define an "order of level m" to be the lattice $\mathfrak{o}'$ contained in $\mathfrak{o}$ such that

$$\mathfrak{o}'_p=\begin{cases} \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in M_2(\boldsymbol{Z}_p)\,\middle|\, c\in m\boldsymbol{Z}_p\right\} & \text{if } p\nmid e, \\ \mathfrak{o}_p & \text{if } p\,|\,e. \end{cases} \tag{3.1}$$

We put then

$$\Gamma'_m=\{\gamma\in\mathfrak{o}'\,|\,N(\gamma)=1\}, \tag{3.2}$$

$$\Gamma_m=\left\{\gamma\in\Gamma'_m\,\middle|\,\mu_p(\gamma)\equiv\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}(\mathrm{mod}\ m\mathfrak{o}_p)\ \text{for all}\ p\,|\,m\right\}. \tag{3.3}$$

These groups will be denoted simply by Γ' and Γ when m is fixed. To define Hecke operators, let Y_m, or simply Y, be the set of all α in $B_+^\times\cap\mathfrak{o}'$ such that $\mu_q(\alpha)\equiv\begin{pmatrix} a & * \\ * & * \end{pmatrix}(\mathrm{mod}\ m\mathfrak{o}_q)$ for every prime factor q of m with an integer a prime to m. Given a Dirichlet character φ modulo m, we define a map $\varphi_Y: Y\to\boldsymbol{C}^\times$ by $\varphi_Y(\alpha)=\varphi(a)^{-1}$ for such α and a. Then, for a positive integer h, we denote by $S_h(\Gamma',\ \varphi)$ the vector space of all cusp forms g on $\mathfrak{H}$ such that $g\|_h\gamma=\varphi_Y(\gamma)g$ for all $\gamma\in\Gamma'$. Now, for each prime number p, there is an element β of Y such that

$$\Gamma'\beta\Gamma' = \{\xi \in Y \mid N(\xi) = p\}.$$

Let $\Gamma'\beta\Gamma' = \cup_\lambda \Gamma'\beta_\lambda$ be a disjoint coset decomposition. Then a Hecke operator $T(p)$ acting on $\mathcal{S}_h(\Gamma', \varphi)$ is defined by

$$f \mid T(p) = p^{(h/2)-1} \sum_\lambda \varphi_Y(\beta_\lambda)^{-1} f \|_h \beta_\lambda \qquad (f \in \mathcal{S}_h(\Gamma', \varphi)). \tag{3.4}$$

This is meaningful for all primes p including those dividing me. (For details, see [15, II, § 1]. The present definition is different from that of [15] by a factor $p^{(h/2)-1}$.) If $B = M_2(\boldsymbol{Q})$, Γ'_m and Γ_m coincide with the groups

$$\Gamma_0(m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\boldsymbol{Z}) \,\middle|\, c \equiv 0 \pmod{m} \right\},$$

$$\Gamma_1(m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(m) \,\middle|\, a \equiv d \equiv 1 \pmod{m} \right\}.$$

In this case, we write $\mathcal{S}(m, \varphi, h)$ for $\mathcal{S}_h(\Gamma'_m, \varphi)$. Then (3.4) coincides with the classical $T(p)$ defined by Hecke.

Let N be a positive integer divisible by 4, χ a character modulo N such that $\chi(-1) = 1$, and κ an odd positive integer. We then denote by $\mathcal{S}(N, \chi, \kappa/2)$ the space of cusp forms f satisfying

$$f(\gamma z) = \chi(d) j(\gamma, z)^\kappa f(z) \qquad \text{for every} \quad \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_0(N),$$

where j is defined by (1.12).

To state a special case of Theorem 2.2, we consider θ of (2.11) with $q=1$ and $t=1$. Thus it has a simpler form

$$\theta(z, w; \eta) = y^{1/2} v^{-2k} \sum_{\alpha \in V} \overline{\eta(\alpha)} [\alpha, \bar{w}]^k e(R[\alpha, z, w]) \tag{3.5}$$
$$(z \in \mathfrak{H}, w \in \mathfrak{H}, y = \mathrm{Im}(z), v = \mathrm{Im}(w)).$$

We now take an explicitly defined η as follows. Let $\mathfrak{o}^*$ be the maximal order in B such that $\mathfrak{o}_p^* = \mathfrak{o}_p$ for all p prime to m and

$$\mathfrak{o}_p^* = \left\{ \alpha \in B_p \,\middle|\, \mu_p(\alpha) = \begin{pmatrix} a & b/m \\ cm & d \end{pmatrix} \text{ with } a, b, c, d \in \boldsymbol{Z}_p \right\}$$

for every $p \mid m$. Given $\alpha \in \mathfrak{o}^*$, we can find an integer b such that

$$\mu_p(\alpha) \equiv \begin{pmatrix} * & b/m \\ * & * \end{pmatrix} \pmod{\mathfrak{o}'_p} \qquad \text{for every} \quad p \mid m.$$

For a Dirichlet character ψ modulo m, we then define a locally constant function η_ψ on V by $\eta_\psi(\alpha) = \psi(b)$ for such α contained in $\mathfrak{o}^* \cap V$ and $\eta_\psi(\alpha) = 0$ for $\alpha \notin \mathfrak{o}^* \cap V$. (We put $\psi(b) = 0$ for $(b, m) \neq 1$.) For simplicity, we fix ψ throughout the rest of this section and write simply η for η_ψ.

THEOREM 3.1. *Let m be a positive integer prime to the discriminant e of B, and ψ a character modulo m. For $f\in\mathcal{S}_{2k}(\Gamma'_m, \psi^2)$ with a positive integer k, put*

$$L(z, f)=(2\mu(\Gamma\backslash\mathfrak{H}))^{-1}\textstyle\sum_{\alpha\in R(\Gamma)}\eta(\alpha)|N(\alpha)|^{-1/2}P(f, \alpha, \Gamma)e(-N(\alpha)z) \tag{3.6}$$

with $\Gamma=\Gamma_m$ and with the above η, where $P(f, \alpha, \Gamma)$ and $R(\Gamma)$ are defined as in (2.5) and Theorem 2.2. Suppose $\psi(-1)=(-1)^k$. Then $L(z, f)$ as a function of z belongs to $\mathcal{S}(N, \chi, (2k+1)/2)$, where N is $2me$ or $4me$ according as me is even or odd, and $\chi(d)=\left(\frac{-1}{d}\right)^k\psi(d)$.

PROOF. By Proposition 2.3, we have

$$\begin{aligned} L(z, f)&=\mu(D)^{-1}\int_D\overline{\theta(z, w; \eta)}f(w)\,\mathrm{Im}(w)^{2k}d\mu(w) \\ &=\langle\theta(z, w; \eta), f\rangle \qquad (D=\Gamma\backslash\mathfrak{H}). \end{aligned} \tag{3.7}$$

Let $L=\mathfrak{o}'\cap V$. Consider the matrix S relative to this L as defined at the end of § 2. Observe that $\det(S)=2m^2e^2$, $\{S\}\equiv N\{S^{-1}\}\equiv 0 \pmod 2$, and $(N/2)S^{-1}$ is integral. Therefore, from (2.26), we obtain $\theta(\gamma z, w; \eta)=\overline{\chi(d)}\,\overline{j(\gamma, z)}^{2k+1}\theta(z, w; \eta)$ for every $\gamma=\begin{pmatrix} * & * \\ c & d\end{pmatrix}\in\Gamma_0(N)$ when $0<d\equiv 1 \pmod 4$. Since $\Gamma_0(N)$ is generated by such γ and -1, the formula is true for all elements of $\Gamma_0(N)$. On the other hand, we can easily verify that $\eta(\delta\alpha\delta^{-1})=\psi_Y(\delta)^{-2}\eta(\alpha)$ for every $\delta\in\Gamma'$, and hence, by (2.13), we have

$$\theta(z, \delta w, \eta)=\psi_Y(\delta)^2 J(\delta, w)^{2k}\theta(z, w; \eta) \qquad \text{for every} \quad \delta\in\Gamma'. \tag{3.8}$$

Therefore we obtain our assertion from Theorem 2.2 and Proposition 2.3.

REMARK. Series $L(z, f)$ and θ can be defined for an arbitrary character ψ modulo m. It can easily be seen, however, that they are equal to 0 if $\psi(-1)\neq(-1)^k$.

For each prime number p, a Hecke operator $T(p^2)$ acting on $\mathcal{S}(N, \chi, (2k+1)/2)$ was defined in [8]. If $g(z)=\sum_{n=1}^\infty a(n)e(nz)\in\mathcal{S}(N, \chi, (2k+1)/2)$, we have $g|T(p^2)=\sum_{n=1}^\infty b(n)e(nz)$ with

$$b(n)=a(p^2n)+\chi'(p)\left(\frac{n}{p}\right)p^{k-1}a(n)+\chi(p^2)p^{2k-1}a(p^{-2}n), \tag{3.9}$$

where $\chi'(n)=\left(\frac{-1}{n}\right)^k\chi(n)$; we understand that $a(p^{-2}n)=0$ if $n\notin p^2\boldsymbol{Z}$ and $\chi'(n)=\chi(n)=0$ if n is not prime to N (see [8, Theorem 1.7]).

THEOREM 3.2. *For every odd prime p and for every $f\in\mathcal{S}_{2k}(\Gamma'_m, \psi^2)$, we have*

$$L(z, f)|T(p^2)=L(z, f|T(p)).$$

The same relation holds also for $p=2$ if me is even.

PROOF. Define a locally constant function η' on V by

$$\eta'(\alpha)=\chi'(p)p^{k-1}\left(\frac{-N(\alpha)}{p}\right)\eta(\alpha)+p^{k}\chi(p^{2})\eta(\alpha/p)+p^{k-1}\eta(p\alpha)$$

if $N(\alpha)\in \boldsymbol{Z}$ and $\eta'(\alpha)=0$ if $N(\alpha)\notin \boldsymbol{Z}$, where $\chi'(n)=\psi(n)$ if n is prime to $2me$ and $\chi'(n)=0$ otherwise. In view of (3.9) and (2.8), we see easily that $L(z, f)|T(p^{2})$ is given by the right-hand side of (3.6) with η replaced by η'. Let us first assume p is prime to $2me$. Then we have

$$L(z, f|T(p))=\langle\theta(z, w;\eta), f|T(p)\rangle=\psi(p)^{2}\langle\theta(z, w;\eta)|T(p), f\rangle$$

(cf. [15, II, Lemma 1.3]). Taking $\{\beta_{\lambda}|0\leqq\lambda\leqq p\}$ as in (3.4), we have, by (2.13),

$$\theta(z, w;\eta)|T(p)=\sum_{\lambda,\alpha}\psi_{Y}(\beta_{\lambda})^{-2}p^{2k-1}y^{1/2}v^{-2k}\overline{\eta(\beta_{\lambda}\alpha\beta_{\lambda}^{\iota})}[\alpha, \bar{w}]^{k}e(p^{2}R[\alpha, z, w]).$$

Substituting $p^{-1}\alpha$ for α, we find that $\theta(z, w;\eta)|T(p)=\theta(z, w;\eta^{*})$ with

$$\eta^{*}(\alpha)=p^{k-1}\sum_{\lambda=0}^{p}\psi_{Y}(\beta_{\lambda})^{2}\eta(\beta_{\lambda}\alpha\beta_{\lambda}^{-1}).$$

Observe that $\eta'(\gamma\alpha\gamma^{-1})=\psi_{Y}(\gamma)^{-2}\eta'(\alpha)$ and $\eta^{*}(\gamma\alpha\gamma^{-1})=\psi_{Y}(\gamma)^{-2}\eta^{*}(\alpha)$ for $\gamma\in\Gamma'$, and hence $L(z, f)|T(p^{2})$ and $L(z, f|T(p))$ are given by the right-hand side of (3.6) with η replaced by η' and $\psi(p)^{2}\eta^{*}$, respectively. We shall therefore prove the desired equality by showing

$$\eta'(\alpha)=\psi(p)^{2}\eta^{*}(\alpha) \tag{3.10}$$

for all $\alpha\in V$ with $N(\alpha)<0$. For this purpose, we first observe that $\eta'(\alpha)\neq 0$ or $\eta^{*}(\alpha)\neq 0$ only if $N(\alpha)\in\boldsymbol{Z}$ and $p\alpha\in\mathfrak{o}^{*}$.

LEMMA 3.3. *Let p be a prime not dividing $2me$, and let*

$$W=\{\beta\in V\cap p^{-1}\mathfrak{o}^{*}|0>N(\beta)\in\boldsymbol{Z}\}.$$

Then for every $\beta\in W$, there exists an element γ of Γ such that $\alpha=\gamma\beta\gamma^{-1}$ belongs to the following three types:

(1) $\mu_{p}(\alpha)=\begin{pmatrix} a & pb \\ c/p & -a\end{pmatrix}$ *with* $a, b, c\in\boldsymbol{Z}_{p}$, $c\notin p\boldsymbol{Z}_{p}$;

(2) $\mu_{p}(\alpha)=\begin{pmatrix} a & b \\ c & -a\end{pmatrix}$ *with* $a, b, c\in\boldsymbol{Z}_{p}$, $c\notin p\boldsymbol{Z}_{p}$;

(3) $\alpha\in p\mathfrak{o}^{*}$.

PROOF. We first note a simple fact: for every $x\in\boldsymbol{Z}_{p}$, there exist elements δ and ε of Γ such that

$$\mu_{p}(\delta)\equiv\begin{pmatrix} 1 & 0 \\ x & 1\end{pmatrix},\qquad \mu_{p}(\varepsilon)\equiv\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\qquad (\mathrm{mod}\ p\mathfrak{o}_{p}). \tag{3.11}$$

This follows from the strong approximation theorem due to Eichler [1, Satz 5]. Now, given $\alpha\in W$, suppose $\alpha\notin\mathfrak{o}_p$. Put $N(\alpha)=-n$ and $p\mu_p(\alpha)=\begin{pmatrix} a & b \\ c & -a\end{pmatrix}$. Then $a^2+bc=p^2n$. Suppose $p\mid b$. Then $p\mid a$ but $p\nmid c$ since $\alpha\notin\mathfrak{o}_p$, so that $p^2\mid b$. Thus α has form (1). If $p\nmid b$, choose $x\in\boldsymbol{Z}$ so that $a\equiv bx \pmod{p\boldsymbol{Z}_p}$ and take $\delta\in\Gamma$ as in (3.11). Then

$$p\mu_p(\delta\alpha\delta^{-1})\equiv\begin{pmatrix} a-bx & b \\ * & *\end{pmatrix} \pmod{p\mathfrak{o}_p}.$$

Replacing α by $\delta\alpha\delta^{-1}$, we may assume that $p\mid a$. Then $p^2\mid c$. Take ε as in (3.11). Then we find that $\varepsilon\alpha\varepsilon^{-1}$ has form (1). Next suppose $\alpha\in\mathfrak{o}_p$, $\notin p\mathfrak{o}_p$. Put $\mu_p(\alpha)=\begin{pmatrix} a & b \\ c & -a\end{pmatrix}$. Take δ as in (3.11). Then $\mu_p(\delta\alpha\delta^{-1})=\begin{pmatrix} * & * \\ c' & *\end{pmatrix}$ with $c'\equiv c+2ax-bx^2 \pmod{p\boldsymbol{Z}_p}$. Since $\alpha\notin p\mathfrak{o}_p$ and $p>2$, we can choose x so that $c'\notin p\boldsymbol{Z}_p$. Then $\delta\alpha\delta^{-1}$ has form (2). Finally if $\alpha\in p^{-1}\mathfrak{o}^*$ and $\alpha\in p\mathfrak{o}_p$, then $\alpha\in p\mathfrak{o}^*$, which completes the proof.

We are going to prove (3.10) for the elements α of types (1), (2), (3) of the above lemma. First let α be of type (1). Then $\eta'(\alpha)=p^{k-1}\eta(p\alpha)=p^{k-1}\psi(p)\eta(\alpha)$. We can take $\{\beta_\lambda\}$ of (3.4) so that

$$\mu_p(\beta_\lambda)\equiv\begin{cases}\begin{pmatrix} 1 & \lambda \\ 0 & p\end{pmatrix} & (0\leqq\lambda<p) \\ \begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix} & (\lambda=p)\end{cases} \pmod{p^2\mathfrak{o}'_p} \tag{3.12}$$

Then we find that $\beta_\lambda\alpha\beta_\lambda^{-1}\in\mathfrak{o}^*$ if and only if $\lambda=0$. Therefore $\eta^*(\alpha)=p^{k-1}\psi_Y(\beta_0)^2\cdot\eta(\beta_0\alpha\beta_0^{-1})$. For a prime factor q of m, let $\mu_q(\alpha)=\begin{pmatrix} a & b/m \\ cm & -a\end{pmatrix}$ and $\mu_q(\beta_0)=\begin{pmatrix} r & s \\ tm & u\end{pmatrix}$. Then

$$\mu_q(\beta_0\alpha\beta_0^{-1})\equiv\begin{pmatrix} * & r^2b/(pm) \\ * & *\end{pmatrix} \pmod{\mathfrak{o}'_q}, \tag{3.13}$$

and hence $\eta(\beta_0\alpha\beta_0^{-1})=\psi(p)^{-1}\psi_Y(\beta_0)^{-2}\eta(\alpha)$. This shows that $\eta^*(\alpha)=p^{k-1}\psi(p)^{-1}\eta(\alpha)$. Next suppose α is of type (2); let $n=-N(\alpha)$. Then

$$\eta'(\alpha)=\psi(p)p^{k-1}\left(\frac{n}{p}\right)\eta(\alpha)+p^{k-1}\psi(p)\eta(\alpha).$$

Put $\mu_p(\alpha)=\begin{pmatrix} a & b \\ c & -a\end{pmatrix}$. Then $\beta_\lambda\alpha\beta_\lambda^{-1}\in\mathfrak{o}^*$ if and only if $b-2a\lambda-c\lambda^2\in p\boldsymbol{Z}_p$, $\lambda\neq p$. The number of such λ is $\left(\frac{n}{p}\right)+1$. We have again $\eta(\beta_\lambda\alpha\beta_\lambda^{-1})=\psi(p)^{-1}\psi_Y(\beta_\lambda)^{-2}\eta(\alpha)$, so that $\eta^*(\alpha)=p^{k-1}\left(1+\left(\frac{n}{p}\right)\right)\psi(p)^{-1}\eta(\alpha)$, which proves (3.10). Finally suppose $\alpha\in p\mathfrak{o}^*$. Then $p^2|N(\alpha)$, and

$$\eta'(\alpha)=p^k\psi(p^2)\eta(\alpha/p)+p^{k-1}\eta(p\alpha)=(p+1)p^{k-1}\psi(p)\eta(\alpha).$$

In this case $\beta_\lambda\alpha\beta_\lambda^{-1}\in\mathfrak{o}^*$ for all λ, so that $\eta^*(\alpha)=p^{k-1}(p+1)\psi(p)^{-1}\eta(\alpha)$. Thus (3.10) is true for all $\alpha\in W$. This proves the equality of Theorem 3.2 for p not dividing $2me$.

Suppose $p|m$. Then $\eta'(\alpha)=p^{k-1}\eta(p\alpha)$ if $N(\alpha)\in \boldsymbol{Z}$. Let δ be an element of Y such that $N(\delta)=p$ and $\mu_q(\delta)\equiv\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}$ (mod $p^2m\mathfrak{o}_q$) for all prime factors q of m. Then we have $f|T(p)=p^{k-1}\sum_{\lambda=0}^{p-1}f\|\delta_\lambda$ with $\{\delta_\lambda\}$ such that $\Gamma\delta\Gamma=\bigcup_{j=0}^{p-1}\Gamma\delta_\lambda$. Let $\beta=\delta^\iota$ and $\Gamma\beta\Gamma=\bigcup_{\lambda=0}^{p-1}\Gamma\beta_\lambda$. Then we have

$$\begin{aligned}L(z, f|T(p))&=\langle\theta(z, w;\eta), f|T(p)\rangle\\ &=p^{k-1}\textstyle\sum_{\lambda=0}^{p-1}\langle\theta(z, w;\eta)\|\beta_\lambda, f\rangle\end{aligned}$$

by [7, Prop. 3.39], and hence, by (2.13),

$$L(z, f|T(p))=\langle\theta(z, w;\eta^*), f\rangle$$

with $\eta^*(\alpha)=p^{k-1}\sum_{\lambda=0}^{p-1}\eta(\beta_\lambda\alpha\beta_\lambda^{-1})$. Now we can take β_λ so that $\mu_q(\beta_\lambda)\equiv\begin{pmatrix}p & 0\\ \lambda m & 1\end{pmatrix}$ (mod $p^2m\mathfrak{o}_q'$) for all prime factors q of m. For the same reason as in the case $p\nmid m$, it is sufficient to prove that $\eta^*(\alpha)=\eta'(\alpha)$ for α in a suitably chosen $R(\Gamma)$. We may naturally assume that $N(\alpha)\in\boldsymbol{Z}$. First suppose $\alpha\in\mathfrak{o}^*$ and put $\mu_p(\alpha)=\begin{pmatrix}* & r/m\\ * & *\end{pmatrix}$. Then $\eta'(\alpha)=0$ and $\mu_p(\beta_\lambda\alpha\beta_\lambda^{-1})\equiv\begin{pmatrix}* & pr/m\\ * & *\end{pmatrix}$ (mod $p\mathfrak{o}_p$), so that $\eta^*(\alpha)=0$.

Next suppose $\alpha\notin\mathfrak{o}^*$ and put $p\cdot\mu_p(\alpha)=\begin{pmatrix}a & b/m\\ cm & -a\end{pmatrix}$. If $b\in p\boldsymbol{Z}_p$, we have again $\eta^*(\alpha)=\eta(\alpha)=0$ for the same reason. Therefore assume $b\notin p\boldsymbol{Z}_p$. Take $x\in\boldsymbol{Z}$ so that $a-bx\in p\boldsymbol{Z}_p$, and take $\gamma\in\Gamma$ so that $\mu_p(\gamma)\equiv\begin{pmatrix}1 & 0\\ xm & 1\end{pmatrix}$ (mod $m\mathfrak{o}_p^*$). Then $p\cdot\mu_p(\gamma\alpha\gamma^{-1})=\begin{pmatrix}a' & b'/m\\ * & *\end{pmatrix}$ with $b'\notin p\boldsymbol{Z}_p$, $a'\in p\boldsymbol{Z}_p$. Replacing α by $\gamma\alpha\gamma^{-1}$, we may assume that $a\in p\boldsymbol{Z}_p$. Since $a^2+bc\in p^2\boldsymbol{Z}$, we see that $p^2|c$. Now a direct calculation shows that $\beta_\lambda\alpha\beta_\lambda^{-1}\in\mathfrak{o}^*$ if and only if $\lambda=0$, and

$$\mu_p(\beta_0\alpha\beta_0^{-1})\equiv\begin{pmatrix}a/p & b/m\\ cm/p^2 & -a/p\end{pmatrix}\quad(\mathrm{mod}\ \mathfrak{o}_p').$$

A similar congruence can be found for each prime factor q of m. We then obtain $\eta^*(\alpha)=p^{k-1}\eta(p\alpha)=\eta'(\alpha)$.

Let us finally consider the case $p|e$. We have again $\eta'(\alpha)=p^{k-1}\eta(p\alpha)$. Take $\beta\in\mathfrak{o}'$ so that $N(\beta)=p$. Then $f|T(p)=p^{k-1}\psi_Y(\beta)^{-2}f\|\beta$, so that

$$L(z, f|T(p))=p^{k-1}\phi_Y(\beta)^{-2}\langle\theta(z, w; \eta), f\|\beta\rangle$$
$$=p^{k-1}\phi_Y(\beta)^{-2}\langle\theta(z, w; \eta)\|\beta^{-1}, f\rangle=\langle\theta(z, w; \eta^*), f\rangle$$

with $\eta^*(\alpha)=p^{k-1}\phi_Y(\beta)^{-2}\eta(\beta^{-1}\alpha\beta)$. A relation similar to (3.13) shows that $\eta(\beta^{-1}\alpha\beta)=\phi_Y(\beta)^2\eta(p\alpha)$ and hence $\eta^*=\eta'$. This completes the proof of Theorem 3.2.

As mentioned in the introduction, Theorems 3.1 and 3.2 were given by Shintani in the case $B=M_2(\boldsymbol{Q})$ (with minor differences in formulation; see [16, Theorem 2]). His proof of the commutativity with Hecke operators relies on the theory of binary quadratic forms. Here we have presented a shorter and simpler proof which requires only local computations.

4. Main theorem on the periods.

Still with $F=\boldsymbol{Q}$, for an arbitrary congruence subgroup Γ of $B^\times$, let $\mathcal{A}_0(\Gamma)$ denote the field of all Γ-invariant (meromorphic) automorphic functions on $\mathfrak{H}$ which take algebraic values at CM-points on $\mathfrak{H}$. Take an element g of $\mathcal{A}_0(\Gamma)$ other than the constants. We call an element f of $\mathcal{M}_h(\Gamma)$ arithmetic (or $\overline{\boldsymbol{Q}}$-rational) if $\pi^h(dg/dz)^{-h}f^2$ belongs to $\mathcal{A}_0(\Gamma)$. (The arithmeticity can be defined in the general case with an arbitrary F. For details, see [12, § 7], [13, §§ 4, 5], [15, II, § 2].) It can be shown that $\mathcal{S}_h(\Gamma'_m, \psi)$ is spanned by arithmetic elements (see [15, II, Lemma 2.3]); moreover Hecke operators send the set of arithmetic elements into itself.

We say that an element f of $\mathcal{S}_h(\Gamma)$ is *primitive* if there exists m and φ with the following properties:

(4.1) $f\in\mathcal{S}_h(\Gamma'_m, \varphi)$; *$f$ is an eigenform of all Hecke operators $T(p)$ (of level m);*

(4.2) *if l is a divisor of m smaller than m and g is an element of $\mathcal{S}_h(\Gamma'_l, \varphi)$ which is an eigenform of $T(p)$ for almost all p, then the eigenvalues of $T(p)$ for g are different from those for f for infinitely many p.*

Then f is said to be of type (m, e, φ, h), where e is the discriminant of B defined in § 3. Now the theory of Jacquet-Langlands in [2] (which sharpens that of Shimizu [5]) combined with the result of Miyake [3] shows the following facts.

(4.3) *If f is a primitive form of type (m, e, φ, h), then there is a primitive form g of type $(me, 1, \varphi, h)$ with the same eigenvalues of Hecke operators Moreover, if f_1 is an element of $\mathcal{S}_h(\Gamma'_m, \varphi)$ which is an eigenform of Hecke operators $T(p)$ for almost all p with the same eigenvalues as those for f, then f_1 is a constant multiple of f.*

As a consequence, every primitive form has a constant multiple which is arithmetic.

We shall now define the "fundamental periods" of a primitive form. This requires the cohomology group associated with $\mathcal{S}_h(\Gamma)$. We first identify B_R with $M_2(\boldsymbol{R})$ and consider B a subring of $M_2(\boldsymbol{R})$. Then

$$V_R = V\otimes_Q \boldsymbol{R} = \{x\in M_2(\boldsymbol{R}) \mid \mathrm{Tr}(x)=0\}$$

for V defined by (1.13). Let $\begin{pmatrix} r & s \\ t & -r \end{pmatrix}$ be the variable element of V_R. For each integer $\kappa\geqq 0$, we denote by $\mathcal{P}_R^\kappa$ the vector space over $\boldsymbol{R}$ of all $\boldsymbol{R}$-valued homogeneous polynomial functions $\mathfrak{h}$ on V_R of degree κ such that $(\partial^2/\partial r^2+4\partial^2/\partial s\partial t)\mathfrak{h}=0$. (This is similar to $\mathcal{P}_\lambda$ of Section 1, which was defined at the archimedean prime *ramified* in B.) We put then $\mathcal{P}_C^\kappa=\mathcal{P}_R^\kappa\otimes_R \boldsymbol{C}$ and define a representation ρ_κ of $GL_2(\boldsymbol{R})$ on $\mathcal{P}_C^\kappa$ by

$$[\rho_\kappa(\gamma)\mathfrak{h}](\xi)=\mathfrak{h}(\gamma^\iota\xi\gamma) \qquad (\mathfrak{h}\in\mathcal{P}_C^\kappa,\ \gamma\in GL_2(\boldsymbol{R}),\ \xi\in V_R). \tag{4.4}$$

For the same reason as in Lemma 1.1, ρ_κ is equivalent to the representation of $GL_2(\boldsymbol{R})$ by symmetric tensors of degree 2κ. Obviously $\mathcal{P}_R^\kappa$ is stable under ρ_κ. If Γ is a congruence subgroup of $B^\times$, we can let Γ act on $\mathcal{P}_R^\kappa$ through ρ_κ. Therefore we can define the (first) cohomology group

$$H(\Gamma, \mathcal{P}_R^\kappa)=Z(\Gamma, \mathcal{P}_R^\kappa)/B(\Gamma, \mathcal{P}_R^\kappa)$$

with $Z(\Gamma, \mathcal{P}_R^\kappa)$ and $B(\Gamma, \mathcal{P}_R^\kappa)$ given as follows (cf. [6], [7, Ch. 8]). An element $\mathfrak{x}$ of $Z(\Gamma, \mathcal{P}_R^\kappa)$, called a *cocycle*, is a $\mathcal{P}_R^\kappa$-valued function on Γ such that

$$\mathfrak{x}(\gamma\delta)=\mathfrak{x}(\gamma)+\rho_\kappa(\gamma)\mathfrak{x}(\delta) \qquad \textit{for every}\ \ \gamma,\delta\in\Gamma, \tag{4.5}$$

$$\mathfrak{x}(\sigma)\in[1-\rho_\kappa(\sigma)]\mathcal{P}_R^\kappa \ \ \textit{for every parabolic element } \sigma \textit{ of } \Gamma. \tag{4.6}$$

$B(\Gamma, \mathcal{P}_R^\kappa)$ consists of all $\mathfrak{x}$, called *coboundaries*, for which there exists an element $\mathfrak{y}$ of $\mathcal{P}_R^\kappa$ such that

$$\mathfrak{x}(\gamma)=[1-\rho_\kappa(\gamma)]\mathfrak{y} \qquad \textit{for all}\ \ \gamma\in\Gamma. \tag{4.7}$$

In the present setting, however, it is more convenient to view such a cocycle or a coboundary as an $\boldsymbol{R}$-valued function $\mathfrak{x}(\gamma, \xi)$ with variables $\gamma\in\Gamma$ and $\xi\in V_R$. Then (4.5), (4.6), and (4.7) can be written as

$$\mathfrak{x}(\gamma\delta, \xi)=\mathfrak{x}(\gamma, \xi)+\mathfrak{x}(\delta, \gamma^{-1}\xi\gamma) \qquad (\gamma,\delta\in\Gamma;\ \xi\in V_R); \tag{4.5$'$}$$

$$\mathfrak{x}(\sigma, \xi)=\mathfrak{z}_\sigma(\xi)-\mathfrak{z}_\sigma(\sigma^{-1}\xi\sigma) \ \textit{with}\ \mathfrak{z}_\sigma\in\mathcal{P}_R^\kappa \ \textit{for each parabolic element } \sigma \textit{ of } \Gamma; \tag{4.6$'$}$$

$$\mathfrak{x}(\gamma, \xi)=\mathfrak{y}(\xi)-\mathfrak{y}(\gamma^{-1}\xi\gamma) \qquad \textit{with}\ \ \mathfrak{y}\in\mathcal{P}_R^\kappa. \tag{4.7$'$}$$

We can similarly define the modules Z, B, H with $\mathcal{P}_C^\kappa$ instead of $\mathcal{P}_R^\kappa$; obviously $H(\Gamma, \mathcal{P}_C^\kappa)=H(\Gamma, \mathcal{P}_R^\kappa)\otimes_R \boldsymbol{C}$. For an element $\mathfrak{a}$ of $Z(\Gamma, \mathcal{P}_C^\kappa)$ or $H(\Gamma, \mathcal{P}_C^\kappa)$, its complex conjugate, real part, and imaginary part can be defined in an obvious way; they are denoted by $\bar{\mathfrak{a}}$, $\mathrm{Re}(\mathfrak{a})$, and $\mathrm{Im}(\mathfrak{a})$. If $\mathfrak{a}\in Z(\Gamma, \mathcal{P}_C^\nu)$, we denote by $\mathrm{cl}(\mathfrak{a})$ its cohomology class.

Let us now investigate certain integrals attached to the elements of $\mathcal{S}_{2k}(\Gamma)$ and their periods. We first observe that for every $w\in\mathfrak{H}$, $[\xi, w]^h$ as a function of $\xi\in V_R$ belongs to $\mathcal{P}_C^h$. Given $f\in\mathcal{S}_{2k}(\Gamma)$, $v\in\mathfrak{H}$, $\xi\in V$, and $\gamma\in\Gamma$, put

$$X(z, \xi, f, v)=\int_v^z[\xi, w]^{k-1}f(w)dw \qquad (z\in\mathfrak{H}),$$

$$\mathfrak{x}(\gamma, \xi, f, v)=X(\gamma v, \xi, f, v).$$

These as functions of ξ belong to $\mathcal{P}_C^{k-1}$. It can easily be seen that

$$X(\gamma z, \xi, f, v)=X(z, \gamma^{-1}\xi\gamma, f, v)+\mathfrak{x}(\gamma, \xi, f, v);$$

$\mathfrak{x}(\gamma, \xi, f, v)$ as a function of (γ, ξ) belongs to $Z(\Gamma, \mathcal{P}_C^{k-1})$; moreover, its cohomology class is independent of v. (For the treatment in a more general case, the reader is referred to [6] and [7, Chapter 8].) We put then $c[f]=\mathrm{cl}(\mathfrak{x}(\gamma, \xi, f, v))$.

PROPOSITION 4.1. *The map $f\mapsto\mathrm{Re}(c[f])$ gives an $\boldsymbol{R}$-linear isomorphism of $\mathcal{S}_{2k}(\Gamma)$ onto $H(\Gamma, \mathcal{P}_R^{k-1})$.*

This is a special case of [6, Théorème 1] and of [7, Theorem 8.4]. In fact, we can assign to each $p\in\mathcal{P}_C^{k-1}$ a homogeneous function p^* on $\boldsymbol{R}^2$ by $p^*(x)=p(\varepsilon^{-1}x\cdot{}^tx)$ for $x\in\boldsymbol{R}^2$ as in the proof of Lemma 1.1. Obviously $\mathrm{Re}(p^*)=(\mathrm{Re}(p))^*$. Therefore $f\mapsto\mathrm{Re}(c[f])$ is essentially the same as the maps of [6, Théorème 1] and [7, Theorem 8.4].

To simplify our notation, with fixed Γ and k, let us write $H(\boldsymbol{R})$ and $H(\boldsymbol{C})$ for $H(\Gamma, \mathcal{P}_R^{k-1})$ and $H(\Gamma, \mathcal{P}_C^{k-1})$, and further denote by $H(\bar{\boldsymbol{Q}})$ the submodule of $H(\boldsymbol{C})$ consisting of all cohomology classes represented by the cocycles $\mathfrak{x}$ such that $\mathfrak{x}(\gamma, \xi)\in\bar{\boldsymbol{Q}}$ for all $\gamma\in\Gamma$ and all $\xi\in V$.

PROPOSITION 4.2. *There exists a $\boldsymbol{C}$-valued $\boldsymbol{C}$-bilinear alternating form A on $H(\boldsymbol{C})$ with the following properties:*

(4.8) $i\mu(D)\langle f, g\rangle=A(\overline{c[f]}, c[g])$ *for* $f, g\in\mathcal{S}_{2k}(\Gamma)$, *where* $D=\Gamma\backslash H$;

(4.9) $i\mu(D)\{\langle f, g\rangle-\langle g, f\rangle\}=4A(\mathrm{Re}(c[f]), \mathrm{Re}(c[g]))$ *for* $f, g\in\mathcal{S}_{2k}(\Gamma)$;

(4.10) *A is $\boldsymbol{R}$-valued on $H(\boldsymbol{R})$ and $\bar{\boldsymbol{Q}}$-valued on $H(\bar{\boldsymbol{Q}})$.*

PROOF. This is an easy consequence of [6]. In fact, by virtue of Proposition 4.1, we can define an $\boldsymbol{R}$-valued alternating form A on $H(\boldsymbol{R})$ so that (4.9) holds. As shown in [6, §4], $A(\mathrm{cl}(\mathfrak{a}), \mathrm{cl}(\mathfrak{b}))$ can be explicitly expressed in terms of $\mathfrak{a}(\gamma)$ and $\mathfrak{b}(\gamma)$ with finitely many elements γ of Γ and some quantities determined at the cusps. If we extend A to $H(\boldsymbol{C})$ $\boldsymbol{C}$-linearly, then the computation of [6, § 4] shows that (4.8) holds. Assertion (4.10) can be proved by the same argument as in the proof of [6, Théorème 2].

Let us now consider the case where Γ is the group Γ_m defined by (3.3). For each prime number p, we can find an element β of Y_m such that $N(\beta)=p$ and $\mu_q(\beta)\equiv\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}$ (mod $m\mathfrak{o}_q$) for all prime factors q of m. Let $\Gamma\beta\Gamma=\bigcup_\lambda \Gamma\beta_\lambda$ be a disjoint coset decomposition. Then it can be shown that $f|T(p)=p^{k-1}\sum_\lambda f\|_{2k}\beta_\lambda$ for every $f\in\mathcal{S}_{2k}(\Gamma)$. Now we can define the action of $\Gamma\beta\Gamma$ on $H(\boldsymbol{C})$ and on $H(\boldsymbol{R})$ in a purely algebraic way as in [7, § 8.3]; we denote this action also by $T(p)$. Then we have $c[f|T(p)]=c[f]|T(p)$. This is proved in [7, Prop. 8.5] with $\mathrm{Re}(c)$ instead of c; the proof can easily be seen to be valid for c. We can also find an element δ of $\mathfrak{o}$ such that $N(\delta)=-1$ and $\mu_q(\delta)\equiv\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$ (mod $m\mathfrak{o}_q$) for all prime factors q of m. Then $\delta^2\in\Gamma$, $\delta\Gamma\delta^{-1}=\Gamma$, $\delta\Gamma'\delta^{-1}=\Gamma'$ and $\varphi_Y(\delta\alpha\delta^{-1})=\varphi_Y(\alpha)$ for all $\alpha\in Y$, where Γ' is defined by (3.2) and φ is a character modulo m. Now, for $f\in\mathcal{S}_{2k}(\Gamma)$, we put

$$(f|\delta)(w)=J(\delta,\ w)^{-2k}\overline{f(\delta\bar{w})}\qquad (w\in\mathfrak{H}). \tag{4.11}$$

Then $f|\delta\in\mathcal{S}_{2k}(\Gamma)$; moreover $f|\delta\in\mathcal{S}_{2k}(\Gamma',\bar{\varphi})$ if $f\in\mathcal{S}_{2k}(\Gamma',\varphi)$; further $(f|\delta)|T(p)=(f|T(p))|\delta$ for every prime p. We can also define the action of δ on $Z(\Gamma,\mathfrak{P}_C^{k-1})$ by

$$(\mathfrak{x}|\delta)(\gamma,\ \xi)=(-1)^k\mathfrak{x}(\delta^{-1}\gamma\delta,\ \delta^{-1}\xi\delta), \tag{4.12}$$

which induces an action on $H(\boldsymbol{C})$ in a natural way. A direct calculation shows that

$$c[f]|\delta=\overline{c[f|\delta]}. \tag{4.13}$$

Let f be a primitive form of type $(m,\ e,\ \varphi,\ 2k)$ which is arithmetic, and let $f|T(p)=\lambda_p f$ with $\lambda_p\in\boldsymbol{C}$ for each prime p. We have $(f|\delta)|T(p)=\bar{\lambda}_p(f|\delta)$. Now, as to the nature of $\{\lambda_p\}_p$, the following two cases can occur.

Case I. $\lambda_p=\bar{\lambda}_p$ for all p. (This is so if $\lambda_p=\bar{\lambda}_p$ for almost all p.)
Case II. $\lambda_p\neq\bar{\lambda}_p$ for infinitely many p.

In Case I, we have $f|\delta=bf$ with a constant b. Then $b\bar{b}=1$. By [15, II, Lemma 4.2], $f|\delta$ is arithmetic, so that $b\in\overline{\boldsymbol{Q}}$. Taking $a\in\overline{\boldsymbol{Q}}$ so that $b=a/\bar{a}$ and replacing f by af, we may assume that $f|\delta=f$. Therefore we shall always assume $f|\delta=f$ in Case I in the following treatment. In Case II, f and $f|\delta$ are linearly independent over $\boldsymbol{C}$. Put, in either case,

$$U_C=\{z\in H(\boldsymbol{C})\,|\,z|T(p)=\lambda_p z \text{ for all } p\},$$

$$U=U_C\cap H(\overline{\boldsymbol{Q}}).$$

Obviously $U_C = U \otimes_{\bar{Q}} \boldsymbol{C}$.

LEMMA 4.3. *If f is a primitive form as above, $c[f]$ and $c[f]|\delta$ form a basis of U_C over $\boldsymbol{C}$. Furthermore, U has a basis $\{\mathfrak{a}, \mathfrak{b}\}$ over $\bar{\boldsymbol{Q}}$ such that $\mathfrak{a}|\delta = \mathfrak{a}$ and $\mathfrak{b}|\delta = -\mathfrak{b}$.*

PROOF. In Case I, Proposition 4.1 shows that $\mathrm{Re}(c[f])$ and $\mathrm{Im}(c[f])$ form a basis of U_C over $\boldsymbol{C}$, and hence $c[f]$ and $\overline{c[f]}$ $(=c[f]|\delta)$ form a basis of U_C over $\boldsymbol{C}$. In Case II, we see again from Proposition 4.1 that $\mathrm{Re}(c[f])$, $\mathrm{Im}(c[f])$, $\mathrm{Re}(c[f|\delta])$, $\mathrm{Im}(c[f|\delta])$ are linearly independent over $\boldsymbol{C}$, and hence $c[f]$ and $\overline{c[f|\delta]}$ $(=c[f]|\delta)$ form a basis of U_C over $\boldsymbol{C}$. In either case, δ has eigenvalues ± 1 on U_C with multiplicity one, which proves the second assertion.

With a fixed arithmetic primitive element f as above, we take $\mathfrak{a}$ and $\mathfrak{b}$ as in Lemma 4.3. Since $c[f] \in U_C$, we can put

$$c[f] = u_+(f)\mathfrak{a} + u_-(f)\mathfrak{b} \tag{4.14}$$

with complex numbers $u_+(f)$ and $u_-(f)$. We call these numbers *the fundamental periods of f*. They are determined by $\{\lambda_p\}_p$ up to algebraic factors. We have

$$\overline{c[f|\delta]} = c[f]|\delta = u_+(f)\mathfrak{a} - u_-(f)\mathfrak{b}, \tag{4.15}$$

so that $u_+(f)u_-(f) \neq 0$ in view of Lemma 4.3. In Case I, U is stable under the complex conjugation, and hence we can take $\mathfrak{a}$ and $\mathfrak{b}$ to be real; then $u_+(f)$ is real and $u_-(f)$ is pure imaginary.

THEOREM 4.4. *If f is an arithmetic primitive element as above, $\pi\langle f, f\rangle / [u_+(f)\overline{u_-(f)}]$ is an algebraic number.*

PROOF. Put $u = u_+(f)$ and $v = u_-(f)$ for simplicity. By (4.8), we have

$$i\mu(D)\langle f, f\rangle = A(\overline{u\mathfrak{a}} + \overline{v\mathfrak{b}},\ u\mathfrak{a} + v\mathfrak{b}),$$

$$i\mu(D)\langle f|\delta, f|\delta\rangle = A(u\mathfrak{a} - v\mathfrak{b},\ \overline{u\mathfrak{a}} - \overline{v\mathfrak{b}}).$$

Since $\langle f, f\rangle = \langle f|\delta, f|\delta\rangle$, we have

$$i\mu(D)\langle f, f\rangle = \bar{u}vA(\bar{\mathfrak{a}}, \mathfrak{b}) + u\bar{v}A(\bar{\mathfrak{b}}, \mathfrak{a}). \tag{4.16}$$

In Case I, $\mathfrak{a}$, $\mathfrak{b}$, u, and iv are real, so that

$$i\mu(D)\langle f, f\rangle = 2u\bar{v}A(\bar{\mathfrak{b}}, \mathfrak{a}). \tag{4.17}$$

In Case II, we have

$$0 = i\mu(D)\langle f|\delta, f\rangle = A(u\mathfrak{a} - v\mathfrak{b},\ u\mathfrak{a} + v\mathfrak{b}) = 2uvA(\mathfrak{a}, \mathfrak{b}). \tag{4.18}$$

Now $\mathrm{Re}(c[f+f|\delta])=u\mathfrak{a}+\bar{u}\bar{\mathfrak{a}}$ and $\mathrm{Re}(c[f-f|\delta])=v\mathfrak{b}+\bar{v}\bar{\mathfrak{b}}$. Since $\langle f+f|\delta, f-f|\delta\rangle =0$, we obtain, from (4.9) and (4.18),

$$0=A(u\mathfrak{a}+\bar{u}\bar{\mathfrak{a}}, v\mathfrak{b}+\bar{v}\bar{\mathfrak{b}})=u\bar{v}A(\mathfrak{a}, \bar{\mathfrak{b}})+\bar{u}vA(\bar{\mathfrak{a}}, \mathfrak{b}).$$

This combined with (4.16) yields again (4.17), which proves our assertion, since $\mu(D)/\pi$ and $A(\mathfrak{a}, \bar{\mathfrak{b}})$ are algebraic.

PROPOSITION 4.5. *For an arithmetic primitive f of type $(m, e, \psi^2, 2k)$ with $e>1$, put $2\mu(\Gamma\backslash\mathfrak{H})L(z, f)=\sum_{n=1}^{\infty}a_n e(nz)$ with $\Gamma=\Gamma_m$ and with $L(z, f)$ of (3.6). Then $a_n/u_+(f)\in\bar{\boldsymbol{Q}}$ for all n.*

PROOF. Let τ denote the order of $\Gamma\cap\{\pm 1\}$. Given $\alpha\in R(\Gamma)$ such that $n=-N(\alpha)$, let γ be an element of Γ_α which generates $\Gamma_\alpha\{\pm 1\}/\{\pm 1\}$. Define an isomorphism ω of $\boldsymbol{R}^\times$ onto H_α as in (2.19). In the present situation, we have

$$\beta\alpha\beta^{-1}=\begin{pmatrix} n^{1/2} & 0 \\ 0 & -n^{1/2}\end{pmatrix}, \qquad \omega(s)=\beta^{-1}\begin{pmatrix} s & 0 \\ 0 & s^{-1}\end{pmatrix}\beta \qquad (s\in\boldsymbol{R}^\times)$$

with $\beta\in SL_2(\boldsymbol{R})$. Changing γ for γ^{-1} if necessary, we may assume that $\gamma=\omega(t)$ with $t>0$. Then (2.20) together with (2.22) shows that

$$P(f, \alpha, \Gamma)=-(2n^{1/2}/\tau)\int_v^{\gamma v}[\alpha, w]^{k-1}f(w)dw,$$

with $v\in\mathfrak{H}$. Let $\mathfrak{x}$ be a cocycle in the class $c[f]$. Since $\gamma\alpha=\alpha\gamma$, we see that $\mathfrak{x}(\gamma, \alpha)$ depends only on $c[f]$, and

$$n^{-1/2}P(f, \alpha, \Gamma)=-(2/\tau)\mathfrak{x}(\gamma, \alpha).$$

Put $\alpha_1=\delta^{-1}\alpha\delta$, $\gamma_1=\delta^{-1}\gamma\delta$. Then $\eta(\alpha_1)=\psi(-1)\eta(\alpha)$ and

$$n^{-1/2}P(f, \alpha_1, \Gamma)=-(2/\tau)\mathfrak{x}(\gamma_1, \alpha_1)=-(2/\tau)(\mathfrak{x}|\delta)(\gamma, \alpha).$$

Since $\psi(-1)=(-1)^k$, we have

$$\eta(\alpha)n^{-1/2}P(f, \alpha, \Gamma)+\eta(\alpha_1)n^{-1/2}P(f, \alpha_1, \Gamma) = -(2/\tau)\eta(\alpha)[\mathfrak{x}(\gamma, \alpha)+(\mathfrak{x}|\delta)(\gamma, \alpha)]. \tag{4.19}$$

If α and α_1 are not conjugate under Γ, this sum is a part of the coefficient a_n. Suppose $\varepsilon\alpha\varepsilon^{-1}=\alpha_1$ with $\varepsilon\in\Gamma$; then $\psi(-1)\eta(\alpha)=\eta(\alpha_1)=\eta(\alpha)$, so that $\eta(\alpha)=0$ or $\psi(-1)=1$; thus (4.19) is 0 or its half contributes to a_n. Now, with a cocycle $\mathfrak{a}_0$ in the class $\mathfrak{a}$ as in (4.14), we have

$$\mathfrak{x}(\gamma, \alpha)+(\mathfrak{x}|\delta)(\gamma, \alpha)=2u_+(f)\mathfrak{a}_0(\gamma, \alpha).$$

Since $\mathfrak{a}_0(\gamma, \alpha)$ is algebraic, we see that (4.19) is $u_+(f)$ times an algebraic number, which completes the proof.

For an elliptic modular form $g(z)=\sum_{n=1}^{\infty} b_n e(nz)$ and a Dirichlet character ξ, put

$$D(s, g, \xi)=\sum_{n=1}^{\infty} \xi(n) b_n n^{-s}.$$

Suppose that g is a primitive element of type $(N, 1, \varphi, k)$ with $b_1=1$. As shown in [10], there exist two constants $v_+(g)$ and $v_-(g)$ with the following properties: *for* $t\in \boldsymbol{Z}$, $0<t<k$, *one has*

$$\pi^{-t}D(t, g, \xi)\sim\begin{cases} v_+(g) & \textit{if } \xi(-1)=(-1)^t, \\ v_-(g) & \textit{if } \xi(-1)=(-1)^{t-1}, \end{cases} \tag{4.20}$$

$$\pi\langle g, g\rangle\sim v_+(g)v_-(g). \tag{4.21}$$

Here and henceforth, we write $a\sim b$ for two complex numbers a and b if $b\neq 0$ and $a/b\in\overline{\boldsymbol{Q}}$.

LEMMA 4.6. *If k is even, $v_+(g)\sim\overline{v_+(g)}$ and $v_-(g)\sim\overline{v_-(g)}$; if k is odd, $v_+(g)\sim\overline{v_-(g)}$.*

PROOF. Put $g'=\sum_{n=1}^{\infty}\bar{b}_n e(nz)$. Since $b_n=\varphi(n)\bar{b}_n$ for $(n, N)=1$, we have

$$D(t, g, \xi)=a\cdot D(t, g', \xi\varphi) \tag{4.22}$$

for $0<t<k$ with $a\in\overline{\boldsymbol{Q}}$ (depending on t). If $k>2$, the quantities of (4.22) are not 0 for $t=k-1$ by [9, Prop. 2] (cf. also [11, Prop. 4.16]). If $k=2$, we have to choose ξ so that (4.22) is not 0 for $t=1$, but this is guaranteed by [10, Theorem 2]. In any case, we have $v_+(g')\sim v_+(g)$ and $v_-(g')\sim v_-(g)$ if k is even, and $v_+(g')\sim v_-(g)$ if k is odd. As shown in [10, Theorem 1], $\overline{v_+(g)}\sim v_+(g')$ and $\overline{v_-(g)}\sim v_-(g')$, and hence we obtain our lemma.

We are now ready to state and prove our main theorem.

THEOREM 4.7. *Let ψ be a character modulo m such that $\psi(-1)=(-1)^k$, and f an arithmetic primitive form of type $(m, e, \psi^2, 2k)$ with $e>1$. Further let g be the primitive form of type $(me, 1, \psi^2, 2k)$ whose first Fourier coefficient is 1 and whose eigenvalues for Hecke operators are the same as those for f. Suppose $L(z, f)$ defined by (3.6) is not identically equal to 0. Then both $u_+(f)/v_+(g)$ and $u_-(f)/v_-(g)$ are algebraic numbers.*

PROOF. Put $h(z)=u_+(f)^{-1}\mu(\Gamma_m\backslash\mathfrak{H})L(f, z)$. By Proposition 4.5, h is $\overline{\boldsymbol{Q}}$-rational, and by Theorems 3.1 and 3.2, $h\in\mathcal{S}(N, \chi, (2k+1)/2)$ and $h|T(p^2)=\lambda_p h$ for all odd primes p. We have also

$$u_+(f)h(z)=\int_D\overline{\theta(z, w; \eta)}f(w)\operatorname{Im}(w)^{2k}d\mu(w) \qquad (D=\Gamma_m\backslash\mathfrak{H})$$

as in (3.7). Put

$$q(w)=\int_{D_1} h(z)\theta(z,\ w\ ;\ \eta)\,\mathrm{Im}\,(z)^{k+(1/2)}d\mu(z) \qquad (D_1=\Gamma_1(N)\backslash\mathfrak{H}).$$

Then

$$\mu(D_1)u_+(f)\langle h,\ h\rangle=\int_{D_1}\overline{h(z)}\int_D\overline{\theta(z,\ w\ ;\ \eta)}f(w)\,\mathrm{Im}\,(w)^{2k}\,\mathrm{Im}\,(z)^{k+(1/2)}d\mu(w)d\mu(z)$$
$$=\mu(D)\langle q,\ f\rangle\,.$$

By Proposition 2.6, q is $\overline{\boldsymbol{Q}}$-rational and by (3.8) belongs to $\mathcal{S}_{2k}(\Gamma_m)$. Since f is primitive, we have $q=bf+r$ with $b\in\overline{\boldsymbol{Q}}$ and an element r of $\mathcal{S}_{2k}(\Gamma_m)$ orthogonal to f. Thus

$$\mu(D_1)u_+(f)\langle h,\ h\rangle=\mu(D)\bar{b}\langle f,\ f\rangle\sim u_+(f)\overline{u_-(f)}\,.$$

Now Theorem 1 of [14] asserts that $\pi\langle h,\ h\rangle\sim v_-(g)$, and hence $u_-(f)\sim\overline{v_-(g)}\sim v_-(g)$ by Lemma 4.6. On the other hand, $\langle f,\ f\rangle\sim\langle g,\ g\rangle$ by [15, II, Theorem 3.8], and hence $u_+(f)\overline{u_-(f)}\sim v_+(g)v_-(g)$ by Theorem 4.4 and (4.21), so that $u_+(f)\sim v_+(g)$. This completes the proof.

Let us now add a few concluding remarks. In the case $e=1$ (i.e. if $B=M_2(\boldsymbol{Q})$), we may put $f=g$. Then it can be shown that $u_+(f)\sim v_+(f)$ and $u_-(f)\sim v_-(f)$ more directly by the same technique as in the proof of [10, Theorem 3].

Recently, Ribet proved in [4] that the jacobian of $\Gamma_1'\backslash\mathfrak{H}$ is isogenous over $\boldsymbol{Q}$ to the primitive part of the jacobian of $\Gamma_0(e)\backslash\mathfrak{H}$. We can derive from this the conclusion of the above theorem when $k=1$ and $m=1$. Conversely, if we could prove the rationality of $u_+(f)/v_+(g)$ and $u_-(f)/v_-(g)$ over a specific number field, that would extend the result of Ribet to the case of arbitrary m.

In the above theorem, we assumed that m is prime to e and $L(z,\ f)\neq 0$. It is naturally desirable to remove these conditions.

We treated in this section only the case $F=\boldsymbol{Q}$. It seems that our methods can be generalized to a totally indefinite B over a field F of higher degree. The case of a partially definite B (i.e. the case with $r<n$), however, is not so transparent. The periods of the elements of $\mathcal{S}_{2k,2\lambda}(\Gamma)$ in this case seem to be related to the inner products of the nonholomorphic pullback of a Hilbert modular form of half-integral weight with holomorphic forms, which are similar to the inner products considered in [15, II, Theorems 3.6 and 3.7].

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