博士学位論文

決済システムの理論分析

平成 26 年 1 月

東京大学大学院経済学研究科

早川　仁

博士論文

論文題目　Theoretical Analyses on Settlement System

決済システムの理論分析

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早川　仁

Contents

Introduction

1 Background of the Study

In Japan, settlement systems are in the midst of transition. Table 1 shows major events for the transition. The transition started at January 2001, when the interbank funds transfer system of the Bank of Japan Financial Network System(BOJ-NET) abolished designated-time net settlement(DTNS) to make the shift to real-time gross settlement(RTGS). In its DTNS systems, settlements were executed once a day. The shift to RTGS improved its service by enabling participants to make each payment at any point during operating hour. Under the adopted RTGS system, each obligation must be settled by a transfer of an offsetting amount of settlement fund through accounts in the BOJ. Since insufficient availability of settlement fund possibly causes settlements to delay, this system tends to require large volume of settlement fund in order to facilitate smooth settlement. Required settlement fund is fulfilled either with reserve of each participant or through intraday lending provided by the BOJ. Consequently, settlements executed with large volume of funds come with cost in each form. When participants need to make additional reserve solely for the purpose of their settlements, they incur opportunity cost for that amount. Daylight overdraft associated with the provision of intraday lending primarily lets the BOJ be vulnerable to risk of default of participants, which partly turns to cost for participants in the form of provision of collateral. Those costs associated with settlements in the RTGS system were thought to be considerable, and were expected to be mitigated by subsequent environmental/functional changes relevant to the system[1].

Certain part of settlements have moved to sub-platforms of the BOJ-NET. In January 2003, the Japan Securities Clearing Corporation(JSCC) started clearing cash transactions on stock exchanges. In May 2004, the JASDEC DVP Clearing Corporation(JDCC) started clearing service for off-exchange transactions. In May 2005, the Japan Government Bond Clearing Corporation(JGBCC) commenced operation as a central clearing counterparty for the government bond. These central clearing counterparties provide settlement service themselves by offsetting obligations that were to be settled in the BOJ-NET, while obligations that could not be offset are settled in the BOJ-NET via accounts of those central clearing entities. In October 2013, the JSCC and the JGBCC merged together.

The BOJ-NET itself has modified its RTGS system. In November 2011, the BOJ introduced a liquidity saving mechanism that effectively added offsetting service into the existent RTGS system. The liquidity saving mechanism consists of two mechanism; queueing mechanism and offset mechanism. The queueing mechanism enables payments to be put into the central queue instead of immediate settlement. The offset mechanism searches combination of payments that are possible to be offset and executes offsetting.

[1]Discussion within the BOJ associated with cost of RTGS is documented, for example, in "Proposal for the Next-Generation RTGS Project of the BOJ-NET Funds Transfer System" in December 2, 2005, available on the BOJ website.

Year, Month	Events
2001 Jan.	the BOJ-NET shifted from DTNS to RTGS system
2003 Jan.	the JSCC started clearing transactions on stock exchanges
2004 May	the JDCC started clearing service for off-exchange transactions
2005 May	the JGBCC started clearing for the government bond
2011 Nov.	the BOJ-NET introduced a liquidity saving mechanism into its RTGS system
2013 Oct.	the JSCC and the JGBCC merged

Table 1: Major Events for Japan's Settlement Systems

Shift from DTNS to RTGS, and further to modified RTGS systems is observed not solely in Japan but is a world-wide trend since 1980s. It is documented[2] that 116 out of 139 countries (83 percent) have adopted RTGS until 2010. However, it is not sufficiently understood on how efficiently those RTGS systems work in themselves, nor together with relevant central clearing entities or with modifications within their systems.

This research proposes a theoretical approach to examine RTGS systems, providing several theoretical analyses on how efficiently RTGS systems would work in itself, or in combination with external sub-platforms, or with internal modifications on those systems themselves. Our research contributes to the ongoing search for efficiently functioning settlement system.

Conceptually, we can suppose two extreme types of settlement, each is to serve as a reference when we view real-world settlement systems. One is "centralized" settlement, in which all obligations or trades are collected and settled without any transfer of settlement fund, just as supposed in the traditional Walrasian Market. The other extreme is "decentralized" settlement where any unit of obligation needs to be settled with transfer of corresponding amount of settlement funds. Each settlement system in reality is to be placed somewhere between these two extremes.

Traditionally adopted designated-time net settlement(DTNS) is much closer to centralized settlement, than recently adopted real-time gross settlement(RTGS). RTGS systems get closer to centralized settlement when they are accompanied with sub-platforms that provide central clearing, or when liquidity saving mechanisms are introduced to the systems.

Along with this view, we divide our tasks into two parts. The first is to examine decentralized settlement, which is to intend to analyze RTGS systems separately from any sub-platform and without introduction of liquidity saving mechanisms. The second is to examine effects of partly combining centralized settlement into decentralized settlement, which is to analyze effects of relevant sub-platforms or introduction of liquidity saving mechanisms.

We analyze those types of settlements in reference to settlement efficiency. Observing complex nature of network of obligations settled in each settlement system[3], one of the key

[2] World Bank (2013)

[3] The topology for real-world networks were examined in Soramaki, Bech, Arnold, Glass, and Beyeler

determinants of settlement efficiency would be expected to be structural factors of network of obligations. Although simulation studies[4] would help examine settlement efficiency for each structure of network, theoretical clarification on how and which network factors are relevant is crucial in the face of potentially wide variety of network structures in real-world settlement systems.

Our aim is to provide theoretical analyses on settlement efficiency especially in reference to structure of network of obligations for each of our settlement type.

2 Scope of the Study

We focus on the phase of settlement, omitting explicit examination on how those obligations have been formed. More concretely, what we examine in this research is how efficiently settlement would be executed among banks that have supposed to be endowed certain obligations. Though it would be an important issue, we do not argue how settlement efficiency in turn possibly affect formation of obligations. In this sense, our analyses should be interpreted to provide short-term analyses, settlements for each day. Longer-term analyses are our important future tasks, which should deepen our argument into how settlement efficiency matters for real-economic factors; investment, consumption, that are out of scope of this study.

In our analyses, we mainly discuss settlement efficiency, little mentioning risk of settlement. Risk of settlement would be a crucial issue for DTNS systems contemplating that unsettled exposure could be considerably large under its much less frequency nature of settlement; typically once a day. In contrast, risk sourced from unsettled exposure would be less crucial for RTGS systems whose basic aim is to settle each obligation in real-time basis.

In our former part of analysis on decentralized settlement without any type of clearing, that type of risk is just ignored since it would expect to be much less crucial. However, we admit that in our latter analysis on decentralized settlement partly combined with each type of clearing, importance of risk aspect would tend to get larger when more obligations are to be offset supposing offsetting more obligations would require more time to collect those relevant obligations. There, our analysis should be interpreted as that settlements in sub-platforms are appropriately controlled with respect to risk, or liquidity saving mechanisms work for sufficiently small part of settlements.

3 Settlement fund, Money, Liquidity

We have adopted a neutral terminology "settlement fund" so as to refer to what settle obligations in settlement systems. In a general usage, "money" can be its substitutive terminology. "Liquidity", potentially a longer-reach word, can also refer to settlement funds utilized in settlement systems.

(2007) for the Fedwire, Rordam and Bech (2009) for the Danish interbank system, Imakubo and Soejima (2010) for the BOJ-NET. See Newman (2003) for argument of complex networks

[4] For example, see Bank of Finland (2007).

In the literature of "money", micro-foundation of its existence has been studied with monetary search models[5]. Although the literature has common interest with ours in examining decentralized settlement, there is difference in focus. The main purpose of monetary search literature is to show existence of monetary equilibrium where "money" circulates as medium of exchange, together with non-monetary equilibrium. Accordingly, network structure is maintained simple enough to allow detailed analysis on incentive to accept "money" as medium of exchange. In contrast, the objective of our research is to examine circulation of "settlement funds", or "money" in relation to structure of network. Accordingly, we treat much wider structures of network, while almost implicitly assuming relevant economy is in monetary equilibrium. In the literature, "money" as medium of exchange is analyzed on the basis of several assumptions; long-term relationship is hard to be formed, and also double coincidence of wants is not likely. Those assumptions are basically held also in our analyses on settlement system. The assumptions would be consistent with observation that financial institutions in interbank settlement systems do not necessarily form long-term relationship regarding their daily payments[6] with each other, and double coincidence of wants would be relatively less likely under large number of participants.

The concept of "liquidity" has particular strength in examining how difficulty of payment for individual subject spreads to the other subjects through letting financing "settlement fund" be harder, which would be crucial in researches of financial crises. There, "liquidity" can refer not only to volume of settlement funds available in each domain, but also to how easily settlement funds are to be obtained through each market or other routes. In our research, we confine us on examining settlement efficiency, not on risk aspect as already mentioned. Accordingly for our research, "liquidity" is interpreted as almost the same as "settlement fund" in relevant settlement system.

In the following chapters, we interchangeably use "settlement fund", "money", and "liquidity" if not otherwise mentioned.

4 Approach on Settlement Efficiency

We regard the major factors for settlement efficiency as both cost of financing settlement fund and cost of settlement delay[7]. Delay of settlement would primarily harm each recipient through limiting availability of funds, for each settlement purpose, or for investment purpose, etc. Associated cost would be incurred also by each sender through explicit contract in a form of additional fee or others, or more implicitly through damaging reputation as an efficient payment processor[8].

Aggregate cost of financing settlement funds becomes larger when more settlement funds are required for each settlement system. Requirement of settlement funds gets larger

[5] Kiyotaki and Wright (1989) paved the way to following researches; e.g., Kiyotaki and Wright (1993), Trejos and Wright (1995), or Lagos and Wright (2005)

[6] Observe that certain part of obligations settled in interbank settlement systems are formed by their non-financial customers, where each counterpart of obligation is not controlled by relevant financial institution.

[7] The view to grasp settlement efficiency from these two aspects has been repeatedly presented in the literature. For example, see Bech, Preisig, and Soramaki (2008).

[8] See Angelini (1998), Bech and Garratt (2003)

when each unit of settlement fund is recycled(re-utilized) less often. Then, how much each unit of settlement fund is recycled is primarily dependent on “relative” timing among the relevant settlements. In contrast, cost of settlement delay is primarily dependent on each “absolute” timing of settlement in the way that slower settlement processing means larger cost of settlement delay.

In each chapter, we either focus solely on “relative” timing of settlement or on both of “relative” and “absolute” timing.

5 Overview

Chapter 1 provides analyses on decentralized settlement without any combination of centralized settlement. Chapter 2 and 3 for analyses on decentralized settlement combined with centralized settlement in each way. Chapter 2 examines role of sub-platforms that provide offsetting service, or role of central clearing counterparties(CCP). Chapter 3 examines effect of introduction of liquidity saving mechanism(LSM).

From the view of which aspect of timing is examined, chapter 1 and 2 focus exclusively on “relative” timing aspect, that enables to examine general network structures. Chapter 3 focuses both on “relative” and “absolute” timing aspect together with detailed analyses on incentive of participants while analysis is limited in less general network structure.

5.1 chapter 1

Chapter 1 titled as “Complexity of Payment Network” serves as the basis for the rest of the chapters. A network model is demonstrated that is to capture settlements under decentralized settlement. The distinct feature of the model is explicit treatment of relative timing of settlement. The model allows flexible examination on how relative timing of settlements would affect settlement efficiency for a general network. We explicitly examine two scenarios on how relative timing of settlement is to be realized. One is “optimal scenario”, where relative timing is optimally formed regarding settlement efficiency. The other is “worst scenario”, where it is formed in the way settlement efficiency is least optimal. Our purpose is to approach from the two polar scenarios to realistic situations that would lie somewhere between the two.

In our analysis, “interconnectedness” of payment network is interpreted along with certain type of decomposition of network in a way that network is more “interconnected” when the decomposition leads to more decomposed networks. Our finding is stated in relation to “interconnectedness” of network in that sense. For the optimal scenario, what we found is that “interconnectedness” of network possibly brings negative spillover effect in the sense that required amount of settlement fund would be less when the network of obligations were separated along with supposed “decomposition”. The negative spillover is shown to be characterized with our invented concept on network; “arrow-twist” property. For the worst scenario, what we found is that “interconnectedness” of network possibly brings positive spillover effect. The positive spillover is shown to be characterized with our another invented concept; “vertex-twist” property.

Interestingly, our two concepts have certain relation to each other in a way that existence of “arrow-twist” property also implies “vertex-twist” property but not necessarily for the opposite direction. It indicates that whether there exists positive or negative spillover in relation to our “interconnectedness” of network cannot always be scenario independent. Actually, networks that hold “arrow-twist” property allow to emerge both of negative spillover effect and positive spillover effect.

5.2 chapter 2

Chapter 2 titled as “Does Central Counterparty Reduce Liquidity Requirement?” applies our basic analysis in Chapter 1 to examine decentralized settlement combined with its sub-platform. Combined sub-platform is termed as Central Clearing Counterparty(CCP). CCP is introduced as a provider of central clearing service itself for a particular class of securities, and also a participant for relevant interbank settlement system with its own account in the central bank. Implication of introducing CCPs is shown to be clarified in reference to this dual nature of its role.

We show that the total effect of introducing a CCP is decomposed into two types of effect; “routing effect” and “netting effect”, where the former effect is associated with its participant aspect, while the latter associated with its aspect as clearing service provider. Each of our two conceptualized “routing effect” and “netting effect” is shown to be scenario-dependent. We specifically take up two scenarios introduced in Chapter 1; “best scenario” and “worst scenario”.

For “routing effect”, it is simply shown to have always positive effect under the best scenario, while have always negative effect under the worst scenario. Intuitively, adding a participant that is placed as an an mediator for multiple payments under the best scenario tends to promote more efficient recycle of settlement fund by “connecting” otherwise “disconnected” route for settlement fund to be transfered. Under the worst scenario, additional participant would possibly act as an additional “stop” for settlement fund.

For “netting effect”, it is possibly positive and negative under the best scenario, while it is always positive under the worst scenario. The reason of possible negative effect under the best scenario is that realizing netting for a cycle of payments can inhibit more efficient recycle of settlement fund within a larger cycle of payment, by separating the cycle of payments to be settled with settlement fund into smaller cycles of payments.

From the view of total effect, we can summarize our analysis as follows. It is shown that under the best scenario, positive “routing effect” can possibly be countered by negative ”netting effect”, while negative “routing effect” can possibly be mitigated or overcome by positive ”netting effect” under the worst scenario.

5.3 chapter 3

Chapter 3 titled as “Liquidity Saving Mechanism under Interconnected Payment Network” provides an analysis on liquidity saving mechanism(LSM) that serves as internal offsetting mechanism within RTGS systems. In comparison to roles served by CCP, LSM has the same feature in providing central clearing. It basically lets us apply the analysis in Chapter 2 to that on LSM, but the analysis turns not sufficient for the purpose of this chapter. Our focus is now not on issues arisen solely by offsetting service. We analyze effect of offsetting service provided by LSM together with the existence of CCP. The key ingredient of analysis in this chapter is to treat two types of obligations differently: obligations among participant banks, and those to CCP. Concretely, each payment to CCP by a participant bank appears as a liquidity shock at the middle of the period in a day, while the other type of obligations among participant banks are endowed at the beginning of the day. Examining heterogeneity on probability of relevant liquidity shock is to unravel further negative effect associated with central clearing, that could not be revealed in Chapter 2. We are to term effect we are to reveal as "indirect" negative effect, in contrast to "direct" negative effect already shown in Chapter 2. The reason of our terminologies is that "indirect" negative effect requires change of certain actions in reaction to clearing itself, while "direct" negative effect does not.

The assumption of heterogeneous liquidity shock can be interpreted to be sourced by difference in types of customers among participant banks; how much their customers are likely to have obligations settled in CCP . Heterogeneous liquidity shock is shown to have its consequence in affecting choice of timing of payments among participant banks. In this respect, an additional key feature in this chapter is explicit examination of incentive of banks on choice of "absolute" timing of payment. Participant banks are to choose timing of their payments in the face of trade-off between cost of delaying payment and cost of intraday borrowing. Heterogeneity of liquidity shock effectively let banks either "patient" or "impatient" as higher(lower) probability of liquidity shock lets banks be "patient"("impatient"), in the sense that "patient" banks tend to make their payments in latter timing compared to "impatient" banks.

Our indirect negative effect is shown ready to arise when cost structure lies in a certain range, where liquidity cost is relatively expensive compared to cost of delaying payment. Suppose LSM had not been introduced under the situation, existence of "patient" bank served to spillover positive effect in a way to let "impatient" banks effectively to turn to be "patient" to make their payments in latter timing, which served to decrease relatively expensive liquidity(borrowing) cost by promoting more efficient recycle of settlement fund. The key observation in our analysis is that such positive effect spillovers only through connected network of obligations. Indirect negative effect brought by introducing LSM is understood to work by dismissing that positive spillover. Introducing LSM serves to "cut" otherwise connected network of obligations, which let positive effect by each "patient" bank to spillover only among "shorter" networks, dismissing positive effect for "isolated" networks that with no "patient bank".

For our analysis, we take up a specific class of network that is to capture real-world payment network, in the face of would-be intractability on the general class of network adopted in Chapter 1, 2. What this chapter focuses is a theoretical class of payment network we invent so as to capture properties of networks documented under the terminology of "core-periphery" structure[9]. Each "core-periphery" network consists of "core" banks and "periphery" banks where "core" banks are more interconnected each other. Within our class of "core-periphery" structure, networks are characterized with "density" regarding how "core" banks are connected to "periphery" banks. Along with the notion of "density", it is shown that both indirect and direct negative effect tends to be larger under more "dense" network.

[9]Real world payment networks were examined from the view of "core-periphery" structure in Soramaki et al. (2007), Rordam and Bech (2009) and Imakubo and Soejima (2010).

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Chapter 1.

Compexity of Payment Network

Abstract

A graph-theoretic framework is developed to study decentralized settlement in a general payment network. This paper argues settlement efficiency through examining how much settlement fund needs to be provided to settle all given obligations. Observing that required amount of settlement fund depends on in which order those obligations are settled, we focus on a pair of problems that derives its lower-bound and upper-bound, each formalized as a numbering problem on flow network. Our main finding is that twist nature of underlying directed graph (who obliged to whom) is a key factor to form settlement efficiency. The twist nature is captured through our original concepts; *arrow-twisted*, and *vertex-twisted*. Lower-bound of required settlement fund tends to be larger when underlying directed graph is twisted in *arrow-twisted* sense, while upper-bound tends to be smaller when it is twisted in *vertex-twisted* sense.

Keywords: settlement, payment network, interconnected financial system, graph-theoretic model
JEL classification: D53, D85, G20

1 Introduction

The paper proposes a graph-theoretic framework to study decentralized settlement in a general payment network. We formalize and examine a pair of problems that have important applications in interbank settlement systems.

Following simple example hints our framework, also quickly introduces our problems.

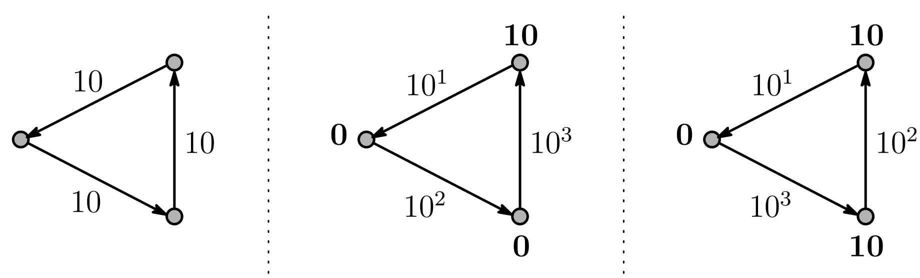

Figure 1:

The environment for our problems is summarized in the left of Figure 1, which shows a distribution of obligations among economic subjects. Each of the three vertices specifies an economic subject, or a participant in the settlement system. Each of the three arrows traces each relation of obligation between two subjects. Each of the numbers indicates individual amount of each obligation. To summarize the environment, each of the three subjects has one obligation and one claim, each with 10 amount. We suppose each obligation needs to be settled via a transfer of settlement fund in each specified amount. For our analysis, we only allow each unit of obligation to be settled at one time, prohibiting settlement in multiple times.

Under these suppositions, we investigate settlement efficiency, or how much settlement fund needs to be provided to settle all the obligations. We notice that relative order of settlement is crucial. In the middle of Figure 1, each order of settlement is expressed by each number written in the upper-right of each amount of obligation. In this order of settlement, only the subject on the top needs to input settlement fund in the amount of 10, as indicated in boldface. The two other subjects need not input settlement fund because they can recirculate the payments they receive. The figure is interpreted to express a possible settlement procedure for our given distribution of obligations. Under this settlement procedure, 10 amount of settlement are required in total. In contrast, under a different settlement procedure as depicted in the right of Figure 1, total required amount of settlement is now 20 as confirmed similarly.

Supposing any relative order of settlement be possible to realize, the paper specifically examines lower-bound and upper-bound of total required amount of settlement fund. The lower-bound would be attained when order is optimally chosen by a central planner, that is each central bank for the case of settlement in interbank settlement systems. The upper-bound would be possible when order is formed under ill-coordination among subjects. The paper examines these specified problems in a general setting.

Settlement efficiency is one of critical concern in recent interbank settlement systems. Traditionally, interbank settlement systems processed transactions on a net basis; payments are collected and settled only at certain designated time –typically once a day– , and participant banks make net payments; the difference between payments received and payments owed. Realizing that net settlement systems are prone to cascades of defaults, many of interbank settlement systems now adopt real-time gross settlement (RTGS) systems that settle each payment on an individual basis.

Though RTGS systems reduce the risk of cascades of defaults compared to net settlement systems, it tends to require considerable settlement fund. When participants hold insufficient funds for settlement, typically central banks provide settlement fund through intraday lending. For well-functioning of settlement systems, it is crucial to provide sufficient settlement fund. The critical question there is how much settlement fund is required for settlements in each interbank settlement system. This study investigates the question by presenting two formal problems pertaining to each lower-bound and upper-bound of required fund. Accordingly, it is a benchmark for further research into that issue.

One of the main contribution of this paper is to provide a mathematical framework for decentralized settlement. The framework enables to express every possible settlement procedures, which allows us to argue settlement efficiency. Eisenberg and Noe (2001) provided a different framework for decentralized settlement. His framework is not to examine settlement efficiency, by effectively supposing settlement always with infinitely small unit of installment. There, settlement efficiency is assumed highest possible. Our framework is able to examine settlement in arbitrary unit of installment each as a different environment, that means settlement in infinitely small unit is encompassed as one environment. Rotemberg (2011) firstly pointed out in a persuading manner that settlement efficiency gets worse depending on which settlement procedure is realized. The paper focused on certain class of network structure whose underlying directed graph is to be Euler graph, and assumed specific behavioral pattern of subjects. Our research proceeds to a different direction in examining both lower-bound and upper-bound of required settlement fund supposing much wider settlement procedures are possible to realize. In our direction, we show several key network factors contribute to form settlement efficiency.

Our main finding is that twist nature of underlying directed graph is a key factor to form settlement efficiency. The twist nature never appears in the class of network Rotemberg (2011) examined, and it is captured with our original concepts; *arrow-twisted* and *vertex-twisted*.

Here we take up several examples to see how those concepts are defined, how they matter for our problems, and also what they imply in economic sense. Figure 2 presents three example environments for our problems.

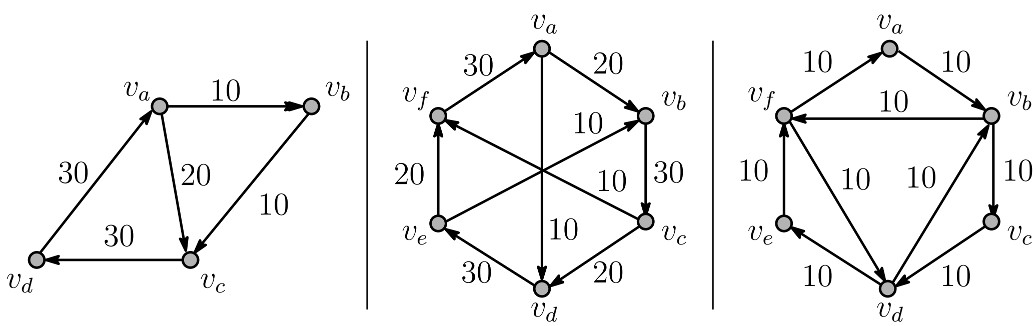

Figure 2: The left of the figure has no relevance neither to *arrow-twisted* nor *vertex-twisted*. The middle is relevant to *arrow-twisted*, while the right is relevant to *vertex-twisted*

We start by an environment expressed in the left of the figure, which has no relevance neither to *arrow-twisted* nor to *vertex-twisted*. Role of *arrow-twisted* and *vertex-twisted* is explained in comparison with this example. There are four subjects v_a, v_b, v_c, and v_d. Five obligations have been formed among those subjects. The lower-bound of required settlement fund is derived as 30, that is realized with a settlement procedure in the left of Figure 3. The upper-bound is 70, that is realized with that in the right of the same figure.

Now hypothetically decompose distribution of obligations into two distributions as shown in Figure 4. In latter analysis, we formally define our decomposition. Here notice that total amount of "decomposed" obligations for each subjects are equal to the original amount. For example, original obligation by v_d to v_a is 30, which is divided into 20 and 10. Now derive lower-bound and upper-bound for each decomposed distributions of obligations. There we hypothetically suppose each of divided obligations were settled in each unit. The lower-bound for each of the decomposed distributions of obligations is 20, and 10. while the upper-bound is 40, 30 each. For this example, we confirm that mere summation of the lower-bound, upper-bound comes back to each for the original distribution of obligations, as confirmed as $20 + 10 = 30$, while $40 + 30 = 70$.

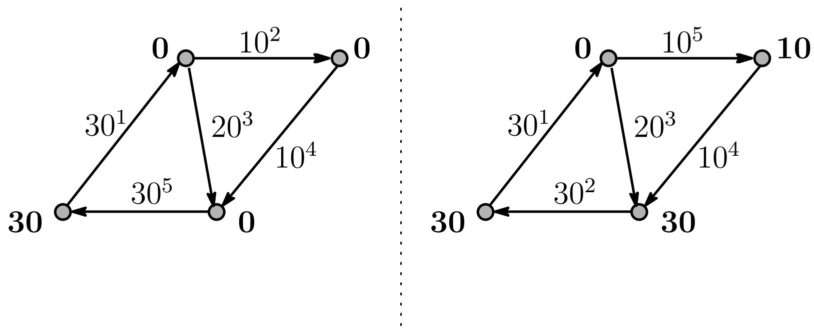

Figure 3: The left settlement procedure attains the lower-bound 30 of required settlement fund for distribution of obligations shown in the left of Figure 2, while the right attains the upper-bound 70.

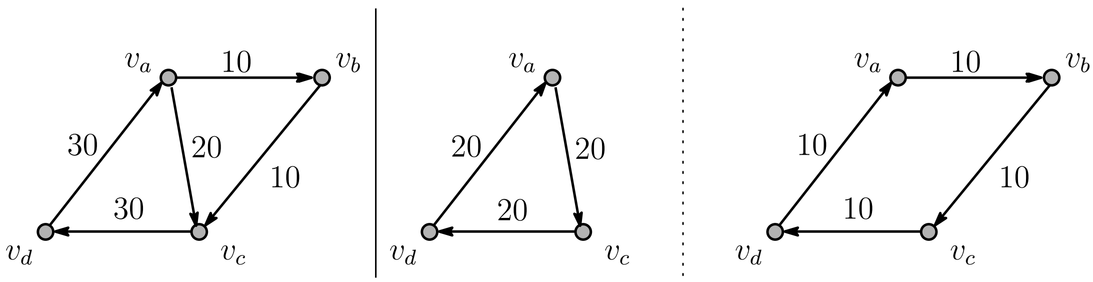

Figure 4: Original distribution of obligations shown in the left of the figure is decomposed into that in the middle, and that in the right.

Let us move to the case expressed with the middle of Figure 2, which is relevant to *arrow-twisted*. The lower-bound is 40 as in Figure 5, and we are going to see how *arrow-twisted* contributes to form that value. Now we have a decomposition as shown in Figure

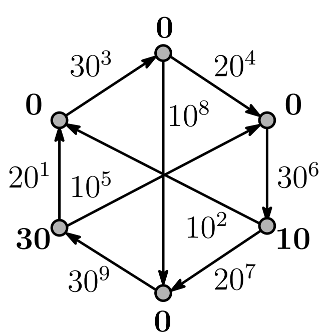

Figure 5: Example settlement procedure that attains lower-bound 40 for the middle of Figure 2.

6. We confirm that mere summation of the lower-bound for each decomposed distribution of obligations is 30, which is less than 40. The reason is that there emerges inconsistency of synchronization among hypothetically divided obligations. Focus on three obligations, by v_f to v_a, by v_b to v_c, and by v_d to v_e. Observe that for each of the decomposed distributions of obligations, the three obligations are to be settled in an order along with direction indicated by the arrows when each lower-bound is attained. But for the original distribution of obligations, there is no order that is consistent with both of such two

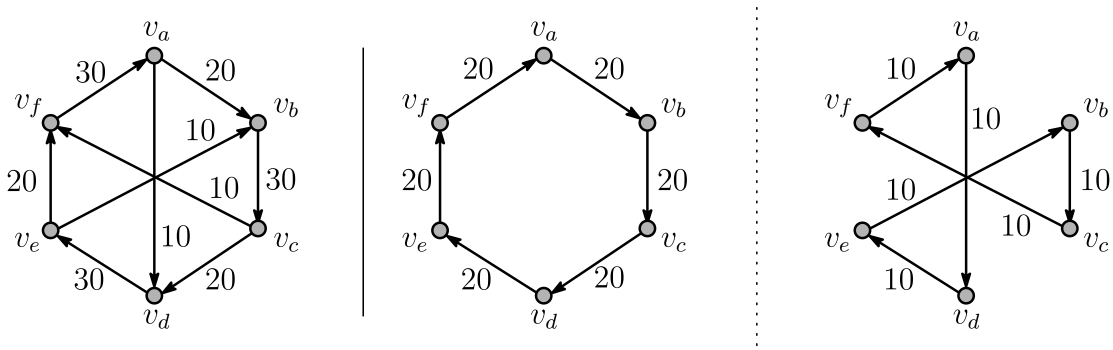

Figure 6: Original distribution of obligations shown in the left of the figure is decomposed into that in the middle, and that in the right.

orders. This inconsistency is thought to generate negative spillover in the sense that the lower-bound is larger than that without the inconsistency. The inconsistency is captured with our notion of *arrow-twisted* in general.

For our distribution of obligations, suppose obligation by v_f to v_a is to be settled in two units, 20 and 10 as shown in the left of Figure 7. Then, the lower-bound is now 30 as shown in the right of the same figure. We confirm that the inconsistency of synchronization is resolved and the lower-bound gets smaller. Further notice that input of settlement fund by v_f remains zero for those specific example settlement procedures. This indicates that inconsistency of synchronization is interpreted to arise from an externality when we suppose subject who inputs settlement fund incur corresponding financing cost, and also subject who owes obligation can choose unit of settlement.

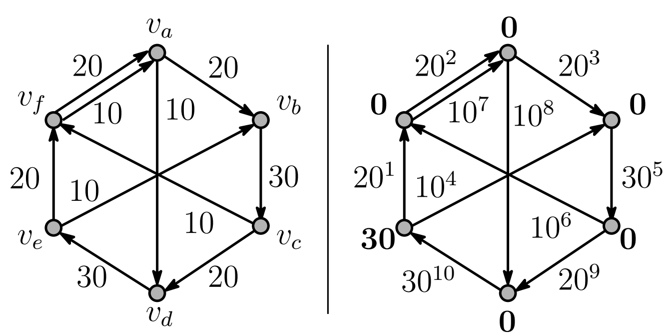

Figure 7: For the left distribution of obligations, the right shows an example settlement procedure that attains its lower-bound 30.

Let us proceed to the case expressed with the right of Figure 2, which is relevant to *vertex-twisted*. The upper-bound is 60 as in Figure 8, and we are going to see how *vertex-twisted* contributes to form that value. We have a decomposition as shown in Figure 9. We confirm that mere summation of the upper-bound for each decomposed distribution of obligations is $70 = 50 + 20$, which is larger than 60. The reason is that there emerges inconsistency of synchronization now regarding subjects. For each of the decomposed distribution of obligations, settlement needs to be executed along with the reverse of direction indicated by each arrows. Now for the original distribution of obligations, focus on

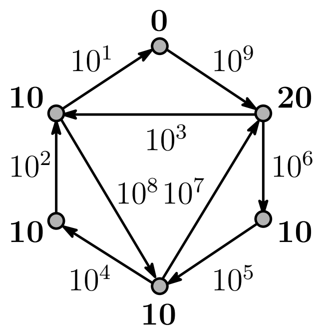

Figure 8: Example settlement procedure that attains upper-bound 60 for the right of Figure 2.

three subjects v_f, v_b, and v_d each has multiple obligations to make and receive. The obligations cannot be settled in a way that each of the three subjects make all their payments before their receipts as much as possible and also consistent with both of the orders for decomposed distributions of obligations each attains upper-bound of required settlement fund. This inconsistency is thought to generate positive spillover in the sense that the upper-bound is smaller than that without the inconsistency. This type of inconsistency of synchronization is captured with our notion of *vertex-twisted* in general. Confirm that

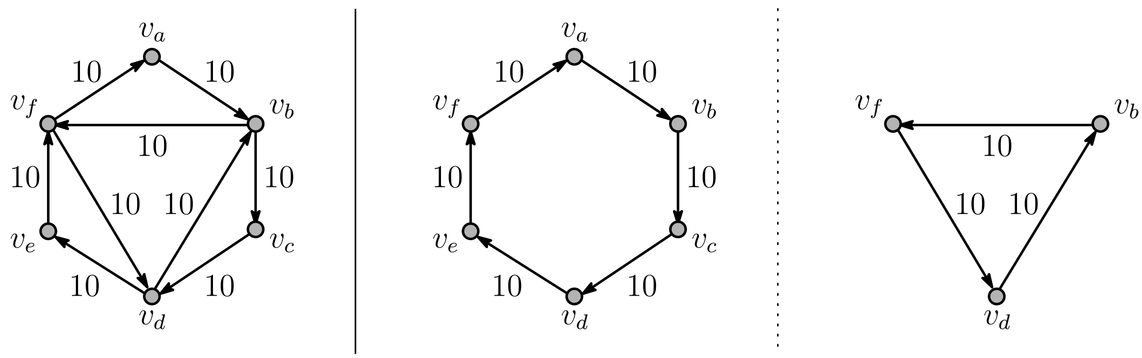

Figure 9: Original distribution of obligations shown in the left of the figure is decomposed into that in the middle, and that in the right.

the inconsistency indicated by *vertex-twisted* is dismissed in Figure 10, where direction of obligation among v_f, v_b, v_d is oppositely formed.

The remainder of this study is organized as follows. Section 2 introduces our framework and supplies definitions essential for statement of our problems, and Section 3 presents our problems formally. Section 4 offers preliminary analyses, introducing the central concept of *closed cycle decomposition* alongside several fundamental results. Section 5 displays the first half of our analyses, where we introduce three network properties—*domination*, *arrow-twisted* and *vertex-twisted*—and show how they help characterize the problems. Those properties combined with others are proposed as the key characteristics for our problems. Section 6 extends our analysis in detail, examining several types of transformation of networks in relation to the key characteristics. Section 7 reviews relevant earlier literature, and Section 8 concludes. The Appendix includes proofs for several theorems and additional results relevant to specific literature.

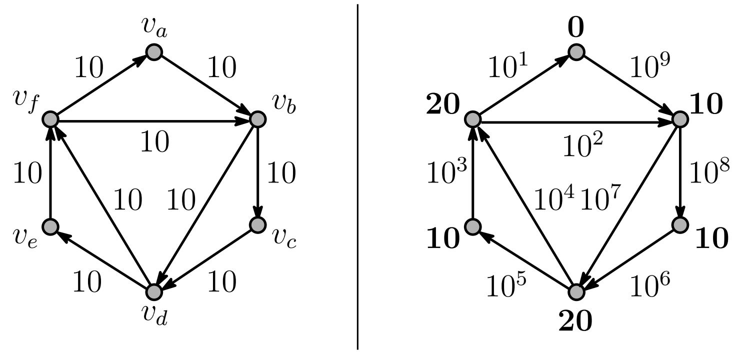

Figure 10: For the left distribution of obligations, the right shows an example settlement procedure that attains its upper-bound 70.

2 Model and Definitions

Our framework consists of five elements, which are expressed with five characters: V, A, f, s, p. The base elements are V and A, where V is a set of *vertices* which expresses economic subjects, while $A = \{(v, w, n) | v, w \in V, n = 1, 2, ..\}$ is a set of *arrows* each of which is an ordered pair of vertices with each index, and expresses payment relation between a pair of subjects. Indices are used to distinguish different payments among the same subjects. If there is no such multiplicity, all the indices are set as 0, and the indices are usually not mentioned in order to avoid notational cumbersome. We do not allow any arrow from and to the same vertex, or exclude payments from and to the same subject. $< V, A >$ constitutes a *directed graph*. An example directed graph is shown in the left of Figure 11.

We are to add additional elements f, s, p to $< V, A >$ to constitute two types of *Networks*; $< V, A, f >$ and $< V, A, f, s, p >$, where $< V, A, f >$ is to express distribution of obligations, and $< V, A, f, s, p >$ is to indicate its relevant settlement procedure. Firstly, $f : A \to R_+$ is called as *flow*, which expresses each amount for each payment. Secondly, $s : A \to \{1, 2, .., |A|\}$ is called as *sequence*, which is one-to-one mapping where $|A|$ denotes the total number of arrows, and economically it expresses relative order of settlement. Lastly, $p : V \to R_{0+}$ is called as *potential*, which expresses amount of settlement fund input by each subject.

We simply term $< V, A, f >$ as *f-Network* and $< V, A, f, s, p >$ as *fsp-Network*. The middle of Figure 11 shows an example of f-Network, and the right of the figure shows an example of fsp-Network constructed by adding s, p to the left f-Network.

In order to state our problems, we are to define two properties for the Networks, each of which is to express each economic assumption. One is termed as *closed* property of f-Network, which is to express amounts to make payment and to receive are balanced for each subject, which we call distribution of obligations are balanced. Given f-Network $< V, A, f >$, aggregate amount of payments to receive for $v \in V$ is denoted as $f_v^I \equiv \sum_{v' \in V} f((v', v))$, while aggregate amount of payments to make for $v \in V$ as $f_v^O \equiv \sum_{v' \in V} f((v, v'))$. Now *closed* property is defined as follows.

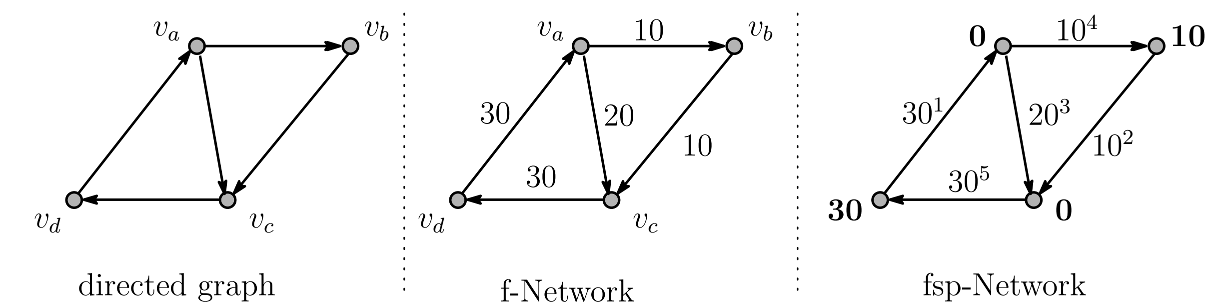

Figure 11: $V = \{v_a, v_b, v_c, v_d\}$, $A = \{(v_a, v_b), (v_a, v_c), (v_b, v_c), (v_c, v_d), (v_d, v_a)\}, f((v_a, v_b)) = f((v_b, v_c)) = 10, f((v_a, v_c)) = 20, f((v_c, v_d)) = f((v_d, v_a)) = 30, s((v_a, v_b)) = 4, s((v_a, v_c)) = 3, s((v_b, v_c)) = 2, s((v_c, v_d)) = 5, s((v_d, v_a)) = 1, p(v_a) = p(v_c) = 0, p(v_b) = 10, p(v_d) = 30$

Definition 1. closed property: balanced distribution of obligations
f-Network $< V, A, f >$ is *closed* if $f_v^I = f_v^O$ for every $v \in V$.

The middle of Figure 11 is an example of *closed* f-Network.

The other property is *e-covered*(exact covered) property for fsp-Network, which is to express settlement procedure is to be proper in a sense that each subject input sufficient amount of settlement fund as well as any of input settlement fund is not to be redundant under attached order of settlement. Given fsp-Network $< V, A, f, s, p >$, suppose periods proceed as $t = 0, 1, .., |A|$ where relative order, or sequence s corresponds to each period t in a way that payments to be executed at the beginning of period t are $arg_a s(a) = t$. Aggregate periodical payments to receive for $v \in V$ at period t is denoted as $f_{v,t}^I = \sum_{v' \in V} 1_{\{s(v',v)=t\}} f((v', v))$, while that to make is denoted as $f_{v,t}^O = \sum_{v' \in V} 1_{\{s(v,v')=t\}} f((v, v'))$. Then periodical holding of money for each subject $v \in V$ at the last of period t is denoted as $p^t(v) = p^{t-1}(v) + (f_{v,t}^I - f_{v,t}^O)$ for $t = 1, 2, .., |A|$ and $p^0 = p(v)$. settlement fund input by each subject is sufficient when every periodical holding is sufficient. Sufficiency condition is defined as *covered* property. $< V, A, f, s, p >$ is *covered* if $p^t(v) \geq 0$ for every $v \in V$ and every $t = 0, 1, .., |A|$. Property of *e-covered* is defined as *covered* property added with property of no redundant settlement fund, as stated below.

Definition 2. e-covered property: proper settlement procedure
fsp-Network $< V, A, f, s, p >$ is *e-covered* (exact covered) if
(no shortage) $< V, A, f, s, p >$ is *covered*, and
(no redundancy) there is no other $p' : V \to R_{0+}$ such that $< V, A, f, s, p' >$ is *covered*, and $p'(v) \leq p(v)$ for every $v \in V$, and $p'(v') < p(v')$ for some $v' \in V$.

The right of Figure 11 shows an example of *e-covered* fsp-Network.

For $< V, A, f, s >$ on closed $< V, A, f >$, e-covered $< V, A, f, s, p >$ is uniquely derived. When $< V, A, f, s, p >$ is e-covered, we term *circulation* for $< V, A, f, s >$ is $\sum_{v \in V} p(v)$.

3 Payment Circulation Problem

We define our problem to derive lower-bound of required settlement fund as *minimum Payment Circulation Problem (min PCP)* which is formally stated as follows;

(min PCP in original form)

Given a closed f-Network $< V, A, f >$,
$\min_{s,p} \sum_{v \in V} p(v)$,
s.t. fsp-Network $< V, A, f, s, p >$ *is covered.*

Our problem to derive upper-bound of required settlement fund is formalized as *maximum Payment Circulation Problem (max PCP)* as follows;

(max PCP in original form)

Given a closed f-Network $N^f =< V, A, f >$,
$\max_{s,p} \sum_{v \in V} p(v)$,
s.t. fsp-Network $< V, A, f, s, p >$ *is e-covered.*

We term value derived by each min/max PCP as *min/max circulation.*

Though this paper focuses on these min/max PCP, we can view that the problems belong to a further abstract problem;

Find a set of fsp-Networks which satisfies condition X*.*

In the latter literature section, economic researches in the field of settlement system, emergence of money, and currency area are reviewed in reference to this general form, and our contribution is stated along with the view.

3.1 First thought on the min/max PCP

Let us take up a candidate of approach to derive min/max circulation, which is only partially successful, and whose failure motivates our approach.

Suppose our input is f-Network shown in the left of Figure 12. Then, we have that the middle of the figure shows an fsp-Network which attains the minimum circulation, while the right of the figure is that for the maximum circulation. First for the case of the minimum, we observe that sequence is taken in a way that number is increasing along with direction indicated by the arrows. Actually, start by v_d, move to v_a, then v_c, back to v_d, sequence for corresponding arrows is increasing as 1, 3, 5. For the different route: from v_d, v_a, v_b, v_c, and back to v_d, sequence is also increasing as 1, 2, 4, 5. For the case of the maximum, we observe that sequence is taken in a way that number is increasing along with the opposite direction. Confirm that starting by v_d and move to v_c, v_a back to v_d, sequence is 1, 2, 3. It is similarly confirmed for the other route.

Conversely, we could formulate each of the above ways of taking sequence as an algorithm to derive each min/max circulation combined with some appropriate detailed procedures. These simple algorithms will actually solve min/max PCP for certain class of f-Networks, but not in general. For example, the algorithms do not work for f-Network as shown in the middle of Figure 2.

Our approach departs from constructing algorithms, instead grasp the problems in a topological manner so that relevant economic contexts are to be revealed.

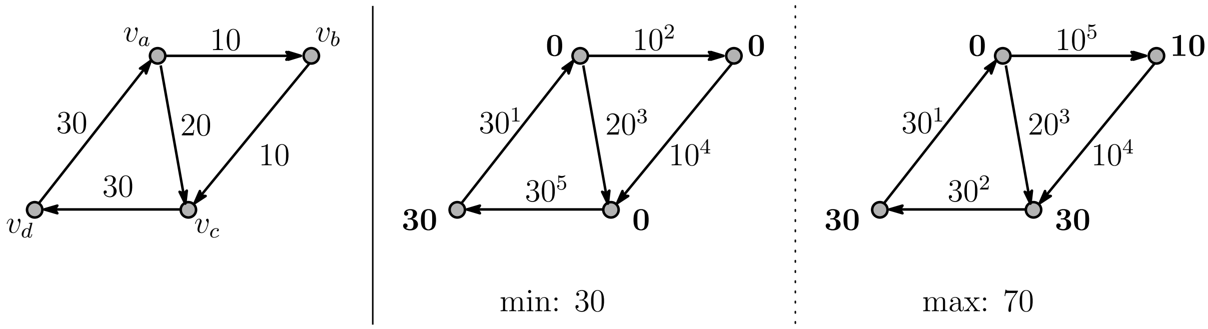

Figure 12: For the left f-Network, the middle fsp-Network attains the minimum circulation, while the right fsp-Network attains the maximum circulation.

4 Preliminary Analysis

This section presents key notions and results for our latter analyses.

We say a f-Network $< V, A, f >$ is not connected when we can divide into $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ such that there is no arrow $(v, v') \in A$ where v and v' belongs to different set with respect to V_1, V_2. In that case, it is apparent if we divide $< V, A, f >$ into two f-Networks along with such V_1, V_2 and associate arrows and flow, we derive min/max circulation for the original f-Network just as summation of that for each divided f-Network. Throughout this article, without loss of generality we focus on f-Networks that are connected.

4.1 Closed Cycle Decomposition

Our approach bases on an observation that closed f-Network can be *decomposed* into several closed f-Networks. *Decomposition* of f-Network is an algebraic notion on f-Networks, which is naturally derived from addition on real number.

Figure 13 shows an example of *decomposition* on f-Network. N^f is *decomposed* into N^f_1 and N^f_2, which is denoted as $N^f = N^f_1 + N^f_2$.

Formally, we term that a f-Network $N^f =< V, A, f >$ is *decomposed* into $\left\{ N^f_k =< V_k, A_k, f_k > \right\}_{k=1,2,..,K}$ if $V = \cup_{1 \leq k \leq K} V_k$ and $A = \cup_{1 \leq k \leq K} A_k$, and $\forall a \in A$, $f(a) = \sum_{k \in K'} f_k(a)$, where $K' = \{k'' | a \in A_{k''}\}$. We denote $N^f = \sum_{k=1}^{K} N^f_k$ for the decomposition.

We find a specific type of *decomposition* is critical for analyses of min/max PCP, which we term as *closed cycle decomposition*. *Decomposition* in Figure 13 is actually a *closed cycle decomposition*, where each of decomposed f-Networks is *closed*, and each consists of

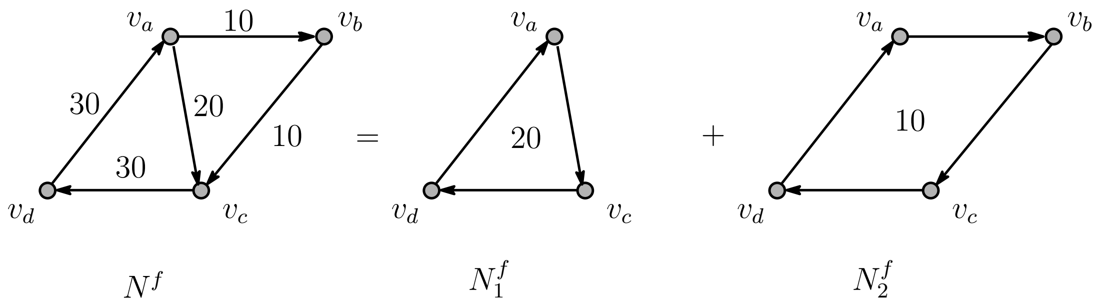

Figure 13: $N^f =< V, A, f >$ is the same as the left of Figure 11. $N_1^f =< V_1, A_1, f_1 >$, where $V_1 = \{v_a, v_c, v_d\}$, $A = \{(v_a, v_c), (v_c, v_d), (v_d, v_a)\}$, $f((v_a, v_c)) = f((v_c, v_d)) = f((v_d, v_a)) = 20$. $N_2^f =< V_2, A_2, f_2 >$, where $V_2 = \{v_a, v_b, v_c, v_d\}$, $A = \{(v_a, v_b), (v_b, v_c)(v_c, v_d), (v_d, v_a)\}$, $f((v_a, v_b)) = f((v_a, v_c)) = f((v_c, v_d)) = f((v_d, v_a)) = 0$. Cycle value for N_1^f is 20, which is expressed in the center of the f-Network, while it is 10 for N_2^f.

one *cycle*. For our formal statement of *closed cycle decomposition*, we define *cycle* and several relevant terminologies.

Given a directed graph $< V, A >$, we denote a set of vertices included in $A' \subseteq A$ as $V_{A'}$, and denote a set of arrows which includes $v \in V$ as A_v. For a directed graph $< V, A >$, $A' \subseteq A$ is a *path* from $v \in V_{A'}$ to $v' \in V_{A'}$ if we can order vertices in $V_{A'}$ such that $(v, v_1, v_2, .., v')$ where each consecutive ordered pair of vertices consists A'. The same arrow is not allowed to appear more than once in a path, but it is allowed for the same vertex. $A' \subseteq A$ is a *cycle* if A' is a path between the same vertex. We say a cycle is *punctured* if it includes the same vertex, and say *non-punctured* if not. For a directed graph G, we denote C_G as the set of cycles included in G, and call it as the cycle set of G.

Our formal definition of *closed cycle decomposition* is as follows.

Definition 3. closed cycle decomposition

A f-Network $N^f =< V, A, f >$ with $G =< V, A >$ is *closed cycle decomposed* into $\left\{N_k^f =< V_k, A_k, f_k >\right\}_{1 \leq k \leq K}$ if

1) $N^f = \sum_{k=1}^K N_k^f$ is a *decomposition*, and

2) $\forall k = 1, 2, ..., K$, each N_k^f consists of mutually different one *cycle* and is also *closed*.

We specifically write $N^f = \sum_{c \in C} < V^c, c, f^c >$ for a closed cycle decomposition with $C \subseteq C_G$, where f^c is referred as cycle value for c. Note that closed cycle decomposition is allowed to include decomposed f-Networks which consist of one *punctured cycle*.

We have following result for *closed cycle decomposition*.

Theorem 1 (Ford and Fulkerson (1962)).

Any closed f-Network can always be closed cycle decomposed.

The theorem ensures that our observation of closed cycle decomposition in Figure 13 is not just luck. However, it is shown that closed cycle decomposition in Figure 13 is a rather special case regarding *uniqueness* of closed cycle decomposition. We say f-Network is uniquely closed cycle decomposed if there is no other closed cycle decomposition for the same f-Network such that cycle sets are different[1]. Uniqueness of closed cycle decomposition is characterized with the notion of *disjoint*. Intuitively, two sets of cycles are *disjoint* if we cannot create the other cycle using the arrows included in the two sets. Formally, given a set of cycles C in some directed graph, we say $C', C'' \subseteq C$ are *disjoint*(regarding C) if $\forall K \in C$ and $K \subset A_{C' \cup C''}$, $K \notin C \setminus (C' \cup C'')$. We say C is a *disjoint set* if $\forall C', C'' \subseteq C$, C' and C'' are disjoint.

Corollary 1 (Uniqueness of closed cycle decomposition)**.**
Given a closed f-Network $N^f =< V, A, f >$ *with* $G =< V, A >$, N^f *is uniquely closed cycle decomposed if and only if* C_G *is a disjoint set.*

We turn to *decomposition* on fsp-Network, which is similarly defined with that on f-Network. Figure 14 and 15 are examples of *decomposition* on fsp-Network.

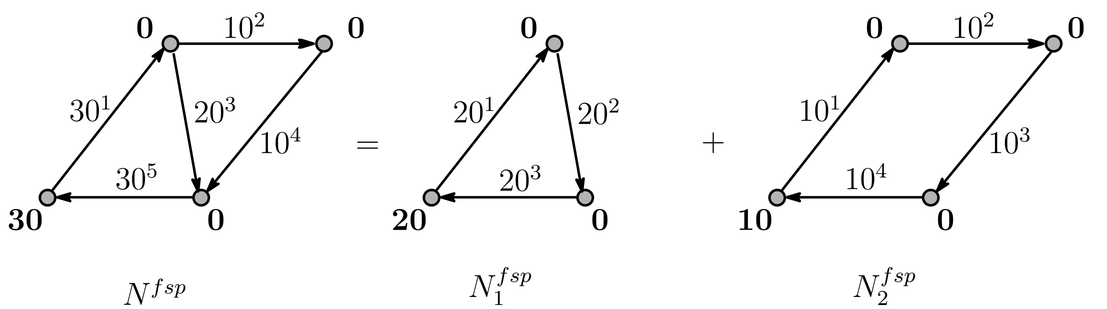

Figure 14: example for decomposition on fsp-Network

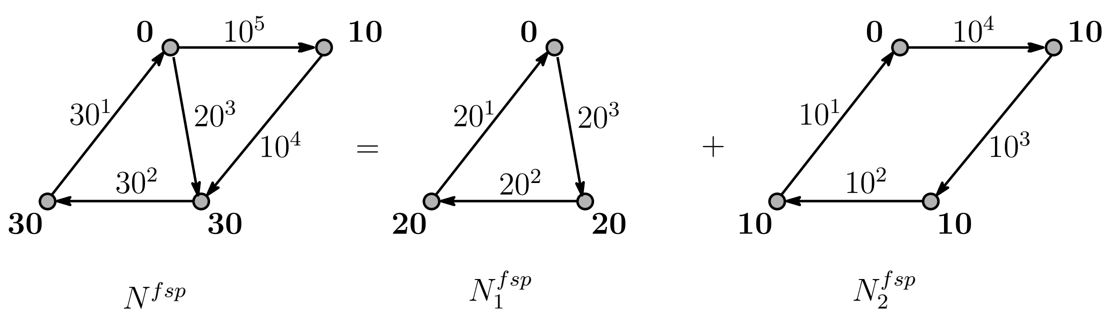

Figure 15: example for decomposition on fsp-Network

Formally, we say fsp-Network $N^{fsp} =< V, A, f, s, p >$ is *decomposed* into fsp-Networks $\left\{N_k^{fsp} =< V_k, A_k, f_k, s_k, p_k >\right\}_{k=1,2,..,K}$ if
(1) $< V, A, f >$ is decomposed into $\{< V_k, A_k, f_k >\}_{k=1,2,..,K}$,
(2) each sequence s_k is consistent with s in the sense that the ordering is preserved, and
(3) $\sum_k p_k(v) = p(v)$ for every $v \in V$.

[1]For uniqueness of closed cycle decomposition, we ignore trivial differences such that difference is only on sets of vertices, such that whether isolated vertices are included or not

When a decomposition on fsp-Network $N^{fsp} =< V, A, f, s, p >$ is also a closed cycle decomposition on corresponding f-Network, we write as: $< V, A, f, s, p >= \sum_{c \in C} < V^c, c, f^c, s^c, p^c >$.

Notice that in Figure 14 and 15, e-covered fsp-Network is decomposed into fsp-Networks which are all e-covered.

Theorem 2. *decomposition on e-covered fsp-Network*

Given a closed f-Network $< V, A, f >$, for any e-covered fsp-Network $N^{fsp} =< V, A, f, s, p >$, there exists decomposition $N^{fsp} = \sum_{c \in C} < V^c, c, f^c, s^c, p^c >$ such that $< V^c, c, f^c, s^c, p^c >$ is e-covered for every $c \in C$.

Proof. See Appendix A.2. □

5 Key Characteristics for the min/max PCP

Purpose of this section is to reveal key characteristics for the min/max PCP. Rearranging the original min/max PCP utilizing *closed cycle decomposition* leads us to find them.

Notice that in Figure 14, fsp-Network that attains min circulation is expressed with *decomposed* fsp-Networks. Figure 15 shows that for the case of maximum. Stepping further, the next two theorems show that each of the min/max PCP is rewritten in a form to choose fsp-Networks for *closed cycle decomposed* f-Networks of given f-Network, while the original form is just to choose fsp-Network for given f-Network.

Theorem 3. *min PCP in decomposed form*

Given a closed f-Network $N^f =< V, A, f >$, the following problem gives the same value with the min PCP on N^f;

$\min_{s, C \in C_{N^f}, \{f^c\}_{c \in C}} \sum_{c \in C} \sum_{v \in V^c} p^c(v)$.

s.t. $N^f = \sum_{c \in C} < V^c, c, f^c >$ *is a closed cycle decomposition,*

$< V^c, c, f^c, s^c, p^c >$ *is e-covered for every* $c \in C$, *and*

$< V, A, f, s, p >= \sum_{c \in C} < V^c, c, f^c, s^c, p^c >$

Proof. see the appendix A.3. □

Theorem 4. *max PCP in decomposed form*

Given a closed f-Network $N^f =< V, A, f >$, the following problem gives the same value with the max PCP on N^f;

$\max_{s, C \in C_{N^f}, \{f^c\}_{c \in C}} \sum_{c \in C} \sum_{v \in V^c} p^c(v)$.

s.t. $N^f = \sum_{c \in C} < V^c, c, f^c >$ *is a closed cycle decomposition,*

$< V^c, c, f^c, s^c, p^c >$ *is e-covered for every* $c \in C$, *and*

$< V, A, f, s, p >= \sum_{c \in C} < V^c, c, f^c, s^c, p^c >$ *is e-covered.*

Proof. See Appendix A.4. □

The above decomposed forms of the min/max PCP need to be rearranged so as to reveal their own worth. Each of the decomposed form problems is to be separated into *decomposition choice part* and *sequence choice part*. We first define a sub-problem for each min/max PCP, each of which corresponds to the *sequence choice part*.

(sub-problem for min PCP)

Given a closed f-Network N^f and its closed cycle decomposition that is characterized with $C \in C_{N^f}$ and $\{f^c\}_{c\in C}$,

$\min_{\{s^c,p^c\}_{c\in C}} \sum_{c\in C}(\sum_{v\in V^c} p^c(v) - f^c)$

s.t. $< V^c, c, f^c, s^c, p^c >$ *is exact covered for every* $c \in C$, *and*

$< V, A, f, s, p >= \sum_{c\in C} < V^c, c, f^c, s^c, p^c >$.

(sub-problem for max PCP)

Given a closed f-Network N^f and its closed cycle decomposition that is characterized with $C \in C_{N^f}$ and $\{f^c\}_{c\in C}$,

$\min_{\{s^c,p^c\}_{c\in C}} \sum_{c\in C}((|c| - 1)f^c - \sum_{v\in V^c} p^c(v))$,

s.t. $< V^c, c, f^c, s^c, p^c >$ *is e-covered for every* $c \in C$, *and*

$< V, A, f, s, p >= \sum_{c\in C} < V^c, c, f^c, s^c, p^c >$ *is e-covered.*

, where $|c|$ denotes the number of arrows which constitute cycle c.

Denote each value as $R^{min/max}(N^f, C, \{f^c\}_{c\in C})$.

Next lemma ensures $R^{max}(.)$ has some value for any closed cycle decomposition.

Lemma 1.

Given a closed f-Network $< V, A, f >$ and its closed cycle decomposition $< V, A, f >= \sum_{c\in C} < V^c, c, f^c >$ Then we can always take fsp-Network $< V, A, f, s, p >$ and associated e-covered fsp-Networks $\{< V^c, c, f^c, s^c, p^c >\}_{c\in C}$ such that $< V, A, f, s, p >= \sum_{c\in C} < V^c, c, f^c, s^c, p^c >$ is e-covered.

Proof. Take arbitrary v-number $s_v : V \to \{1, 2, .., |V|\}$. Denote each set of vertices $V_k = arg_{v\in V} s_v(v) = k$ for $k = 1, 2, .., |V|$. Take sequence on arrows $a_k \in A$ that starts from $v \in V_k$ so that $\sum_1^{k-1} |V_{k-1} < s(a_k) < \sum_1^{k} V_k$. Such sequence s let us take p^c that each decomposed fsp-Networks is e-covered. Now for each vertex $v \in V$, take any two out-arrows $a' = (v, v'), a'' = (v, v'') \in A$. Then, there is no in-arrow $a''' = (v''', v) \in A$ such that $s(a') < s(a''') < s(a'')$. It is true for any two out-arrows. When we take $p(v) = \sum_{c\in C} p^c(v)$ for each $v \in V$, it says that the combined fsp-Network $< V, A, f, s, p >$ is also e-covered. □

Now we rewrite the min/max PCP in each separated form.

(min PCP in separated form)

Given a closed f-Network $N^f =< V, A, f >$,

$\min_{C\in C_{N^f},\{f^c\}_{c\in C}} \sum_{c\in C} f^c + R^{min}(N^f, C, \{f^c\}_{c\in C})$

s.t. $N^f = \sum_{c\in C} < V^c, c, f^c >$ *is a closed cycle decomposition.*

(max PCP in separated form)

Given a closed f-Network $N^f =< V, A, f >$,

$\max_{C\in C_{N^f},\{f^c\}_{c\in C}} \sum_{c\in C}(|c| - 1)f^c - R^{max}(N^f, C, \{f^c\}_{c\in C})$

s.t.$N^f = \sum_{c\in C} < V^c, c, f^c >$ *is a closed cycle decomposition*

For a given closed f-Network N^f, let $x^{min/max}(N^f)$ denote each min/max circulation.

In the rest of this sections, we are to reveal certain network properties help characterize each of the decomposition choice part and the sequence choice part of min/max PCP. We show *arrow-twist* property is key to the sequence choice part of min PCP, while *vertex-twist* property to the same part of max PCP, and *domination* property is to the decomposition choice part both for min/max PCP.

5.1 Property of *arrow-twist* and min PCP

Let f-Network shown in the left of Figure 16 be our input for the min PCP. We know that fsp-Network shown in the right of the figure realizes its min circulation 40. The min circulation is derived with a closed cycle decomposition shown in Figure 17. We confirm that the residual is solved as 10. In search of sources of the non-zero residual, we focus on two of the decomposed cycles. For each of the cycles, take sequence in a way that it is increasing along with direction indicated by its arrows. Suppose we start by the arrow (v_f, v_a). Then, focusing on two of the other arrows (v_b, v_c) and (v_d, v_e), in the left cycle sequence for (v_b, v_c) needs to be smaller than (v_d, v_e) while it is opposite for the right cycle. That is the source of non-zero value for residual, which is captured with the notion of *arrow-twisted*. Below we formally define *arrow-twisted* and related notions.

For Networks which include $G =< V, A >$ and sequence s on A, let cycle c consists of arrows $(a_1, a_2, ..a_n, a_{n+1} = a_1)$ where $a_k = (v_k, v_{k+1})$ for $k = 1, 2, .., n$, then the arrow-reverse number is defined as $r^{atwi}(c, s) = \sum_{k=1}^{n} 1_{\{s(a_k)>s(a_{k+1})\}}$. When there exist multiple ways to index arrows for a cycle and accordingly multiple values of *arrow-reverse number* (which is possible when it is punctured), set arrow-reverse number as the minimum among those. We say cycles in $C \subseteq C_G$ are in *arrow-twisted* relation, or just say they are *arrow-twisted* if we cannot take any sequence s such that $r^{atwi}(c, s) = 1$ for every $c \in C$. We say cycles in $C \subseteq C_G$ are *minimum arrow-twisted* when there exists no *arrow-twisted* cycles $C' \subset C$. Going back to Figure 17, the two of the decomposed cycles are *arrow-twisted* and *minimum arrow-twisted*. Note that *minimum arrow-twisted* cycles are not always a pair, as confirmed in Figure 18.

Property of *arrow-twist* for given f-Network among its cycle sets refers to whole relations of *arrow-twisted* among its sets of cycles as well as their *arrow-reverse numbers*.

For *arrow-twist* property, following lemma is fundamental for our analyses.

Lemma 2. *arrow-twisted and* $R^{min}(.)$

Given a closed f-Network N^f *and its closed cycle decomposition that is characterized with* $C \in C_{N^f}$ *and* $\{f^c\}_{c \in C}$,

$R^{min}(N^f, C, \{f^c\}_{c\in C}) = 0$ *if and only if* C *is not arrow-twisted.*

Proof. When C is not arrow-twisted, then we can always take sequence so that arrow-reverse number for every $c \in C$ is one. It lets us take $\sum_{v \in V^c} p^c(v) = f^c$ for every $c \in C$. Conversely, when $R^{min}(.) = 0$, we can always take arrow-reverse number is one for all $c \in C$ under any sequence that realises $R^{min}(.) = 0$. □

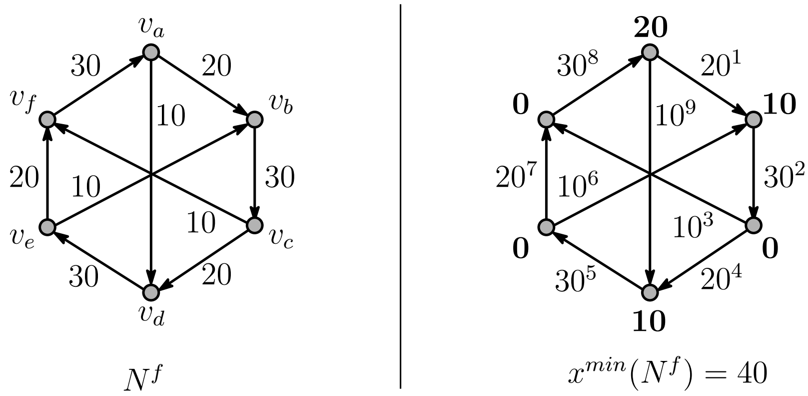

Figure 16:

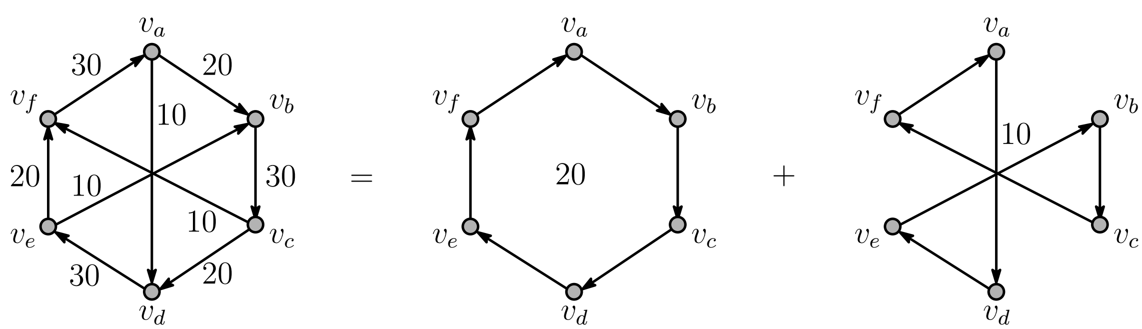

Figure 17:

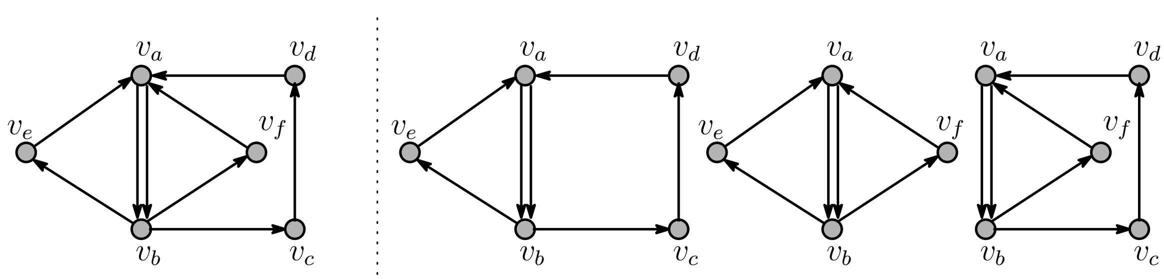

Figure 18: Example for arrow-twisted

For the left directed graph, the rest three cycles are minimum arrow-twisted.

We have a basic result for the case of *disjoint*.

Lemma 3.

For given closed f-Network $< V, A, f >$ with $G =< V, A >$ and its cycles $C \subseteq C_G$, we have

C is not disjoint if C is arrow-twisted.

Proof. For a closed f-Network, suppose some of its cycles C is *arrow-twisted*. Denote A as the set of arrows which constitutes C. Since C is *arrow-twisted*, we can take at least two cycles using arrows in A such that the two cycles have at least two common arrows which are not successive. Otherwise it is straight that we take sequence in a way that arrow-reverse number is all one for every possible cycles in A, which let C be not arrow-twisted.

If two cycles have two common arrows which are not successive, we can immediately take another cycle using part of arrows both from the two cycles. It says C is not disjoint, which ends our proof. □

The next theorem shows that min circulation is derived in a straight way for the case of *disjoint*.

Theorem 5. *min PCP for f-Network whose cycle set is disjoint*

For a closed f-Network $N^f =< V, A, f >$ with $G =< V, A >$, if C_G is disjoint, then

$x^{min}(N^f) = \sum_{c \in C_G} f^c$, *and*

with its closed cycle decomposition $N^f = \sum_{c \in C_G} < V^c, c, f^c >$.

Proof. From Lemma 2 and 3, we need not consider into sequence choice part for the min PCP in separated form. Further, Corollary 1 states that there exists only one *closed cycle decomposition* for the case of *disjoint*, which ends our proof. □

We show our additional results after introducing *domination* property in latter part.

5.2 Property of *vertex-twist* and max PCP

Let f-Network shown in the left of Figure 19 be our input for the max PCP. We know that fsp-Network shown in the right of the figure realizes its max circulation 110. The max circulation is derived with a closed cycle decomposition shown in Figure 20. We confirm that the residual is solved as -10. Let us focus on two of the decomposed cycles. For each of the cycle, take sequence in a way that it is increasing along with direction indicated by its arrows. Suppose we start by the arrow (v_f, v_a). Now examine in which order each vertex appears under supposed sequence. Focusing on the three of the vertices v_f, v_b, v_d, in the left cycle sequence for v_b needs to come before v_d while it is opposite for the right cycle. That is the reason for non-zero value for residual, and it is captured with the notion of *vertex-twisted*. Below we define *vertex-twisted* and related notions.

We prepare a different type of sequence for our model. For $< V, A >$, define *vertex-sequence* (sequence for vertex) $s_v : V \rightarrow \{1, 2, .., |V|\}$ as one-to-one mapping.

Let cycle c consists of $v_1 v_2 .. v_n v_{n+1}$ with $v_{n+1} = v_1$, then *vertex-reverse number* is defined as $r^{vtwi}(c, s_v) = \sum_{k=1}^{n} 1_{\{s_v(v_k) > s_v(v_{k+1})\}}$. When there exist multiple ways to index vertices for a cycle and accordingly multiple values of *vertex-reverse number* (which is possible when it is punctured), set *vertex-reverse number* for the cycle as the minimum among those. We say cycles in $C \subseteq C_G$ are in *vertex-twisted* relation, or just say they are *vertex-twisted* if we cannot take any vertex-sequence s^v such that $r^{vtwi}(c, s) = 1$ for every $c \in C$. We say cycles in $C \subseteq C_G$ are *minimum vertex-twisted* when there exists no vertex-twisted cycles $C' \subset C$.

Note that although any punctured cycle as in Figure 21 is *vertex-twisted* by itself, notion of *vertex-twisted* is not trivial in the sense that cycles which are not punctured can be also *vertex-twisted* as already shown in Figure 20, and as in Figure 22.

Property of *vertex-twist* for given f-Network refers to whole relations of *vertex-twisted* among its sets of cycles as well as their *vertex-reverse numbers*.

Notice that if cycles in C are *arrow-twisted*, then they are also *vertex-twisted* as stated in the following Lemma 4. The reverse is not always true as easily confirmed.

Lemma 4. *arrow-twisted and vertex-twisted*
Given $G =< V, A >$ and C_G, then for any $C \subseteq C_G$,
C are vertex-twisted if C are arrow-twisted.

Proof. Suppose C is not *vertex-twisted*. Then the definition says that we can take *vertex-sequence* s_v on vertices in C such that *vertex-reverse number* is one for all $c \in C$. Take *sequence* s^c on arrows for each $c \in C$ such that $s^c((v, v')) = s_v(v)$. We have $r^{atwi}(c, s^c) = 1$ for every $c \in C$. Since we can always take a *sequence* for arrows on C such that it is consistent with all the s^c, we know C is not *arrow-twisted*. □

For *vertex-twist* property, the following result is fundamental for our analyses.

Lemma 5. *vertex-twisted and $R^{max}(.)$*
Given a closed f-Network N^f and its closed cycle decomposition that is characterized with $C \in C_{N^f}$ and $\{f^c\}_{c \in C}$,
$R^{max}(N^f, C, \{f^c\}_{c\in C}) = 0$ if and only if C is not vertex-twisted.

Proof. When C is not *vertex-twisted*, then from its definition, we can always take *vertex-sequence* s_v on vertices in C such that *vertex-reverse number* is $|c| - 1$ for all $c \in C$. Denote each set of vertices $V_k = arg_{v \in V} s_v(v) = k$ for $k = 1, 2, .., |V|$. take sequence on arrows $a_k \in A$ that start from $v \in V_k$ so that $\sum_1^{k-1} |V_{k-1} < s(a_k) < \sum_1^k V_k$. Since there exist no vertex-twisted, such sequence s let us take p^c that each decomposed fsp-Networks is e-covered and $\sum_{v \in V^c} p^c(v) = (|c| - 1)f^c$. What needs to be shown is that combined fsp-Network with the decomposed fsp-Network is e-covered. For each vertex $v \in V$, take any two out-arrows $a' = (v, v'), a'' = (v, v'') \in A$. Then, there is no in-arrow $a''' = (v''', v) \in A$ such that $s(a') < s(a''') < s(a'')$. It is true for any two out-arrows. It says that the combined fsp-Network is e-covered.

For the converse direction, take a sequence s that realizes $R^{max}(N^f, C, \{f^c\}_{c\in C}) = 0$. Under the sequence s, for each cycle $c \in C$ with its set of vertices V^c, take a vertex $v^c \in V^c$ such that $s((v, v^c)) = argmin_{a \in c} s(a)$ and call it *head-vertex* for c. Then, for every vertices $v' \in V^c \setminus v^c$ with its arrow $(v', v) \in c$, there is no arrow $a = (v'', v') \in C$ such that $s(a) < s((v', v))$ since otherwise it immediately leads to $R^{max}(N^f, C, \{f^c\}_{c\in C}) > 0$.

It is true for every cycles $c \in C$. Then, we can naturally define partial order $<$ on $v \in V^c$ from sequence s in a way that each *head-vertex* is largest and gets smaller along with the direction opposite to that indicated by the arrows. We can always take vertex-sequence consistent with the order $<$, and under such vertex-sequence, vertex-reverse number is 1 for every cycles $c \in C$. It says C is not vertex-twisted.

$\square$

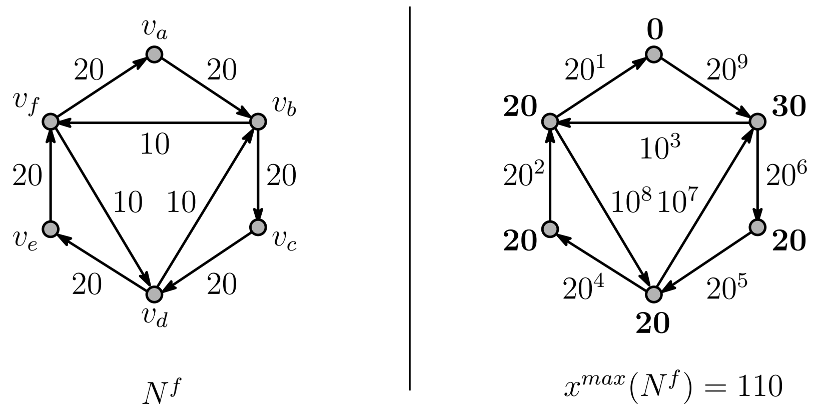

Figure 19:

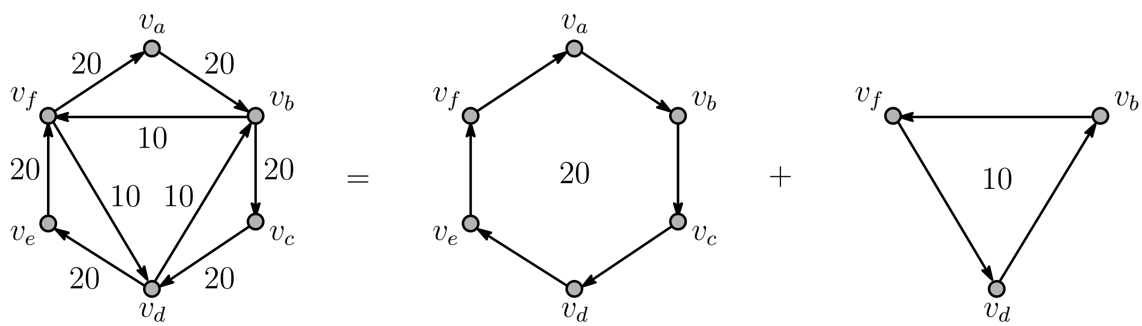

Figure 20:

We have a basic result for the case of *disjoint*.

Lemma 6.

For given closed f-Network $< V, A, f >$ with $G =< V, A >$ and its cycles $C \subseteq C_G$, we have

C is not disjoint if C is vertex-twisted.

Proof. For a closed f-Network, suppose some of its cycles C is *vertex-twisted*. When there exists any punctured cycle, it is immediate C is not disjoint. Suppose not. Denote A as the set of arrows which constitutes C. Since C is *vertex-twisted*, we can take at least two cycles using arrows in A such that the two cycles have at least two common vertices which are not included in successive common arrows of the two cycles. Otherwise it is straight that we take sequence in a way that vertex-reverse number is all one for every possible cycles in A, which let C be not vertex-twisted.

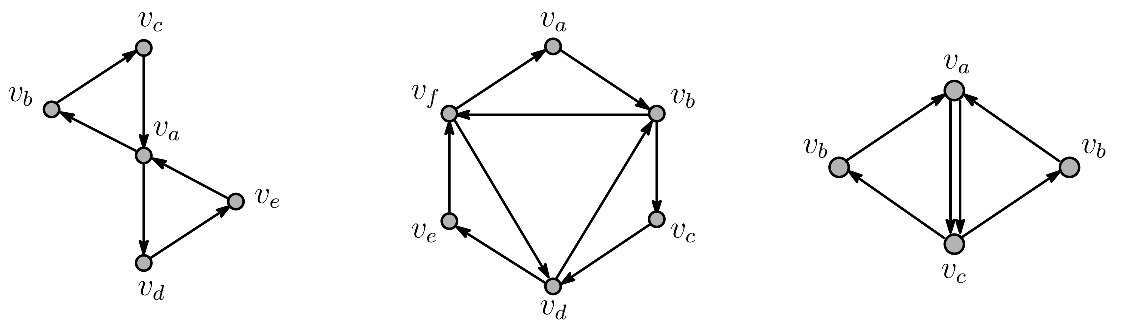

Figure 21: example for vertex-twisted cycle : Each of the above directed graph constitutes a punctured cycle.

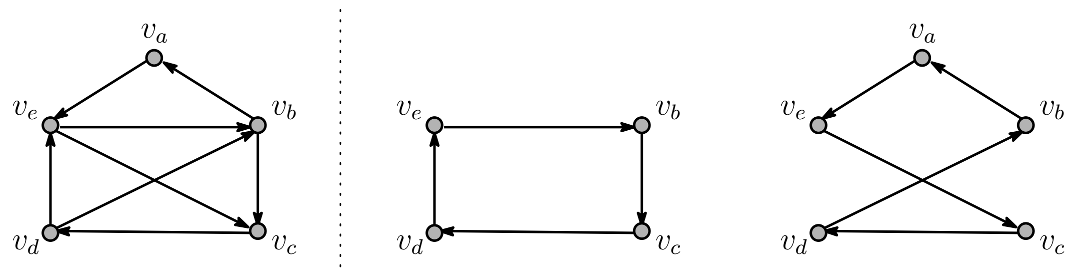

Figure 22: example for vertex-twisted cycles : for directed graph at the left, two cycles in the right are *vertex-twisted*.

If two cycles have such two common vertices, we can immediately take another cycle using part of arrows both from the two cycles. It says C is not disjoint, which ends our proof. □

The next theorem shows that max PCP are derived in a straight way for f-Networks which are disjoint.

Theorem 6. *max PCP for f-Network whose cycle set is disjoint*
For a closed f-Network $N^f =< V, A, f >$ with $G =< V, A >$, if C_G is disjoint, then
$x^{max}(N^f) = \sum_{c \in C_G}(|c| - 1)f^c$
with its closed cycle decomposition $N^f = \sum_{c \in C_G} < V^c, c, f^c >$.

Proof. From Lemma 6, when C is *disjoint*, then C is not vertex-twisted. Further, Corollary 1 states that there exists only one *closed cycle decomposition* with cycles C_G for the case of *disjoint*. From Lemma 5 and with the same procedure of taking sequence shown in its proof, we always take sequence and potential so that associated fsp-Network is e-covered and has $\sum_{c \in C_G}(|c| - 1)f^c$.

□

We show our additional results after introducing *domination* property.

5.3 Property of *domination* and min/max PCP

For a closed f-Network shown in the left of Figure 23, min circulation is derived as 30, which for example is realized with a fsp-Network in the right of the figure.

In Figure 24, observe that the same f-Network is closed cycle decomposed into different number of closed f-Networks. The minimum circulation is derived with the decomposition that has smaller number of cycles, that is captured with notion of *domination* formally defined below.

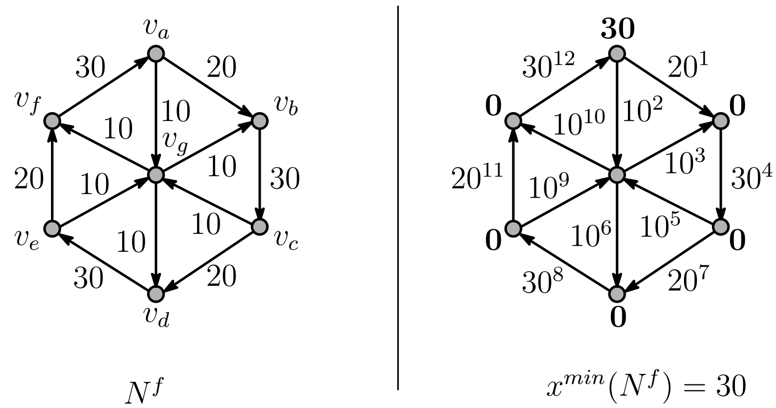

Figure 23:

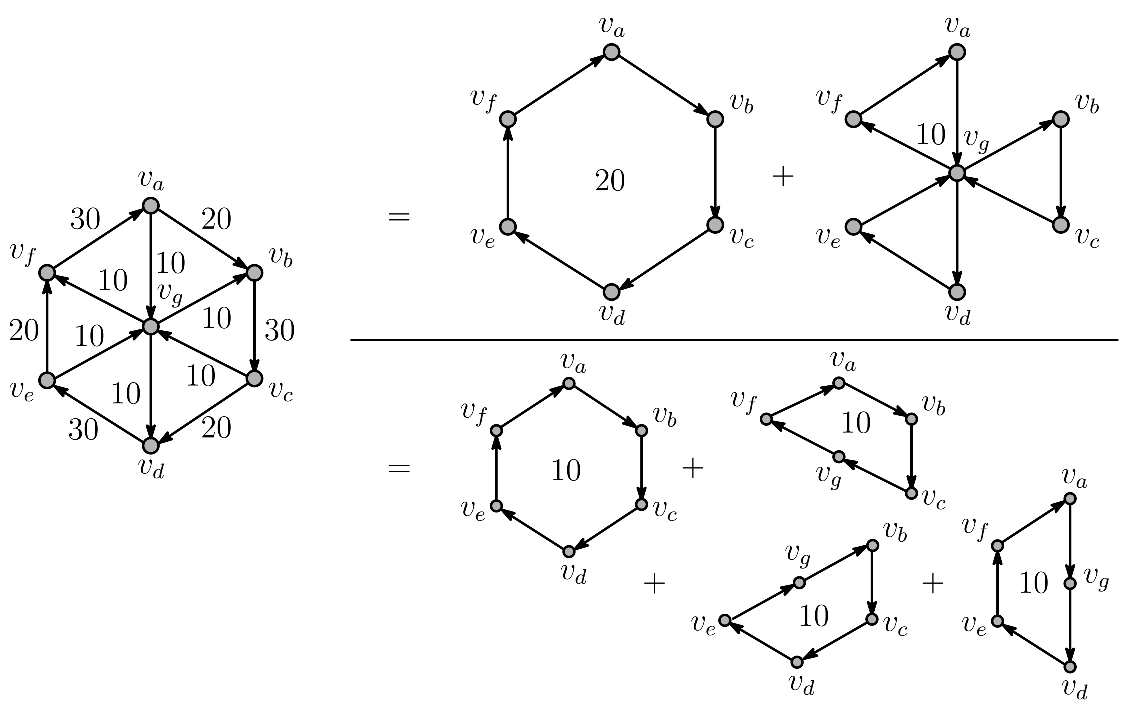

Figure 24:

Given a directed graph G and its cycle set C_G, a set of cycles $C \subseteq C_G$ is termed as *dominated* by another set of cycles $C' \subseteq C_G$ when there exist the same f-Network N^f

and closed cycle decomposition $N^f = \sum_{c\in C} < V^c, c, f^c >= \sum_{c'\in C'} < V^{c'}, c', f^{c'} >$ with $\sum_{c\in C} f^c < \sum_{c'\in C'} f^{c'}$. Note that *dominated* is well-defined since closed cycle decomposition with the same cycle set C leads to unique value of $\sum_{c\in C} f^c$ for a given closed cycle decomposition, though there exist room for the choice of flow for each cycle. We especially say $c \in C_G$ *singular dominates* $C' \subseteq C_G$ if c dominates C'. We say $C \subseteq C_G$ is *undominated* in C_G if there is no $C' \subseteq C_G$ which dominates C. We say a set of cycle C has *no domination* if there exists no $C', C'' \subseteq C$ such that C' dominates C''. A set of cycles C is *undominated* in C_G if C is not *domintated* by any $C' \subseteq C_G$.

Note that any punctured cycle *dominates* the set of its component non-punctured cycles. For example in Figure 24, a punctured cycle $v_a v_g v_b v_c v_g v_d v_e v_g v_f v_a$ *dominates* $\{v_a v_g v_f v_a, v_c v_g v_b v_c, v_e v_g v_d v_e\}$. Though number of decomposed cycles seemingly a determinant of *dominated* relation, Figure 25 shows it is not always true.

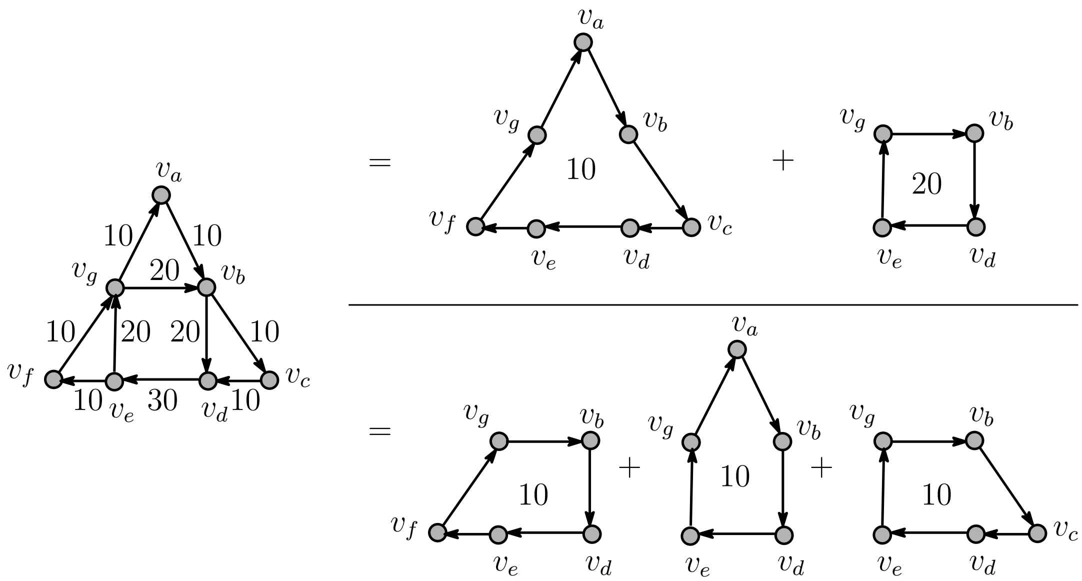

Figure 25:

Theorem 7. *min circulation under no arrow-twisted*

Given a closed f-Network $N^f =< V, A, f >$ on $G =< V, A >$, if there exist no arrow-twisted cycles in C_G, then,

$x^{min}(N^f) = \sum_{c\in C} f^c$ *with any closed cycle decomposition* $N^f = \sum_{c\in C} < V^c, c, f^c >$ *such that $C \subseteq C_G$ is undominated in C_G.*

Proof. Lemma 2 ensures $R^{min}(.)$ is always zero. Definition of *undominated* ensures our choice of closed cycle decomposition.

□

Theorem 8. *max circulation under no vertex-twisted*

Given a closed f-Network $N^f =< V, A, f >$ on $G =< V, A >$, if there exist no vertex-twisted cycle (punctured cycle) nor vertex-twisted cycles in C_G, then,

$x^{max}_{N^f} = \sum_{c \in C}(|c| - 1)f^c$

with any closed cycle decomposition $N^f = \sum_{c \in C} < V^c, c, f^c >$ *such that* $C \subseteq C_G$ *is undominated in* C_G*.*

Proof. Take a closed cycle decomposition $C \in C_{N^f}$, $\{f^c\}_{c \in C}$ that is undominated in C_G. By taking sequence for given f-Network shown in the proof of Lemma 5. We can take e-covered fsp-Networks for decomposed f-Networks with p^c so that $\sum_{v \in V^c} p^c(v) = \sum_{c \in C}(|c| - 1)f^c$ for every $c \in C$ and combined fsp-Network is e-covered. □

Note that f-Networks with punctured cycles are excluded from the above theorem, since any punctured cycle is *vertex-twisted* by itself. The next theorem allows punctured cycles but not for the other vertex-twisted cycles. For $G =< V, A >$, denote $C^p_G \subset C_G$ as the set of punctured cycles for G, and denote $C^{np}_G \subseteq C_G \setminus C^p_G$ as the set of non-punctured cycles for G.

Theorem 9. *max circulation under no vertex-twisted except for punctured cycles*

Given a closed f-Network $N^f =< V, A, f >$ *on* $G =< V, A >$*, if there exists no vertex-twisted cycles, then,*

$x^{max}_{N^f} = \sum_{c \in C}(|c| - 1)f^c$

with any closed cycle decomposition $N^f = \sum_{c \in C} < V^c, c, f^c >$ *such that* $C \subseteq C_G$ *is undominated in* C^{np}_G*.*

Proof. When there exits no *vertex-twisted* cycles in C^{np}_G, take closed cycle decomposition for C^{np}_G with undominated cycles. Suppose we can make a punctured cycle c from two of the cycles c', c''. Consider *closed cycle decomposition* that has c instead of c' and c'' for the amount of flow z. In separated form, it always increase the former part by z since $|c| = |c'| + |c''|$. However, since c itself is *vertex-twisted*, it always $R^{max}(.)$ part at least by z. We confirm that taking into account a punctured cycle generated by the two cycles never increase circulation of that without it. Any punctured cycle is able to be constituted by iterating combination of two cycles, which always leads to the same result.

Among *closed cycle decomposition* within C^{np}_G, the largest circulation is realized with cycles which are *undominated* from its definition. It ends our proof. □

Property of *domination* for given f-Network refers to whole relations of *dominated* among its sets of cycles.

5.4 The Key Characteristics

We have seen properties of *arrow-twist*, *vertex-twist*, and *domination* help characterize the min/max PCP. In addition, notice that number of vertices is also an important property for the max PCP, which we capture with a notion of *weighted.* For a closed f-Network $N^f =< V, A, f >$, given a closed cycle decomposition $N^f = \sum_{c \in C} < V^c, c, f^c >$, we call each cycle $c \in C$ is *weighted* by $|V^c - 1|$. Property of *weight* for given f-Network refers to *weighted* amounts for decomposed cycles for every possible *closed cycle decompositions.*

For our following analyses, we call *arrow-twist*, *vertex-twist*, *domination*, and *weight* are the key characteristics of the min/max PCP.

6 Effects of Network Transformations

The previous section introduced key characteristics and showed relevant basic results. This section examines in more detail how those characteristics work. For that purpose, we specifically examine how min/max circulation are affected when our input, f-Network is transformed into some other f-Network in various manner.

6.1 Definitions of Local Operations, Semi-Global Operations

We take up five types of transformations as shown in Figure 26. Suppose our original input for min/max PCP is that shown in the left of the figure, which consists one cycle $v_av_bv_cv_dv_ev_f$ with flow 10. For the f-Network, shown in the right part of the figure are transformed f-Networks, each of which is derived through following *operations*: *arrow separation* on an arrow (v_a, v_b) into (v_a, v_g) and (v_g, v_b) with added vertex v_g for the upper-left, *arrow slicing* on $(v_a, v_b, 0)$ into $(v_a, v_b, 0)$ and $(v_a, v_b, 1)$ for the upper-middle, *vertex contraction* on v_a, v_d to v_a for the upper-right, *cycle addition* $v_av_dv_ev_bv_cv_fv_a$ with flow 20 for the lower-right, *cycle separation* on cycle $v_av_bv_cv_dv_ev_fv_a$ in the amount of 10 for the lower-right. We call first three of the operations; *arrow separation, arrow slicing, vertex contraction* as *local operations*, while *semi-global operations* for the latter two.

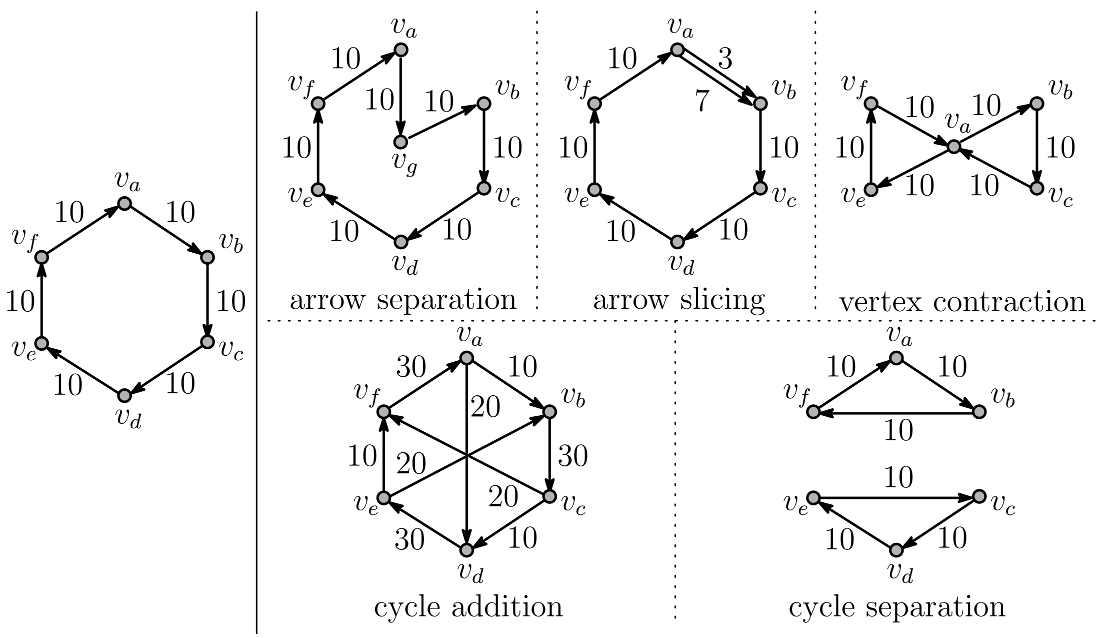

Figure 26:

Formally, given a f-Network $< V, A, f >$, each operation is defined as follows. We say *arrow separation* on $a = (v, v') \in A$ into a' and a'' with v'' to have $< V, A', f' >$ when $A' = A \cup a' \cup a'' \setminus a$ and $a' = (v, v''), a'' = (v'', v')$ with $f'(a') = f'(a'') = f(a)$, while $f'(a''') = f(a''')$ for every $a''' \in A \setminus a$. We say *arrow slicing* on $a \in A$ into a and a' to have $< V, A', f' >$ when arrows a, a' are between the same vertices, $A' = A \cup a'$, $f(a) = f'(a) + f'(a')$, and $f'(a'') = f(a'')$ for every $a'' \in A \setminus a$. We say *vertex contraction* for $N^f =< V, A, f >$ on $v, v' \in V$ to v to have $< V', A', f' >$ when $V' = V \setminus v'$, and all the arrows from or to v' in A are replaced by arrows from or to v in A', and f' is determined accordingly. We exclude vertices v, v' such that both $(v, v'), (v', v)$ exists in A. We say *cycle addition* $c \subseteq A$ with its flow f^c on $N^f =< V, A, f >$ to $N^{f'} =< V', A', f' >$ when $V' = V \cup V_c$, $A' = A \cup A_c$, and $f'(a) = f(a) + f^c$ for every $a \in c$ and $f'(a) = f(a)$ otherwise. Note that *flow increase* is a special case of *cycle addition*. We say a closed graph $G =< V, A >$ is *separated cycle* graph if any two cycles $c, c' \in C_G$ have no common vertex. We say *cycle separation* for $N^f =< V, A, f >$ on cycle $c \subseteq C_{<V,A>}$ in the amount of $f^k (\leq \min_{a \in c} f(a))$ to have $N^{f'} =< V, A', f' >$ when for some $a_1 = (v_1, v_1'), a_2 = (v_2, v_2') \in c$ such that $v_1' \neq v_2$ and $v_2' \neq v_1$, we take $a_1' = (v_1, v_2'), a_2' = (v_2, v_1')$, and $f'(a_1) = f(a_1) - f^k$, $f'(a_2) = f(a_2) - f^k$, $f'(a_1') = f(a_1') + f^k$, $f'(a_2') = f(a_2') + f^k$.

Note that those five operations are sufficient to examine network transformations in the following sense. For any given two closed f-Networks, we can always attain some closed f-Network from each of the two f-Networks through combinations of *arrow slicing, vertex contraction, cycle addition*. Notice that though *arrow separation* and *cycle separation* are redundant there, each has its own worth. *Arrow separation* reveals simpler cases within *vertex contraction*, while *cycle separation* reveals effects which are not directly captured by each of the other operations alone.

For latter reference, Table 6.1 summarizes our results in this section. It shows how each of the operations affects min/max circulation in total, as well as how each of the key characteristics contributes to the effect. For example, we read that *arrow separation* has no effect regarding min circulation, while it tends to increase max circulation through affecting *weight* property.

		min. circulation			max. circulation			
		dom.	a-twi.	total	wei.	dom.	v-twi.	total
Local	arrow sep.	$-$	$-$	$-_{10}$	$\uparrow$	$-$	$-$	$\uparrow_{11}$
	arrow slice.	$\downarrow$	$\downarrow$	$\downarrow_{12}$	$-$	$\downarrow$	$\uparrow$	$-_{13}$
	vertex cont.	$\downarrow$	$\downarrow$	$\downarrow_{14}$	$\downarrow$	$\uparrow$	$\downarrow$	$\downarrow_{15}$
semi-Global	cycle add.	$\Downarrow_{16}$	$\Uparrow_{18}$	$\Downarrow\Uparrow$	$-$	$\Uparrow_{17}$	$\Downarrow_{17,19}$	$\Downarrow$
	cycle sep.	$\uparrow_{22}$	$\downarrow_{23}$	$\downarrow\uparrow$	$-$	$\downarrow_{24}$	$\uparrow_{25}$	$\downarrow\uparrow$

Table 1: Effects of Operations in relation to the Key Characteristics
"$\uparrow$"("$\downarrow$") and "$\Uparrow$"("$\Downarrow$") show that the corresponding *operation* has weakly increasing(decreasing) effect through the corresponding property on either problem, while "$-$" means no effect. "$\Uparrow$","$\Downarrow$" especially express possibility of "multiplier effect". Numbers in the cells show those for related theorems.

6.2 Effects of Local Operations

arrow separation

For the min PCP, arrow separation has no effect.

Theorem 10. *no effect of arrow separation on min circulation*

Given a closed f-Network $N^f =< V, A, f >$, for any arrow separation on $a \in A$ to have $N^{f'}$, we have
$x^{min}(N^{f'}) = x^{min}(N^f)$.

Proof. For any sequence on N^f, we can take a sequence on $N^{f'}$ such that the relative orders are all the same and separated arrows have successive numbers. Under the sequence, circulation for $N^{f'}$ is the same as that on N^f. Conversely, for any sequence on $N^{f'}$, we can take sequence on N^f such that the relative orders are the same when we correspond either of the two of separated arrows in $N^{f'}$ to arrow a in N^f. Under that sequence, circulation for N^f is equal to or smaller than that for $N^{f'}$. □

Observe that *arrow separation* has no effect on either property of *domination* or *arrow-twist*, which leads to no effect on min circulation.

Theorem 11. *increasing effect of arrow separation on max circulation*
Given a closed f-Network $N^f =< V, A, f >$, for any arrow separation on $a \in A$ to have $N^{f'}$, we have
$x^{max}(N^{f'}) = x^{max}(N^f) + f(a)$.

Proof. Denote arrow separation on $a = (v, v')$ into $a' = (v, v'')$ and $a'' = (v'', v''')$. Given a sequence which realizes the maximum circulation for N^f, it is always possible to take sequence s for $N^{f'}$ such that$s(a') > s(a'')$ while the other orderings are unchanged. Circulation under the sequence s for $N^{f'}$ is $x^{max}(N^f) + f(a)$. Conversely, suppose there exists sequence for $N^{f'}$ such that its circulation is larger than $x^{max}(N^f) + f(a)$. Then, we can always take sequence for N^f such that orderings are the same when we correspond either of a' or a'' to a. It decreases circulation by at most $f(a)$, which contradicts max circulation for N^f is $x^{max}(N^f)$. □

Observe that *arrow separation* has no effect on either property of *domination* or *vertex-twist* but has effect on *weight* property, which is the source of increase of max circulation.

arrow slicing

Theorem 12. *decreasing effect of arrow slicing on min circulation*
Given a closed f-Network $N^f =< V, A, f >$, for any arrow slicing on $a \in A$ to have $N^{f'}$, we have
$x^{min}(N^{f'}) \leq x^{min}(N^f)$.

Proof. Given a sequence which realizes the minimum circulation for the original f-Network, take sequence for arrow-sliced f-Network so that sliced arrows have successive number, and maintain the ordering for the other arrows. It never increase the circulation. □

Arrow slicing has decreasing effect on min circulation both through affecting property of *domination* and *arrow-twist*. Figure 27 shows effect through *domination*, and Figure 28 shows effect through *arrow-twist*.

We observe that decreasing effect through *domination* property can be partly cancelled out through *arrow-twist* property by generating new *arrow-twisted* cycles, as shown in Figure 29.

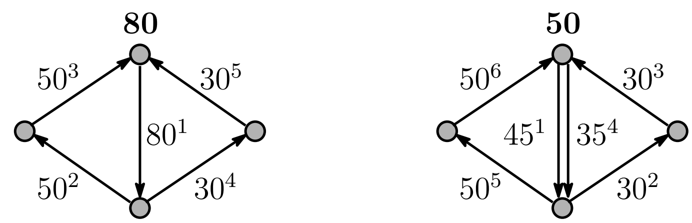

Figure 27: arrow slicing (decrease of min circulation through *domination*)

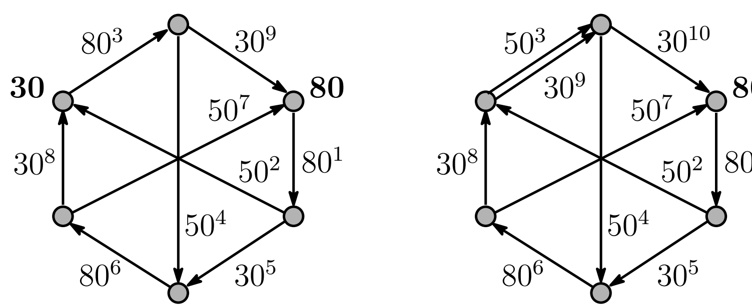

Figure 28: arrow slicing (decrease of min circulation through *arrow-twist*)

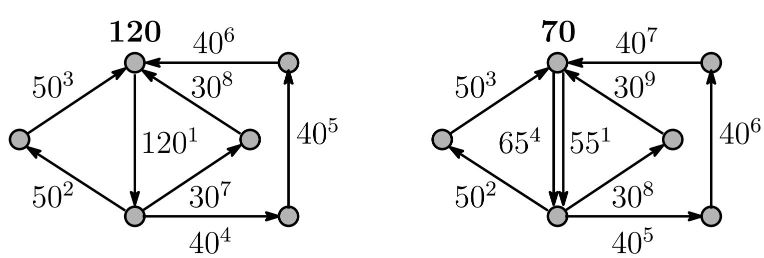

Figure 29: arrow slicing (partly cancel-out effect on min circulation through *arrow-twist*): Observe that there is no *arrow-twisted* cycles in the left fsp-Network. In the right shows an arrow-sliced fsp-Network, whose directed graph is shown in Figure 18, where there emerges a cycle which *dominates* cycles in the left fsp-Network as well as *arrow-twisted* cycles. It realizes the minimum circulation 70, which is larger than the maximum flow 65. The difference amounts to cancel-out effect by arrow-twist.

Theorem 13. *no effect of arrow slicing on max circulation*

Given a closed f-Network $N^f =< V, A, f >$, for any arrow slicing on $a \in A$ to have $N^{f'} =< V, A', f' >$, we have

$x^{max}(N^{f'}) = x^{max}(N^f)$.

Proof. See Appendix A.5. □

We observe that *arrow separation* affect *vertex-twist* and *domination* in opposite direction, and we can interpret the effects are canceled out regarding max circulation. More clearly, when we confine us to non-punctured cycles, we observe that *arrow slicing* never affect *v-twit* nor *domination*, which amounts to no effect on max circulation in total.

vertex contraction

Theorem 14. *decreasing effect of vertex contraction on min circulation*

Given a closed f-Network $N^f =< V, A, f >$, for any vertex contraction for N^f on $v, v' \in V$ to have $N^{f'}$, we have

$x^{min}(N^{f'}) \leq x^{min}(N^f)$.

Proof. The original sequence for the generated f-Network let the associated fsp-Network still covered. □

Vertex contraction has decreasing effect both through affecting *domination* and *arrow-twist*. Figure 30 shows effect through *domination*, where *vertex contraction* generates a cycle which dominates existent cycles. Figure 31 shows effect through *arrow-twist*, where *vertex contraction* let *arrow-reverse number* for two *arrow-twisted* cycles be less.

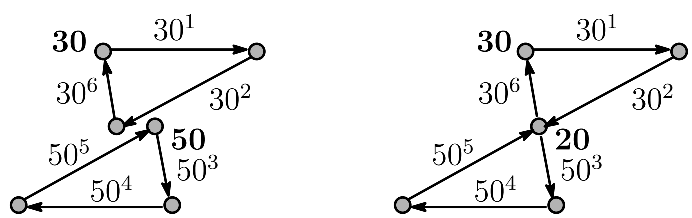

Figure 30: Vertex Contraction (decrease of min circulation through *domination*)

Theorem 15. *decreasing effect of vertex contraction on max circulation*

Given a closed f-Network $N^f =< V, A, f >$, for any vertex contraction on $v, v' \in V$ to have $N^{f'} =< V, A', f' >$, we have

$x^{max}(N^{f'}) \leq x^{max}(N^f)$.

Proof. We say $s : A \rightarrow \{1, 2, .., |A|\}$ and $s' : A' \rightarrow \{1, 2, .., |A'|\}$ are the same sequence when $s(a) = s'(a)$ for every $a \in A$ supposing each of $v, v' \in V$ is equal with $v \in V'$. For any same sequence s, s', take associated exact covered fsp-Network $< V, A, f, s, p >$ and $< V, A', f', s', p' >$, then we have $\sum_{v'' \in V \setminus (v,v')} p(v'') = \sum_{v'' \in V \setminus v} p'(v'')$, and $p(v) + p(v') \geq p'(v)$. □

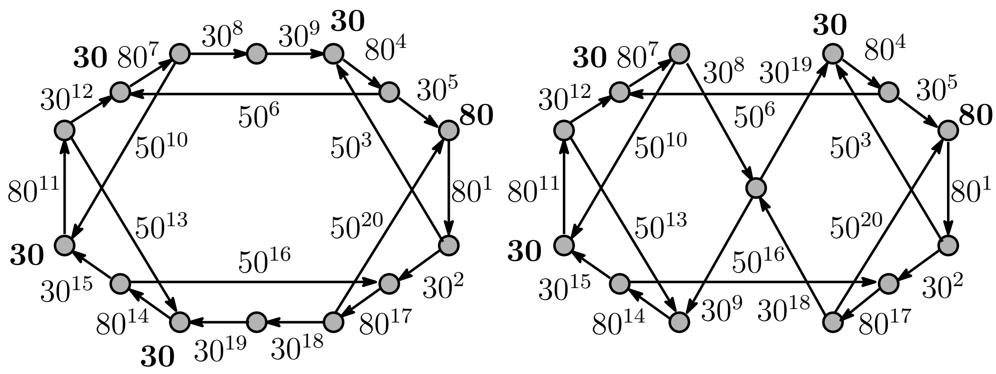

Figure 31: Vertex Contraction (decrease of min circulation through *arrow-twist*)

Vertex contraction affects *vertex-twist*, which itself has decreasing effect of max circulation as show in Figure 32, while its decreasing effect can be canceled out through affecting *domination* as shown in Figure 33. Notice that cycle set is unaffected in the former example.

When we confine us to non-punctured cycles, we interpret that *vertex contraction* never affect *domination*.

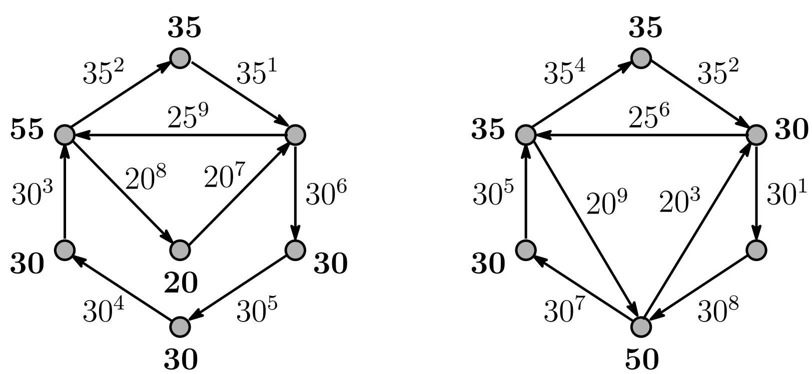

Figure 32: vertex contraction (decrease of max circulation through *vertex-twist*)

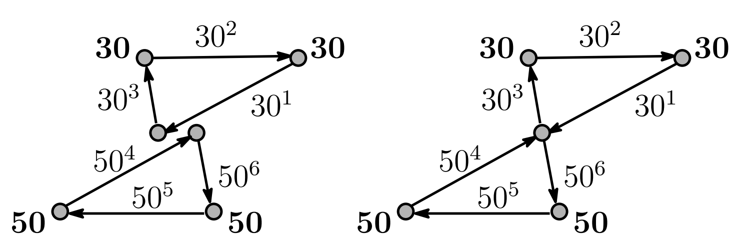

Figure 33: vertex contraction (no effect on max circulation)

6.3 Effects of Semi-Global Operations

cycle addition

We especially take up two special cases for the cycle addition which clarify heterogeneity of effects through *domination* and *arrow-twist*.

We say a directed graph $< V, A >$ is a *separated-cycles graph* when it consists of cycles which have no common vertex each other.

Theorem 16. *decreasing multiplier effect of cycle addition on min circulation*

Given a closed f-Network $N^f =< V, A, f >$ *where* $< V, A >$ *is a separated-cycles graph, make cycle addition* c *with* f^c *to* $N^{f'} =< V', A', f' >$ *so that* c *has at least one common vertex with* n *cycles but has no common arrows with any of the cycles. Then,*

$x^{min}(N^{f'}) - x^{min}(N^f) - f^c = -f^c * n + \sum_{c' \in C} \max\{f^c - f(c'), 0\}$.

Proof. It is straight since the cycle addition always add a punctured cycle which consists of $n + 1$ cycles. □

We say multiplier for the above N^f and $N^{f'}$ is $m = (x^{min}(N^{f'}) - x^{min}(N^f) - f^c)/f^c$.

We know that the multiplier is as large as n if f^c is sufficiently small, but it decreases as f^c gets larger, which is $\frac{\sum_{c' \in C} f^{c'}}{f^c}$ if $f^c > \max_{c' \in C} f^{c'}$.

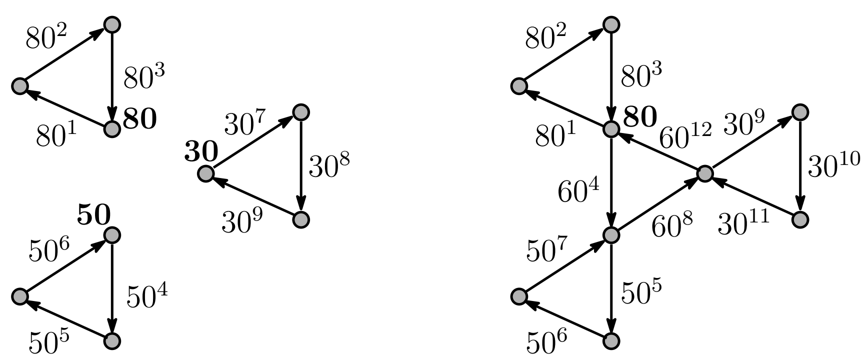

Figure 34: cycle addition (decreasing multiplier effect on min circulation through *domination*): min circulation decreases by $60 * 3 - \max\{60 - 80, 0\} - \max\{60 - 50, 0\} - \max\{60 - 30, 0\} - 60 = 80$.

The following theorem contrasts effect of *cycle addition* on max circulation to that on min circulation.

Theorem 17. *no effect of cycle addition on max circulation: separated non-punctured graph*

Given a closed f-Network $N^f =< V, A, f >$ *where* $< V, A >$ *is a separated non-punctured cycle graph, make cycle addition non-punctured cycle* c *with* f^c *to* $N^{f'} =< V', A', f' >$ *so that* c *has* $k \leq 2$ *common vertices with* n *cycles but has no common arrows with any of the cycles. Then,*

$x^{max}(N^{f'}) = x^{max}(N^f) + (|c| - 1)f^c$.

Proof. Denote $G =< V, A >$, $G' =< V', A' >$. When $k \leq 2$, there never appear vertex-twisted cycles other than punctured cycles including c. Theorem 9 ensures that the maximum circulation for $N^{f'}$ is mere summation of that of N^f and $(|c| - 1)f^c$. □

Next theorem shows effects through *arrow-twist* on min circulation, which has opposite effect of *domination*. The theorem also shows that *arrow-twist* is actually a determining property on min circulation for certain cases.

Theorem 18. *increasing multiplier effect of cycle addition on min circulation*

Given a closed f-Network which consists of only one non-punctured cycle $N^f =< V, c, f^c >$*. make cycle addition non-punctured cycle* c' *with* $f^{c'}$ *to have* $N^{f'} =< V', A', f' >$ *so that it has n arrow-reverse number with c and there exist no punctured cycle. Then,*

$x^{min}(N^{f'}) - x^{min}(N^f) - f^c = n * min\left\{f^c, f^{c'}\right\}$

Proof. We say two arrow-twisted cycles c, c' are *base n arrow-twisted* cycles if each cycle consists $2 * (n + 2)$ arrows, and they have $(n + 2)$ common vertices so that they have n arrow-twists. The left graph of Figure 16 includes base 1 arrow-twisted cycles; $\{v_a v_b v_c v_d v_e v_f v_a, v_a v_d v_e v_b v_c v_f v_a\}$.

Suppose cycle addition is executed so that the added cycle is base n arrow-twisted with the other cycle. Then, closed cycle decomposition for $N^{f'}$ is realized either with $\{c, c'\}$, or with cycles which consists of 4 arrows; two arrows from common arrows of c, c' and each one arrow from cycles c, c', and either or both of c, c'. It is straight the minimum circulation is derived as above. For cycles which are n arrow-twisted but not base n arrow-twisted, our previous results for *Local Operations* ensures that constructed f-Network can always be transformed into that with two base n arrow-twisted cycles while the minimum circulation is unchanged. □

Note that the multiplier is as large as n if $f^{c'}$ is small enough, but decreases as $f^{c'}$ is larger, which is $\frac{n*f^c}{f^{c'}}$ if $f^{c'} > f^c$.

The theorem indicates how sequence needs to be taken to attain min circulation under existence of *arrow-twisted* cycles. As confirmed in the right of Figure 35, cycle with larger flow is endowed priority to the other in the sense that sequence is increasing along with the former cycle while not for the latter.

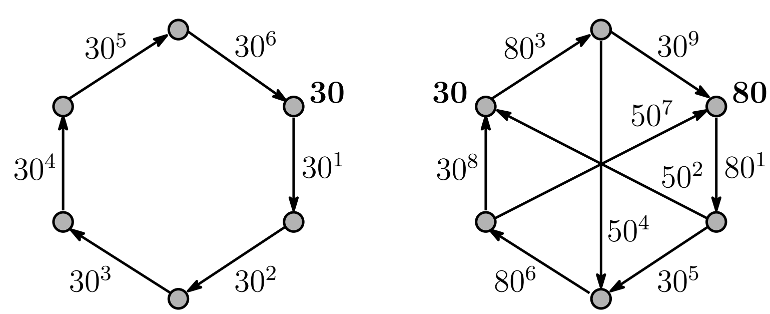

Figure 35: cycle addition (increasing multiplier effect on min circulation through *arrow-twist*): cycle addition with its flow 50 lets minimum circulation increase by $1 * \min\{30, 50\} + 50 = 80$.

There exists more complicated case where both properties of *domination* and *arrow-twist* take part, as shown in Figure 36.

Next we see effects of *vertex-twist* on max circulation. We take up a special class of f-Networks. Given $G =< V, A >$, we say two cycles $c, c' \in C_G$ are n *opposite cycles* for $n =$

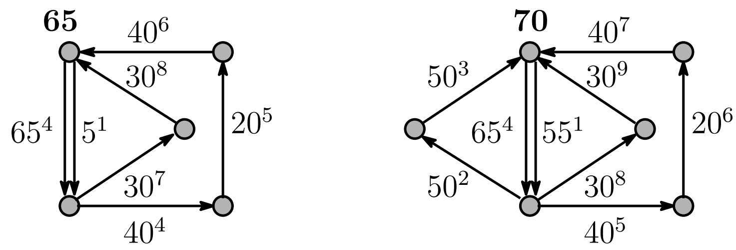

Figure 36: cycle addition (effect on min circulation through both *domination* and *arrow-twist*): cycle addition with its flow 50 lets minimum circulation decrease by 50(minus the added cycle value) through domination part, but arrow-twist partly cancels out the decrease by 5.

$1, 2, ..$ if c, c' have $n+2$ common vertices and no common arrow, and the common vertices appear exactly the opposite order. For example, in the left of Figure 19, $v_a v_b v_c v_d v_f v_a$ and $v_b v_f v_d$ are 1 *opposite cycles.*

Theorem 19. *decreasing multiplier effect of cycle addition on max circulation: n opposite cycles*

Given a closed f-Network which consists of only one non-punctured cycle $N^f =< V, c, f^c >$, make cycle addition non-punctured cycle c' with $f^{c'}$ to have $N^{f'}$ so that it has either c, c' are n opposite cycles. Then,

$x^{max}(N^{f'}) - x^{max}(N^f) - (|c'| - 1)f^{c'} = -n * \min\left\{f^c, f^{c'}\right\}$

Proof. When the cycles are n opposite cycles, Theorem 9 says that we only need to examine a closed cycle decomposition where all cycles are non-punctured. We have such a closed cycle decomposition which consists of $n+2$ cycles with flow $\min\left\{f^c, f^{c'}\right\}$, and either of the cycle c, c' with flow $|f^c - f^{c'}|$. Without loss of generality, suppose $f^c \leq f^{c'}$. For $n+2$ cycles with f^c, sum of maximum number of reverse equals to the number of cycles $n+2$. We have $x^{max}_{N^f} = f^c*(|c|+|c'|-n-2)+(f^{c'}-f^c)(|c'|-1) = (|c|-1)f^c+(|c'|-1)f^{c'}-n*f^c$. □

Figure 37 is an example for the decreasing multiplier effect. Note that the multiplier is as large as n if $f^{c'}$ is small enough, but decreases as $f^{c'}$ is larger, which is $\frac{n*f^c}{f^{c'}}$ if $f^{c'} > f(c)$.

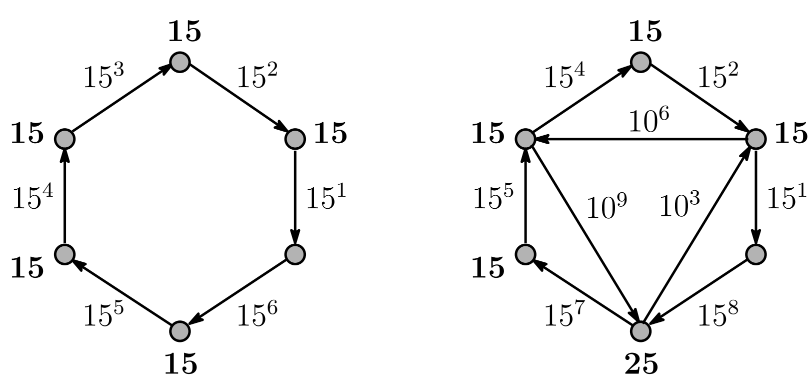

Figure 37: cycle addition (decreasing multiplier effect on max circulation through *vertex-twist*: cycle addition with its flow 10 lets the max circulation for the combined f-Network turn to be $15 * 5 + 10 * 2 - 1 * \min\{10, 20\} = 85$, which is less than $15 * 5 + 10 * 2$, the summation of max circulation for two combined f-Networks.

As a special case of *cycle addition*, we define *flow increase*. For a closed f-Network $N^f =< V, A, f >$, we say *flow increase* on N^f to $N^{f'} =< V, A, f' >$ when $f'(a) \geq f(a)$ for every $a \in A$ and there exists $a' \in A$ such that$f'(a') > f(a')$, and $N^{f'}$ is still closed.

Theorem 20. *Regime Change for min circulation*

For a closed f-Network $N^f =< V, A, f >$, suppose $< V, A, f, s, p >$ realizes the min circulation. We consider flow increase on N^f to $N^{f'} =< V, A, f' >$. Then,

there exists f' such that the minimum circulation is not attained with the original sequence s if and only if there exists arrow-twisted cycles.

Proof. When there exists no arrow-twisted cycles, the same sequence always gives the minimum circulation for any flow increase since flow increase never change domination. Next, suppose there exist arrow-twisted cycles for $< V, A, f >$, then for each sequence which realizes the minimum circulation we always find a cycle c where there exists more than one reverse. Take such sequence s and some other sequence s' which lets reverse for the cycle c be one. Circulation for s is less than that for s' by certain amount with the original flow f. When we increase flow for the cycle c, difference of circulations between under s and s' gets smaller in proportion to the increase. There exits some point where the original amount of difference is canceled-out and more increase lets circulation for s be larger than that for s', which completes our proof. $\square$

Regime Change for min circulation is only through *arrow-twist* since *domination* is unaffected by *flow increase*. Figure 38 shows an example.

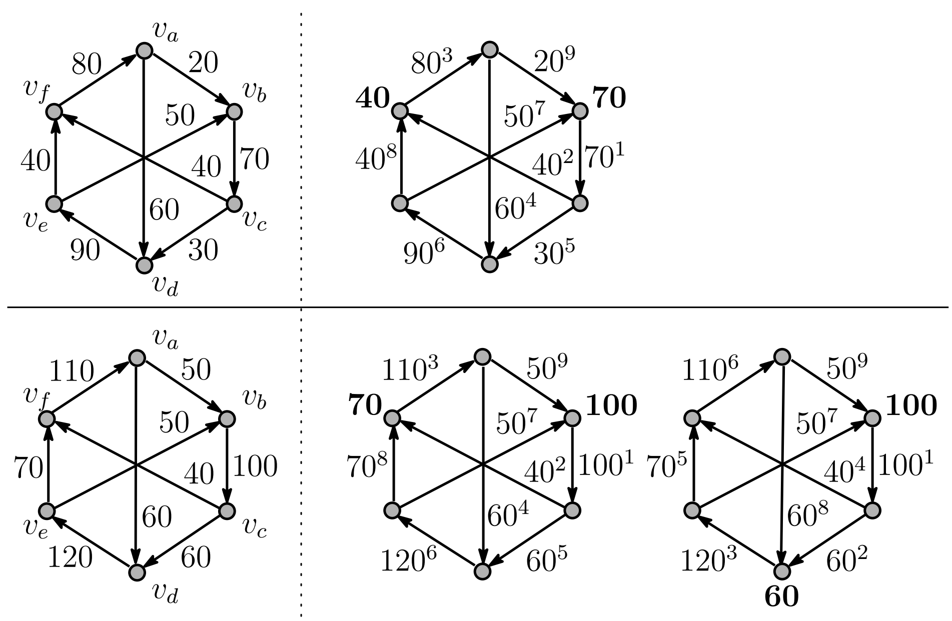

Figure 38: Regime Change; flows are increased for $v_a v_b v_c v_d v_e v_f v_a$ by 30 (from the upper left to the lower left f-Network). The upper right realizes min circulation for the original f-Network. The lower middle is an exact covered fsp-Network for the new f-Network with the same sequence. The lower right realizes the minimum circulation for the new f-Network, where the sum of potentials is actually smaller than that of the lower middle by 10.

Theorem 21. *Regime Change for max PCP*

For a closed f-Network $N^f =< V, A, f >$ with $G =< V, A >$, suppose $< V, A, f, s, p >$ realizes max circulation. We consider flow increase on N^f to $N^{f'} =< V, A, f' >$. Then,

1). For any f', exact covered fsp-Network $< V, A, f', s, p' >$ always realizes the maximum circulation for $N^{f'}$ if there exists no vertex-twisted cycles.

2). There exists f' such that the maximum circulation is not attained with the original sequence s if there exists arrow-twisted cycles in C_G.

Proof. 1) is straight from the definition of vertex-twisted. 2) is the same as the counterpart theorem for min PCP. □

Figure 39 is an example for the Regime Change when vertex-twisted but not arrow-twisted cycles exist. Note that Regime Change may not occur depending on sequence for this case.

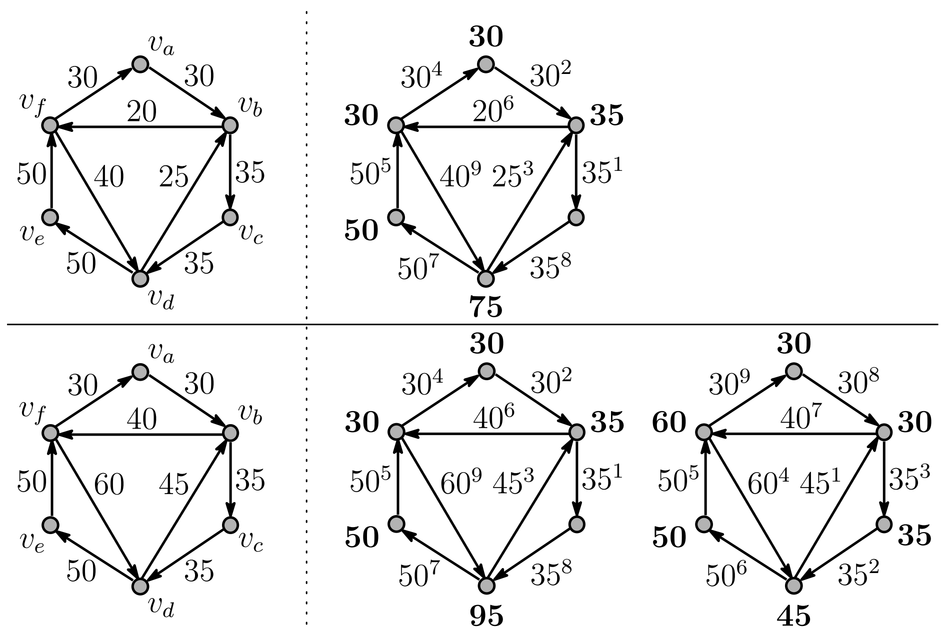

Figure 39: Regime Change; flows are increased for $v_b v_f v_d v_b$ by 20 (from the upper left to the lower left f-Network). The upper right realizes the maximum circulation for the original f-Network. The lower middle is an exact covered fsp-Network for the new f-Network with the same sequence. The lower right realizes the maximum circulation for the new f-Network, where the sum of potentials is confirmed as larger than that of the lower middle by 10

cycle separation

Theorem 22. *increasing effect of cycle separation on min circulation: one cycle*

Given a closed f-Network N^f which consists of a cycle c with its flow f^c. Make cycle separation on c in the amount of f^k to have a closed f-Network $N^{f'}$. Then,

$x^{min}(N^{f'}) - x^{min}(N^f) = f^k$.

Proof. Cycle separation add two more cycles, and the all three cycles are disjoint and not arrow-twisted. It is straight minimum circulation increases by f^k. □

We prepare terminologies for the next theorem. For n arrow-twisted cycles $c = v_1..v_kv_1, c'$ and its common arrows $A = c \cap c'$, replace $a = (v_{k'}, v_{k'+1}) \in A$ by $a' = (v, v') \notin A$ for c so that we have $c'' = v_1..v_{k'-1}vv'v_{k'+2}..v_kv_1$ is a cycle. We say $a \in A$ *contributes* its arrow-twistedness when c', c'' is no more n arrow-twisted. We say c, c' are *quasi-base* n arrow-twisted cycles if every common arrows contribute its arrow-twistedness.

Theorem 23. *decreasing effect of cycle separation on min circulation: quasi-base arrow-twisted non-punctured cycles*

Take a closed f-Network N^f such that $N^f =< V^c, c, f^c > + < V^{c'}, c', f^{c'} >$, where c, c' are quasi-base arrow-twisted non-punctured cycles. Suppose $f^c < f^{c'}$. Make cycle separation on c in the amount of $f^k (\leq f^c)$ so that separation is not on the common arrows, and each separated cycle has at least 2 common arrows with c'. Denote the generated f-Network as $N^{f'}$. Then,

$x^{min}(N^{f'}) - x^{min}(N^f) = -f^k$

Proof. Before the cycle separation, since the cycles are n arrow-twisted, the minimum circulation is $f(c') + (n+1) * f(c)$. c has $n+1$ reverse number under the minimum circulation. After the separation, sum of reverse numbers for the separated cycles is n under sequences which realizes the new minimum circulation. It is not the case when either separated cycle has at most 1 common arrow with c', since reverse number for each cycle cannot be less than 1 under any sequence. □

Decreasing effect of cycle separation is through decreasing the arrow-reverse number. Effects of cycle separations are shown in Figure 40, 41.

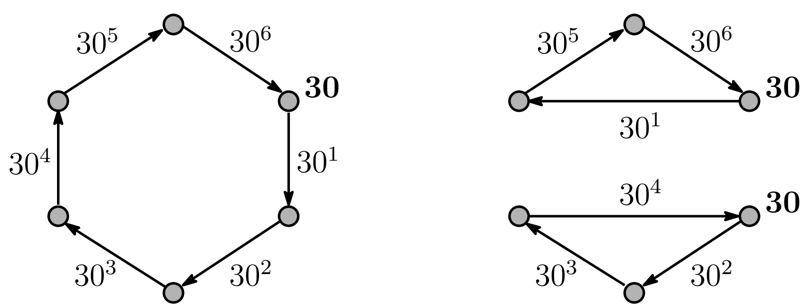

Figure 40: cycle separation (increasing effect on min circulation through *domination*)

Theorem 24. *decreasing effect of cycle separation on max circulation: one cycle*

Given a closed f-Network N^f which consists of a cycle c with its flow f^c. Make cycle separation on c in the amount of $f^k (\leq f^c)$ to have a closed f-Network $N^{f'}$. Then,

$x^{max}(N^{f'}) - x^{max}(N^f) = -f^k$.

Proof. It is straight and omitted. □

Theorem 25. *increasing effect of cycle separation on max circulation: opposite non-punctured cycles, quasi-base arrow-twisted non-punctured cycles*

Take a closed f-Network N^f such that $N^f =< V^c, c, f^c > + < V^{c'}, c', f^{c'} >$, where c, c' are either opposite non-punctured cycles or quasi-base arrow-twisted non-punctured cycles. Suppose $f^c < f^{c'}$.

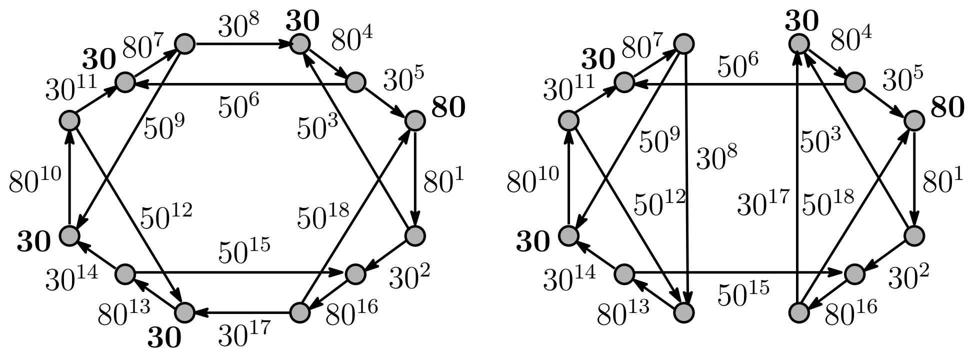

Figure 41: cycle separation (decreasing effect on min circulation through *arrow-twist*)

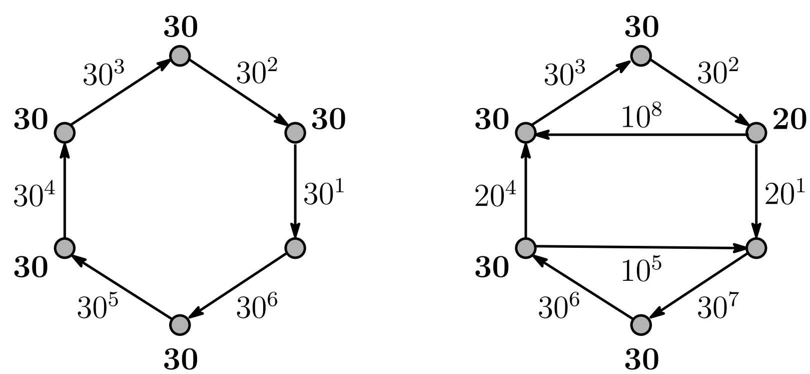

Figure 42: cycle separation (decreasing effect on max circulation through *weight*)

Make cycle separation on c in the amount of $f^k(\leq f^c)$ so that separation is not on the common arrows, and each separated cycle has at least 2 common arrows with c' when cycles are quasi-base arrow-twisted, while each separated cycle has at least 3 common vertices with c' for the case of opposite cycles. Denote the generated f-Network as $N^{f'}$. Then,

$$x^{max}(N^{f'}) - x^{max}(N^f) = f^k$$

Proof. We can prove exactly the same as the case for the min PCP for the case of arrow-twisted. For the case of opposite cycles, we have $x^{max}(N^f) = (|c'|-1)f^{c'} + (|c|-n)f^c$ from Theorem 9. By the separation, since separated cycles has at least 3 common vertices, $x^{max}_{N^{f'}} = (|c'|-1)f^{c'} + (|c|-n)(f^c - f^k) + (|c|-n+1)f^k$. It leads to $x^{max}(N^{f'}) = (|c'|-1)f^{c'} + (|c|-n)f^c + f^k$. □

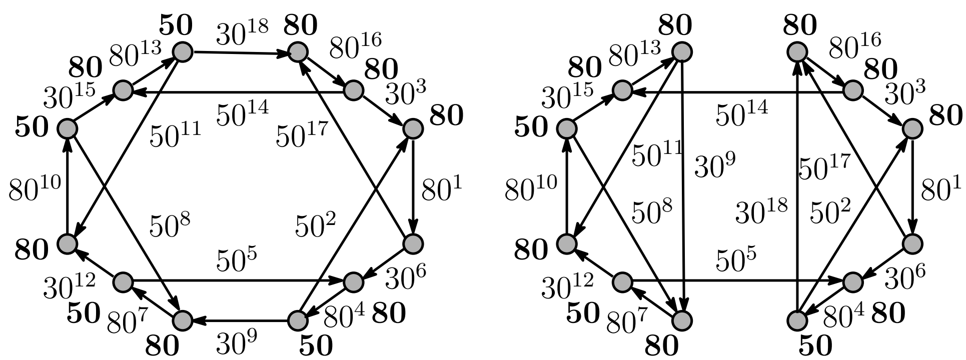

Figure 43: cycle separation (increasing effect on max circulation through *arrow-twist*)

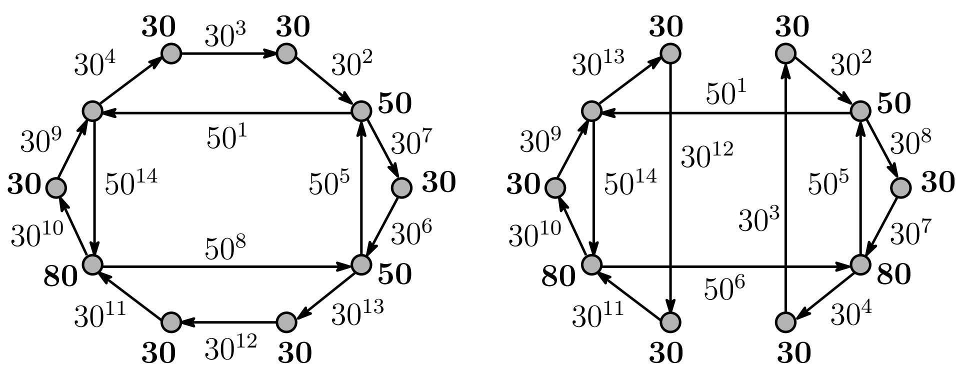

Figure 44: cycle separation (increasing effect on max circulation through *vertex-twist*)

7 Related Literature

Our contribution lies on the cross-point of financial economics and graph theory.

Quesnay (1758) provided the basis for graph-theoretic analysis on payment networks[2]. Mainly for the analysis of the reproduction of goods, he analyzed a simpler class of payment networks, which is embedded in our general model as a special case where payments

[2]For network analysis in the other field of economics, Jackson (2008) provides a wide survey.

are among three representative subjects (Proprietary, Productive, Sterile). Our notion of *closed* for payment flows is a generalized expression for one of the assumptions in the Tables.

The following main contributions to the graph-theoretic analysis on payment networks were accomplished in three fields: settlement systems, emergence of money, and currency area. As already stated in our introductory section, in the field of settlement systems, Eisenberg and Noe (2001) provided a mathematical framework to examine payment network but without element of order of settlement. Rotemberg (2011) firstly pointed out in a persuading manner that order of settlement possibly matters for liquidity problem in the field of settlement systems, taking up a specific class within Euler graph, examining order of settlement under certain exogenous way of decentralized decision pattern. Along with this literature, this paper provides a framework that treats a general class of payment network, and enables to examine every possible order of settlements for given distribution of obligations.

In the literature which covers the emergence of money, focus has been rather in showing the reason of existence of monetary substance, not much in examining a general network structure. Kiyotaki and Wright (1989) examined the circulation of the "medium of exchange" on"Wicksellian Triangle"[3], whose network structure is within the simplest cases in our model. Many of the studies that followed (Kiyotaki and Wright (1993),Trejos and Wright (1995),Lagos and Wright (2005)) adopted networks with one cycle. Yasutomi (2000) in section 4 focused on diversity of network and the related "strength" of money to the diversity. This was a major step in examining a class of graphs which contains multiple cycles, though still without *twisted* relations in the terminology of this paper. His study is interpreted to relate multiplicity of *closed cycle decomposition* to the emergence of "strong" or "weak" money. From the view of the *min PCP*, one of its important results is rephrased as "strong" money emerges only when the maximum flow is equal to the *minimum circulation*. In its historical analysis on the multiplicity of currency, Kuroda (2003) compared two types of graphs: the pyramid type and the horizontal type. His study is also related to the notion of *closed cycle decomposition*. He examined situations where each different type of money circulates within each set of *decomposed* cycles.

There are researches which focus on specific properties of payment network. In the analysis of financial contagion, "connectedness" or "connectivity" of network is shown to be a useful notion, as in the case of Allen and Gale (2000), Freixas, Parigi, and Rochet (2000), Lagunoff and Schreft (2001),Cifuentes, Shin, and Ferrucci (2005), Nier, Yang, Yorulmazer, and Alentorn (2007), Caballero and Simsek (2009),Gai and Kapadia (2010), Castiglionesi and Navarro (2008), and Allen, Babus, and Carletti (2010). Among them, "density" of network is proposed as an analytical tool in Zawadowski (2011). In the field of settlement system, the existence of a cycle itself is known as a potential source of gridlock(Beck and Sorämaki (2001)). Our original concepts of *arrow-twisted*, *vertex-twisted* properties provide a different aspect of payment network from those papers.

Related to properties of network in a broad sense, several concepts of properties have been proposed in the literature of social network[4]. Our analysis implies that properties relevant to "payment network" in the field of economics are not necessarily the same as those proposed for "social network", but possibly requires its own concepts.

[3] a cycle with 3 subjects and 3 arrows among them

[4] See Jackson (2008) for analysis on social network.

From the graph-theoretic view, the *min/max PCP* can be thought of as a variant of "numbering" problem which contains aspects of "flow" problem[5]. The closest "numbering" problem would be the Bandwidth problem[6], where the objective being minimized is determined only with its sequence, while in the *min/max PCP*, the objective is more indirect, in the way that the amount of each flow takes part in. The "flow" aspect of the *min/max PCP* lets us utilize the method of *decomposition*, which has been shown as useful for some "flow" problem[7]. But its "numbering" aspect leads us to a distinct approach– examination of relation among *decomposed* cycles.

8 Concluding Remarks

We set up a graph-theoretic framework to express circulation of settlement fund, or money under network structures. From the view of the equation of exchange[8], its distinguishing characteristic is to capture relative[9] velocity of money through its element of *sequence*. This allows us to examine relation between velocity and quantity for money circulation. The framework has potential for wide applications in financial economics: in the field of settlement system, emergence of money, or currency area.

Under the framework, we have presented a pair of graph-theoretic problems that are fundamental for analysis on gross settlement systems. The problems have unique characteristics from the view of network problems in that the problems have both "numbering" and "flow" aspect. Utilizing a cycle decomposition approach, we have specified several key network properties for the problems. We have examined the effects of network transformations to show how each network property affects minimum and maximum circulation each.

Because we have concentrated on determining the general properties of the problems, it is not discussed which classes of payment networks well captures networks in our real world. Our future task is to specify appropriate classes of payment networks to execute more detailed analysis.

Turning to the graph-theoretic view, we conducted a qualitative characterization of the problems. One of the remaining tasks is to probe whether the problem is NP hard or not. It is known that although many types of "flow" problems are not NP hard, many "numbering" problems, including Bandwidth Problem, are NP hard. In our view, though the min/max PCP would be as "easy" as many "flow" problems regarding *domination* property, they will still contain "difficulty" in relation to *arrow-twist* and *vertex-twist* property. The latter two properties would be the key for the probing.

[5] Diaz, Petit, and Serna (2002) provides a survey for "numbering" problems in the view of graph layout on some dimension. For "flow" problems, see Ahuja, Magnanti, and Orlin (1993)

[6] See Chinn, Chvatalova, Dewdney, and Gibbs (1982) for the Bandwidth Problem.

[7] Goldberg, Plotkin, and Tardos (1991)

[8] Fisher (1911)

[9] The framework is not to capture absolute velocity of money. The framework clarifies the notional difference between *absolute* and *relative* velocity of money itself.

A Appendix

A.1 Local Minimum/Maximum

Given a covered fsp-Network $N^{fsp} =< V, A, f, s, p >$, we say s attains *local minimum* for $A' \subseteq A$ when we cannot take s' such that 1). s' gives the same ordering with s for $A \setminus A'$, and 2). for exact covered fsp-Network $< V, A, f, s', p' >$, $\sum_{v \in V} p'(v) < \sum_{v \in V} p(v)$. We similarly define *local maximum*.

The following theorem is useful.

Theorem 26. *local minimum/maximum*

Given a closed f-Network $N^f =< V, A, f >$, if an exact covered fsp-Network $N^{fsp} =< V, A, f, s, p >$ realizes the minimum/maximum circulation for N^f, then, s attains local minimum/maximum for every $A' \subset A$.

The theorem is straight from the definition of min/max PCP.

A.2 Proof of Theorem 2.

For vertices which have multiple inflows and/or outflows, we execute "unbundling" (see Figure 45). First "unbundle" some vertex to several "hypothetical" vertices such that each "hypothetical" vertex has one inflow and one outflow, and has exact amount of potential for each corresponding sequences. We can always execute such "unbundling", and the derived "unbundled" fsp-Network is also exact covered. We can continue this "unbundling" until each fsp-Network has no vertex which has multiple inflows and/or outflows, and any derived fsp-Network consists of exact covered fsp-Networks, each with one cycle.

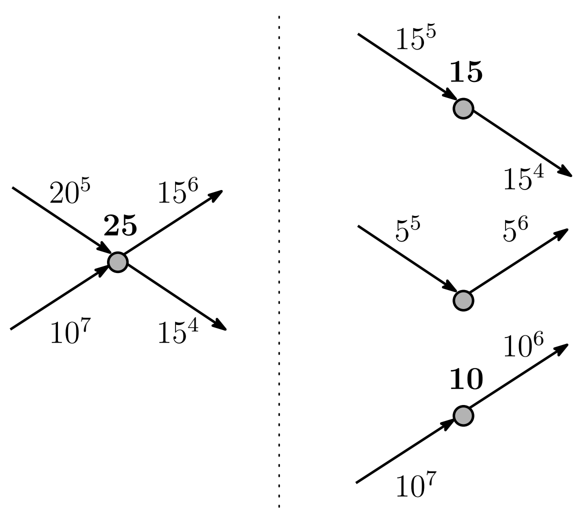

Figure 45: Example of "unbundling" procedure. "Unbundle" vertex in the left to hypothetical vertices in the right

A.3 Proof of Theorem 3.

Theorem 2 ensures that our search for fsp-Networks on the basis of *closed cycle decomposed* f-Networks always include right fsp-Networks in the sense they realize min circulation. What remains to be shown is that we correctly choose right fsp-Networks by minimizing circulation for *closed cycle decomposed* f-Networks. Next lemma ensures that part.

Lemma 7.

Given a closed f-Network $N^f =< V, A, f >$,

For any closed cycle decomposition $N^f = \sum_{c \in C} < V^c, c, f^c >$, *if* $< V^c, c, f^c, s^c, p^c >$ *is exact covered for every* $c \in C$, *and we can take* $s : V \to \{1, 2, .., |A|\}$ *which is consistent of* $\{s^c\}_{c \in C}$, *then*

$< V, A, f, s, p >= \sum_{c \in C} < V^c, c, f^c, s^c, p^c >$ *is covered .*

Proof. As long as $\{s^c\}_{c \in C}$ is consistent with s, it is straight that combining covered fsp-Networks always emerge a covered fsp-Network. □

The lemma states that our search on the basis of *closed cycle decomposed* f-Networks never let us find smaller circulation than "true" min circulation. Combining Theorem 2 and Lemma 7, we complete our proof.

A.4 Proof of Theorem 4.

Our proof is similar to that for Theorem 3. Theorem 2 ensures that our search for fsp-Networks on the basis of *closed cycle decomposed* f-Networks always include right fsp-Networks in the sense they realize max circulation. What remains to be shown is that we correctly choose right fsp-Networks by maximizing circulation for *closed cycle decomposed* f-Networks. Since we confine us to search cases where combined fsp-Networks become e-covered, Lemma 7 ensures our search on the basis of *closed cycle decomposed* f-Networks never let us reach larger circulation than "true" max circulation.

A.5 Proof of Theorem 13

We show that the maximum circulation for N^f is always attained for $N^{f'}$, and also maximum circulation for $N^{f'}$ is always attained for N^f. For the former part, given N^{fsp} which attains the maximum circulation for N^f, endow successive sequence for the two sliced arrow with the ordering of the arrows are unchanged with N^{fsp}, which ensures the same amount of circulation. For the latter part, we examine local maximum for sliced vertices $a, a' \in A'$ based on Theorem 26.

Suppose there exits sequence s for $N^{f'}$ such that it attains max circulation for $N^{f'}$. Take a new sequence s' for $N^{f'}$ such that a, a' has successive order in a way that $s(a) = s'(a)$ and $s'(a') = s'(a)+1$, and orderings among $A \setminus a'$ is the same between two sequences s and s'. Then, circulation realized with s' and $N^{f'}$ is equal to or smaller than that with s and N^f. When circulation gets smaller, take another sequence s'' such that $s(a') = s''(a')$ and $s''(a) = s''(a') + 1$ and orderings among $A \setminus a$ is the same between two sequence s and s''. It never changes circulation. It is clarified by dividing the above step to take a new sequence in two steps.

Consider we take a new sequence s' on $N^{f'}$ through following steps. The first step is to remove arrow a from fs-Network $N^{f'}$ with s. Take a temporal fs-Network which maintains all the orderings among $A \setminus a$. The second step is to add arrow a with a s'.

We have following lemma.

Lemma 8.

When circulation gets smaller for fs-Network with s' and $N^{f'}$ that that with s and N^f,

1) circulation of the temporal fs-Network needs to get less than that with s and $N^{f'}$, and

2) circulation of fs-Network with s' and $N^{f'}$ is equal to or larger than that of the temporal fs-Network, though the difference is less than that between circulation of the temporal fs-Network and that of s and $N^{f'}$.

Proof. Firstly, if circulation for the temporal fs-Network become larger, the sequence s contradicts to our assumption that it leads to max circulation. It is confirmed that we can take another sequence where a has the last order while maintaining the other orderings, which attains the same circulation by the temporal fs-Network.

Secondly, circulation for the temporal fs-Network is not larger than that for fs-Network with N^f and s'. It is because in that case when we further remove arrow a' from the temporal fs-Network, circulation gets larger, which leads to a contradiction as above. □

Now when we take the dividing steps on a', circulation of temporal fs-Network needs to the same as that with s and $N^{f'}$. It comes from the latter part of Lemma 8, which amounts to state that increasing flow for a' alone did not increase its circulation under s immediately. It tells that removing a' never decreases circulation, and we already confirmed it never increases.

Further, combining with the former part of the same lemma, we know that circulation of temporal fs-Network needs to be the same as that with s'' and $N^{f'}$. The lemma amounts to state that increasing flow only on a never decrease circulation, and we confirm that increase of circulation led to a contradiction of our assumption that s and $N^{f'}$ attains max circulation.

When two sliced arrows have successive ordering, we have no reason to distinguish sliced arrows from the original arrow regarding circulation, which completes our proof.

A.6 Results relevant to Rotemberg (2011)

We maintain notations in Rotemberg (2011) regarding its target class of network.

The next corollary shows that Rotemberg (2011) treated one of the simplest classes of f-Network with no arrow-twisted cycles.

Corollary 2. *case for Rotemberg (2011)*

For a closed f-Network $N^f =< V, A, f >$ which is in a class of C_N^K with flow z [10] *, we have*

$x_{N^f}^{min} = z$.

[10] We obey the definition of Rotemberg (2011) on C_N^K. N subjects indexed by $i \in [0, 1, .., N-1]$ are arrayed in a circle so that $N-1$ is followed by firm 0. Each subject i has payment in the amount of z to subjects $i+j$ with $j \leq K$, where the addition is taken modulo N. $2K \leq N-1$ is assumed. C_N^K is an associated f-Network.

Proof. Since C_N^K is based on an Euler graph, we can take a cycle c which consists of all the arrows. Further, since flow for each arrow is equal in the amount of z, we can take a closed cycle decomposition with unique undominated cycle c with flow $f(c) = z$. Since Euler graph has no arrow-twisted cycles, the minimum circulation is realized with c, and the derived value is z.

□

The next corollary shows that Rotemberg (2011) treated one of the simplest classes of f-Network with no vertex-twisted cycles.

Corollary 3. *case for Rotemberg (2011)*

For a closed f-Network $N^f = < V, A, f >$ *which is in a class of* C_N^K *with flow* z *, suppose* $N/(K!)$ *is integer. Then, we have*

$x_{N^f}^{max} = z \sum_{k=1}^{K} k * (\frac{N}{k} - 1)$

Proof. When $N/(K!)$ is integer, there exists no vertex-twisted cycles within C_G^{np} for any associated graph G. Further, we have $C_{N^f}^{nd,np} = C_{N^f}^{ud,np}$. We can take $C \in C^{nd,np}$ such that $|C| = \sum_{k=1}^{K} k$ and each $c \in C$ consists of $\frac{N}{k}$ vertices for $k = 1, 2, ..K$. Considering into weight $\frac{N}{k} - 1$ for each cycle with k vertices, we have the value as stated in the theorem. □

Note that if $N/(K!)$ is not integer, then there exists vertex-twisted cycles within C_G^{np}. We proceed to examine effects of several network transformation on each maximum circulation.

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Chapter 2.

Does Central Counterparty Reduce Liquidity Requirement?

Abstract

This paper examines whether central clearing of debt or equities economizes use of settlement funds, or liquidity, within interbank settlement systems. A model is developed based on a graph-theoretic framework proposed by Hayakawa (2013a). Our model is to capture how central clearing counterparty (CCP) transform structure of relevant payment network, and our analysis examines whether required liquidity is reduced or not through those transformations.

Effect of adding a CCP is shown to be decomposed into two inter-relating effects: *(central) routing effect* and *(central) netting effect*. The former refers to effects when CCP had not provided any netting service, while the latter corresponds to effects of netting service. Whether each of the effects is positive or negative is shown to depend on exogenously given settlement environment.

Adding a CCP has positive effect through *routing effect* under good environment by serving best possible route for liquidity to circulate, while it is negative under bad environment by serving as an additional stop for liquidity. Netting effect under bad environment is shown to alleviate negative effect by *routing effect*. In contrast, *netting effect* under good environment can be either positive or negative. Each of the positive and negative effect is examined for each specific class of network. *Netting effect* under good environment tends to be negative when nettings are executed for obligations that had helped better circulation of liquidity, while it tends to be positive when netted obligations had required additional liquidity on their own settlements.

JEL classification: G20, E42, D85
Keywords: settlement system, central counterparty, interconnected financial network, graph theoretic Approach

1 Introduction

A central clearing counterparty (CCP) for bonds or equities substitutes itself as a seller to every buyers and a buyer to every sellers. One of major expected roles of CCP is to reduce cost of settlement through economizing settlement fund required for its associated payments. This role needs to be evaluated from two aspects. First, CCP is a *provider* of settlement service in a form of netting; settling obligations by offsetting mutual obligation opposing with each other. Second, much less argued role of CCP is that CCP itself is a *participant* of interbank settlement systems, where CCP has its account that is used for settlements of its substituting obligations. CCP lets sellers and buyers to transfer settlement funds through account of the CCP.

Role of CCP is typically argued along with the former role, focusing on amounts of obligations that would have been netted. There are two essential differences that distinguish our view from the typical view; one is on its role and the other is its focus. In our view, focusing only on how much obligations could be offset by CCP is not sufficient even when we examine only the role of CCP as a provider of netting service. Observe that the same unit of liquidity is possibly transfered multiple times to settle long sequence of obligations under interconnected payment network. Offsetting part of obligations would affect circulation of liquidity by eliminating part of "route" to be transferred. When we turn to the role as a participant of settlement system, we also observe it possibly alters circulation of liquidity by letting "routes" for liquidity be combined, or by standing as an additional "stop" of liquidity.

In order to clarify our point, we start by an example that shows how the role as a participant is to be examined, then proceed to show how the role as an offsetting service provider is accordingly grasped.

Suppose there exists four financial institutions a, b, d, e and one CCP, which have formed a distribution of obligations as expressed in the left of the Figure 1. It shows that each of the two institutions b, d has an obligation in the amount of 10 to the CCP, and the CCP in turn has two obligations in the amount of 10 to institutions a, e. Suppose CCP deals with equity trading and the four obligations are associated with equity trading. The remained two obligations from a to b and from e to d are supposed to be associated with the other financial contracts. In order to see how the existence of CCP affect distribution of obligations and associated liquidity circulation, we need to prescribe how obligations would have been formed without CCP. For our example, suppose each of the four obligations handled by the CCP were associated with trading of one unit of equity with its price 10. Further supposing any of the financial institutions are indifferent to their counterparts for their tradings and equity had been traded in market situation, we can presume two possible situations without CCP as expressed in the middle and the right of Figure 1. The middle one corresponds to the situation where b bought equity from a, while d bought from e. The right one shows the situation where b bought from e, and d did from a.

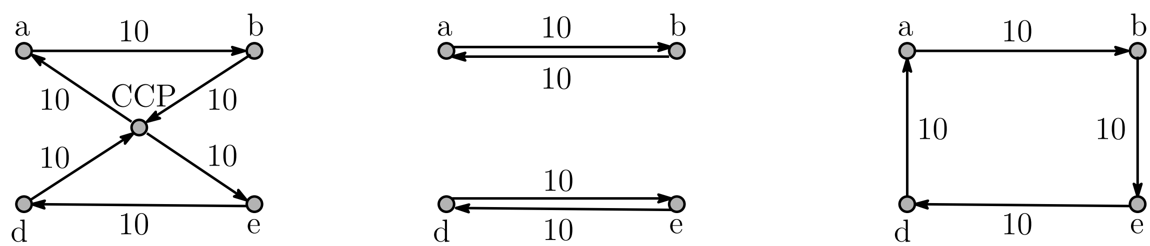

Figure 1: example distribution of obligations

The left of the figure shows an example distribution of obligations with CCP. The middle and the right of the figure show distributions of obligations that can be presumed to realize under situation without CCP.

Suppose any bilateral netting between financial institutions is not possible. The rest of our task is to examine how each of the obligations are to be settled and compare how much liquidity is likely to be required. Before proceeding to the task, let us see how role of netting service by CCP is to be examined in consistent with the view presented here.

Now for the previous example, suppose two financial institutions f, g are added with the same one CCP (denoted as C) and four institutions a, b, d, e, and they have formed a distribution of obligations as expressed in the left of the Figure 2. Each of the added financial institutions f and g has one obligation to CCP in the amount of 10 and the same amount of payment from CCP. Each pair of obligations can be netted by the CCP. Suppose the CCP had netted the obligations, then the situation would be as expressed in the right of the Figure 2, which is the same as the previous example shown in the left of Figure 1 when we remove institutions f and g that are effectively irrelevant for examination of circulation of liquidity.

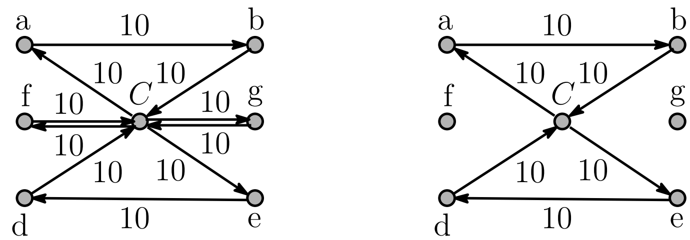

Figure 2: netting service by CCP

The left of the figure shows a distribution of obligations before CCP provides netting service, and the right of the figure shows that after netting is executed.

Back to the situation in the left of the figure, we can presume how obligations would have been formed when the CCP had not existed in a similar way to the previous example. Figure 3 shows some of such presumed distributions of obligations under market situation.

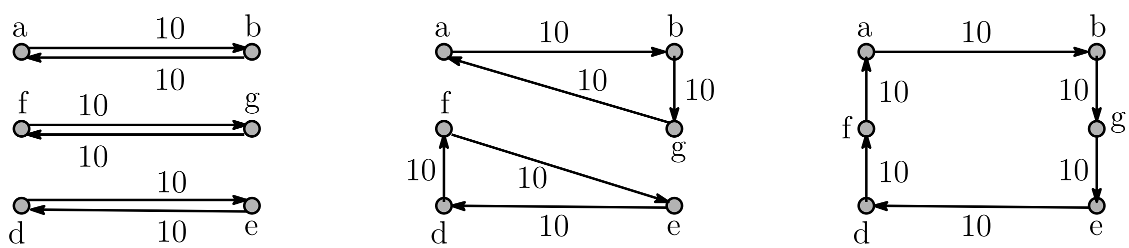

Figure 3: some of presumed distributions of obligations without CCP (for the left of Figure 2)

Refer to the situation expressed in the left of Figure 2 as a pre-netted CCP situation, and to the right of the figure as its associated netted CCP situation. For a pre-netted situation, consider its associated netted situation. Supposing our given situation is the netted situation as in the left of Figure 1, then role of CCP is only as a participant for settlement system, and the effect is analyzed by comparing with its associated presumed distributions of obligations under market situation as expressed in the middle and the right of Figure 1. In this sense, we are to refer the effect as *(central) routing effect* for our original pre-netted situation. For our original pre-netted situation as shown in Figure 2, total effect of adding CCP is captured by comparing itself with its associated presumed distributions of obligations under market situation as some of which are shown in Figure 3. We view that role of CCP as a provider of netting service is captured through comparing presumed market situations for the pre-netted situation and those for the netted situation. We refer to the effect as *(central) netting effect*. In our view, *netting effect* is effectively gasped as difference of total effect and routing effect.

In order to examine *routing effect* and *netting effect* of CCP, we now need to argue how each given distribution of obligations are to be settled, and how much liquidity is required for the settlement. Specifically, what matters is in which order given obligations are to be settled. For example, for a distribution of obligations shown in the left of Figure 1, we can take relative order of settlement as shown in the left or the right of Figure 4. There, order of settlement is expressed with each smaller number attached with each obligation. For the left of the figure, obligations of a to b is firstly settled. It requires a to input 10 amount of liquidity for its settlement, which is expressed with the number in boldface attached to the vertex for the institution a. Secondly settled obligation is that of b to CCP. Notice that b does not need input additional liquidity for the settlement since it has already received liquidity from a and can utilize it for the settlement, which is similarly expressed with the number in boldface that shows 0 attached to the vertex for the institution b. Proceeding in this way, we can describe how much liquidity input is required for each institution under the order of settlement. We are to term the figure that expresses order of settlement and associated required liquidity in combination with given distribution of obligations shows a settlement procedure for the given distribution of obligations. Each of the left and the right of Figure 4 shows a settlement procedure for given distribution of obligations expressed in the left of Figure 1. Now focusing on total amount of required liquidity for each settlement procedure, we confirm it is 10 for the left of the figure, while it is 40 for the right of the figure.

In literature of interbank settlement systems, it is argued that timing of each settlement within each day is determined in a decentralized manner. Formal arguments in several papers [1] showed that order of settlement possibly depends on several parameter values, that can be captured as settlement environment. We suppose certain exogenous settlement environment determines possible settlement procedures. We specifically suppose two polar cases; good (settlement) environment and bad (settlement) environment. We further suppose under good environment, settlement procedures need to be those realize minimum possible total amount of required liquidity among every possible settlement procedures, while under bad environment, settlement procedures need to be those realize maximum possible total amount of required liquidity. For our given distribution of obligations in the left of Figure 1, a settlement procedure in the left of Figure 3 is possible under good environment while that in the right of the figure is possible under bad environment. Note that there we suppose each of the obligations need to be settled at one time, or it is not allowed to settle dividing each given payment into multiple units. This assumption is to capture settlements in interbank settlement systems, which we maintain throughout our analysis.

[1] Bech and Garratt (2003) proposed games between two players to examine incentive regarding settlement timing. It modeled situations players face trade-off between cost of delaying payment and cost of financing intra-day liquidity. Those parameters are shown to affect timing of payment in various manner for each transaction scheme. The similar incentive structure is adopted with larger number of players by Martin and McAndrews (2008), Martin and McAndrews (2010) or Hayakawa (2013b). Hayakawa (2013b) explicitly examined amount of required liquidity in relation to those parameter values to show required

Figure 4: Example settlement procedures (for the left of Figure 1)

Our comparison between each CCP situation and its associated market situation is executed under the same environment, which is either good or bad. Figure 5 shows each possible settlement procedure under good environment for each distribution of obligations shown in Figure 1 , while Figure 6 shows those under bad environment.

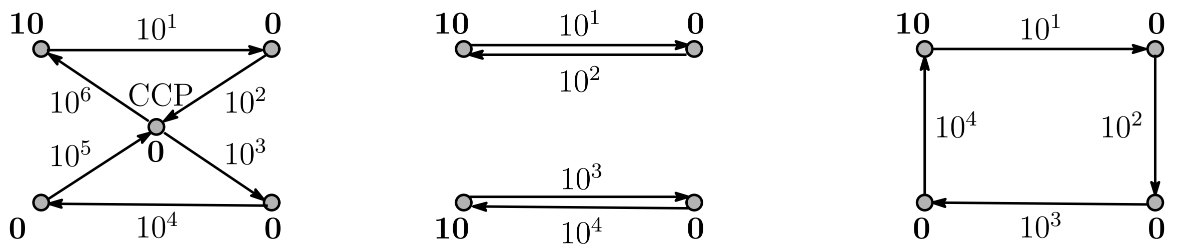

Figure 5: settlement procedures for Figure 1 under good environment

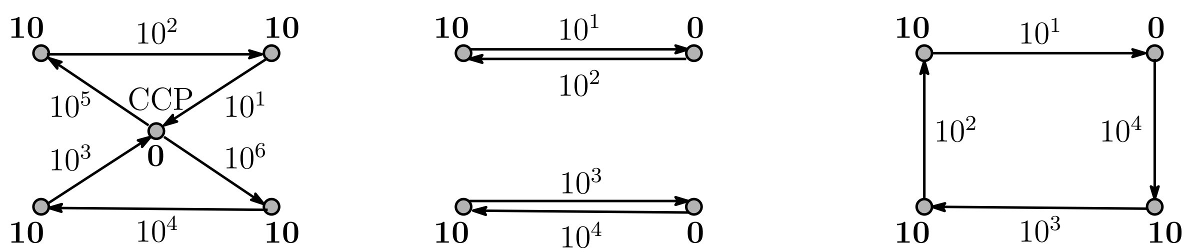

Figure 6: settlement procedures for Figure 1 under bad environment

As it is shown in Figure 5, under good environment CCP situation requires 10 amount of liquidity, while market situation shown in the right of the figure is to require the same 10 amount of liquidity while the middle of the figure is to require 20. Notice that the required amount for CCP situation equals to the minimum among those required by settlement procedures for its associated market situation. We will see this observation is generally satisfied. Remember that examination of this example amounts to examine *routing effect* of associated pre-netted CCP situations. Under good environment, we will show that *routing effect* has always positive effect in the sense total amount of required liquidity is never larger for the case with CCP than the case without CCP. We interpret that additional CCP amounts to serve best possible route for liquidity to circulate. Under

liquidity tends to be larger when relative cost of delaying payment is smaller.

bad environment, in contrast , Figure 6 indicates *routing effect* is not positive. It will be shown to be generally true. Additional CCP now serves as an additional stop for liquidity that tends to increase required liquidity.

With our previous example, we proceed to see how *netting effect* would be. Suppose our given situation is pre-netted CCP situation as shown in the left of Figure 2. Since its associated netted situation effectively equals to that shown in the left of Figure 1, *routing effect* of the additional CCP is as examined above. We see whether *netting effect* works to complement *routing effect* or has its opposing effect. Remember Figure 3 shows some of possible distributions of obligations under its associated market situation. First under bad environment, Figure 7 shows each possible settlement procedure for those in Figure 3. For our examination of *netting effect*, compare those of Figure 7 with the left and the middle of Figure 6. We confirm that required liquidity tends to be larger for Figure 7. This observation will shown to be generally true in a more careful manner, which states that *netting effect* under bad environment is positive. When we consider total effect of CCP under bad environment, negative effect through *routing effect* is alleviated by positive effect through *netting effect* and possibly positive effect dominates.

Next we proceed to see *netting effect* under good environment. For our example, we compare Figure 8 with the middle and the right of Figure 5. For this case, we just point out the left of Figure 8 presents a settlement procedure that requires 30, which is strictly larger than that required by each of the two settlement procedures in Figure 5. This example shows *netting effect* can be positive, but it is shown *netting effect* can also be negative under good environment.

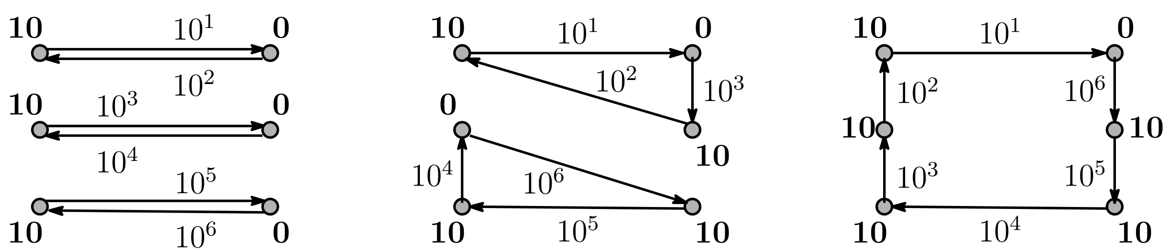

Figure 7: settlement procedures for those in Figure 3 under bad environment

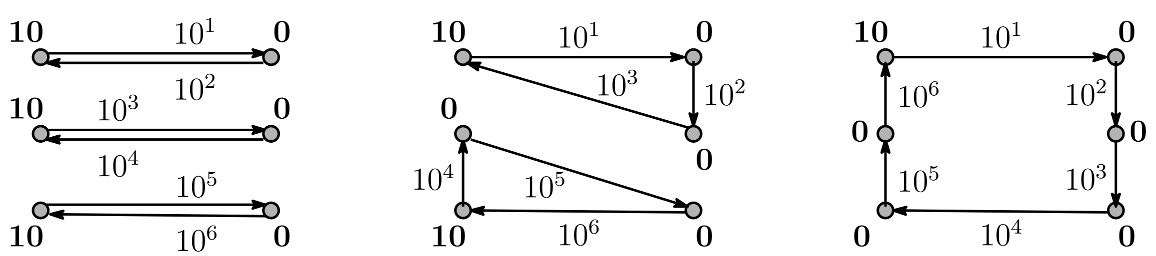

Figure 8: settlement procedures for those in Figure 3 under good environment

Suppose our pre-netted situation is that shown in the left of Figure 9 with its netted situation is shown in the right of the figure. For the pre-netted situation, the left of Figure 10 shows a possible presumed distribution of obligations under market situation, while for the netted situation, the right of the figure shows one such possible distribution of obligations. Under good environment, it is easily confirmed that for the left of Figure 10, required liquidity is just 10 while it is 30 for the right of the figure. Further, we confirm for the netted situation there is no associated presumed distribution of obligations in market situation that requires less than that 30 amount of liquidity. It indicates that *netting effect* is negative for this example. We will show *netting effect* has negative effect for certain class of network.

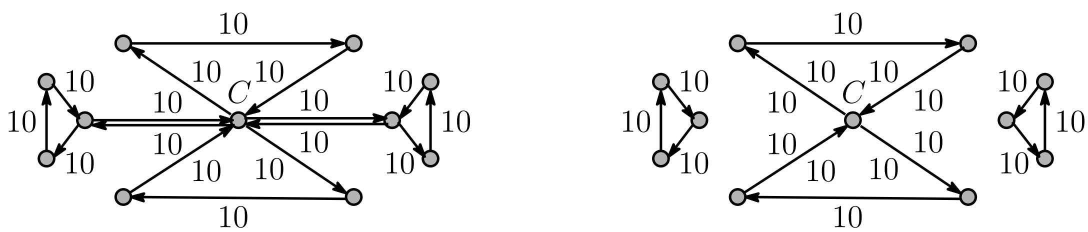

Figure 9: netting service by CCP

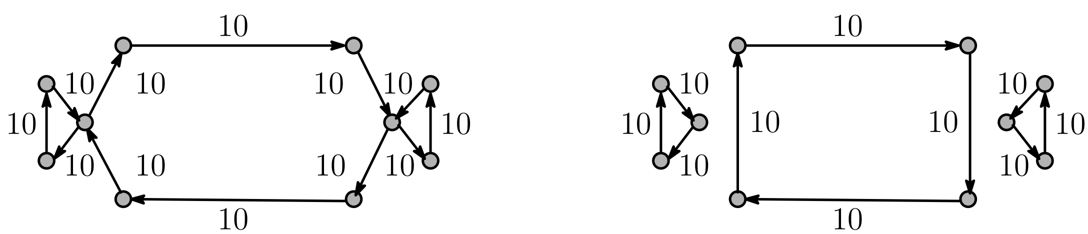

Figure 10: without CCP (for the left of Figure 9)

For our analysis in a general setting, this paper adopts a graph-theoretic framework proposed by Hayakawa (2013a). Further we utilize a pair of liquidity problems called as the min/max PC Problem firstly proposed and examined in the same paper. The min(max) PC Problem derives minimum(maximum) amount of required liquidity for given distribution of obligations. Our main contribution is to apply the min/max PC Problem to analysis of CCP. Our analysis effectively examines effects of certain type of network transformation on the min/max PC Problem. Hayakawa (2013a) examined effects of each basic network transformation, and this paper is to examine effects of certain combination of those basic transformations.

1.1 Interbank Settlement Systems

We provide some background for our analysis. Our examination of role of CCP is executed within interbank settlement systems. Interbank settlement systems are typically operated by central banks, where payments are settled through accounts in central banks, and their participants are various financial institutions including banks, securities companies, and also CCPs.

In interbank settlement systems, it is not unusual that payment networks are highly complex[2]. Our model is constructed as a general depiction of those complex networks.

Traditionally, interbank settlement systems processed transactions on a net basis; payments are collected and settled only at certain designated time –typically once a day–, and participant banks make net payments; the difference between payments received and payments owed. Realizing that net settlement systems are prone to cascades of defaults, many of interbank settlement systems now adopt real-time gross settlement (RTGS) systems that settle each payment on an individual basis. Our framework is to capture circulation of liquidity under RTGS systems[3].

Though RTGS systems reduce the risk of cascades of defaults compared to net settlement systems, it tends to require considerable liquidity. When participants hold insufficient funds for settlement, typically central banks provide intraday liquidity. For well-functioning of settlement systems, it is crucial to provide sufficient liquidity. The critical question there is how much liquidity is required for settlements in each interbank settlement system. The study investigates the question focusing on whether and how CCPs affect required liquidity as a whole.

1.2 Relevant Literature on role of CCP

Roles of CCP have been studied from two aspects: one is from liquidity view for CCP that deal with bonds or equities, and the other is from risk view for CCP for derivative contracts. For the latter risk view, Duffie and Zhu (2011) examined whether CCPs decrease counterparty risk[4]. For the former aspect that is relevant to this paper, it is a wide consensus that CCP helps reduce required liquidity through providing netting service on their own[5]. There, what is presumed to matter is amount of obligations that are netted by CCP. This paper is to challenge the presumption by explicitly analyzing amount of liquidity required for settlement. Our analysis shows that CCP possibly affect amount of required liquidity not only through its netting service, but through the role as a participant of interbank settlement systems. The former effect is captured with *(central) netting effect*, while the latter with *(central) routing effect*. The paper further points out *netting effect* should not be examined solely through amount of netted obligations, but needs to be analyzed considering into how liquidity circulate to settle both those directly relevant obligations and also obligations that are not directly relevant to CCP.

For the rest of the paper, section 2 prepares our framework and the min/max PC Problem based on Hayakawa (2013a). Section 3 shows our model setting. Section 4

[2] About 500 financial institutions participate in the BOJ-NET, which settles approximately 50000 transactions daily. Complexity of payment networks in interbank settlement systems is studied empirically by Imakubo and Soejima (2010) for the BOJ-NET, Soramaki, Bech, Arnold, Glass, and Beyeler (2007) for the Fedwire, and Rordam and Bech (2009) for Danish interbank money flow.

[3] Many of real-world RTGS systems partially combine net settlement systems for the purpose of liquidity saving. Our model captures a "pure" RTGS system without any such liquidity-saving mechanism. Effects of liquidity saving mechanisms are analyzed in Martin and McAndrews (2008), and Martin and McAndrews (2010).

[4] Roles of CCP from counterparty risk in derivative markets are also argued in Bliss and Steigerwald (2006), Bliss and Kaufman (2006), and Pirrong (2009)

[5] Hills, Rule, Parkinson, and Young (1999) provided examples for netting effects by CCP, comparing CCP with bilateral trading situation.

presents several definitions and useful observations for our analysis. Section 5 is our main part, which shows several general results that were hinted in our introductory section. Section 6 is for our concluding remarks. Some of relevant definitions and results are shown in Appendix.

2 Framework

2.1 distribution of obligations, settlement procedure

The following definitions rest on Hayakawa (2013a).

Our framework consists of five elements, which are expressed with five characters: V, A, f, s, p. The base elements are V and A, where V is a set of *vertices* which expresses financial institutions, while $A = \{(v, w, n)|v, w \in V, n = 1, 2, ..\}$ is a set of *arrows* each of which is an ordered pair of vertices with each index, and expresses payment relation between a pair of financial institutions. Indices are used to distinguish different payments among the same institution. If there is no such multiplicity, all the indices are set as 0, and the indices are usually not mentioned in order to avoid notational cumbersome. We do not allow any arrow from and to the same vertex, or exclude payments from and to the same institutions. $< V, A >$ constitutes a *directed graph*.

We are to introduce additional elements f, s, p to $< V, A >$ to constitute *distribution of obligations* $< V, A, f >$, and *settlement procedure* $< V, A, f, s, p >$. In our analysis, *distribution of obligations* $< V, A, f >$ indicates how obligations have been formed among financial institutions, while *settlement procedure* $< V, A, f, s, p >$ indicates in which order and with which liquidity those obligations are settled. Firstly, $f : A \to R_+$ is called as *flow*, which expresses each amount for each payment. Secondly, $s : A \to \{1, 2, .., |A|\}$ is called as *sequence*, which is one-to-one mapping where $|A|$ denotes the total number of arrows, and it indicates relative order of settlement. Lastly, $p : V \to R_{0+}$ is called as *potential*, which expresses each initial holding of settlement funds, or liquidity for each subject that is utilized for settlements.

The left of Figure 11 shows an example for distribution of obligations, and the right of the figure shows an example of settlement procedure constructed by adding s, p to the distribution of obligations in the left.

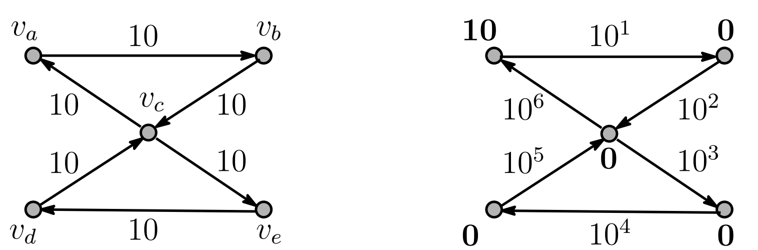

Figure 11: $V = \{v_a, v_b, v_c, v_d, v_e\}$, $A = \{(v_a, v_b), (v_b, v_c), (v_c, v_a), (v_c, v_e), (v_e, v_d), (v_d, v_c)\}$, $f(a) = 10$ for every $a \in A$, $s((v_a, v_b)) = 1, s((v_b, v_c)) = 2, s((v_c, v_a)) = 6, s((v_c, v_f)) = 3, s((v_e, v_d)) = 4, s((v_d, v_c)) = 5, p(v_a) = 10, p(v_b) = p(v_c) = p(v_d) = p(v_e) = 0$

We confine us in a specific situation throughout the paper, where obligations are balanced for each financial institution. Specifically, for each financial institution, total amount of payments to make and to receive are balanced. In other words, obligations are universally in gridlock situation. We prepare some notations. Given a distribution of obligations $< V, A, f >$, aggregate amount of payments to receive for $v \in V$ is denoted as $f_v^I \equiv \sum_{v' \in V} f((v', v))$, while aggregate amount of payments to make for $v \in V$ as $f_v^O \equiv \sum_{v' \in V} f((v, v'))$.

Definition 1. balanced obligations
distribution of obligations $< V, A, f >$ is *balanced* if $f_v^I = f_v^O$ for every $v \in V$.

The left of Figure 11 actually shows a balanced distribution of obligations.

Given a distribution of obligations, we set up a problem that examines how much settlement funds, or liquidity is required to settle all the obligations. We are to capture the amount through constructing associated settlement procedures. For this purpose, we require settlement procedure $< V, A, f, s, p >$ needs to be feasible in the sense specified liquidity with potential p is sufficient for settle all the obligations, and also there is no redundant liquidity that is never used for any of the settlements. For those formal statements, we first prepare several notations. Given a settlement procedure $< V, A, f, s, p >$, suppose periods proceed as $t = 0, 1, .., |A|$ where relative order, or sequence s corresponds to each period t in a way that payments to be executed at the beginning of period t are $arg_a s(a) = t$. Aggregate periodical payments to receive for $v \in V$ at period t is denoted as $f_{v,t}^I = \sum_{v' \in V} 1_{\{s(v',v)=t\}} f((v', v))$, while that to make is denoted as $f_{v,t}^O = \sum_{v' \in V} 1_{\{s(v,v')=t\}} f((v, v'))$. Then periodical holding of liquidity for each subject $v \in V$ at the last of period t is denoted as $p^t(v) = p^{t-1}(v) + (f_{v,t}^I - f_{v,t}^O)$ for $t = 1, 2, .., |A|$ and $p^0 = p(v)$. Initial holding of liquidity is sufficient when every periodical holding is sufficient.

Definition 2. feasible settlement procedure
Settlement procedure $< V, A, f, s, p >$ is *feasible* if $p^t(v) \geq 0$ for every $v \in V$ and every $t = 0, 1, .., |A|$.

Definition 3. proper settlement procedure
Settlement procedure $< V, A, f, s, p >$ is *proper* if
(feasibility) $< V, A, f, s, p >$ is *feasible*, and
(no redundancy) there is no other $p' : V \to R_{0+}$ such that $< V, A, f, s, p' >$ is *feasible*, and $p'(v) \leq p(v)$ for every $v \in V$, and $p'(v') < p(v')$ for some $v' \in V$.

Note that for balanced distribution of obligations $< V, A, f >$, when relevant sequence s is given, settlement procedure $< V, A, f, s, p >$ is uniquely derived. When settlement procedure $< V, A, f, s, p >$ is proper, we term $\sum_{v \in V} p(v)$ as *circulation* for $< V, A, f, s >$.

2.2 min/max PC Problem

Below shows the min/max PC Problem proposed by Hayakawa (2013a).

The minimum Payment Circulation Problem (min PCP) is;

(min PCP)
Given a balanced distribution of obligations $< V, A, f >$,
$\min_{s,p} \sum_{v \in V} p(v)$,
s.t. $< V, A, f, s, p >$ *is a proper settlement procedure.*

The maximum Payment Circulation Problem (max PCP) is;

(max PCP)
Given a balanced distribution of obligations $N^f =< V, A, f >$,
$\max_{s,p} \sum_{v \in V} p(v)$,
s.t. $< V, A, f, s, p >$ *is a proper settlement procedure.*

We term solution for each min/max PCP as *min/max circulation*.

Note that in the above problems, each obligation is not allowed to be settled in multiple units, but needs to be settled at one time. The setting is to capture reality in interbank settlement systems, where settlement in multiple-units is possible but not widely executed.
The min/max PC Problem is to be analyzed along with our model.

3 Model

3.1 financial system, and generation

In order to examine role of CCP, we suppose our economy starts by following *hypothetical* situations. There exist several types of securities, and financial institutions have already determined how much to sell or buy each of the securities, but not yet to be determined is whether CCP is utilized or trades are executed bilaterally for each of the securities. The situation is expressed with a distribution of obligations $\tilde{N}^f =< V, A, f >$ with *egg vertices* $V^e \subset V$, where each of V^e is a "hypothetical entity" that either turns to a CCP or turns to a market where its associated payments become bilateral payments among the relevant institutions. For each distribution of obligations $\tilde{N}^f =< V, A, f >$ with $V^e \subset V$, we introduce a *differentiation function* $d : V^e \to D$, where $D = \{market, CCP\}$ as *differentiation potency*. Function $d : V^e \to D$ specifies to which each of the hypothetical entities V^e turns. We specifically term each corresponding triple $(\tilde{N}^f, V^e, d)$ as a *financial system*. For any financial system $(\tilde{N}^f, V^e, d)$, we require $\tilde{N}^f$ is *balanced*.
Our economy starts with each given *financial system*, and it prescribes which distribution of obligations is possibly derived in the following period.

For given financial system, we make *Separative* assumption, which is to exclude unrealistic situations where hypothetical entity engages in trades with some other hypothetical entity.

Assumption 1. Separative

Given a financial system $(\tilde{N}^f, V^e, d)$,
there exists no $v, v' \in V^e$ such that $(v, v') \in A$ or $(v', v) \in A$

When $d(v) = market$ for some hypothetical entity $v \in V^e$ and the entity has multiple payments to make and receive, then there are multiple ways for bilateral payments. In order to capture possible combinations for those bilateral payments in a consistent way, we introduce *Equal flow* assumption for financial systems. *Equal flow* assumption requires obligations relevant to each hypothetical entity are to be formed in the same unit. *Equal flow* assumption is interpreted that each hypothetical entity handles one security class with homogeneous price for the same unit. When trades for the security are executed bilaterally, financial institutions are supposed to be indifferent for identity of their counterparts. The assumption helps simplify our analysis. Below shows its formal definition.

Assumption 2. Equal flow

Given a financial system $(\tilde{N}^f, V^e, d)$,
for every $v \in V^e$, for every $a, a' \in A_v$, $f(a) = f(a')$,
where A_v is a set of arrows such that for every $(v_s, v_e) \in A_v$, either $v_s = v$ or $v_e = v$.

Now we describe which distributions of obligations are possible to be derived from each given financial system. For a financial system, $(\tilde{N}^f =< V, A, f >, V^e, d)$, suppose $d(v) = market$ for $v \in V^e$. Define one-to-one mapping $m_v : V_v^I \to V_v^O$ as a *matching* for $v \in V^e$, where we denote $V_v^I = \{v' \in V | (v', v) \in A\}$ $V_v^O = \{v' \in V | (v, v') \in A\}$. Each matching specifies combination of bilateral payments, and given matchings for all the relevant hypothetical entities, we have one distribution of obligations.

Next suppose $d(v) = CCP$, $v \in V^e$ for $(\tilde{N}^f =< V, A, f >, V^e, d)$. We suppose CCPs provide netting service. Specifically, we eliminate any pair of arrows $(v', v), (v, v') \in A$ to have $< V, A', f' : A' \to R_+ >$, where $f'(a) = f(a)$ for every $a \in A'$. After the elimination, remained obligations relevant to the hypothetical entity v are to be interpreted as those of CCP.

Together with the above assumptions, it has been specified which distributions of obligations are possible for given financial system. For generation $< V, A, f >$, we denote $V^{CCP} \subset V$ as subjects generated as CCPs.

3.2 settlement environment

We suppose *settlement environment* indicates possible settlement procedures for each generated distribution of obligations. Settlement environment ω is assumed either good or bad as $\omega \in \{good, bad\}$. For our analysis, the min(max) PC Problem is to derive required amount of liquidity under good(bad) environment.

Good environment is supposed to correspond to the situation where coordination is well-formed, while bad environment is supposed to cause ill-formed coordination. This paper adopt direct supposition on which settlement procedures are derived under each environment. For a generated distribution of obligations, good environment requires settlement procedures to indicate minimum possible total amount of liquidity for their settlements, while for bad environment settlement procedures need to indicate maximum possible amount of liquidity. Note that considering into its coordination nature, we do not suppose any probability on settlement environment. Also, which settlement environment is unveiled is independent of which distribution of obligations has been generated.

For each generated distribution of obligations N^f, let $x^{min}(N^f)$ denotes value derived through min PC Problem and term it as the minimum(min) circulation of N^f, while denote $x^{max}(N^f)$ for its maximum(max) circulation.

Figure 12 summarizes our model and relevant notations.

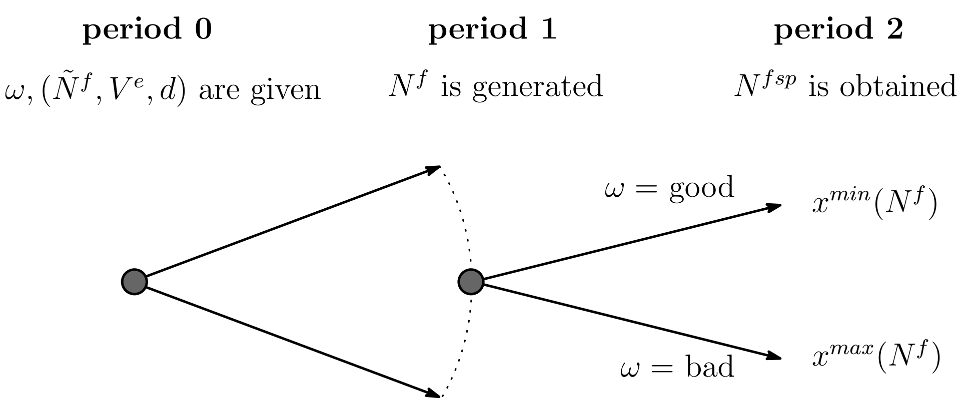

Figure 12: Summary of the model

4 Preliminary Analysis

In this section, we provide several results and observations useful for our analyses. The following lemma ensures our examination through the min/max PC Problem.

Lemma 1.
For any financial system, any of its generated distribution of obligations $< V, A, f >$ *is balanced.*

Proof. We have supposed distribution of obligations $\tilde{N}^f$ for financial system $(\tilde{N}^f, V^e, d)$ is always balanced. Generation never change either amount of total inflow and total outflow for each vertex, which ensures every generation maintains balanced property and completes our proof. □

We define *locally net-out*, *net-out* for our observation and analysis.

Definition 4. locally net-out, net-out
Given a distribution of obligations $< V, A, f >$, we say obligations for $v \in V$ are *locally net-out* if there exists no $v' \in V$ such that $(v, v') \in A$ and $(v', v) \in A$. Further we say financial system $(\tilde{N}^f, V^e, d)$ is *net-out* when obligations for $v \in V^e$ are *locally net-out* for every $v \in V^e$.

Next observation shows how obligations relevant to CCPs are formed by generation.

Observation 1 (locally net-out for CCP)**.**
For any given financial system and any of its generated distribution of obligations $< V, A, f >$ with CCPs $V^{CCP} \subset V$,
obligations for $v \in V^{CCP}$ are locally net-out for every $v \in V^{CCP}$.

For our examination of effect of additional CCP, we prepare definitions. Given two financial systems $(\tilde{N}^f, V^e, d)$ and $(\tilde{N}^f, V^e, d')$ where the only difference is on d, d', we denote $d' >_{CCP} d$ when there exists non-empty $V' \subseteq V^e$, such that for every $v \in V'$ $d'(v) = CCP, d(v) = market$ and for every $v' \in V^e \setminus V'$, $d(v') = d'(v')$. Further, we denote $d -_{CCP} d' = V'$ for this case.
Adding CCPs for a financial system $(\tilde{N}^f, V^e, d)$ is to have another financial system $(\tilde{N}^f, V^e, d')$ such that $d' >_{CCP} d$. There, adding a CCP requires $d' -_{CCP} d = v \in V^e$.

4.1 routing effect and netting effect

We view effect of CCP is conceptually decomposed into two effects: *central routing effect* and *central netting effect*. Our results are stated along with this view. For two financial system $(\tilde{N}^f, V^e, d)$ and $(\tilde{N}^f, V^e, d')$ where $d -_{CCP} d' = v$, take a distribution of obligations $\tilde{N}^f_{net}$ by locally net-out with respect to v, and take two financial systems $(\tilde{N}^f_{net}, V^e, d)$ and $(\tilde{N}^f_{net}, V^e, d')$.
Comparison of the two derived financial systems $(\tilde{N}^f_{net}, V^e, d)$ and $(\tilde{N}^f_{net}, V^e, d')$ amounts to examine effects of adding a CCP supposing net-out obligations had not existed from the beginning. It exclusively examines effects of CCP as an additional subject in the middle of each relevant obligation. We term it examines *central routing effect* of CCP. Comparison of the original two financial systems is to examine total effects, and we view that the difference of the total effect and *central routing effect* expresses effects through netting service provided by CCP. In that sense we say the difference is brought by *central netting effect* of CCP.
Our analysis starts by examining *central routing effect* separated from *central netting effect*.

We say two financial systems $(\tilde{N}^f, V^e, d)$ and $(\tilde{N}^{f'}, V^e, d')$ have *common obligations* with respect to $V \subseteq V^e$ when for every $v \in V$,
a) $A_v = A'_v$, where $A_v = \{(v, v')|v' \in A\} \cup \{(v', v)|v' \in A\}$,
b) for every $a \in A_v = A'_v$, $f(a) = f'(a)$, and
c) $d(v) = d(v')$ for every $v \in V$.
For our comparison of two generations, we introduce definition of *consistent generation*.

Definition 5 (consistent generation)**.**
Given two financial systems $(\tilde{N}^f, V^e, d)$ and $(\tilde{N}^{f'}, V^e, d')$ that have common obligations with respect to $V \subseteq V^e$, take a generation N^f of $(\tilde{N}^f, V^e, d)$ and $N^{f'}$ of $(\tilde{N}^{f'}, V^e, d')$,
we say two generations N^f and $N^{f'}$ are *consistent* when relevant matchings are the same for every $v \in V$.

Next observation is fundamental for analysis on central routing effect.

Observation 2 (unique consistent generation)**.**

For two financial systems $(\tilde{N}^f, V^e, d_+)$ and $(\tilde{N}^f, V^e, d)$ where $d_+ -_{CCP} d = V'$, take a generation N^f of $(\tilde{N}^f, V^e, d)$, we can always take its unique consistent generation N^f_+ of $(\tilde{N}^f, V^e, d_+)$.

For analysis of central netting effect, we introduce definition of *addition of obligation pair*.

Definition 6.

For a financial system $(\tilde{N}^f =< V, A, f >, V^e, d)$, we say adding an *obligation pair* between $v \in V \setminus V^e$ and $v' \in V^e$ to have another financial system $(\tilde{N}^{f'} =< V', A', f' >, V^e, d)$ when

a) $V' = V$,

b) $A' = A \cup \{(v, v'), (v', v)\}$,

c) $f'((v, v')) = f'((v', v))$ with $f'(a) = f(a)$ for every $a \in A$, and $f'((v, v')) = f'((v'', v'))$ if $\exists v'' \in V$ such that $(v'', v') \in A$,

5 Analysis

We start by focusing on central routing effect.

5.1 Central Routing Effect

We have following clear-cut result for central routing effect.

Theorem 1 (Central Routing Effect: general)**.** *Given two net-out financial systems $(\tilde{N}^f, V^e, d_+)$ and $(\tilde{N}^f, V^e, d)$ where $d_+ >_{CCP} d$, for any generation N^f of $(\tilde{N}^f, V^e, d)$ and its consistent generation N^f_+ of $(\tilde{N}^f, V^e, d_+)$, we have*

$x^{min}(N^f_+) \leq x^{min}(N^f)$, *and*

$x^{max}(N^f_+) > x^{max}(N^f)$,

Proof. We always derive N^f_+ from N^f with certain combination of network transformations. The relevant network transformations are *arrow separation* and *vertex contraction*, whose definitions are shown with several relevant results in Appendix 7.1

We observe *arrow separation* on appropriate arrows followed by *vertex contraction* on appropriate vertices on N^f derives N^f_+. From the result that both *arrow separation* and *vertex contraction* never increase min circulation for any $< V, A, f >$, our result for the min circulation is straight.

For the result for max circulation, suppose make *arrow separation* on two of the arrows $a, a' \in A$ for some $< V, A, f >$ by adding two vertices $v, v' \in V$. It increases max circulation by $f(a) + f(a')$. For our case, we only need to focus on the case $f(a) = f(a')$, then the increase of max circulation is $2f(a)$. Now take *vertex contraction* on added vertices v, v', then it at most decreases max circulation by $f(a)$. This derives our result for max circulation. □

Our introductory example in Figure 1 is relevant to the above theorem. The theorem shows opposite direction of routing effect regarding settlement environment. Under good environment, additional CCP tends to enhance cross circulation of liquidity, and effectively serves best possible route for liquidity to circulate. In contrast, under bad environment, additional CCP serves as an additional stop for liquidity which always increase liquidity needs.

5.2 Central Netting Effect

For analysis of central netting effect, we examine effects of adding obligations for net-out financial system in a way that all the added obligations would be net-out with CCP. Such operation has no effect when all the hypothetical institutions turn to be CCPs, but not readily known for the other cases.

We first examine how max circulation is affected. The next theorem shows effect of max circulation tends to be positive. We prepare notations for the theorem. For a financial system $(\tilde{N}^f, V^e, d)$, take a generation N^f. Let $[N^f]_v$ for some $v \in V^e$ denote a set of generations of the same financial system that are generated under the same matchings with N^f except for v.

Theorem 2 (Central Netting Effect: positive for bad env.).
For a financial system $(\tilde{N}^f =< \tilde{V}, \tilde{A}, \tilde{f} >, V^e, d)$ *with* $d(v) = Market$ *for* $v \in V^e$*, take another financial system* $(\tilde{N}^{f'}, V^e, d)$ *by adding an obligation pair between* v *and some other* $v' \in \tilde{V}$*.*

Then, denoting a generation N^f *of the original financial system* $(\tilde{N}^f, V^e, d)$ *and any of its consistent generation* $N^{f'}$ *of the derived financial system* $(\tilde{N}^{f'}, V^e, d)$*, we have;*
there always exists a surjection $F : [N^{f'}]_v \to [N^f]_v$ *such that*
$x^{max}(N^{f'}) \geq x^{max}(F(N^{f'}))$ *for any* $N^f \in [N^f]_v$*.*

Proof. For each generation $N^f =< V, A, f >$ of the original financial system $(\tilde{N}^f, V^e, d)$, take an arbitrary arrow $a = (v_a, v_b) \in A$ such that $v_a, v_b \neq v'$ and $(v_a, v), (v, v_b) \in \tilde{A}$. For N^f, we can take its consistent generation $N^{f'}$ of the derived financial system by taking arrow separation on a with additional vertex v_c, then taking vertex contraction on v_c, v' into v'. As we already showed, combination of arrow separation and vertex contraction never decrease max circulation.

Conversely, for any generation of the derived financial system, we can always have a consistent generation of the original financial system by executing the reverse of the above procedure. When added obligations pair had not existed for given generation of the derived financial system, it is by itself its consistent generation of the original financial system. If it is not the case, execute reverse of vertex contraction on each vertex $v \in \tilde{V} \setminus V^e$ that are attached any obligation pair so that one vertex has only added obligation pair while the other vertex has only obligations that have not been added. Then execute reverse of arrow separation by removing a vertex that has the added obligation pair. We always have a consistent generation of the original financial system, which completes our proof.

□

Our comparison between Figure 7 and the left and the right of 6 in our introductory section is relevant to the above theorem. We already saw that central routing effect is negative under bad environment. The above theorem states that the negative effect tends to be alleviated by positive effect through central netting effect.

In contrast, we do not have such general result for the effect under good environment since both positive and negative effect are possible.

We first see for the case of positive effect. For the statement we prepare definition of *isolated* vertex. For given distribution of obligations $N^f =< V, A, f >$, $v \in V$ is an *isolated* vertex within N^f if there is no obligation neither from v nor to v.

Theorem 3 (Central Netting Effect: positive for good env.)**.**

Given a net-out financial system $(\tilde{N}^f =< \tilde{V}, \tilde{A}, \tilde{f} >, V^e, d)$ *with* $d(v) = Market$ *for* $v \in V^e$ *and isolated vertices* $V^i \subset \tilde{V}$*, take another financial system* $(\tilde{N}^{f'}, V^e, d)$ *by adding* $2K$ *number of obligation pairs between* v *and a set of isolated vertices* V^i *with integer* $K \geq 1$*,* $|V^i| = 2K$ *and each* $v' \in V^i$ *attached an obligation pair.*

Then, for any generation N^f *of the original financial system* $(\tilde{N}^f, V^e, d)$*, we always take its consistent generation* $N^{f'}$ *of the derived financial system* $(\tilde{N}^{f'}, V^e, d)$ *that satisfies;*

$x^{min}(N^{f'}) = x^{min}(N^f) + K * f((v', v))$ *for arbitrary* $v' \in V^i$*.*

Proof. It is straight that we can take additional K cycles with the added obligations relevant to the isolated vertices. The result is immediate since the obligations for the cycles have the same amount $f((v', v))$ with arbitrary $v' \in V^i$. □

Comparison between Figure 8 and the left and the right of Figure 5 in our introductory section is relevant to the theorem. The theorem states that when central counterparty is to net-out obligations between "isolated" subjects, it surely decreases required liquidity under good settlement environment. In that sense, central netting effect is positive for the case.

In contrast, next theorem shows that negative effect is also possible. We prepare to define *island* for a set of vertices. For given distribution of obligations $N^f =< V, A, f >$, $V' \in V$ forms an *island* within N^f if V' constitutes only one cycle, and for every $v' \in V'$, there is no $v \in V \setminus V'$ such that $(v, v') \in A$ or $(v', v) \in A$.

Theorem 4 (Central Netting Effect: negative for good env.)**.**

For a net-out financial system $(\tilde{N}^f =< \tilde{V}, \tilde{A}, \tilde{f} >, V^e, d)$ *with* $d(v) = Market$ *for* $v \in V^e$ *and* K *number of islands* $\{V_1, V_2, .., V_K\}$ *with integer* $K \geq 1$*, take another financial system* $(\tilde{N}^{f'}, V^e, d)$ *by adding* $K + 1$ *number of obligation pairs between* v *and each one vertex* $v_k \in V_k$ *for* $k = 1, 2, .., K + 1$*. Denote flow for each island* $k = 1, 2, .., K + 1$ *as* f^k*.*

Then, for any generation N^f *of the original financial system* $(\tilde{N}^f, V^e, d)$*, we always take its consistent generation* $N^{f'}$ *of the derived financial system* $(\tilde{N}^{f'}, V^e, d)$ *that satisfies;*

$x^{min}(N^{f'}) = x^{min}(N^f) - K * \min\left\{\min_k f^k, f((v', v))\right\}$ *for arbitrary* $v' \in \{v_1, .., v_{K+1}\}$*.*

Proof. It is straight that we can take one large cycle with obligations for the islands and the added obligations pairs. The result is immediate since the obligations for the cycles have the same amount $f((v', v))$ with arbitrary $v' \in V^i$. □

Figure 9 and Figure 10 are relevant to the theorem. The theorem shows that positive effect through central routing effect can be countered by negative effect through central netting effect.

5.3 Specifications of CCP

In our analysis, no special constraint on CCP has been supposed regarding its relevant order of settlement. But several additional constraints would help our analysis be more realistic. We especially take up two types of specifications each works as each additional constraint for the min/max PC Problem.

The first specification requires CCP never input its own liquidity to settle its obligations. We term such CCP is "passive". When CCP needs to be "passive", our problems turn to be as follows.

Problem with "passive" CCPs

Given a balanced distribution of obligations $< V, A, f >$ with a set of CCP $V^{CCP} \subset V$,

$\min(\max)_{s,p} \sum_{v \in V} p(v)$,

s.t., associated settlement procedure $< V, A, f, s, p >$ is proper, and

$p(v) = 0$ for every $v \in V^{CCP}$.

It is immediate that the change of the problem does not affect min circulation, but possibly affect max circulation. For the same balanced distribution of obligations $< V, A, f >$, the altered maximization problem tends to derive smaller value than the original maximization problem. But it does not always have strict effect. For example, the altered maximization problem for the left of Figure 1 still derives the same amount that derived by the original max problem, as confirmed with the right of Figure 4. Under the settlement procedure shown in that figure, CCP is passive since its potential is shown to be zero. We do not go into detail when the two maximization problems derive different values, but just point out that the two problems derives the same value when there is only one CCP supposed. The two can be different when there are more than one CCP supposed. For our analysis, this observation states that CCP can serve as an additional stop even when it is passive, and negative effect through *routing effect* under bad environment is not trivial.

Next further suppose CCP needs to receive all the relevant payments before any of its making payment. We say such CCP is "passive and synchronous", and when all the CCPs need to be "passive and synchronous", then our original problems turn to be as follows.

Problem with "passive and individually synchronous" CCPs

Given a financial system $(\tilde{N}^f, V^e, d)$ and generated distribution of obligations $< V, A, f >$,

$\min(\max)_{s,p} \sum_{v \in V} p(v)$,

s.t., associated settlement procedure $< V, A, f, s, p >$ is proper, and

for every $v \in V^{CCP}$, $\max_{(v',v) \in A} s((v', v)) < \min_{(v,v'') \in A} s((v, v''))$.

The change of the problem greatly affect minimum amount especially when relevant CCPs have multiple obligations. In our analysis, when CCP had not need to be "synchronous", adding CCP has positive effect through "routing effect" by serving best possible route by letting liquidity cross circulate. But when CCP needs to be "synchronous", such cross circulation is never possible. In contrast, additional "synchronous" assumption does not much harm for max circulation, since max circulation is more likely to be realized for settlement procedures where many vertices are in synchronous situation.

Note that our analysis on *routing effect* is accordingly understood with each of the specifications on CCP, but either specification does not affect *netting effect* since it effectively compares between generations under market situations.

6 Concluding Remarks

The paper examined whether CCP helps to reduce liquidity requirement within interbank settlement systems. Challenging a prevailing view that CCP is better to economize liquidity when it nets out more obligations, it provides analysis along with the view that existence of CCP possibly affects circulation of liquidity that settle obligations not only directly relevant to CCP but also those not directly relevant to CCP. The analysis showed total effect of CCP is well understood through two types of effects: *central routing effect* and *central netting effect*. The former effect exclusively captures effects of CCP as an additional participant within interbank settlement systems. The latter effect refers to effects of CCP as a provider of netting service. The decomposition clarified our argument by presenting rather clear-cut results through *central routing effect*, which were shown to be combined with *central netting effect* that affects in more subtle ways.

The analysis of this paper fully rests on the framework developed by Hayakawa (2013a). The framework well captures circulation of money as medium of exchange in an abstract manner, and our analysis specifically examined within interbank settlement systems. Since in the context of interbank settlement systems, neither production nor consumption is considered to have the primary importance, our analysis maintained its clarity by ignoring those aspects.

Though we confined us to suppose our analysis is to examine role of central clearing counterparty for debts or securities, it would be applicable to various types of central counterparties that stand between payers and receivers of monetary subjects. Focusing on how circulation of money is affected, additional retailers between consumers and producers, or even existence of intermediate subcontractors among a chain of subcontractors would also possibly serve such role. Our analysis would hopefully provide a step toward those applications.

7 Appendix

7.1 Relevant definitions and theorems

Following definitions and results are from Hayakawa (2013a).

Take a balanced distribution of obligations $< V, A, f >$. We say *arrow separation* on $a = (v, v') \in A$ into a' and a'' with v'' to have $< V, A', f' >$ when $A' = A \cup a' \cup a'' \setminus a$ and $a' = (v, v''), a'' = (v'', v')$ with $f'(a') = f'(a'') = f(a)$, while $f'(a''') = f(a''')$ for every $a''' \in A \setminus a$.

We say *vertex contraction* for $N^f =< V, A, f >$ on $v, v' \in V$ to v to have $< V', A', f' >$ when $V' = V \setminus v'$, and all the arrows from or to v' in A are replaced by arrows from or to v in A', and f' is determined accordingly. We do not allow *vertex contraction* for $N^f =< V, A, f >$ on $v, v' \in V$ when both $(v, v'), (v', v)$ are included in A.

Theorem 5. *arrow separation*
Given a balanced distribution of obligations $N^f =< V, A, f >$, for any arrow separation on $a \in A$ to have $N^{f'}$, we have
$x^{min}(N^{f'}) = x^{min}(N^f)$.
$x^{max}(N^{f'}) = x^{max}(N^f) + f(a)$.

Theorem 6. *vertex contraction*
Given a balanced distribution of obligations $N^f =< V, A, f >$, for any vertex contraction for N^f on $v, v' \in V$ to have $N^{f'}$, we have
$x^{min}(N^{f'}) \leq x^{min}(N^f)$.
$x^{max}(N^{f'}) \leq x^{max}(N^f)$.

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Chapter 3.

Liquidity Saving Mechanism under Interconnected Payment Network

Abstract

We develop a model of real-time gross settlement system to study welfare effect of liquidity saving mechanism(LSM) under strategic context. We firstly examine implications of limited nature of netting ability of LSM in a form that LSM provides at most *partial netting* service. Analysis of partial netting service is enabled with our invented class on structure of underlying payment network; *core-periphery* structure. Each network within the class is characterized with *density*. We find that *partial netting* service by LSM possibly has negative welfare effects through effectively letting underlying connected payment network be disconnected each other. The negative effects are shown to be more likely for less *dense* network.

JEL classification: G20, E42, C72
Keywords:
settlement system, payment network, liquidity recycle, interconnected financial network, liquidity saving mechanism

1 Introduction

Huge volume of transactions are settled in interbank settlement systems, which are also referred as large-value payment systems(LVPS). Well functioning of the payment systems is critical to the stability of financial systems, and naturally an important policy concern.

Many LVPSs recently moved from net settlement system to real-time gross settlement system(RTGS). In net settlement systems, settlements are executed only at some designated time typically once a day, where participant banks make net payments; the difference between payments received and payments owed. Realizing that net settlement systems could generate large volume of unsettled exposure that let their participants be prone to default risk, many of LVPSs now adopt RTGS systems that settle each payment on an individual basis aiming to reduce unsettled exposure. Although RTGS systems are better to reduce risk in that sense, but has its own problem that it tends to require large amount of liquidity. Most LVPSs are operated by central banks and payments are made with reserves in the accounts of central banks. When reserve of a payer is short of its payment, each central bank typically provides intraday liquidity with some type of fee. Recognizing that providing and preparing sufficient liquidity is costly for central banks, Many LVPSs move to modify RTGS systems so that they reduce required amount of liquidity. Such internal modifications or improvements within the systems are referred under the term of "liquidity saving mechanisms". Typical liquidity saving mechanism works in a way that it allows payments be "queued" instead of immediate payment, and offsets them when possible. Liquidity saving mechanism with more than bilateral netting is not fully incorporated in many LVPSs[1].

The paper provides a model of RTGS system and studies welfare effects of a liquidity saving mechanism which serves partial netting. We show possible negative welfare effects that source from the limited nature of netting ability of LSM.

In order to examine partial netting service by LSM, we develop and adopt *core/periphery* structure for our underlying payment network. Our *core/periphery structure* is to capture characteristics of real world payment networks. Imakubo and Soejima (2010) executed network analysis on payment flows in Japan's interbank money market, and pointed out that some banks form a core network which serves as hubs for peripheral banks, which is termed as core/periphery structure. Similar observation was also reported in Soramaki, Bech, Arnold, Glass, and Beyeler (2007) for the case of Fedwire, in Rordam and Bech (2009) for Danish interbank money flow. In our *core/periphery* payment networks, participants are classified into two groups: core banks and periphery banks, where the former serves as hubs to the latter. The key notion for our analysis is the notion of *density*. Roughly speaking, periphery banks are more tightly connected each other in more *dense* network. Severeness of negative effect of LSM is shown to have significant relation with how dense supposed payment network is.

In our model, strategic incentives for participants are influenced by two types of costs: one is cost associated with delaying payments, and the other is borrowing cost of liquidity. Earlier payment than the other banks means more likely bearing borrowing cost, while delaying payment tends to decrease borrowing cost while it incurs larger delay cost. Welfare consists of aggregate liquidity cost and aggregate delaying cost.

Effect of LSM is examined for two types of situations, or parameter values: One is termed as *Liquidity non-Precious* Regime, while the other as *Liquidity Precious* Regime. Literally, liquidity is not relatively precious, or, liquidity cost is relatively smaller in *Liquidity non-Precious* Regime than in *Liquidity Precious* Regime. We examine how introduction of LSM would affect welfare level for each regime. In our analysis, we assume optimal policy making on setting intraday lending/borrowing fee by the CB before LSM is introduced, and examine welfare effect of introducing LSM under the same parameter values. In this sense, our analysis should be interpreted as certain shock analysis, not as longer-term analysis where policy has been sufficiently adjusted to situation brought by shock. This approach intends to shed its clearest light on our negative effect, which would possibly be persistent with its smaller scale in longer-term.

Under *Liquidity non-Precious* Regime, intraday borrowing fee has been set sufficiently low so as to let participants make payments early to avoid delay cost. Introducing LSM only serves to economize liquidity cost by netting some of the payments that were to be settled earliest with positive amount of liquidity transfer.

In contrast, under *Liquidity Precious* Regime, fee has been optimally set so as to encourage recycle of liquidity, where smallest possible number of participants make payment earliest while the others wait to recycle received liquidity, which is to economize liquidity cost instead of incurring larger delay cost. Introducing LSM serves to let each unit of

[1]The Bank of Japan provides bilateral netting service basically available throughout its operating hour, while provides multilateral netting service only at some designated time basis.

liquidity recycle within smaller number of payments, which tends to increase aggregate amount of required liquidity. Negative effect of LSM by discouraging efficient recycle is mitigated by decrease of delay cost through netting. When payment network is less *dense*, such negative effect is more likely to dominate. In addition to the negative effect that directly affect length of recycle for each unit of liquidity, this paper points out possible indirect negative effect, that arises through dismissing positive spillover effect. In the presence of heterogeneous prospect of facing liquidity needs, each path of payments can serve as a path to transmit positive spillover effect, in that it works to suppress collectively undesirable tendency to make earlier payment. Introducing LSM tends to dismiss that positive spillover effect by disconnecting payments through netting. Since netting by LSM serves to disconnect underlying payment network into larger number of sub-networks for less dense payment network, the indirect negative effect is more likely to arise for less dense network. Note that if netting is executed for all the payments, positive effect of netting overwhelms its negative effects. The negative effects matter only for partial netting.

Here we show two example payment networks that are within our core/periphery class.

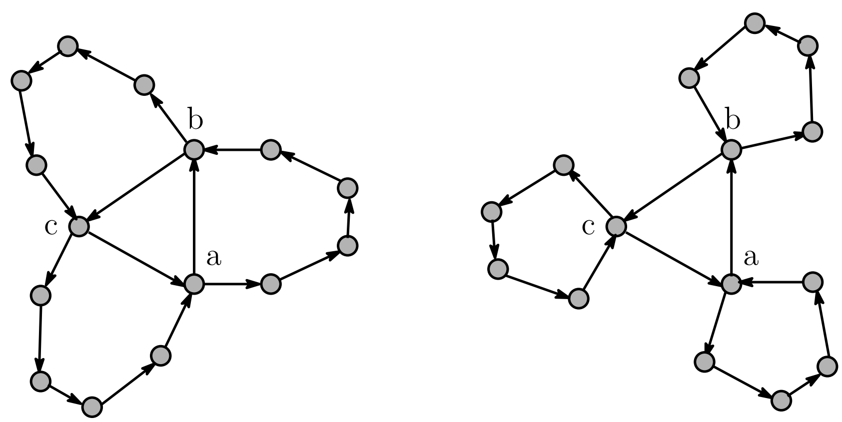

Figure 1: Two payment networks within core/periphery class

For each of the two payment networks, $I_{core} = \{a, b, c\}$, and the others are periphery banks.

In each of the left and the right of Figure 1, each vertex shows a bank, each arrow shows a payment to make along with its direction, and each of the payments is supposed to be one unit of amount. Each bank has either one pair of payments to make and to receive, or two pairs of those. We are to refer to banks that have two pairs of those as *core* banks, that are denoted as a, b, c in each of the figure, and the rest of the banks are periphery banks. Observe that for each payment network, the minimum amount of settlement funds, or liquidity to settle all the payments is one unit of liquidity[2].

Our introduction of LSM effectively serves to eliminate payments among core banks, those are three payments among banks a, b, and ,c for cases here. Our elimination of payments by LSM only among core banks comes from our supposition of core/periphery network where payments among core banks are more interconnected than those between periphery banks. Though such elimination itself is not automatically served but is to be

[2]Each payment network shown in Figure 1 constitutes an Euler graph

realized as an outcome of decisions by relevant banks, introduction of LSM would let each payment network be as in Figure 2 when it were activated.

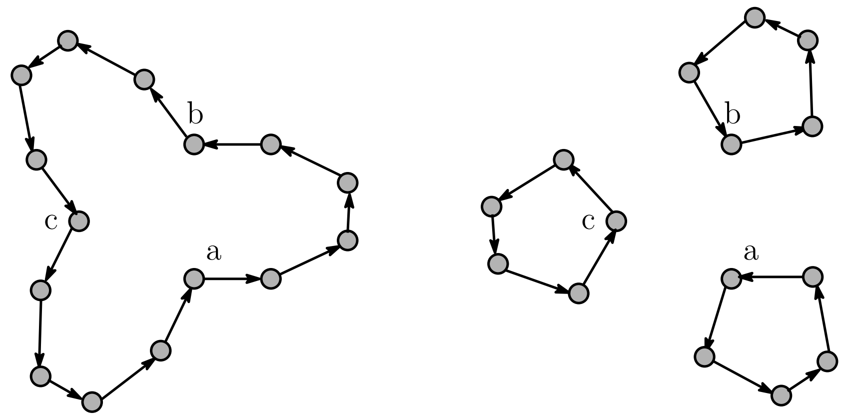

Figure 2: Tow payment networks under LSM
payments among core banks a, b, c have eliminated through utilizing LSM

Now observe again how much liquidity is required to settle each of the payment networks in Figure 2. The minimum required amount for the left of the figure is still one unit of liquidity, while it is now three units of liquidity for the right of the figure. This simple observation corresponds to our *direct negative* effect. We will show that under parameter values within *Liquidity Precious Regime*, amount of required liquidity relevant to our social welfare is the minimum possible under without LSM, and introduction of LSM would have negative effect through increasing the minimum required liquidity. In addition, for parameter values within *Liquidity Precious Regime*, introduction of LSM possibly has additional negative effect, that is to be termed as *indirect negative effect*. The main finding of this paper is relevant to the *indirect negative effect*.

For the case of the right of Figure 2, direct negative effect corresponds to the fact that at least one unit of liquidity is required for each of the separated cycle of payments, but indirect negative effect effectively requires more than each minimum amount of liquidity for some of the separated cycles of payments. Indirect effect never occurs for the case of the left of the figure. Indirect effect will be shown to be effective only when network is to be separated by activation of LSM. Since under our incentive scheme, behaviour of each bank is strongly affected by behaviours of banks that directly make payment to it, behaviour of single bank can spillover to every banks that are within the same connected payment network. Under our heterogeneous type assumption, parameter values are shown to exist where presence of a few "good" type bank lets economize required liquidity. Introduction of LSM and activation of that serves to dismiss such positive spillover effect by separating given payment network to generate cycles of payments where there is no such "good" type bank.

Note that under parameter values under *Liquidity non-Precious Regime*, payments had effectively required maximum possible amount of liquidity without LSM, and introduction of LSM are shown to always have positive effect when it is activated by surely decreasing amount of required liquidity.

The indirect negative effect is firstly expressed in this paper, while the direct negative effect has been shown for more general class of network in our previous paper. Hayakawa (2013a) provided a mathematical framework that treats payment networks formally and in a more general manner. Further that paper proposed and examined a pair of graph-theoretic problems that derive minimum/maximum amount of liquidity required to settle all the given payments. Our direct negative effect is to be examined how each solution is affected through certain type of network transformation. The paper actually examined through network transformation of (reverse of) *cycle addition*, and showed its effect under a general situation. The direct negative effect was also examined in a specific context regarding role of central clearing counterparty in Hayakawa (2013b). In the both papers, all the analyses were executed ignoring potentially relevant incentive aspects. In that sense, contribution of this paper along with these papers lies in providing a strategic situation relevant to the mathematical problem, and showing it brings issues of strategy-oriented effect with respect to network transformation.

Under RTGS systems, the aspect of timing game in our model was explicitly modeled in Bech and Garratt (2003) with two players. Based on its timing game structure, Willison (2005), Martin and McAndrews (2008), and Martin and McAndrews (2010) studied effects of liquidity saving mechanism. They studied RTGS systems where effectively full netting is executed when participants make payments at the same period. Our main departure from the studies is that our paper focuses on effect of partial netting instead of full netting. There are several other modeling differences in each way. Willison (2005) took up full netting situation with complete structure of payment networks. The paper examined when offset is to be executed where offset is possible for two periods(morning, afternoon), whereas our paper supposes only earliest offset. It shows that offset of payments only in the afternoon is worse than that works all day(both morning and afternoon), since only late offset let participants delay payments through decreasing borrowing cost of late payment. Martin and McAndrews (2008) and Martin and McAndrews (2010) dealt with full-netting situation with cycle structures of payment networks. They examined effects of queuing arrangements of liquidity saving mechanisms. It is shown that existence of queue has strategic effects, which largely serves to increase welfare. Our model is to focus more on effect brought by netting itself.

In a different stream of research, Rotemberg (2011) studied how efficiently liquidity is utilized under RTGS systems. The paper focused on "pure" RTGS systems, where there is no intraday liquidity provided. In that situation, participants have no way but to wait until each payment is made when they are short of liquidity. Consequently, a small unit of initial liquidity endowment settle large amount of payments. In our terminology, the paper was to focus on situation where liquidity is recycled fairly well. Our paper is to provide a mechanism when and how such well-recycle situation could be formed through strategic interactions. The main point of his paper is to show that under such well-recycle situation, there still exists possibility of inefficiency, which is caused by route choice of participants. Our model encompass the same feature. Rotemberg (2011) took up a specific class of Euler graph to show the inefficiency, while we focus on a different class of Euler graph. Different from the class in his paper, the class in our paper allows us to express heterogeneity with respect to the amount of payments to be made, which is essential for our analysis of partial netting. The crucial departure is that his paper focused on effect of non-strategic route choice under interconnected payment network, but we examine strategic effects of payment timing choice in addition to route choice.

RTGS systems were also studied in several other papers. Angelini (1998) examined behaviors of banks regarding when to make payment under the existence of less strategic interaction among banks, and show equilibria include excessive delay of payments. Roberds (1999) compared net settlement system and gross settlement system focusing on how each system affects risk-shifting behaviors of participants.

In the rest of the paper, section 2 presents the environment, define our game and equilibrium as well as welfare. Section 3 examine RTGS without LSM. Section 4 is our main part, which analyzes effects of our LSM. Section 5 is for our concluding remarks. Appendix shows some of the proofs.

2 Set-up

2.1 The Base Environment

Our set up bases on Martin and McAndrews (2008) with significant departure as described later.

In the economy, there exist N risk neutral agents. These agents are called banks. There is also a nonstrategic agent, which represents settlement institutions. The economy lasts a day, which consists of four periods: early-morning, late-morning, early-afternoon, late-afternoon, which we simply refer period 1, 2, 3, and 4. As described later in detail, banks have certain amount of payments to make and receive among banks in a day. At the beginning of the day, all the banks have zero reserve. Each bank makes its payment either with borrowing from the Central Bank or with what they have received. They are allowed to borrow only in the early-morning and the late-afternoon period for these payments. In the late-morning and the early-afternoon period, banks can only use reserve which they hold at the beginning of each period, or which they receive within each period.

Three factors affect the decision of banks regarding when to send their payments. First, banks must pay cost x when they borrow one unit of reserve from the CB. Second, banks must pay each delay cost when they make payments after the early-morning period. Delay costs are denoted as $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, each of which is incurred when payments are made at the late-morning, early-afternoon, late-afternoon, and, no payment made within a day. Third, banks receive a liquidity shock with probability σ_i, which is also the ratio of banks that actually receive the shock.

Banks that receive their payments before they make payments have excess reserves at the moment. It is assumed that these reserves cannot be lent to the other banks so that banks have no interest in excess borrowing. Excess borrowing is precluded by passive lending by the CB. In either the early-morning or the late-afternoon period, when only a bank decides to make payment with zero reserve, the bank is to borrow reserve for sure. In contrast, when multiple banks decide to make their payments at the same period, it is not always the case that all the banks borrow reserve. In that case, we suppose when payments are sent unilaterally to the payment system, they arrive in an unexpected order excluding the same arrival, and payments are to be executed along with the order. Reserve is borrowed only if necessary. For example, suppose two banks belong to a cycle with length two, which means two banks are to make payment to each other, and both decide to make payments at the same period with zero reserve. Here, either of the two banks is randomly chosen to make its payment first and needs to borrow, while the other makes its payment afterward with reserve it receives. In this example, the bank that has borrowed reserve will hold the reserve at the end of the period while the other bank does not. There, observe that the borrowed reserve is to come back to who borrowed within the same period.

Liquidity shock is a pair of payments to and from the non-strategic agent. At the end of the late-morning period, banks who face liquidity shock must make one unit of payment to the nonstrategic agent. In turn, they receive one unit of payment at the beginning of the late-afternoon period. There exists heterogeneity regarding level of liquidity shock: each bank faces liquidity shock either with probability σ^l or σ^h, with $0 < \sigma^l < \sigma^h < 1$. The heterogeneity is described later in detail, along with our description of payment network structure. Payments to and from the non-strategic agent cannot be delayed, and the CB will lend banks whenever they face liquidity shock under zero reserve. Table 1 shows definitions of the parameters.

All the banks need to decide whether to make payment at the early-morning period. If they decide to make payment, there is no more to be decided. When banks decide not to make payment at the early-morning period, it needs to be decided whether to make payment or not at each of the following periods. Since banks cannot borrow neither at the late-morning nor at the early-afternoon period, banks that decide to make payment at those periods are to make payment only when they have received their payments.

The timing of the events are summarized in Figure 6. First, nature chooses the banks that receive a liquidity shock. Then, banks make their decisions. After that, payments for the early-morning are executed. Then, late-morning payments are executed with delay cost γ_1. At the end of the late-morning and the beginning of the early-afternoon period, payments between banks and nonstrategic agents are made. Then, payments for early-afternoon are made with delay cost γ_2. Payments for late-afternoon are made with delay cost γ_3, and if payments are not executed in a day, it costs γ_4. Banks pay borrowing cost and return the amount of borrowing to the CB.

Table 1: Parameters

x	Cost of borrowing
$0 < \gamma_1 < \gamma_2 \leq \gamma_3 < \gamma_4$	Delay costs
$0 < \sigma^l < \sigma^h < 1$	Probability of a liquidity shock

2.1.1 departure from set-up in Martin and McAndrews (2008)

Our set-up resembles that of Martin and McAndrews (2008) in time structure, cost structure as well as how liquidity shock is incorporated. This paper departs from that paper crucially in three points.

The first is borrowing scheme for payments made in the same period. In their paper, when every payments in the same cycle are decided to be made, it was assumed to be offset without liquidity input, consequently no borrowing is required. This paper

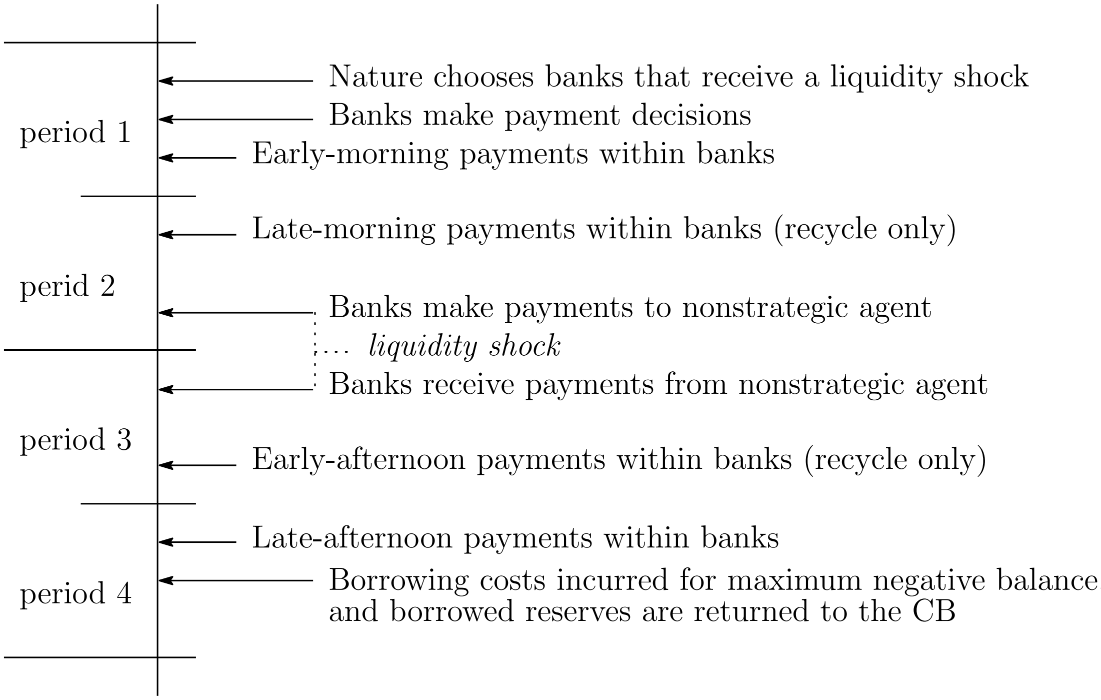

Figure 3: Timeline

supposes at least one unit of liquidity is to be required even under such simultaneous making payments. Our setting requires explicit activation of LSM for obligations to be offset. Secondly, class of network of payments is different. This paper examines a class of core/periphery network that is soon introduced, while it was a class of cycle network. Thirdly, liquidity shock is supposed to be heterogeneous among banks in this paper, while homogeneous liquidity shock was supposed in their paper. This is a significant departure since we will see heterogeneity of liquidity shock is necessary for our indirect negative effect to emerge.

2.2 Definitions and Notations

2.2.1 Core/periphery network and LSM

We suppose a specific class of networks for payments among banks, which is defined as $K-$ core/periphery networks. We denote the set of banks as I.

Definition 1. $K-$ core/periphery networks

We call a payment network as a $K-$ (general) core/periphery network among banks $I = I_{core} \cup I_{per}$ with core banks $I_{core} = \{i_{core,j}\}_{j=1,2,..K}$ and periphery banks $I_{per} = \{I_{per,j}\}_{j=1,2,..,K}$ when
(1) Each core bank has two payment to make where one is to a core bank and the other is to a periphery bank, and has two payments to receive where again one is from a core bank and the other is from a periphery bank,
(2) payments among K core banks forms a cycle with their K payments,
(3) Each periphery bank has each one payment to make and to receive, and,
(4) For each $k = 1, 2, ..K$, payments among $I_{per,k}$ constitutes a path, where there exists a

periphery bank $j \in I_{per,k}$ who receives a payment from a core bank and makes a payment to $j' \in I_{per,k}$, and there exists another bank $j' \in I_{per,k}$ who receives a payment from a bank in $I_{per,k}$ but makes a payment to a core bank, while all the other banks in $I_{per,k}$ makes and receive a payment among $I_{per,k}$.

Let Ψ denote a set of core/periphery networks, and Ψ^K as a set of K-c/p networks. Throughout this paper, we especially focus on core/periphery networks where there exists no *vertex-twisted* relation[3] between a cycle with core banks and any of the other cycles for a core/periphery network. This is a technical assumption to ensure our explicit calculation of welfare.

We suppose there exists $L \in \{1, 2, .., K\}$ where each of $I_{per,L}$ faces a liquidity shock with probability σ^l, and the other periphery banks and all the core banks face with probability σ^h. This special setting on location of banks with respect to each liquidity shock is chosen to highlight *spillover effect* shown in the latter analyses. Note that existence of *spillover effect* does not rest on the special location setting, but on the setting of heterogeneous liquidity shock together with *connected* payment network, as we latter confirm.

Denote the average liquidity shock probability as $\tilde{\sigma} = \frac{1}{N}(N^h\sigma^h + N^l\sigma^l)$, where $N^l = |I_{per,L}|, N^h = N - N^l$.

We define *density* for each given $K-$ c/p network.

Definition 2. density

For a $K-$ c/p network $\psi \in \Psi^K$, remove a cycle of payments among core banks. The rest of the payments now form d disconnected cycles of payments, and we call each cycle as a *core-separated* cycle. Let $d(\psi)$ denote density for ψ, which indicates the number of *core-separated* cycles.

For $\psi, \psi' \in \Psi^K$ with some $K \geq 2$, we say ψ is more *dense* than ψ' if $d(\psi) < d(\psi')$.

For $\psi \in \Psi^K$, we have $d(\psi) \in \{1, 2, .., K\}$. Figure 4 shows three c/p networks with different density for the case of $K = 3$. Note that for $K-$ core/periphery networks that have no *vertex-twisted* relation, we always have at least one network $\psi \in \Psi^K$ that satisfies $d(\psi) = n$ for every $n = 1, 2, .., K$.

2.2.2 Game and Equilibrium

Let A_i denote the set of available actions for bank $i \in I$. For the statement of available actions, we only focus on meaningful actions in the sense they are not eliminated by simple backward induction. At period 4, when a bank has not made payment, it is always preferred to make payment since otherwise very high cost is supposed to be burdened. So, all the banks choose to make payment at period 4. At period 3, if a bank has not made payment and receive payment, it is always desirable to make payment since otherwise it only increase delay cost from γ_2 to γ_3. So, all the banks choose to recycle liquidity at the period. Either of the choice both in period 1 and 2 is not simply eliminated.

[3]See Hayakawa (2013a) for the notion of *vertex-twisted*. Intuitively, there exists no *vertex-twisted* relation when direction of a cycle of payments among core banks is consistent with direction of any other cycle of payments.

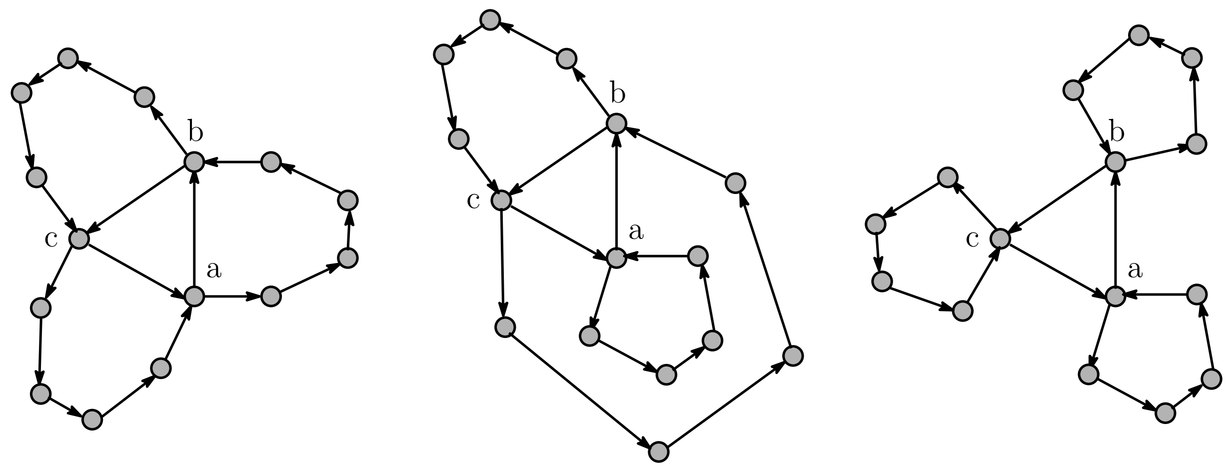

Figure 4: Examples for 3− c/p network

$I_{core} = \{a, b, c\}$, and the others are periphery banks for both networks. *Density* is the minimum(most dense) for the left, while it is the maximum(least dense) for the right, and that for the middle is between the two.

Available actions for each periphery bank $j \in I_{per}$ is now defined as $A_j = \{P, R, H\}$, where P indicates making payment at period 1, R is for not making payment at period 1 but recycle at period 2, and H is for not making payment at period 1 and not recycle at period 2. We only focus on pure strategies, $s_i \in A$ for each bank $i \in I$. Throughout the paper, we suppose banks chooses P when it is indifferent to R or H, and chooses R if it is indifferent to H.

For core banks, it needs to be described whom to make/recycle payments for each of payment[4]. Although our analysis will not require detailed examination about those route choices, here we explicitly shows available actions that incorporate route choices. Available actions for core banks $i \in I_{core}$ is defined as
$A_i = \{P, P_cR, P_pR, R_{cc}, R_{cp}, P_cH, P_pH, H_{cc}, H_{cp}\}$, where P indicates to make both payments at period 1, $P_cR(P_pR)$ is to make payment to a core(periphery) bank at period 1 and recycle at period 2 for the other, R_{cc} indicates not to make both payments at period 1 but recycle both payments at period 2 and 3 in a way that a payment received from a core bank goes to a core bank while a payment from a periphery bank goes to a periphery bank, R_{cp} indicates to recycle in a way that a payment received from a core bank goes to a periphery bank. P_cH, P_pH, H_{cc}, H_{cp} are defined in the same way. H_{cc}, H_{cp} are for recycle only in period 3 in each specified route. Note that some possible actions are excluded[5] from the reason that they are inessential for our analysis.

Let $S = \times_{i \in I} s_i \in \mathbf{S}$ denote an strategy profile. Denote $S_{-i} = \times_{i' \in I \setminus i} s_{i'}$.

Given a structure of payment network $\psi \in \Psi$, payoff function for bank i is denoted as $\pi_i : A_i \times \psi \to R$, which is equal to the negative of the expected settlement cost function, $c_i : A_i \times \psi \to R$, that is, $\pi_i(s_i, S_{-i}, \psi) = -c_i(s_i, S_{-i}, \psi)$. We consider Nash equilibrium

[4]Note that one of the other ways of modeling actions under multiple routes is unconditional way of choices: for example, to specify which to recycle first and next. We do not adopt that approach from an analytical reason. Since our model is not to express timing of receipts within recycle period, equilibrium is not well defined by the approach.

[5]Excluded actions are, for example, to recycle either payment and to hold the other.

for this game.

Definition 3. Nash Equilibrium

A strategy profile $S^* = (s_i^*, S_{-i}^*) \in \mathbf{S}$ is a Nash equilibrium for the LMWR(liquidity management with recycle) game with given structure of payment network ψ if and only if

$$E[\pi_i(s_i^*, S_{-i}^*, \psi)] \geq E[\pi_i(s_i', S_{-i}^*, \psi)] \tag{1}$$

for every $s_i \in A_i$ and $i \in I$.

2.2.3 Welfare

We suppose all the social cost associated with banks' decisions are all internalized in cost parameters, or, we assume no externality. However, in reality, payment delay would have negative externality: for example, wide payment delay may signal insufficient ability of the CB to control monetary system. Our model should be interpreted in a way that all such delay costs are somehow incorporated.

We define social welfare under strategy profile S as the negative of socially valued aggregate costs:

$$W(S) = -max(D(\sum_{i\in I} m_i)|S) * r - \sum_{i\in I}\sum_{t=1}^{4} \gamma_{i,t} \tag{2}$$

There, m_i for bank $i \in I$ denotes total units of borrowing of bank i in a day. Under strategy profile $S = (s_i, S_{-i})$, $\gamma_{i,t} = \gamma_t$ if bank i pays delay cost γ_t, and $\gamma_{i,t} = 0$ otherwise. r denotes market interest rate, which is treated as an indicator of lending cost incurred by the CB. $D(.|S)$ denotes a distribution of the amounts of borrowing in a day under strategy profile S. $max(D(.|S))$ takes the maximum value for the distribution of $D(.|S)$. $max(D(.|S)) * r$ express the maximum lending cost of liquidity incurred by the CB, where in this model the CB is supposed to prepare the maximum expected amount for intraday lending since shortage of intraday lending would cause high delay cost γ_4.

Note that in reality the CB does not need to prepare liquidity for its intraday lending since those lending are executed in central bank accounts. Our supposition of preparation of liquidity is interpreted as to mitigate risk associated with lending. It is further supposed that cost to mitigate lending risk is increasing with expected amount of lending, though our model does not explicitly deal with lending risk.

Since transfer among banks and the CB are canceled out, there is no x shown for the welfare function.

We define first-best strategy profile.

Definition 4. First-Best

A strategy profile S attains first-best if and only if

$$W(S) \geq W(S') \tag{3}$$

for every $S' \in \mathbf{S}$.

We allow $x > r$ for our analysis, though our main analysis is for the case of $x \leq r$, since we suppose existence of arbitrage will not sustain the situation of $x > r$ for long.

For our welfare analysis, we make two assumptions throughout the paper.

Assumption 1.
$\gamma_1 + \sigma^h x < \gamma_2$.

This assumption lets banks to prefer recycling liquidity to holding liquidity at period 2. The assumption is based on our supposition that the CB would increase γ_2 with some penalty so that the above relation is attained, since it improves welfare. It helps our analysis simpler[6].

Assumption 2.
$\frac{K}{N} < \tilde{\sigma} < \frac{N-K}{N}$.

This assumption is a technical requirement to explicitly derive welfare.

3 RTGS without LSM

We focus on two types of strategy profiles.
1.(all-early) $s_i = P$ for every $i \in I$.
2.(1-recycle) $s_j = P$ for some $j \in I_{per}$, $s_{j'} = R$ for every $j' \neq j \in I_{per}$, and $s_i \in \{R_{cc}, R_{cp}\}$ for every $i \in I_{core}$.

For the second type 1-recycle strategy profile, we confine us in strategy profiles where route choice is appropriate in the sense that any of the banks that chooses to recycle successfully recycle. We can always have such strategy profile since any c/p network constitutes an Euler graph.

Denote S^1, S^2 as each representative strategy profile for strategy profiles within *all-early*, and *1-recycle*. We have $W(S^1) = -(N+K)r$[7], and $W(S^2) = -((1+\sigma^l N^l + \sigma^h N^h)r + (N+K-1)\gamma_1)$.

Proposition 1. *first-best strategy profile*
S^1 attains first-best if $\frac{K}{N} < \tilde{\sigma} \leq \frac{N+K-1}{N}(1-\frac{\gamma_1}{r})$. while S^2 attains first-best if $\frac{N+K-1}{N}(1-\frac{\gamma_1}{r}) < \tilde{\sigma} < \frac{N-K}{N}$.

Proof. From Lemma 1, we are suffice to compare $W(S^1)$ and $W(S^2)$. $W(S^1) \geq W(S^2)$ iff $(N+K)r - (1+N\tilde{\sigma})r - (N+K-1)\gamma_1 \geq 0$, which is, $(N+K-1-N\tilde{\sigma})r > (N+K-1)\gamma_1$. We have, $\tilde{\sigma} \leq \frac{N+K-1}{N}(1-\frac{\gamma_1}{r})$. Combining with Assumption 2, we complete our proof. □

Lemma 1.
Under Assumption 1, 2, either of S^1 or S^2 attains first-best.

[6]Assumption 1 excludes issues associated with multiple equilibria that are not essential for analysis in this paper.

[7]Under our assumptions, liquidity shock is sufficiently high so that it is possible each unit of payment requires each unit of liquidity input.

Proof. First, consider a strategy profile where holding of payments$(H, P_cH, P_pH, H_{cc}, H_{cp})$ is included. Replace all those actions with corresponding recycle actions $(R, P_cR, P_pR, R_{cc}, R_{cp})$ always derives equal to or larger welfare through decrease of cost of delaying since $r < \gamma_2$.

Next compare two strategy profiles where no holding action is included and only choices of routes(described by subscript in P_cR, or R_{cc}) are different. Suppose recycle is successful for one strategy profile and not for the other. The former always attains better welfare since failure of recycle increases cost of delaying payments.

Now consider S^2, where one payment is made at period 1. Replace either of the actions such that two payments are made at period 1. It decreases welfare as long as it generates additional liquidity cost in total. Continue the replacement until it derives a strategy profile which include k payments made at period 1. One more step replacement to $k+1$ increases welfare when it no more add additional liquidity cost. Suppose it does, then the replacement to $k+2$ always increase welfare. Since S^2 is a strategy profile that derives minimum liquidity cost among strategy profiles where recycle is successful until period 2, we end our proof. □

For a core bank $i \in I_{core}$, we call banks who are to make payments to i as payers to i. We denote a pair of strategies of payers j, j' to a core bank i as $s_i^p = \{s_j, s_{j'}\}$. For a core bank payer to bank i, we denote only relevant strategy, regarding whether to make payment to bank i (P), or not (R). So we have $s_i^p \in \{\{P,P\}, \{P,R\}, \{R,R\}\}$.

We examine payoff of core bank i under a strategy profile S_{-i}, where it surely receives payment until period 2.

If $s_i^p = \{P,P\}$, then,

$$\begin{aligned}
\pi_i(P, S_{-i}, \psi) &= -(\frac{1}{6}2x + \frac{1}{2}x + \frac{1}{3}\sigma^h x) = -\frac{5}{6}x - \frac{1}{3}\sigma^h x, && (4)\\
\pi_i(P_cR, S_{-i}, \psi) = \pi_i(P_pR, S_{-i}, \psi) &= -\frac{1}{3}x - \frac{2}{3}\sigma^h x - \gamma_1, && (5)\\
\pi_i(P_cH, S_{-i}, \psi) = \pi_i(P_pH, S_{-i}, \psi) &= -\frac{1}{3}x - \gamma_2, && (6)\\
\pi_i(R_{cc}, S_{-i}, \psi) = \pi_i(R_{cp}, S_{-i}, \psi) &= -\sigma^h x - 2\gamma_1, && (7)\\
\pi_i(H_{cc}, S_{-i}, \psi) = \pi_i(H_{cp}, S_{-i}, \psi) &= -2\gamma_2, && (8)\\
& && (9)
\end{aligned}$$

If $s_i^p = \{P,R\}$, then

$$\begin{aligned}
\pi_i(P, S_{-i}, \psi) &= -\frac{1}{3}2x - \frac{2}{3}x = -\frac{4}{3}x, && (10)\\
\pi_i(P_cR, S_{-i}, \psi) = \pi_i(P_pR, S_{-i}, \psi) &= -\frac{1}{2}x - \frac{1}{2}\sigma^h x - \gamma_1 && (11)\\
\pi_i(P_cH, S_{-i}, \psi) = \pi_i(P_pH, S_{-i}, \psi) &= -\frac{1}{2}x - \gamma_2 && (12)\\
\pi_i(R_{cc}, S_{-i}, \psi) = \pi_i(R_{cp}, S_{-i}, \psi) &= -2\gamma_1 - \sigma^h x, && (13)\\
\pi_i(H_{cc}, S_{-i}, \psi) = \pi_i(H_{cp}, S_{-i}, \psi) &= -2\gamma_2, && (14)\\
& && (15)
\end{aligned}$$

If $s_i^p = \{R, R\}$, then

$$\begin{aligned}
\pi_i(P, S_{-i}, \psi) \quad & = -2x, & (16)\\
\pi_i(P_cR, S_{-i}, \psi) = \pi_i(P_pR, S_{-i}, \psi) \quad & = -x - \gamma_1 & (17)\\
\pi_i(P_cH, S_{-i}, \psi) = \pi_i(P_pH, S_{-i}, \psi) \quad & = -x - \gamma_2 & (18)\\
\pi_i(R_{cc}, S_{-i}, \psi) = \pi_i(R_{cp}, S_{-i}, \psi) \quad & = -2\gamma_1 - \sigma^h x, & (19)\\
\pi_i(H_{cc}, S_{-i}, \psi) = \pi_i(H_{cp}, S_{-i}, \psi) \quad & = -2\gamma_2, & (20)\\
& & (21)
\end{aligned}$$

For periphery bank $i \in I_{per}$, we similarly define s_i^p. If $s_i^p = P$, then

$$\begin{aligned}
\pi_i(P, S_{-i}, \psi) \quad & = -\frac{1}{2}x - \frac{1}{2}\sigma^k x, & (22)\\
\pi_i(R, S_{-i}, \psi) \quad & = -\gamma_1 - \sigma^k x, & (23)\\
\pi_i(H, S_{-i}, \psi) \quad & = -\gamma_2, & (24)
\end{aligned}$$

If $s_i^p \neq P$, then

$$\begin{aligned}
\pi_i(P, S_{-i}, \psi) \quad & = -x - \sigma^k x * 1_{\{S_{-i} \notin S^a_{-i}(\psi)\}}, & (25)\\
\pi_i(R, S_{-i}, \psi) \quad & = -\sigma^k x - \gamma_1 * 1_{\{S_{-i} \in S^b_{-i}(\psi)\}} - \gamma_2 * 1_{\{S_{-i} \notin S^b_{-i}(\psi)\}}, & (26)\\
\pi_i(H, S_{-i}, \psi) \quad & = -\gamma_2 - \sigma^k x * 1_{\{S_{-i} \notin S^b_{-i}(\psi)\}} & (27)
\end{aligned}$$

$S^a_{-i}(\psi) \subset S$ is a set of strategy profile where there exists some bank $j \in I$ which is not a payer to i with $s_j = P$, and for every bank j' which locates after j and before i under ψ, $s_{j'} = R$, or simply for every $j \in I \setminus i$, $s_j = R$. We denote $S^b_{-i} = S^a_{-i} \setminus S^R$, where $S^R \in S$ is a strategy profile such that $s_{i'} = R$ for every $i' \in I \setminus i$.

Lemma 2. *equilibrium* [8]
1.(all-early) strategy profile S^1 *is an equilibrium if* $\gamma_1 + \sigma^h x < \gamma_2$, $\frac{1}{2} - \frac{1}{3}\sigma^h < \frac{\gamma_1}{x}$, $\frac{1}{2}(1-\sigma^l) < \frac{\gamma_1}{x}$,
2.(1-recycle) there exists a strategy profile S^2 *which constitutes an equilibrium if* $\gamma_1 + \sigma^h x < \gamma_2$, $1 - \sigma^l < \frac{\gamma_2}{x}$, $\frac{\gamma_1}{x} < \frac{1}{2}(1-\sigma^l)$, *and* $\frac{\gamma_1}{x} < (1 - \sigma^h)$.

Proof. See Appendix A.1. □

For latter convenience, denote the above two conditions for S^1, S^2 to be an equilibrium as Condition (A), (B) as follows.

Condition (A)
$\frac{\gamma_1}{x} + \sigma^h < \frac{\gamma_2}{x}$, $\frac{1}{2} - \frac{1}{3}\sigma^h < \frac{\gamma_1}{x}$, and $\frac{1}{2}(1 - \sigma^l) < \frac{\gamma_1}{x}$,

[8] In Appendix A.2, we present some of the other types of equilibria which have not central importance in this article.

Condition (B)
$\frac{\gamma_1}{x} + \sigma^h < \frac{\gamma_2}{x}$, $1 - \sigma^l < \frac{\gamma_2}{x}$, $\frac{\gamma_1}{x} < \frac{1}{2}(1 - \sigma^l)$, and $\frac{\gamma_1}{x} < (1 - \sigma^h)$.

To see how each condition is relevant for each of our equilibrium, consider best response within our strategy profiles S^1 and S^2. . For periphery banks, we can divide their best response into three types. $(B1)$ mentions *unconditional payment* where best response for a periphery bank i is $s_i = P$ regardless of the other banks choices. $(B2)$ refers to *opportunistic payment* where best response for a periphery bank i is $s_i = P$ only if $s_{i-1} = P$ or there exists no $i' \in I$, $s_{i'} = P$. $(B3)$ is for *sacrificial payment* where best response for a periphery bank i is $s_i = P$ only if there exists no $i' \in I$, $s_{i'} = P$.

We can similarly define possible best response for core banks. $(B1')$ mentions *unconditional payment* where best response for a core bank i is $s_i = \{P, P\}$ regardless of the other banks choices. $(B2')$ refers to *opportunistic payment* where best response for a core bank i is $s_i = \{P, P\}$ if $s_{i-1} = \{P, P\}$ but not if $s_{i-1} = \{R, R\}$. $(B3')$ is for *sacrificial payment* where best response for a periphery bank i is $s_i = \{P, P\}$ either if $s_{i-1} = \{P, P\}$ or $s_{i-1} = \{R, R\}$.

Observation 1 (Best Response)**.**
Under our Assumption 1, and within our strategy profiles S^1 and S^2, and further for S^2, suppose $s_j = P$ for $j+1$ is not a core bank,
best response for periphery bank i with $\sigma_i \in \left\{\sigma^l, \sigma^h\right\}$ is;
$(B1)$ *if* $1 - \sigma_i \leq \frac{\gamma_1}{x}$,
$(B2)$ *if* $\frac{1}{2}(1 - \sigma_i) < \frac{\gamma_1}{x} \leq 1 - \sigma_i$,
$(B3)$ *if* $\frac{\gamma_1}{x} < \frac{1}{2}(1 - \sigma_i)$.
best response for core bank i with $\sigma_i = \sigma^h$ is;
$(B1')$ *if* $1 - \frac{1}{2}\sigma^h \leq \frac{\gamma_1}{x}$,
$(B2')$ *if* $\frac{1}{2}(1 - \frac{2}{3}\sigma^h) < \frac{\gamma_1}{x} \leq 1 - \frac{1}{2}\sigma^h$,
$(B3')$ *if* $\frac{\gamma_1}{x} < \frac{1}{2}(1 - \frac{2}{3}\sigma^h)$.

Figure 5 shows relation between those best responses and our equilibrium strategies for certain parameter values. In the figure, see where each border between $B2$ and $B3$, or $B2'$ and $B3'$ is located for each type of banks. First for periphery banks with low shock, the border is shown to be located on the left side of the value $\frac{1}{2}$. Note that when the border had been placed at that value, it says earliest payment is attractive enough if $\frac{1}{2}x < \gamma_1$. That is true when liquidity shock had not existed. The existence of liquidity shock lets earliest payment be relatively more attractive since earliest payment insures for liquidity shock if a bank actually borrows liquidity and makes payment. The point is that liquidity circulates to come back to itself at the time of liquidity shock under the relevant strategy profiles if a bank has made costly payment at period 1. For periphery banks with high shock, earliest payment is more attractive, and the border is even placed to the left side. But for core banks with high shock, notice that liquidity shock is at most one unit when they have two payments to make and receive. That lets merit of earliest payment as insurance purpose be less than that for periphery banks with high shock regarding marginal attractiveness when they make both two earliest payments, and accordingly the border is located at the right side of the border for them. Note that we do not readily know whether the border for core banks is located at the right side or the left side of that for periphery bank with low shock. The figure shows a case where earliest payment is still attractive enough for core banks compared to periphery banks with low shock.

Back to our result for equilibrium in Lemma 2, each of the conditions works intuitively as follows. First for S^1 to be an equilibrium, $\gamma_1 + \sigma^l x < \gamma_1 + \sigma^h x < \gamma_2$ comes from our Assumption 1, which let banks to prefer recycling liquidity to holding liquidity at period 2. $\frac{1}{2} - \frac{1}{3}\sigma^h < \frac{\gamma_1}{x}$ let core banks to prefer making earliest payment to wait to recycle even when they are to be received payments at period 1. That condition for periphery banks is $\frac{1}{2}(1-\sigma^h) < \frac{1}{2}(1-\sigma^l) < \frac{\gamma_1}{x}$.

Next for S^2 to be an equilibrium, $\gamma_1 + \sigma^l x < \gamma_1 + \sigma^h x < \gamma_2$ comes from the same Assumption 1. $1-\sigma^h < 1-\sigma^l < \frac{\gamma_2}{x}$ ensures periphery banks prefer making payment at period 1 to not doing so when there is no possibility of receiving liquidity until period 2. $\frac{\gamma_1}{x} < \frac{1}{2}(1-\sigma^l)$ let periphery banks with low liquidity shock prefer not making payment at period 1 to doing so. They are to choose recycle liquidity. $\frac{\gamma_1}{x} < (1-\sigma^h) < (1-\sigma^l)$ let periphery banks not prefer unconditional payment.

Note that the figure shows a situation that S^2 constitutes an equilibrium when $(a) = \frac{1}{2}(1-\sigma^h) < \frac{\gamma_1}{x}$. The condition lets best response of periphery banks with high shock be $B2$ or $B1$, which implies that S^2 would no more be an equilibrium if there existed only high shock banks. This observation is elaborated in our statement of *spillover effect* in the next proposition. In the proposition, denote bank $j+1$ as a periphery bank who receives payment from a periphery bank j. It states that under certain parameter values, S^2 is an equilibrium only when the sole payer makes its payment to a bank that has low liquidity shock.

Proposition 2. *spillover effect*

Under Condition (B) and $\frac{1}{2}(1-\sigma^h) \leq \frac{\gamma_1}{x}$, there exists S^2 that constitutes an equilibrium if and only if $s_j^P = P$ for $j+1 \in I_{per,L}$

Proof. When bank $j+1$ is core bank. Then, core banks prefer P_cR or P_pR to R_{cc} or R_{cp} under $\frac{1}{2}(1-\sigma^h) \leq \frac{\gamma_1}{x}$. Similarly, when $j+1$ is a periphery bank with high liquidity shock, it prefer P to R. It never let S^2 be an equilibrium. In contrast, when $j+1$ is a periphery bank with low liquidity shock, it prefers R to P. Since all the other banks except for j also prefer to recycle as much as possible, there exists S^2 which constitutes an equilibrium with appropriate route choice. □

The conditions for spillover effect to exist is shown in Figure 5 for certain parameter values.

We confirmed that spillover effect exists if best response for a periphery bank with low shock is $B3$ and that for a periphery bank with high shock is $B2$. Under the situation, a periphery bank with high shock would make its payment at period 1 if it receives payment at period 1, but existence of low shock periphery bank that chooses to recycle when it receives payment at period 1 excludes an equilibrium where any of high shock periphery banks receives payment at period 1, as long as there exists a low shock periphery bank somewhere in each connected payment network.

We focus on two types of parameter values for our analyses, where each is termed as a *Regime*. Each *Regime* is characterized with a strategy profile that attains first-best and

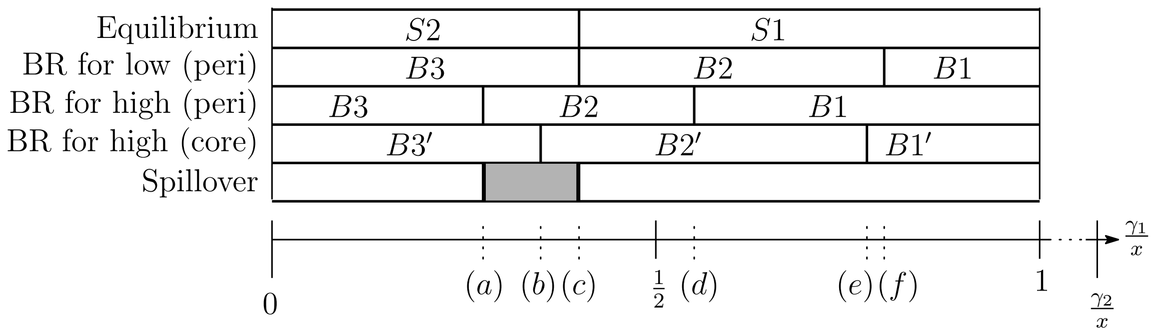

Figure 5: Equilibrium, Best Response, and Spillover Effect

The figure shows equilibrium, best response, spillover for $0 < \frac{\gamma_1}{x} \leq 1$, with $(a) = \frac{1}{2}(1-\sigma^h)$, $(b) = \frac{1}{2}(1-\frac{2}{3}\sigma^h)$, $(c) = \frac{1}{2}(1-\sigma^l)$, $(d) = 1-\sigma^h$, $(e) = 1-\frac{1}{2}\sigma^h$, $(f) = 1-\sigma^l$. Specifically, consistent parameter values are; $\sigma^h = 0.45$, $\sigma^l = 0.2$, and $\frac{\gamma_2}{x} = 1.5$, we have $(a) = 0.275$, $(b) = 0.35$, $(c) = 0.4$, $(d) = 0.55$, $(e) = 0.775$, $(f) = 0.8$.

also constitutes an equilibrium. The notion of *Regime* supposes that fee level x is to be set by the CB so as to maximize welfare given the other parameter values.

Definition 5. Regimes

Liquidity non-Precious Regime refers to parameter values that satisfies the Conditions (A) and $\frac{K}{N} < \tilde{\sigma} \leq \frac{N+K-1}{N}(1-\frac{\gamma_1}{r})$, where strategy profile S^1 is an equilibrium and also attains first-best.

Liquidity Precious Regime refers to parameter values that satisfies the Conditions (B) and $\frac{N+K-1}{N}(1-\frac{\gamma_1}{r}) < \tilde{\sigma} < \frac{N-K}{N}$, where there exists a strategy profile S^2 that is an equilibrium and also attains first-best.

The next lemma ensures that *Liquidity non-Precious Regime* is always attainable with sufficiently low fee level, and *Liquidity Precious Regime* is attainable when marginal delay cost $\frac{\gamma_2}{\gamma_1}$ is sufficiently high.

Lemma 3. *attainability of equilibrium with fee control*

(existence for S^1) Strategy profile S^1 is always realized with sufficiently low fee level x for any parameter values.

(existence for S^2) There exists a strategy profile S^2 that constitutes an equilibrium with appropriate fee level if $\max(2, \frac{1-\sigma^l}{1-\sigma^h}, \frac{2\sigma^h-\sigma^l+1}{1-\sigma^l}, \frac{1}{1-\sigma^h}) < \frac{\gamma_2}{\gamma_1}$.

Proof. It is straight for the former case. For the latter case, condition (B) is rewritten as: $\frac{2\gamma_1}{1-\sigma^l} < x$, $\frac{\gamma_1}{1-\sigma^h} < x$. $x < \frac{\gamma_2}{1-\sigma^l}$, and $x < \frac{\gamma_2-\gamma_1}{\sigma^h}$.

It reduces to the conditions in the lemma. □

We are to analyze effect of LSM given parameters for either of the two Regimes, which amounts to examine rather short-term effect of introduction of LSM supposing either of the two Regimes had been realized. We do not explicitly examine how fee x is optimally set after LSM is introduced. Purpose of this way of examination is to highlight possibility of our indirect negative effect.

4 Effect of LSM

We specifically examine $K-$partial netting with queue for our analysis, with $K \geq 2$. We examine a situation where LSM works only at period 1. All payments chosen to make payment at period 1 are put into *queue*, where less than or equal to K payments that constitutes a cycle within *queued* payments are all netted, or settled without liquidity transfer. The rest of the payments that are not netted at period 1 are executed exactly the same as those are under the situation without LSM, where order of payments are randomly determined and payments are settled with liquidity borrowing when needed. For period 2,3, and 4, payments are dealt with exactly the same as those are under the situation without LSM.

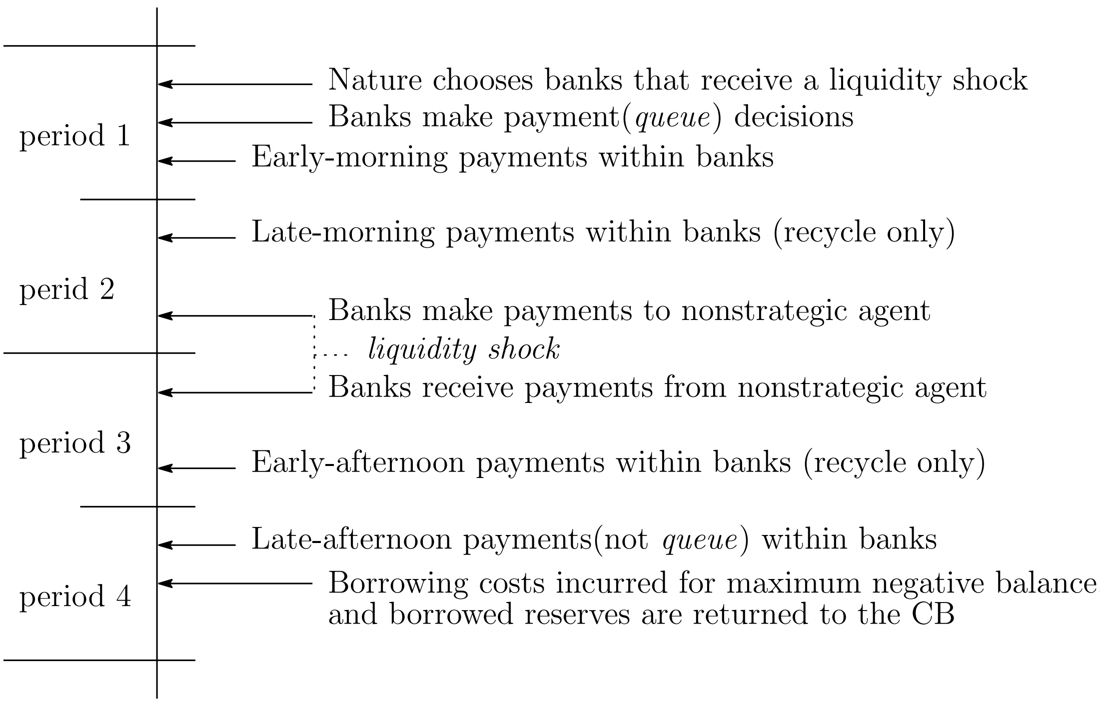

Figure 6: Timeline with LSM

We denote strategies under LSM by replacing P with Q, where Q denotes *queue* at period 1. For $i \in I_{per}$, $A_i = \{Q, R, H\}$. For $i \in I_{core}$, $A_i = \{Q, Q_cR, Q_pR, R_{cc}, R_{cp}, Q_cH, Q_pH, H_{cc}, H_{cp}\}$,

We focus on three types of strategy profiles, each of which is shown to constitutes an equilibrium for certain parameter values.

1. (all-early) $s_i = Q$ for every $i \in I$.
2. (d-recycle) $s_i = Q_cR$ for every $i \in I_{core}$, and there exists just one periphery bank $j \in I_{per}$ with $s_j = Q$ for each core-separated cycle, and for any other periphery bank $j' \in I_{per}$, $s_{j'} = R$.
3. (mixed recycle) Separate I into $I = \cup_{n=1,2,..,d(\psi)} I^n$ so that each constitutes a separated cycle of payments if payments among core banks are to be netted. Denote I^L when $I_{per,L} \subset I^L$. $s_i = Q_cR$ for every $i \in I_{core} \cap I^L$, and $s_{i'} = Q$ for every $i' \in I_{core}/I^L$. There exists one periphery bank $j \in I_{per}$ with $s_j = Q$ for $j \in I_{per} \cap I^L$, while for the other

periphery banks $j' \in I_{per} \cap I^L$, $s_{j'} = R$. For every periphery bank $j'' \in I_{per}$ that does not belong to I^L, $s_{j'} = Q$.

To see the third type of strategy profile (mixed recycle), suppose for each of the three $3-c/p$ network in Figure 7, $I_{per,L}$ is a set of four periphery banks as shown. Figure 8 shows each associated payment network supposing payments among core banks are netted. For the left of the figure, $I^L = I$ since all banks belong to one connected payment network. The third type of strategy profile states that any one periphery bank to take P, while all the other periphery banks to take R. For the middle of the figure, I^L refers to banks that belong to a cycle of payments including core bank b, c. It is required that one periphery bank $j \in I^L$ to take P while the other $j' \in I^L$ to take R. For the other banks that belong to another cycle of payments including a, all the periphery banks to take P, and all the core banks to take PQ_c. For the right figure, in the same way to let $I^L = I_{per,L} \cup b$ and specify strategy profile.

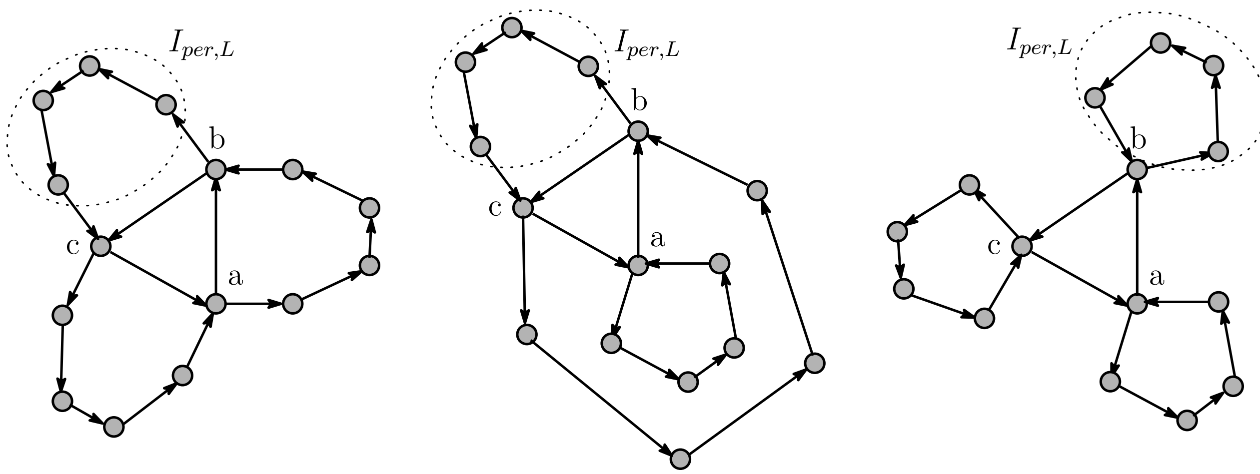

Figure 7: 3− c/p networks

Denote $S^1_{net}, S^2_{net}, S^3_{net}$ for corresponding types of strategy profiles. We have: $W(S^1_{net}) = -Nr$, $W(S^2_{net}) = -(d+\tilde{\sigma}N)r-(N-d)\gamma_1$, $W(S^3_{net}) = -(1+\sigma^l N^l + N^h + \min(0, -n^h(\psi) + \sigma^h N^h))r - (N^l - 1 + n^h(\psi))\gamma_1$. , where $n^h(\psi) = |I^L \setminus I_{per,L}|$.

Note that $n^h(\psi)$ is interpreted as the number of banks with high shock that had enjoyed spillover effect under RTGS without LSM but no more with LSM.

We first examine payoff of core bank i under a strategy profile S_{-i}, where it surely receives payment until period 2. We denote $s^p_i = (x, x')$ for $x \in A_i$ with $i \in I_{core}$, $x' \in A_j$ with $j' \in I_{per}$. To simplify notation, we denote $x, x' \in \{Q, R, H\}$, each expresses relevant strategy for i.

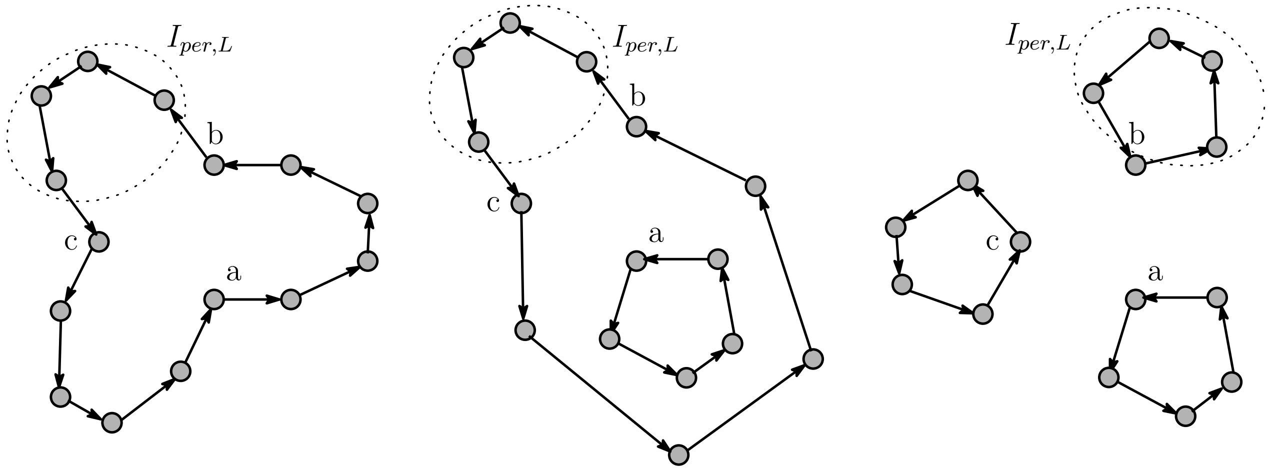

Figure 8: 3− c/p networks under LSM

If $s_i^p = (Q, Q)$, then

$$\begin{aligned}
\pi_i(Q, S_{-i}, \psi) &= -\frac{1}{2}x - \frac{1}{2}\sigma^h x, && (28)\\
\pi_i(Q_c R, S_{-i}, \psi) &= -\sigma^h x - \gamma_1 && (29)\\
\pi_i(Q_p R, S_{-i}, \psi) &= -\frac{1}{3}x - \frac{2}{3}\sigma^h x - \gamma_1 && (30)\\
\pi_i(Q_c H, S_{-i}, \psi) &= -\gamma_2, && (31)\\
\pi_i(Q_p H, S_{-i}, \psi) &= -\frac{1}{3}x - \gamma_2, && (32)\\
\pi_i(R_{cc}, S_{-i}, \psi) = \pi_i(R_{cp}, S_{-i}, \psi) &= -\sigma^h x - 2\gamma_1, && (33)\\
\pi_i(H_{cc}, S_{-i}, \psi) = \pi_i(H_{cp}, S_{-i}, \psi) &= -2\gamma_2, && (34)\\
& && (35)
\end{aligned}$$

If $s_i^p = (Q, R)$, then

$$\begin{aligned}
\pi_i(Q, S_{-i}, \psi) &= -x, && (36)\\
\pi_i(Q_c R, S_{-i}, \psi) &= -\sigma^h x - \gamma_1 && (37)\\
\pi_i(Q_p R, S_{-i}, \psi) &= -x - \gamma_1 && (38)\\
\pi_i(Q_c H, S_{-i}, \psi) &= -\gamma_2, && (39)\\
\pi_i(Q_p H, S_{-i}, \psi) &= -x - \gamma_2, && (40)\\
\pi_i(R_{cc}, S_{-i}, \psi) = \pi_i(R_{cp}, S_{-i}, \psi) &= -\sigma^h x - 2\gamma_1, && (41)\\
\pi_i(H_{cc}, S_{-i}, \psi) = \pi_i(H_{cp}, S_{-i}, \psi) &= -2\gamma_2, && (42)\\
& && (43)
\end{aligned}$$

For periphery banks, payoff function is unchanged when R is replaced with Q.

We confirm each type of strategy profiles constitutes an equilibrium as follows.

Lemma 4. *netting equilibrium*
1. (all-early) S^1_{net} *is an equilibrium if* $\gamma_1 + \sigma^h x < \gamma_2$, $\frac{1}{2}(1-\sigma^l) < \frac{\gamma_1}{x}$.
2. (d-recycle) there exists an equilibrium within S^2_{net} *if* $\gamma_1 + \sigma^h x < \gamma_2$, $\frac{\gamma_1}{x} < \frac{1}{2}(1-\sigma^h)$, *and* $1-\sigma^l < \frac{\gamma_2}{x}$.
3. (mixed recycle) there is an equilibrium within S^3_{net} *if* $\gamma_1 + \sigma^h x < \gamma_2$, $\frac{1}{2}(1-\sigma^h) < \frac{\gamma_1}{x} < \frac{1}{2}(1-\sigma^l)$, $\frac{\gamma_1}{x} < 1-\sigma^h$, *and* $1-\sigma^l < \frac{\gamma_2}{x}$.

Proof. 1. (all-early)
For periphery banks, conditions are unchanged as $\frac{1}{2}(1-\sigma^l) < \frac{\gamma_1}{x}$ with Assumption 1. For core banks, conditions are;
$\frac{1}{2}(1+\sigma^h)x < \sigma^h x + \gamma_1$, $\frac{1}{2}(1+\sigma^h)x < \frac{1}{3}x + \frac{2}{3}\sigma^h x + \gamma_1$, $\frac{1}{2}(1+\sigma^h)x < \gamma_2$, $\frac{1}{2}(1+\sigma^h)x < \frac{1}{3}x + \gamma_2$, $\frac{1}{2}(1+\sigma^h)x < \sigma^h x + 2\gamma_1$, $\frac{1}{2}(1+\sigma^h)x < 2\gamma_2$.
All of these are satisfied with $\frac{1}{2}(1-\sigma^l) < \frac{\gamma_1}{x}$ and Assumption 1. For 2. (d-recycle) and 3. (mixed recycle), core banks have no incentive to deviate under Assumption 1 and $\frac{\gamma_1}{x} < 1-\sigma^h$. For periphery banks, incentive of deviation is almost the same as the case without LSM. The sole difference is additionally to examine situation where there exist only high shock banks for some separated cycles of payments. The conditions are as stated for RTGS without LSM. □

Confirm that when spillover effect had existed, introduction of LSM let (mixed recycle) type of equilibrium emerge. We proceed to formally state effects of LSM for each of the three cases.

Proposition 3. *Effects of LSM 1*
Under Liquidity non-Precious Regime,
a) S^1_{net} *is an equilibrium under RTGS with LSM, and*
b) $W(S^1) < W(S^1_{net})$.

Proof. For part a), it is straight by comparing the two set of conditions. For part b), $W(S^1_{net}) - W(S^1) = -Nr - (-(N+K)r) = Kr > 0$. □

The above proposition states that under Liquidity non-Precious Regime, introduction and use of LSM always improve welfare.

Proposition 4. *Effects of LSM 2.1*
Under Liquidity Precious Regime with $\frac{\gamma_1}{x} < \frac{1}{2}(1-\sigma^h)$,
a) S^2_{net} *is an equilibrium for RTGS with LSM.*
b) $W(S^2) \leq W(S^2_{net})$ *if and only if* $\frac{d(\psi)-1}{K+d(\psi)-1} \leq \frac{\gamma_1}{r}$.

Proof. Again part a) is straight. For part b), $W(S^2_{net}) \geq W(S^2) \Leftrightarrow -(d+\tilde{\sigma}N)r - (N-d)\gamma_1 \geq -((1+\sigma^l N^l + \sigma^h N^h)r + (N+K-1)\gamma_1 \Leftrightarrow (d-1)r \leq (K+d-1)\gamma_1$. □

This proposition states that under *Liquidity Precious Regime*, introduction of LSM does not always improve welfare. Further, welfare is more likely to decrease when payment network is *less dense*. Density matters since activation of netting only allows recycle among remained payments, and *less dense* network means remained payments form larger number of *core-separated* cycles.

Proposition 5. *Effects of LSM 2.2*

Under Liquidity Precious Regime with $\frac{\gamma_1}{x} \geq \frac{1}{2}(1-\sigma^h)$,

a) S^2_{net} *is an equilibrium for RTGS with LSM.*

b) $W(S^2) \leq W(S^2_{net})$ *if and only if* $\frac{(1-\sigma^h)N^h - \max(n^h(\psi)-\sigma^h N^h, 0)}{K+N^h-n^h(\psi)} \leq \frac{\gamma_1}{r}$.

Proof. Part a) is immediate. For part b), $W(S^2_{net}) \geq W(S^2) \Leftrightarrow -(1+\sigma^l N^l + N^h + min(0, -n^h + \sigma^h N^h))r - (N^l - 1 + n^h)\gamma_1 \geq -((1+\sigma^l N^l + \sigma^h N^h)r + (N+K-1)\gamma_1 \Leftrightarrow ((1-\sigma^h)N^h - max(n^h - \sigma^h N^h, 0))r \leq (K + N^h - n^h)\gamma_1$. □

The proposition simply states that introduction of LSM is more likely to have negative effective through dismissing spillover effect when n^h is larger.

5 Concluding Remarks

The paper constructed a model for gross settlement system, and examined welfare effects of LSM that serves partial netting. Our treatment of interconnected payment networks enables analysis of partial netting. We develop and focus on core/periphery structure for payment networks, that is tractable as well as well-captures payment networks in real world.

Effect of LSM was examined separately for two types of parameter values, or, two Regimes: Liquidity non-Precious Regime, and Liquidity Precious Regime. Under Liquidity non-Precious Regime, introducing LSM always have positive welfare effect. It is because policy lets payments be made earliest under those situations with relatively smaller liquidity cost, and introduction of LSM only serves to mitigate those liquidity cost. In contrast, under Liquidity Precious Regime, introducing LSM possibly has negative welfare effect. In those situations with relatively larger liquidity cost, optimal policy had let payments be settled with as smallest liquidity as possible. Introducing and actual usage of LSM brings possible negative effect through two routes. Firstly, activation of LSM serves to shorten possible recycle length with each unit of liquidity and increases total required liquidity. This effect is mitigated by its positive effect through eliminating cost of delaying payments. The less dense the payment network is, the larger the negative effect is. Secondly, activation of LSM possibly serves to dismiss positive spillover effect. In the presence of heterogeneity of prospect of facing liquidity needs, connection of payments had served to transmit positive spillover effect. Banks with higher probability of liquidity needs are more inclined to make earlier payments, but existence of banks with sufficiently low probability of liquidity needs had suppressed their socially undesirable early payments. Introducing and actual usage of LSM serves to disconnect network of payments, which help dismiss positive spillover effect for some of the disconnected network of payments. Severeness of the negative effect simply depends on how many banks turn to be separated to banks with low probability of liquidity needs, which further depends on *density* as well as more detailed factors related to where in each given network those banks are located.

To summarize our analysis, policy implications of this paper are twofold regarding how liquidity is “precious”, and how payment network is formed. When liquidity is not precious enough, fee for liquidity needs to be small enough to encourage earlier payment.

Introduction of LSM is always socially better through reducing liquidity requirement. In contrast, when liquidity is precious enough, it tends to be better to let fee of liquidity be sufficiently high so as to promote efficient liquidity recycle. Introduction of LSM needs to be carefully examined since it may have negative effect. When payment network is less dense in the sense of this paper, negative effect would be more likely both through direct and indirect negative effect.

A Appendix

A.1 Proof of Lemma 2

Proof. 1.(all-early) Core bank i has no incentive to deviate from $s_i = P$ if
$\frac{5}{6}x + \frac{1}{3}\sigma^h x < \frac{1}{3}x + \frac{2}{3}\sigma^h x + \gamma_1$,
$\frac{5}{6}x + \frac{1}{3}\sigma^h x < \frac{1}{3}x + \gamma_2$,
$\frac{5}{6}x + \frac{1}{3}\sigma^h x < \sigma^h x + 2\gamma_1$,
$\frac{5}{6}x + \frac{1}{3}\sigma^h x < 2\gamma_2$,
Note that $\frac{1}{3}x + \frac{2}{3}\sigma^h x + \gamma_1 < \frac{1}{3}x + \gamma_2$ from Assumption 1. $\gamma_1 + \sigma^h x < \gamma_2$. We have, $\frac{1}{2} - \frac{1}{3}\sigma^h < \frac{\gamma_1}{x}$,

Periphery bank i has no incentive to deviate from $s_i = P$ if
$\frac{1}{2}x + \frac{1}{2}\sigma^k x < \gamma_1 + \sigma^k x$,
$\frac{1}{2}x + \frac{1}{2}\sigma^k x < \gamma_2$,
Under Assumption 1, we have $\frac{1}{2}(1 - \sigma^l) < \frac{\gamma_1}{x}$.

2. (1-recycle)

Core bank i has no incentive to deviate from $s_i \in \{R_{cc}, R_{cp}\}$ when $s_i^p = \{R, R\}$ if
$2\gamma_1 + \sigma^k x < 2x$,
$2\gamma_1 + \sigma^k x < x + \gamma_1$,
$2\gamma_1 + \sigma^k x < 2\gamma_2$,
We have; $\frac{\gamma_1}{x} < 1 - \sigma^k$, $\sigma^k < 2\frac{\gamma_2}{x} - 2\frac{\gamma_1}{x}$.
For periphery bank i has no incentive to deviate from $s_i = R$ if
(1) $\gamma_1 + \sigma^k x \leq \gamma_2$ (R is better than H for who takes R),
(2) $\frac{1}{2}(1 + \sigma^k)x > \gamma_1 + \sigma^k x$ (R is better than P for a bank who is to receive payment from a bank which chooses P),
(3)$x \geq \sigma^k x + \gamma_1$($R$ is better than P for banks who is not to receive payment from a bank which chooses P), and,
(4)$x \leq \sigma^k x + \gamma_2$ (P is better than R or H for a bank who takes P)
We have: (1)$\gamma_1 + \sigma^k x \leq \gamma_2$, (2) $\gamma_1 < \frac{1}{2}(1 - \sigma^k)x$, (3)$\gamma_1 < (1 - \sigma^k)x$, (4) $1 - \sigma^k < \frac{\gamma_2}{x}$
Since any of core-periphery payment networks is Euler graph, there exists a route of liquidity recycle where all the payments are made.

□

A.2 Equilibrium under RTGS without LSM

Under RTGS without LSM, we present some types of equilibria which are not taken up in the main part.

Lemma 5. *equilibrium*
1.(high-early, low-recycle) there exists an equilibrium where $s_j = R$ for every $j \in I_{per,L}$, $s_{j'} \in I_{per} \setminus I_{per,L}$, and $s_i \in \{P, P_cR, P_pR, R_{cc}, R_{cp}\}$, if $1 < \frac{\gamma_2}{x}$, $1 - \sigma^h < \frac{\gamma_1}{x} < \frac{1}{2}(1 - \sigma^l)$, and $\sigma^l < \frac{\gamma_2}{x} - \frac{\gamma_1}{x}$
2'(1-recycle) there exists an equilibrium where $s_i = P_pR$ for $i \in I_{core}$, $s_{i'} \in \{R_{cc}, R_{cp}\}$ for every $i' \neq i \in I_{core}$, $s_j = R$ for every $j \in I_{per}$.

Proof. 1.(high-early, low-recycle) For periphery banks not to deviate,
$x < \gamma_1 + \sigma^h x$ (for high to take P rather than R)

$x < \gamma_2$ (for high to take P rather than H)
$\gamma_1 + \sigma^l x < \frac{1}{2}x + \frac{1}{2}\sigma^l x$ (for low to take R rather than P)
$\gamma_1 + \sigma^l x < \gamma_2$ (for low to take R rather than H)
We have, $1 < \frac{\gamma_2}{x}$, $1 - \sigma^h < \frac{\gamma_1}{x} < \frac{1}{2}(1 - \sigma^l)$, $\sigma^l < \frac{\gamma_2}{x} - \frac{\gamma_1}{x}$.

2'(1-recycle)
Core bank i with $s_i = P_p R$ has no incentive to deviate if $1 - \sigma^h < 2\frac{\gamma_2}{x} - \frac{\gamma_1}{x}$. We are suffice to replace the above condition with $1 - \sigma^l < \frac{\gamma_2}{x}$ for the conditions of 2.(1-recycle) in Lemma 2 .

□

Further, 1-recycle equilibrium coexists with the other types of equilibria which are not efficient. We prepare terminologies. For $K-$ c/p network, *separation* procedure is defined as follows. For payments of core banks, match one payment to be received and one payment to be made. We have a pair of matches. Separate each core bank into two banks each have matched payments. When we separate all the core banks into two, we have independent cycles of payments. Each way of separation regarding the matchings derives each set of independent cycles. We say a separation is *core-separation* when it derives a cycle of payments among core banks as one of independent cycles.

For a separation, we denote a set of separated cycles as C, and a set of core(periphery) banks included in a cycle $c \in C$ as $I^c_{core}(I^c_{per})$.

Lemma 6. *multi-recycle*

(multi-recycles) For any separation C excluding core-separation, there exists an equilibrium wherein for every $c \in C$, $s_j = P$ for some $j \in I^c_{per}$, $s_{j'} = R$ for every $j' \neq j \in I^c_{per}$, $s_i = R$ for every I^c_{core}, if $1 - \sigma^l < \frac{\gamma_2}{x}$, $\sigma^h < \frac{\gamma_2}{x} - \frac{\gamma_1}{x}$, and $\frac{\gamma_1}{x} < \frac{1}{2}(1 - \sigma^h)$.

Proof. Under the strategy profile stated in the lemma, all banks successfully make payment until period 2 when each core bank appropriately takes either of R_{cc} or R_{cp}. Then, there is no incentive for each core banks to switch between R_{cc} and R_{cp}. Core bank i has no incentive to deviate from $s_i \in \{R_{cc}, R_{cp}\}$ if
$2\gamma_1 + \sigma^h x < 2x$,
$2\gamma_1 + \sigma^h x < x + \gamma_1$,
$2\gamma_1 + \sigma^h x < 2\gamma_2$,
We have; $\frac{\gamma_1}{x} < 1 - \sigma^k$, $\sigma^k < 2\frac{\gamma_2}{x} - 2\frac{\gamma_1}{x}$.
For periphery bank i has no incentive to deviate from $s_i = R$ if
(1) $\gamma_1 + \sigma^k x \leq \gamma_2$ (R is better than H for who takes R),
(2) $\frac{1}{2}(1 + \sigma^k)x > \gamma_1 + \sigma^k x$ (R is better than P for a bank who is to receive payment from a bank which chooses P),
(3)$x \geq \sigma^k x + \gamma_1$($R$ is better than P for banks who is not to receive payment from a bank which chooses P), and,
(4)$x \leq \sigma^k x + \gamma_2$ (P is better than R or H for a bank who takes P)
We have: (1)$\gamma_1 + \sigma^k x \leq \gamma_2$, (2) $\gamma_1 < \frac{1}{2}(1 - \sigma^k)x$, (3)$\gamma_1 < (1 - \sigma^k)x$, (4) $x < \frac{1}{1-\sigma^k}\gamma_2$

□

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