Single-fluid stability of stationary plasma equilibria with velocity shear and magnetic shear

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(Received 25 June 2009; accepted 23 September 2009; published online 15 October 2009)

By using incompressible single-fluid equations with a generalized Ohm's law neglecting the electron inertia, a linear eigenmode equation for a magnetic field perturbation is derived for stationary equilibria in a slab geometry with velocity and magnetic shears. The general eigenmode equation contains a fourth-order derivative of the perturbation in the highest order and contains Alfvén and whistler mode components for a homogeneous plasma. The ratio of the characteristic ion inertia length to the characteristic inhomogeneity scale length is chosen as a small parameter for expansion. Neglecting whistler mode in the lowest order, the eigenmode equation becomes a second-order differential equation similar to the ideal magnetohydrodynamic eigenmode equation except for the fact that the unperturbed perpendicular velocity contains both electric and ion diamagnetic drifts. A sufficient condition for stability against the Kelvin–Helmholtz instability driven by shear in the ion diamagnetic drift velocity is derived and then applied to tokamaks. © *2009 American Institute of Physics*. [doi:10.1063/1.3247873]

I. INTRODUCTION

The investigation of hydromagnetic stability of plasmas is of interest in such varied fields as the study of fusion plasmas in magnetic confinement devices, the study of magnetospheric plasmas, and the study of space and astrophysical plasmas. Ideal magnetohydrodynamic (MHD) stability of static plasma can be studied by a powerful minimizing principle based on the self-adjointness of the force operator.[1,2] Such an energy principle[1] has been widely used for the investigation of stability of static plasmas in fusion devices[1,2] and magnetospheres.[3,4] However, the existence of plasma flow is known to affect the stability of fusion plasmas. Also Kelvin–Helmholtz instability driven by shear in the flow velocity[5] is important in space and astrophysical plasmas. It is well known that a non-self-adjoint operator appears in such a plasma equilibrium with flow.[6] Therefore, the powerful minimizing principle based on the self-adjointness of the force operator cannot be used to study the stability of a stationary plasma equilibrium with flow. Several different approaches have been made to study ideal MHD stability of plasma equilibrium with flow.[7–13] For fusion plasmas such an approach has been highly developed to include a realistic configuration of the plasma equilibrium.[9–13]

Although the plasma flow velocity in ideal MHD consists of a parallel flow velocity and an $\mathbf{E}\times\mathbf{B}$ drift velocity, the ion diamagnetic drift velocity appears for a nonuniform pressure plasma and adds to the ideal MHD flow velocity when the finite ion inertial length scale is taken into account in single-fluid equations. Thus, single-fluid stability with finite ion inertial length scale is more complicated than ideal MHD stability. Indeed, a fluid instability driven by a shear in the ion diamagnetic drift velocity was recently found[14] and it causes magnetic fluctuations, which may cause heat transport. Such a fluid instability or Kelvin–Helmholtz instability driven by shear in the ion diamagnetic drift velocity is important in fusion plasmas since in magnetic confinement devices there is an inevitable pressure gradient, i.e., the pressure decreasing outward toward the wall. Thus, even if there is no shear in the parallel flow velocity or in the electric drift velocity (the $\mathbf{E}\times\mathbf{B}$ drift velocity), a shear in the ion diamagnetic drift velocity may appear and may cause this fluid instability. This instability may be responsible for residual magnetic fluctuations existent even in a quiescent plasma such as *H*-mode,[15] which is characterized by the appearance of a steep pressure gradient. Although such an instability occurs in a fluid regime, it requires taking into account the finite ion inertial length scale in single-fluid equations, which is beyond ideal MHD.[14]

Therefore, a new approach is necessary to investigate the single-fluid stability against the fluid instability driven by shear in the ion diamagnetic drift velocity. The purpose of this study is then to provide a simple analytic method to investigate the single-fluid stability of plasma equilibria with shear in the ion diamagnetic drift velocity taking into account the finite ion inertial length scale.

In ideal MHD, an initial value approach based on the assumption of time dependent eikonal has shown a stabilizing effect of the toroidal flow on tokamak ballooning instabilities.[10,13] An analytic solution for a circular tokamak equilibrium also gives a stability criterion showing influence of the sheared toroidal flows on tokamak stability.[12] Whereas a realistic three-dimensional geometry is necessary to investigate those influences of equilibrium flow on ideal MHD pressure-driven modes in magnetic confinement devices, a simple slab geometry and the Cartesian coordinate system are used and thus geometrical effect is reduced to a minimum in the present study. Such a simplification is possible since the main focus of this study is the fluid instability of shorter scales taking into account the finite ion inertial length scale, which is assumed to be much smaller than the pressure gradient scale length.

Since the actual plasma in confinement devices such as tokamaks also possesses a magnetic shear (see Fig. 1), which is introduced to gain ideal MHD stability, the inclusion of magnetic shear is essential in order to study the stability of fusion plasmas in confinement devices. Thus, in the present study, a general eigenmode equation is derived for arbitrary stationary equilibria with velocity shear and magnetic shear by using single-fluid equations with a generalized Ohm's law, which includes Hall and electron pressure gradient terms. The obtained single-fluid equations include ion diamagnetic drift velocity, which does not appear in ideal MHD and appears only when the ion inertial length scale is taken into account in single-fluid equations. All kinetic effects are ignored. Although the obtained eigenmode equation contains a fourth-order derivative of a magnetic field perturbation owing to the existence of the finite ion inertia length scale and thus cannot be solved easily, it can be reduced to a second-order differential equation by retaining the lowest order contribution of the finite ion inertial length scale. In order to investigate the stability of stationary plasma equilibria with flow in the framework of single-fluid equations with the generalized Ohm's law, such an approximation enables one to obtain a simple quadratic form and a simple sufficient condition for stability. This sufficient condition for stability is reduced to the ideal MHD stability condition[8] when the unperturbed ion diamagnetic drift velocity is neglected.

The organization of this paper is as follows. The basic configuration of the plasma and basic equations are described in Sec. II. Unperturbed states are described in Sec. III. The general eigenmode equation is derived in Sec. IV. Dispersion relations for uniform equilibrium are obtained within several limits in Sec. V. The eigenmode equation for small ion inertial length limit is derived in Sec. VI. A sufficient condition for stability is obtained in Sec. VII. Discussion and summary are given in Sec. VIII.

II. BASIC CONFIGURATION AND EQUATIONS

One considers a slab geometry, in which unperturbed quantities are functions of x only. The stability of the incompressible plasma in this configuration is described by the following equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0, \tag{1}$$

$$\rho \frac{d\mathbf{V}}{dt} = \mathbf{J} \times \mathbf{B} - \nabla p, \tag{2}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \tag{3}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \tag{4}$$

$$\nabla \cdot \mathbf{V} = 0, \tag{5}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{6}$$

Here, ρ is the plasma mass density, $\mathbf{V}$ is the macroscopic velocity of the plasma, $\mathbf{B}$ is the magnetic field, and p is the plasma pressure. Another equation, which relates the electric field $\mathbf{E}$ to $\mathbf{V}$ and $\mathbf{B}$, is the generalized Ohm's law, which is derived from the equation of motion for the electron fluid. Neglecting the electron inertia term, the generalized Ohm's law becomes

$$-ne(\mathbf{E} + \mathbf{V} \times \mathbf{B}) + \mathbf{J} \times \mathbf{B} - \nabla p_e = 0, \tag{7}$$

where n is the plasma density and p_e is the electron pressure.

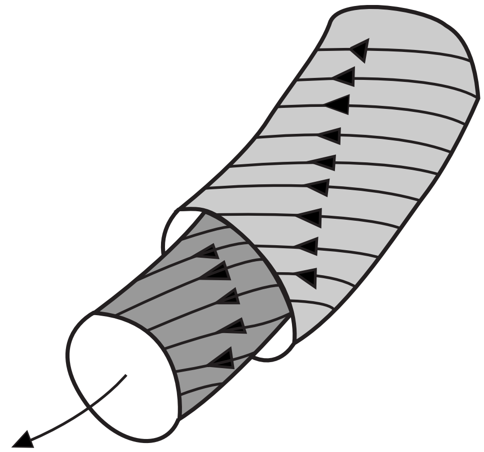

FIG. 1. Magnetic field lines on two different flux surfaces in a tokamak are plotted by solid lines. Dark gray and light gray surfaces are different flux surfaces.

III. UNPERTURBED STATE

In order to allow a shear of the unperturbed magnetic field one assumes generally

$$\mathbf{B}_0(x) = B_{0y}(x)\hat{\mathbf{y}} + B_{0z}(x)\hat{\mathbf{z}}, \tag{8}$$

where the subscript 0 denotes the unperturbed state, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ are unit vectors in the y and z directions, respectively, and subscripts y and z denote y and z components, respectively. It is obvious from Eq. (8) that the field line curvature vector $(\mathbf{b} \cdot \nabla)\mathbf{b} = 0$, where $\mathbf{b} = \mathbf{B}_0 / |\mathbf{B}_0| = \mathbf{B}_0 / B_0$. Therefore, field lines are straight and there are no pressure-driven modes such as interchange or ballooning modes in the present configuration.

Substitution of Eq. (8) into Eq. (3) yields

$$\mathbf{J}_0(x) = J_{0y}(x)\hat{\mathbf{y}} + J_{0z}(x)\hat{\mathbf{z}}, \tag{9}$$

where

$$J_{0y} = -\frac{1}{\mu_0}\frac{dB_{0z}}{dx}, \tag{10}$$

$$J_{0z} = \frac{1}{\mu_0}\frac{dB_{0y}}{dx}. \tag{11}$$

Notice that the y-z plane is a flux surface.

Since $\mathbf{E}_0$ is a function of x only, $\mathbf{E}_0(x)$ is generally expressed from Eq. (4) as

$$\mathbf{E}_0(x) = E_{0x}(x)\hat{\mathbf{x}} + E_{0y}\hat{\mathbf{y}} + E_{0z}\hat{\mathbf{z}}, \tag{12}$$

where E_{0y} and E_{0z} are constants. Since one can remove the effects of E_{0y} and E_{0z} by using a proper coordinate transformation, one assumes for the sake of simplicity $\mathbf{E}_0(x)=E_{0x}(x)\hat{\mathbf{x}}=E_0(x)\hat{\mathbf{x}}$.

Since $\mathbf{J}_0\times\mathbf{B}_0$, ∇p_{e0}, and $\mathbf{E}_0$ are all directed in the x direction, one finds from Eq. (7) that $\mathbf{V}_0\times\mathbf{B}_0$ must also be in the x direction. Therefore, one has

$$\mathbf{V}_0(x) = V_{0y}(x)\hat{\mathbf{y}} + V_{0z}(x)\hat{\mathbf{z}}. \tag{13}$$

Substitution of Eq. (13) into Eq. (2) yields the pressure balance equation $\mathbf{J}_0\times\mathbf{B}_0=\nabla p_0$. Using this pressure balance equation, one obtains from Eq. (7)

$$\mathbf{E}_0 + \mathbf{V}_0\times\mathbf{B}_0 = \frac{1}{n_0 e}\nabla p_{io}, \tag{14}$$

where $p_{i0}=p_0-p_{e0}$ and subscripts i and e denote ion and electron species, respectively. From Eq. (14) one obtains

$$\mathbf{V}_{0\perp} = \frac{\mathbf{E}_0\times\mathbf{B}_0}{B_0^2} - \frac{1}{n_0 e B_0^2}\nabla p_{i0}\times\mathbf{B}_0, \tag{15}$$

where the subscript $\perp$ denotes the component perpendicular to the unperturbed magnetic field.

On the other hand, one obtains from the electron fluid equation of motion neglecting the inertia term

$$\mathbf{V}_{e0\perp} = \frac{\mathbf{E}_0\times\mathbf{B}_0}{B_0^2} + \frac{1}{n_0 e B_0^2}\nabla p_{e0}\times\mathbf{B}_0. \tag{16}$$

Therefore, one has

$$\mathbf{J}_{0\perp} = n_0 e(\mathbf{V}_{i0\perp}-\mathbf{V}_{e0\perp}) \simeq n_0 e(\mathbf{V}_{0\perp}-\mathbf{V}_{e0\perp}) = \frac{1}{B_0^2}\mathbf{B}_0\times\nabla p_0, \tag{17}$$

where $m_e/m_i \ll 1$ was used, with m_i and m_e being ion and electron masses, respectively. This is consistent with the pressure balance equation. The parallel component of the unperturbed current $J_{0\parallel}$ must satisfy $J_{0\parallel}=n_0 e(V_{i0\parallel}-V_{e0\parallel})$, where the subscript $\parallel$ denotes the component parallel to the unperturbed magnetic field. Notice that in the present fluid treatment, $V_{i0\parallel}$ and $V_{e0\parallel}$ can be arbitrarily specified to satisfy $J_{0\parallel}=\mathbf{b}\cdot\mathbf{J}_0=\mu_0^{-1}\mathbf{b}\cdot(\nabla\times\mathbf{B}_0)$.

Let n_N and B_N be characteristic values of $n_0(x)$ and $\mathbf{B}_0(x)$ in the region considered. The n_N and B_N may also be considered normalization constants. Then $n_0(x)$ and $B_0(x)$ can be written as $n_0(x)=n_N\bar{n}_0(x)$ and $\mathbf{B}_0(x)=B_N\bar{\mathbf{B}}_0(x)$, where the overbar represents the normalized quantity, which is an order of one or smaller. Using n_N and B_N, one defines $V_{AN}=B_N/\sqrt{\mu_0\rho_N}\simeq B_N/\sqrt{\mu_0 n_N m_i}$ and the characteristic ion inertia scale length $\lambda_{iN}=V_{AN}/\omega_{iN}$, where ω_{iN} is the ion gyrofrequency defined by B_N. Using V_{AN} and λ_{iN}, n_N can be written as $n_N=B_N/(e\mu_0 V_{AN}\lambda_{iN})$. Let L be the shortest characteristic inhomogeneity scale length, which is the shortest among the scale lengths of $\mathbf{B}_0$, $\mathbf{V}_0$, n_0, and p_{i0}. Then, x can be written by using L as $x=L\bar{x}$.

Since incompressibility is assumed, the sound speed is infinite and hence the velocity must be normalized to V_{AN}. Therefore, from Eq. (15) one obtains

$$\bar{\mathbf{V}}_0 = \bar{\mathbf{V}}_{0\parallel} + \frac{\bar{\mathbf{E}}_0\times\bar{\mathbf{B}}_0}{\bar{B}_0^2} - \frac{1}{2\bar{n}_0\bar{B}_0^2}\frac{\lambda_{iN}}{L}\bar{\nabla}\bar{p}_{i0}\times\bar{\mathbf{B}}_0, \tag{18}$$

where $\bar{\mathbf{E}}_0=\mathbf{E}_0/(V_{AN}B_N)$, $\bar{p}_{i0}=p_{i0}/(B_N^2/2\mu_0)$, and $\bar{\nabla}=\partial/\partial\bar{\mathbf{r}}$, with $\mathbf{r}$ being equal to $x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}$. It is obvious from this equation that the ion diamagnetic drift velocity appears in the flow velocity only when λ_{iN} is retained. Therefore, there is no ion diamagnetic drift velocity in the ideal MHD limit $\lambda_{iN}\to 0$.

The normalized form of the generalized Ohm's law (7) becomes

$$\bar{\mathbf{E}} + \bar{\mathbf{V}}\times\bar{\mathbf{B}} - \frac{\lambda_{iN}}{L}\frac{\bar{\mathbf{J}}\times\bar{\mathbf{B}}}{\bar{n}} + \frac{1}{2\bar{n}}\frac{\lambda_{iN}}{L}\bar{\nabla}\bar{p}_e = 0, \tag{19}$$

where $\bar{\mathbf{J}}=\bar{\nabla}\times\bar{\mathbf{B}}$ and $\bar{p}_e=p_e/(B_N^2/2\mu_0)$. In the ideal MHD limit $\lambda_{iN}\to 0$, one obtains the frozen-in law $\bar{\mathbf{E}}+\bar{\mathbf{V}}\times\bar{\mathbf{B}}=0$.

Since the characteristic ion inertia scale length is given by $\lambda_{iN}=\sqrt{m_i/(\mu_0 n_N e^2)}$, $\lambda_{iN}=2.3$ cm for a typical plasma density $n_N=10^{20}$ m^{-3} in tokamaks. Therefore, for a typical minor radius ~ 1.0 m for tokamaks, λ_{iN}/L is considered to be much smaller or smaller than unity. Therefore, one can consider $\epsilon=\lambda_{iN}/L$ as the small parameter. As is obvious from Eqs. (18) and (19), if one removes terms containing ϵ, one obtains ideal MHD equations. For the characteristic ion Larmor radius ρ_{LiN}, one obtains $\rho_{LiN}/\lambda_{iN}=\sqrt{\beta_{i0N}/2}=\sqrt{\bar{p}_{i0N}/2}$, where β_{i0N} is the characteristic ion plasma beta value. Since β_{i0N} is typically 0.05(5%) for tokamaks, $\rho_{LiN}/\lambda_{iN}\simeq 0.16$. Thus, one has $L>\lambda_{iN}\gg\rho_{LiN}$ for tokamaks. The pressure p in Eq. (2) is generally a pressure tensor and there is a finite ion Larmor radius correction term in the pressure tensor. However, since $L>\lambda_{iN}\gg\rho_{LiN}$ holds for tokamaks, the present analysis, which retains only the ion inertial scale length in Eqs. (18) and (19) and neglects the finite ion Larmor radius term, is justified.

IV. EIGENMODE EQUATION

A. Representation of the linear perturbation

One assumes that any physical quantity $Q(\mathbf{r},t)$ can be expressed as follows:

$$Q(\mathbf{r},t) = Q_0(x) + \tilde{Q}_1(\mathbf{r},t), \tag{20}$$

where $\mathbf{r}=\mathbf{r}(x,y,z)$ and the subscript 1 denotes a linear perturbation. By defining $\mathbf{k}=k_y\hat{\mathbf{y}}+k_z\hat{\mathbf{z}}$, one further assumes that

$$\tilde{Q}_1(\mathbf{r},t) = Q_1(x)\exp[-i(\omega t-\mathbf{k}\cdot\mathbf{r})] = Q_1(x)\exp[i(k_y y+k_z z-\omega t)]. \tag{21}$$

Since the equation of motion for the electron fluid (7) is solved, the present single-fluid equations can describe a whistler wave, which is a right-handed circularly polarized wave rotating in the electron gyration direction. Therefore, the eigenmode equation for a perturbation in the present general MHD equilibria becomes a wave equation for Alfvén or whistler waves when the plasma and the magnetic field are uniform and the wave vector has a component parallel to the unperturbed magnetic field. Thus, the eigenmode equation contains a Laplacian operator. It follows that one can obtain the eigenmode equation straightforwardly by taking x components of the curl of the curl of the equation of motion (2) and the generalized Ohm's law (7) since $\nabla\times\nabla\times\mathbf{Q}=\nabla(\nabla\cdot\mathbf{Q})-\nabla^2\mathbf{Q}$ immediately yields the Laplacian of $\mathbf{Q}$. Taking the curl of Eqs. (2) and (7) also deletes ∇p and ∇p_e terms in those equations.

B. Curl of curl of the equation of motion

First, by taking the curl of Eq. (2), dividing the resultant equation by ρ, and then taking the curl of the resultant equation one obtains

$$\begin{aligned}&\nabla \ln \rho\{\nabla\cdot[(\mathbf{V}\cdot\nabla)\mathbf{V}]\}-\left[\frac{\partial \mathbf{V}}{\partial t}+(\mathbf{V}\cdot\nabla)\mathbf{V}\right]\nabla^2 \ln\rho\\&+\left\{\left[\frac{\partial \mathbf{V}}{\partial t}+(\mathbf{V}\cdot\nabla)\mathbf{V}\right]\cdot\nabla\right\}\nabla \ln\rho\\&-(\nabla\ln\rho\cdot\nabla)\left[\frac{\partial \mathbf{V}}{\partial t}+(\mathbf{V}\cdot\nabla)\mathbf{V}\right]-\frac{\partial}{\partial t}\nabla^2\mathbf{V}\\&+\nabla\{\nabla\cdot[(\mathbf{V}\cdot\nabla)\mathbf{V}]\}-\nabla^2[(\mathbf{V}\cdot\nabla)\mathbf{V}]\\&=\frac{1}{\rho}\nabla\times\nabla\times(\mathbf{J}\times\mathbf{B})+\left(\nabla\frac{1}{\rho}\right)\times[\nabla\times(\mathbf{J}\times\mathbf{B})].\end{aligned} \tag{22}$$

One obtains after some calculation

$$\begin{aligned}&[\nabla\times\nabla\times(\mathbf{J}\times\mathbf{B})]_{1x}\\&=\frac{k^2}{\mu_0}i(\mathbf{k}\cdot\mathbf{B}_0)\widetilde{B}_{1x}+\frac{i}{\mu_0}\\&\quad\times\left\{\left[\frac{d^2}{dx^2}(\mathbf{k}\cdot\mathbf{B}_0)\right]\widetilde{B}_{1x}-(\mathbf{k}\cdot\mathbf{B}_0)\frac{d^2\widetilde{B}_{1x}}{dx^2}\right\},\end{aligned} \tag{23}$$

where subscript 1 on the bracket represents the first order linear perturbations in the bracket and $k^2=k_y^2+k_z^2$. Therefore, taking the x component of the linear perturbation of Eq. (22), one obtains

$$\begin{aligned}&\frac{d\ln\rho_0}{dx}\left\{\left[\frac{d}{dx}(\mathbf{k}\cdot\mathbf{V}_0)\right]V_{1x}+\Omega\frac{dV_{1x}}{dx}\right\}+\Omega\left(\frac{d^2}{dx^2}-k^2\right)V_{1x}\\&+\left[\frac{d^2}{dx^2}(\mathbf{k}\cdot\mathbf{V}_0)\right]V_{1x}\\&=\frac{k^2}{\rho_0\mu_0}(\mathbf{k}\cdot\mathbf{B}_0)B_{1x}\\&\quad+\frac{1}{\rho_0\mu_0}\left\{\left[\frac{d^2}{dx^2}(\mathbf{k}\cdot\mathbf{B}_0)\right]B_{1x}-(\mathbf{k}\cdot\mathbf{B}_0)\frac{d^2B_{1x}}{dx^2}\right\},\end{aligned} \tag{24}$$

where $\Omega=\omega-\mathbf{k}\cdot\mathbf{V}_0$. Equation (24) gives a relation between $V_{1x}(x)$ and $B_{1x}(x)$. Notice that this equation contains terms of $O(\epsilon)$ since $\mathbf{V}_0$ and Ω contain terms of $O(\epsilon)$. When terms of $O(\epsilon)$ are neglected, this equation is valid for the ideal MHD.

For a flute mode satisfying $\mathbf{k}\cdot\mathbf{B}_0=0$, Eq. (24) gives

$$\begin{aligned}&\frac{d\ln\rho_0}{dx}\left\{\left[\frac{d}{dx}(\mathbf{k}\cdot\mathbf{V}_0)\right]V_{1x}+\Omega\frac{dV_{1x}}{dx}\right\}+\Omega\left(\frac{d^2}{dx^2}-k^2\right)V_{1x}\\&\quad+\left[\frac{d^2}{dx^2}(\mathbf{k}\cdot\mathbf{V}_0)\right]V_{1x}=0.\end{aligned} \tag{25}$$

This equation becomes the eigenmode equation for $V_{1x}(x)$ and has been derived previously[14] to show the existence of Kelvin–Helmholtz instability driven by shear in the ion diamagnetic drift velocity. Since there is no Alfén or whistler mode for $\mathbf{k}\cdot\mathbf{B}_0=0$ and also there is no fast magnetosonic mode propagating perpendicularly to the unperturbed magnetic field owing to the incompressible assumption in the present problem, this equation is not a wave equation but an equation describing the vortex motion. Indeed, for the hydrodynamic case, Eq. (25) gives an eigenmode equation. When ρ_0 is constant, Eq. (25) becomes Rayleigh's stability equation[16] in hydrodynamics.

Notice that in Eq. (25), $\mathbf{V}_{0\perp}$ given by Eq. (15) includes the $\mathbf{E}\times\mathbf{B}$ drift and the ion diamagnetic drift. For ideal MHD, $\mathbf{V}_{0\perp}$ becomes the $\mathbf{E}\times\mathbf{B}$ drift velocity only and this eigenmode equation becomes the same as that derived by Chandrsekhar[5] for a configuration in which the flow velocity (the $\mathbf{E}\times\mathbf{B}$ drift velocity) is perpendicular to the unperturbed magnetic field.

C. Curl of curl of the generalized Ohm's law

Next, in order to obtain the eigenmode equation for $B_{1x}(x)$, one needs another equation to relate $V_{1x}(x)$ to $B_{1x}(x)$. This equation can be obtained by taking the curl of the curl of the generalized Ohm's law (7).

Let us define $\mathbf{G}=ne(\mathbf{E}+\mathbf{V}\times\mathbf{B})$. Then, the generalized Ohm's law (7) becomes $\mathbf{G}=\mathbf{J}\times\mathbf{B}-\nabla p_e$. Therefore, one obtains

$$\nabla\times\nabla\times\mathbf{G}=\nabla\times\nabla\times(\mathbf{J}\times\mathbf{B}). \tag{26}$$

By using Eq. (14) one obtains

$$\widetilde{\mathbf{G}}_1=\frac{\widetilde{n}_1}{n_0}\nabla p_{i0}+n_0e(\widetilde{\mathbf{E}}_1+\widetilde{\mathbf{V}}_1\times\mathbf{B}_0+\mathbf{V}_0\times\widetilde{\mathbf{B}}_1). \tag{27}$$

Owing to the quasineutrality of the plasma, one has $\rho=m_in_i+m_en_e\simeq(m_i+m_e)n$. Therefore, taking the linear perturbation of Eq. (1), one obtains

$$\widetilde{n}_1=\frac{1}{i\Omega}\frac{dn_0}{dx}\widetilde{V}_{1x}. \tag{28}$$

Substituting Eq. (28) into Eq. (27) and then taking the curl of the curl of Eq. (27), one obtains

$$
\begin{aligned}
[\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{G}]_{1x}
&= -i\frac{k^2}{\Omega n_0}\frac{dn_0}{dx}\frac{dp_{i0}}{dx}\tilde{V}_{1x} + in_0e\Omega(\boldsymbol{\nabla}\times\mathbf{B})_{1x} \\
&\quad + ie\frac{dn_0}{dx}(\mathbf{k}\cdot\tilde{\mathbf{E}}_1) + in_0e(\mathbf{k}\cdot\mathbf{B}_0)(\boldsymbol{\nabla}\times\mathbf{V})_{1x} \\
&\quad - ie\left[n_0\left(k_y\frac{dB_{0z}}{dx}-k_z\frac{dB_{0y}}{dx}\right)+\frac{dn_0}{dx}\right. \\
&\quad \left.\times(k_yB_{0z}-k_zB_{0y})\right]\tilde{V}_{1x} + ie\left[n_0\left(k_y\frac{dV_{0z}}{dx}-k_z\frac{dV_{0y}}{dx}\right)\right. \\
&\quad \left.+\frac{dn_0}{dx}(k_yV_{0z}-k_zV_{0y})\right]\tilde{B}_{1x}. \qquad (29)
\end{aligned}
$$

Thus, in order to express $[\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{G}]_{1x}$ by using $\tilde{V}_{1x}$ and $\tilde{B}_{1x}$, one needs to express $(\boldsymbol{\nabla}\times\mathbf{B})_{1x}=\mu_0\tilde{J}_{1x}$, $\mathbf{k}\cdot\tilde{\mathbf{E}}_1$, and $(\boldsymbol{\nabla}\times\mathbf{V})_{1x}$ by using $\tilde{V}_{1x}$ and $\tilde{B}_{1x}$. One first takes the x component of the linear perturbation of $\boldsymbol{\nabla}\times\mathbf{G}=\boldsymbol{\nabla}\times(\mathbf{J}\times\mathbf{B})$ and obtains

$$
(\mathbf{k}\cdot\mathbf{B}_0)\tilde{J}_{1x} = (\Omega n_0 e + \mathbf{k}\cdot\mathbf{J}_0)\tilde{B}_{1x} + n_0e(\mathbf{k}\cdot\mathbf{B}_0)\tilde{V}_{1x}. \qquad (30)
$$

Substituting Eq. (27) into $\tilde{\mathbf{G}}_1=\mathbf{J}_0\times\tilde{\mathbf{B}}_1+\tilde{\mathbf{J}}_1\times\mathbf{B}_0-\boldsymbol{\nabla}\tilde{p}_{e1}$ and adding the y component of the resultant equation multiplied by k_y, and the z component of the resultant equation multiplied by k_z, one obtains by using Eq. (30)

$$
\begin{aligned}
\mathbf{k}\cdot\tilde{\mathbf{E}}_1 &= \frac{1}{n_0e}\left[k_y(J_{0z}-n_0eV_{0z}) - k_z(J_{0y}-n_0eV_{0y})\right. \\
&\quad \left. - \frac{k_yB_{0z}-k_zB_{0y}}{\mathbf{k}\cdot\mathbf{B}_0}(\Omega n_0e+\mathbf{k}\cdot\mathbf{J}_0)\right]\tilde{B}_{1x} - \frac{ik^2}{n_0e}\tilde{p}_{e1}. \qquad (31)
\end{aligned}
$$

By taking the x component of the linear perturbation of the curl of Eq. (2), one obtains by using Eq. (30)

$$
\begin{aligned}
(\boldsymbol{\nabla}\times\mathbf{V})_{1x} &= \frac{1}{\Omega}\left[\left(k_y\frac{dV_{0z}}{dx}-k_z\frac{dV_{0y}}{dx}\right)-\frac{n_0e}{\rho_0}(\mathbf{k}\cdot\mathbf{B}_0)\right]\tilde{V}_{1x} \\
&\quad - \frac{n_0e}{\rho_0}\tilde{B}_{1x}. \qquad (32)
\end{aligned}
$$

Taking the x component of the linear perturbation of Eq. (26) and using Eqs. (29)–(32), one obtains

$$
\begin{aligned}
&\left\{-\frac{k^2}{\Omega n_0}\frac{dn_0}{dx}\frac{dp_{i0}}{dx}+\mu_0(n_0e)^2\Omega+\frac{n_0e(\mathbf{k}\cdot\mathbf{B}_0)}{\Omega}\left[\left(k_y\frac{dV_{0z}}{dx}-k_z\frac{dV_{0y}}{dx}\right)-\frac{n_0e}{\rho_0}(\mathbf{k}\cdot\mathbf{B}_0)\right]-e\left[n_0\left(k_y\frac{dB_{0z}}{dx}-k_z\frac{dB_{0y}}{dx}\right)\right.\right. \\
&\quad \left.\left.+\frac{dn_0}{dx}(k_yB_{0z}-k_zB_{0y})\right]\right\}V_{1x} + \left\{\frac{n_0e\Omega\mu_0(\Omega n_0e+\mathbf{k}\cdot\mathbf{J}_0)}{\mathbf{k}\cdot\mathbf{B}_0}+\frac{1}{n_0}\frac{dn_0}{dx}\left[k_yJ_{0z}-k_zJ_{0y}-\frac{k_yB_{0z}-k_zB_{0y}}{\mathbf{k}\cdot\mathbf{B}_0}(\Omega n_0e+\mathbf{k}\cdot\mathbf{J}_0)\right]\right. \\
&\quad \left.-\frac{(n_0e)^2}{\rho_0}(\mathbf{k}\cdot\mathbf{B}_0)+e\left[n_0\left(k_y\frac{dV_{0z}}{dx}-k_z\frac{dV_{0y}}{dx}\right)\right]\right\}B_{1x} - i\frac{1}{n_0}\frac{dn_0}{dx}k^2p_{e1} \\
&\quad = \frac{k^2}{\mu_0}(\mathbf{k}\cdot\mathbf{B}_0)B_{1x}+\frac{1}{\mu_0}\left\{\left[\frac{d^2}{dx^2}(\mathbf{k}\cdot\mathbf{B}_0)\right]B_{1x}-(\mathbf{k}\cdot\mathbf{B}_0)\frac{d^2B_{1x}}{dx^2}\right\}. \qquad (33)
\end{aligned}
$$

One thus has obtained another relationship between $V_{1x}(x)$ and $B_{1x}(x)$. Notice that $p_{e1}(x)$ is also included in this equation. If one multiplies Ω on both sides of this equation, this equation contains $O(\epsilon^1)$, $O(\epsilon^2)$, and $O(\epsilon^3)$ terms.

D. General eigenmode equation

Equations (24) and (33) constitute a set of necessary equations to obtain the eigenmode equation for $B_{1x}(x)$. In order to close this equation, p_{e1} must be expressed by using V_{1x} and B_{1x}. Therefore, some specific assumptions are necessary to close Eqs. (24) and (33). However, it is obvious from Eq. (33) that when $n_0(x)$ is constant, p_{e1} does not appear in Eq. (33) and the coupled set of Eqs. (24) and (33) is closed.

When $\mathbf{k}\cdot\mathbf{B}_0=0$, the eigenmode equation becomes Eq. (25) and there is no need to use Eq. (33) in order to obtain the eigenmode equation. However, for a nonflute mode satisfying $\mathbf{k}\cdot\mathbf{B}_0\neq 0$, the coupled set of Eqs. (24) and (33) must be used to obtain the eigenmode equation for B_{1x}. First, one obtains from Eq. (33)

$$
\begin{aligned}
&R_1(x)V_{1x}+R_2(x)B_{1x}+R_3(x)p_{e1} \\
&\quad = \frac{k^2}{\mu_0}(\mathbf{k}\cdot\mathbf{B}_0)B_{1x}+\frac{1}{\mu_0}\left\{\left[\frac{d^2}{dx^2}(\mathbf{k}\cdot\mathbf{B}_0)\right]B_{1x}\right. \\
&\quad \left.-(\mathbf{k}\cdot\mathbf{B}_0)\frac{d^2B_{1x}}{dx^2}\right\}, \qquad (34)
\end{aligned}
$$

where

$$R_1(x) = -\frac{k^2}{\Omega n_0}\frac{dn_0}{dx}\frac{dp_{i0}}{dx} + \mu_0(n_0e)^2\Omega + \frac{n_0e(\mathbf{k}\cdot\mathbf{B}_0)}{\Omega}\left[\left(k_y\frac{dV_{0z}}{dx} - k_z\frac{dV_{0y}}{dx}\right) - \frac{n_0e}{\rho_0}(\mathbf{k}\cdot\mathbf{B}_0)\right] - e\left[n_0\left(k_y\frac{dB_{0z}}{dx} - k_z\frac{dB_{0y}}{dx}\right) + \frac{dn_0}{dx}(k_yB_{0z} - k_zB_{0y})\right], \quad (35)$$

$$R_2(x) = \frac{n_0e\Omega\mu_0(\Omega n_0e + \mathbf{k}\cdot\mathbf{J}_0)}{\mathbf{k}\cdot\mathbf{B}_0} + \frac{1}{n_0}\frac{dn_0}{dx}\left[k_yJ_{0z} - k_zJ_{0y} - \frac{k_yB_{0z} - k_zB_{0y}}{\mathbf{k}\cdot\mathbf{B}_0}(\Omega n_0e + \mathbf{k}\cdot\mathbf{J}_0)\right] - \frac{(n_0e)^2}{\rho_0}(\mathbf{k}\cdot\mathbf{B}_0) + e\left[n_0\left(k_y\frac{dV_{0z}}{dx} - k_z\frac{dV_{0y}}{dx}\right)\right], \quad (36)$$

$$R_3(x) = -i\frac{1}{n_0}\frac{dn_0}{dx}k^2. \quad (37)$$

From Eq. (34) one obtains

$$V_{1x} = P_1(x)B_{1x} + P_2(x)\frac{d^2B_{1x}}{dx^2} + P_3(x)p_{e1}, \quad (38)$$

where

$$P_1(x) = \frac{1}{R_1}\left\{-R_2 + \frac{1}{\mu_0}\left[k^2(\mathbf{k}\cdot\mathbf{B}_0) + \frac{d^2}{dx^2}(\mathbf{k}\cdot\mathbf{B}_0)\right]\right\}, \quad (39)$$

$$P_2(x) = -\frac{1}{R_1\mu_0}(\mathbf{k}\cdot\mathbf{B}_0), \quad (40)$$

$$P_3(x) = -\frac{R_3}{R_1}. \quad (41)$$

Substituting Eq. (38) into Eq. (24), one obtains

$$T_1\frac{d^4B_{1x}}{dx^4} + T_2\frac{d^3B_{1x}}{dx^3} + T_3\frac{d^2B_{1x}}{dx^2} + T_4\frac{dB_{1x}}{dx} + T_5B_{1x} + T_6\frac{d^2p_{e1}}{dx^2} + T_7\frac{dp_{e1}}{dx} + T_8p_{e1} = 0, \quad (42)$$

where

$$T_1 = \Omega P_2, \quad (43)$$

$$T_2 = \Omega\left(\frac{d\ln\rho_0}{dx}P_2 + 2P_2'\right), \quad (44)$$

$$T_3 = \frac{d\ln\rho_0}{dx}\left\{\left[\frac{d}{dx}(\mathbf{k}\cdot\mathbf{V}_0)\right]P_2 + \Omega P_2'\right\} + \left[\frac{d^2}{dx^2}(\mathbf{k}\cdot\mathbf{V}_0)\right]P_2 + \Omega(P_1 + P_2'' - k^2P_2) + \frac{1}{\rho_0\mu_0}(\mathbf{k}\cdot\mathbf{B}_0), \quad (45)$$

$$T_4 = \Omega\left(\frac{d\ln\rho_0}{dx}P_1 + 2P_1'\right), \quad (46)$$

$$T_5 = \frac{d\ln\rho_0}{dx}\left\{\left[\frac{d}{dx}(\mathbf{k}\cdot\mathbf{V}_0)\right]P_1 + \Omega P_1'\right\} + \left[\frac{d^2}{dx^2}(\mathbf{k}\cdot\mathbf{V}_0)\right]P_1 + \Omega(P_1'' - k^2P_1) - \frac{1}{\rho_0\mu_0}\left\{k^2(\mathbf{k}\cdot\mathbf{B}_0) + \left[\frac{d^2}{dx^2}(\mathbf{k}\cdot\mathbf{B}_0)\right]\right\}, \quad (47)$$

$$T_6 = \Omega P_3, \quad (48)$$

$$T_7 = \Omega\left(\frac{d\ln\rho_0}{dx}P_3 + 2P_3'\right), \quad (49)$$

$$T_8 = \frac{d\ln\rho_0}{dx}\left\{\left[\frac{d}{dx}(\mathbf{k}\cdot\mathbf{V}_0)\right]P_3 + \Omega P_3'\right\} + \left[\frac{d^2}{dx^2}(\mathbf{k}\cdot\mathbf{V}_0)\right]P_3 + \Omega(P_3'' - k^2P_3), \quad (50)$$

where the prime denotes a derivative with respect to x.

Thus, one finds that the general eigenmode equation for an arbitrary equilibrium configuration and a nonflute mode becomes a fourth-order ordinary differential equation with respect to B_{1x}. Notice that for ideal MHD the general eigenmode equation for a nonflute mode becomes a second-order ordinary differential equation[8] since higher-order derivatives associated with smaller scale whistler modes are not necessary. Since p_{e1}, dp_{e1}/dx, and d^2p_{e1}/dx^2 terms are contained in the eigenmode equation (42), this eigenmode equation is not closed unless p_{e1} is expressed in terms of B_{1x}. When $n_0(x)$ is constant, however, T_6, T_7, and T_8 vanish and Eq. (42) is closed within itself. Notice that for a flute mode satisfying $\mathbf{k}\cdot\mathbf{B}_0=0$, T_1 and T_2 vanish and the eigenmode equation becomes a second-order ordinary differential equation with respect to B_{1x}. For the flute mode the eigenmode equation with respect to V_{1x} is also closed as Eq. (25) shows irrespective of $n_0(x)$. Except for those special cases, p_{e1} must be expressed by using B_{1x} in order to close the eigenmode equation (42). In determining p_{e1}, it should be noted that there is a constraint $\nabla\cdot\mathbf{V}_e=0$ since $\nabla\cdot\mathbf{V}=\nabla\cdot\mathbf{J}=0$ holds.

V. DISPERSION RELATION FOR UNIFORM EQUILIBRIUM

In order to investigate the relation of the coupled set of Eqs. (24) and (33) or the eigenmode equation (42) to dispersion relations of Alfvén and whistler modes in a uniform magnetic field, one assumes that $\mathbf{B}_0(x)$ and $\rho_0(x)$ are constant and $\mathbf{V}_0(x)=0$. Then, $V_{1x}(x)$ and $B_{1x}(x)$ can also be Fourier expanded in the x direction to have

$$V_{1x}(x) = V_{1x}'e^{ik_xx}, \quad (51)$$

$$B_{1x}(x) = B_{1x}'e^{ik_xx}. \quad (52)$$

Substitution of Eqs. (51) and (52) into the coupled set of Eqs. (24) and (33) yields the following dispersion relation:

$$\left[\omega^2 - \frac{(\mathbf{k}_p \cdot \mathbf{B}_0)^2}{\rho_0\mu_0}\right]^2 = \frac{\omega^2(\mathbf{k}_p \cdot \mathbf{B}_0)^2 k_p^2}{\mu_0^2(n_0 e)^2}, \tag{53}$$

where the propagation vector $\mathbf{k}_p = \mathbf{k} + k_x\hat{\mathbf{x}}$ and $k_p^2 = k^2 + k_x^2 = k_x^2 + k_y^2 + k_z^2$. Notice that since $\mathbf{B}_0$ is assumed to be in the y-z plane, $\mathbf{k}\cdot\mathbf{B}_0 = \mathbf{k}_p\cdot\mathbf{B}_0$.

By defining $V_A^2 = B_0^2/(\mu_0\rho_0)$, $\lambda_i = V_A/\omega_i$, and $\omega_i = eB_0/m_i$, Eq. (53) can be rewritten as

$$[\omega^2 - (k_\parallel V_A)^2]^2 = \lambda_i^2\omega^2(k_\parallel V_A)^2 k_p^2, \tag{54}$$

where $k_\parallel$ means the component of $\mathbf{k}_p$ parallel to the unperturbed magnetic field. A similar equation, which also includes the electron inertia term, has previously been derived for a uniform plasma from two-fluid equations.[17]

Let us consider various limits of Eq. (54). When $\mathbf{k}_p$ is perpendicular to $\mathbf{B}_0$, $k_\parallel$ is equal to zero. Therefore, Eq. (54) gives simply $\omega=0$. Thus, there is no propagating mode for $\mathbf{k}_p \perp \mathbf{B}_0$. In the ideal MHD limit, this is reasonable since in the present case there is no fast magnetosonic mode propagating perpendicularly to the magnetic field owing to the incompressible assumption. In single-fluid formalism using the generalized Ohm's law, this is also reasonable since Alfvén and whistler modes cannot propagate perpendicularly to $\mathbf{B}_0$.

In the ideal MHD limit, the ion inertial length $\lambda_i \to 0$. Thus, Eq. (54) can be written as

$$\frac{\omega^2}{k_p^2} = V_A^2 \cos^2\theta, \tag{55}$$

where θ is the angle between the unperturbed magnetic field and the direction of propagation vector $\mathbf{k}_p$. This is the same as the dispersion relation of the Alfvén mode.

Next, when $\omega^2 \gg \omega_i^2 \gg (k_\parallel V_A)^2$, Eq. (54) becomes

$$\frac{\omega^2}{k_p^2} = \frac{\omega B_0}{\mu_0 n_0 e}\cos\theta. \tag{56}$$

For $\omega_i \ll \omega \ll \omega_e$ and $\omega \ll \omega_{pe}$, where $\omega_e = eB_0/m_e$ and ω_{pe} is the electron plasma frequency, the well-known cold-plasma dispersion relation is simplified to give

$$n_r^2 = \frac{\omega_{pe}^2}{|\omega\omega_e\cos\theta|}, \tag{57}$$

where

$$n_r = \frac{|\mathbf{k}_p|c}{\omega} \tag{58}$$

is the index of refraction and c is the light speed. This is a simplified whistler mode dispersion relation, which was first used to explain the whistler mode propagation along the earth's magnetic field.[18] This dispersion relation does not depend on ion and electron masses and means that ions are immobile because of $\omega \gg \omega_i$ and electrons have electric drifts. It is obvious that Eqs. (56) and (57) agree. Therefore, for $\omega^2 \gg \omega_i^2 \gg (k_\parallel V_A)^2$, the whistler mode is included in the dispersion relation (54). Although massless electrons are assumed in obtaining Eq. (54), the electric drifts of electrons in the whistler wave electric field are properly described by the present single-fluid equations.

Finally, let us consider the propagation parallel to $\mathbf{B}_0$. In this case, the cold-plasma dispersion relation is obtained by calculating ion and electron contributions to $\tilde{\mathbf{J}}_{\perp 1}$ and is given by Eq. (3-45) of *Spitzer*,[19]

$$\frac{1}{V_A^2}\frac{\omega^2}{k_\parallel^2} = \left(1 \mp \frac{\omega}{\omega_i}\right)\left(1 \pm \frac{\omega}{\omega_e}\right), \tag{59}$$

where $\omega_i/\omega_e \ll 1$ is assumed. In the limit of massless electron ($m_e \to 0$), Eq. (59) becomes

$$\frac{1}{V_A^2}\frac{\omega^2}{k_\parallel^2} = 1 \mp \frac{\omega}{\omega_i}. \tag{60}$$

Equation (60) is further reduced to

$$(\omega^2 - k_\parallel^2 V_A^2)^2 = k_\parallel^4 \lambda_i^2 \omega^2 V_A^2. \tag{61}$$

This dispersion relation for parallel propagation becomes identical with Eq. (54) since $k_\parallel = k_p$ for parallel propagation. Thus, one finds that when the plasma and the magnetic field are uniform and when $\mathbf{k}_p\cdot\mathbf{B}_0 \neq 0$, the present single-fluid equations contain Alfvén and whistler modes.

VI. EIGENMODE EQUATION FOR SMALL ION INERTIA LENGTH LIMIT

The general eigenmode equation (42) for B_{1x}, which is a fourth-order differential equation, is too complicated to solve for practical applications. Therefore, in this section, a simplified eigenmode equation is obtained by taking a small ion inertial length limit. When the plasma and magnetic field are uniform and $\mathbf{k}\cdot\mathbf{B}_0 \neq 0$, this simplification corresponds to retaining only Alfvén mode and dropping whistler mode.

After dividing by $(kV_{AN})^2$, Eq. (33) can be written in the dimensionless form as

$$\left\{\left(\frac{\Omega}{kV_{AN}}\right)^2-\frac{1}{\bar{\rho}_0}(\hat{\mathbf{k}}\cdot\bar{\mathbf{B}}_0)^2+\frac{\lambda_{iN}}{L}\left[\frac{\hat{\mathbf{k}}\cdot\bar{\mathbf{B}}_0}{\bar{n}_0}\left(\hat{k}_y\frac{d\bar{V}_{0z}}{d\bar{x}}-\hat{k}_z\frac{d\bar{V}_{0y}}{d\bar{x}}\right)-\left(\frac{1}{\bar{n}_0}\frac{\Omega}{kV_{AN}}\left(\hat{k}_y\frac{d\bar{B}_{0z}}{d\bar{x}}-\hat{k}_z\frac{d\bar{B}_{0y}}{d\bar{x}}\right)+\frac{1}{\bar{n}_0^2}\frac{\Omega}{kV_{AN}}\frac{d\bar{n}_0}{d\bar{x}}(\hat{k}_y\bar{B}_{0z}-\hat{k}_z\bar{B}_{0y})\right)\right]\right.$$
$$\left.-\left(\frac{\lambda_{iN}}{L}\right)^2\frac{1}{2\bar{n}_0^3}\frac{d\bar{n}_0}{d\bar{x}}\frac{d\bar{p}_{i0}}{d\bar{x}}\right\}\bar{V}_{1x}+\frac{1}{\hat{\mathbf{k}}\cdot\bar{\mathbf{B}}_0}\frac{\Omega}{kV_{AN}}\left\{\left(\frac{\Omega}{kV_{AN}}\right)^2-\frac{1}{\bar{\rho}_0}(\hat{\mathbf{k}}\cdot\bar{\mathbf{B}}_0)^2+\frac{\lambda_{iN}}{L}\left[\frac{\hat{\mathbf{k}}\cdot\bar{\mathbf{B}}_0}{\bar{n}_0}\left(\hat{k}_y\frac{d\bar{V}_{0z}}{d\bar{x}}-\hat{k}_z\frac{d\bar{V}_{0y}}{d\bar{x}}\right)\right.\right.$$
$$-\left(\frac{1}{\bar{n}_0}\frac{\Omega}{kV_{AN}}\left(\hat{k}_y\frac{d\bar{B}_{0z}}{d\bar{x}}-\hat{k}_z\frac{d\bar{B}_{0y}}{d\bar{x}}\right)+\frac{1}{\bar{n}_0^2}\frac{\Omega}{kV_{AN}}\frac{d\bar{n}_0}{d\bar{x}}(\hat{k}_y\bar{B}_{0z}-\hat{k}_z\bar{B}_{0y})\right)\Bigg]+\left(\frac{\lambda_{iN}}{L}\right)^2\left[\frac{\hat{\mathbf{k}}\cdot\bar{\mathbf{B}}_0}{\bar{n}_0^3}\frac{d\bar{n}_0}{d\bar{x}}\left(\hat{k}_y\frac{d\bar{B}_{0y}}{d\bar{x}}+\hat{k}_z\frac{d\bar{B}_{0z}}{d\bar{x}}\right)\right.$$
$$\left.\left.-\frac{1}{\bar{n}_0^3}\frac{d\bar{n}_0}{d\bar{x}}(\hat{k}_y\bar{B}_{0z}-\hat{k}_z\bar{B}_{0y})\hat{\mathbf{k}}\cdot(\bar{\boldsymbol{\nabla}}\times\bar{\mathbf{B}}_0)\right]\right\}\bar{B}_{1x}-\frac{i}{2}\left(\frac{\lambda_{iN}}{L}\right)^2\frac{kL}{\bar{n}_0^3}\frac{d\bar{n}_0}{d\bar{x}}\frac{\Omega}{kV_{AN}}\bar{p}_{e1}$$
$$=\left(\frac{\lambda_{iN}}{L}\right)^2\frac{1}{\bar{n}_0^2}\frac{\Omega}{kV_{AN}}\left\{(kL)^2(\hat{\mathbf{k}}\cdot\bar{\mathbf{B}}_0)\bar{B}_{1x}+\left[\left(\frac{d^2}{d\bar{x}^2}(\hat{\mathbf{k}}\cdot\bar{\mathbf{B}}_0)\right)\bar{B}_{1x}-(\hat{\mathbf{k}}\cdot\bar{\mathbf{B}}_0)\frac{d^2\bar{B}_{1x}}{d\bar{x}^2}\right]\right\}, \tag{62}$$

where $\hat{\mathbf{k}}=\mathbf{k}/k$, $\hat{k}_y=k_y/k$, $\hat{k}_z=k_z/k$, $\bar{V}_{0y}=V_{0y}/V_{AN}$, and $\bar{V}_{0z}=V_{0z}/V_{AN}$. This equation contains $O(\epsilon^1)$, $O(\epsilon^2)$, and $O(\epsilon^3)$ terms.

In the following, one considers low-frequency waves, which satisfy $O(\omega/(kV_{AN}))\sim 1$. Since the characteristic ion inertia length λ_{iN} is smaller than L in tokamaks, $\epsilon=\lambda_{iN}/L$ is considered to be a small parameter. For Kelvin–Helmholtz instability driven by shear in the ion diamagnetic drift velocity, the stability analysis shows that $k_\perp L_p \lesssim 1$ for instability,[14] where L_p is the scale length of the pressure gradient. Since $k=\sqrt{k_\parallel^2+k_\perp^2}\simeq k_\perp$ for tokamaks and $L\le L_p$, one also assumes that kL is smaller than one. For fusion plasmas $\bar{p}_{i0}$ and $\bar{p}_{e1}$ are also orders of one or smaller.

One first writes Eq. (18) as

$$\bar{\mathbf{V}}_0=\bar{\mathbf{V}}_{00}+\epsilon\bar{\mathbf{V}}_{01}, \tag{63}$$

where

$$\bar{\mathbf{V}}_{00}=\bar{\mathbf{V}}_{0\parallel}+\frac{\bar{\mathbf{E}}_0\times\bar{\mathbf{B}}_0}{\bar{B}_0^2}, \tag{64}$$

$$\bar{\mathbf{V}}_{01}=-\frac{1}{2\bar{n}_0\bar{B}_0^2}\bar{\boldsymbol{\nabla}}\bar{p}_{i0}\times\bar{\mathbf{B}}_0. \tag{65}$$

Then, in order to obtain the lowest order approximation, one neglects $O(\epsilon^2)$ and $O(\epsilon^3)$ terms in Eq. (62) and obtains

$$(S_1+\epsilon S_2)\bar{V}_{1x}+[(S_3-\epsilon S_4)S_1+\epsilon S_2 S_3]\bar{B}_{1x}=0, \tag{66}$$

where

$$S_1=(\bar{\omega}-\hat{\mathbf{k}}\cdot\bar{\mathbf{V}}_{00})^2-\frac{1}{\bar{\rho}_0}(\hat{\mathbf{k}}\cdot\bar{\mathbf{B}}_0)^2, \tag{67}$$

$$S_2=-2(\bar{\omega}-\hat{\mathbf{k}}\cdot\bar{\mathbf{V}}_{00})\hat{\mathbf{k}}\cdot\bar{\mathbf{V}}_{01}+\frac{\hat{\mathbf{k}}\cdot\bar{\mathbf{B}}_0}{\bar{n}_0}\times\left(\hat{k}_y\frac{d\bar{V}_{00z}}{d\bar{x}}-\hat{k}_z\frac{d\bar{V}_{00y}}{d\bar{x}}\right)-\frac{(\bar{\omega}-\hat{\mathbf{k}}\cdot\bar{\mathbf{V}}_{00})}{\bar{n}_0}\times\left[\left(\hat{k}_y\frac{d\bar{B}_{0z}}{d\bar{x}}-\hat{k}_z\frac{d\bar{B}_{0y}}{d\bar{x}}\right)+\frac{1}{\bar{n}_0}\frac{d\bar{n}_0}{d\bar{x}}(\hat{k}_y\bar{B}_{0z}-\hat{k}_z\bar{B}_{0y})\right], \tag{68}$$

$$S_3=\frac{1}{\hat{\mathbf{k}}\cdot\bar{\mathbf{B}}_0}(\bar{\omega}-\hat{\mathbf{k}}\cdot\bar{\mathbf{V}}_{00}), \tag{69}$$

$$S_4=\frac{1}{\hat{\mathbf{k}}\cdot\bar{\mathbf{B}}_0}\hat{\mathbf{k}}\cdot\bar{\mathbf{V}}_{01}. \tag{70}$$

Here,

$$\bar{\omega}=\frac{\omega}{kV_{AN}}, \tag{71}$$

$$\bar{V}_{00y}=\frac{\bar{\mathbf{B}}_0\cdot\bar{\mathbf{V}}_0}{\bar{B}_0^2}\bar{B}_{0y}-\frac{\bar{E}_0\bar{B}_{0z}}{\bar{B}_0^2}, \tag{72}$$

$$\bar{V}_{00z}=\frac{\bar{\mathbf{B}}_0\cdot\bar{\mathbf{V}}_0}{\bar{B}_0^2}\bar{B}_{0z}+\frac{\bar{E}_0\bar{B}_{0y}}{\bar{B}_0^2}. \tag{73}$$

Since ϵ is a small parameter, one can, in principle, express $\bar{V}_{1x}$ and $\bar{B}_{1x}$ by the power series of ϵ. Therefore, $\bar{V}_{1x}$ can be written as $\bar{V}_{1x}=\Sigma_{n=0}^{\infty}\epsilon^n\bar{V}_{1xn}$. However, since only $O(\epsilon^0)$ and $O(\epsilon^1)$ terms are retained in Eq. (62) to obtain Eq. (66), it is enough to retain only terms up to ϵ^1 in the power series expansion. Thus, one can write $\bar{V}_{1x}$ and $\bar{B}_{1x}$ as

$$\bar{V}_{1x}=\bar{V}_{1x0}+\epsilon\bar{V}_{1x1}, \tag{74}$$

$$\bar{B}_{1x}=\bar{B}_{1x0}+\epsilon\bar{B}_{1x1}. \tag{75}$$

Substituting Eqs. (74) and (75) into Eq. (66), one obtains

$$(S_1+\epsilon S_2)(\bar{V}_{1x0}+\epsilon\bar{V}_{1x1})+[(S_3-\epsilon S_4)S_1+\epsilon S_2S_3]\times(\bar{B}_{1x0}+\epsilon\bar{B}_{1x1})=0. \quad (76)$$

Since Eq. (76) is a mathematical identity including parameter ϵ, all coefficients of powers of ϵ must vanish. Therefore, by equating the coefficient of ϵ^0 zero, one has

$$S_1(\bar{V}_{1x0}+S_3\bar{B}_{1x0})=0. \quad (77)$$

By equating the coefficient of ϵ^1 zero, one has

$$S_1(\bar{V}_{1x1}+S_3\bar{B}_{1x1}-S_4\bar{B}_{1x0})+S_2(\bar{V}_{1x0}+S_3\bar{B}_{1x0})=0. \quad (78)$$

In an ideal MHD plasma, $S_1=0$ is of special interest since it means that the wave phase velocity along the unperturbed magnetic field, which is calculated using the Doppler shifted frequency, is equal to the Alfvén speed. However, in a real plasma, such a condition, i.e., $S_1=0$, does not have a special significance since the Doppler shifted frequency is $\bar{\omega}-\hat{\mathbf{k}}\cdot\bar{\mathbf{V}}_0$ and not $\bar{\omega}-\hat{\mathbf{k}}\cdot\bar{\mathbf{V}}_{00}$. Therefore, the case of $S_1=0$ does not need special attention and one assumes $S_1\neq 0$. Then, from Eqs. (77) and (78) one has

$$\bar{V}_{1x0}+S_3\bar{B}_{1x0}=0 \quad (79)$$

and

$$\bar{V}_{1x1}+S_3\bar{B}_{1x1}-S_4\bar{B}_{1x0}=0. \quad (80)$$

By multiplying ϵ to Eq. (80) and then adding the resultant equation to Eq. (79), one obtains

$$\bar{V}_{1x}+(S_3-\epsilon S_4)\bar{B}_{1x}+\epsilon^2S_4\bar{B}_{1x1}=0. \quad (81)$$

If the last term proportional to ϵ^2 is neglected in this equation in accord with the omission of terms of $O(\epsilon^2)$ and higher orders in Eqs. (66), (74), and (75), one has

$$\bar{V}_{1x}=-\frac{\Omega}{\hat{\mathbf{k}}\cdot\bar{\mathbf{B}}_0kV_{AN}}\bar{B}_{1x}. \quad (82)$$

The denormalized form of Eq. (82) is

$$V_{1x}=-\frac{\Omega}{\mathbf{k}\cdot\mathbf{B}_0}B_{1x}. \quad (83)$$

This is a simplified relation between $V_{1x}(x)$ and $B_{1x}(x)$, when terms of order ϵ^2 and higher orders are neglected.

If one defines

$$h(x)=\frac{B_{1x}(x)}{\mathbf{k}\cdot\mathbf{B}_0}=-\frac{V_{1x}(x)}{\Omega}, \quad (84)$$

substitution of Eq. (83) into Eq. (24) yields

$$\frac{d}{dx}\left\{\rho_0\left[\Omega^2-\frac{(\mathbf{k}\cdot\mathbf{B}_0)^2}{\rho_0\mu_0}\right]\frac{dh}{dx}\right\}-\rho_0k^2\left[\Omega^2-\frac{(\mathbf{k}\cdot\mathbf{B}_0)^2}{\rho_0\mu_0}\right]h=0. \quad (85)$$

This is the eigenmode equation for $B_{1x}(x)$ when terms of order ϵ^2 and ϵ^3 are neglected in Eq. (62). When $\mathbf{k}\cdot\mathbf{B}_0=0$, Eq. (85) becomes identical with Eq. (25). The eigenmode equation (85) is the same as the eigenmode equation for ideal incompressible MHD (Ref. 8) except for the fact that $\mathbf{V}_{0\perp}$ in Eq. (85) includes the $\mathbf{E}\times\mathbf{B}$ drift velocity and the ion diamagnetic drift velocity, which is absent in ideal MHD.

For a uniform equilibrium without a flow, substitution of Eq. (52) into Eq. (85) yields

$$k_p^2(\omega^2-k_\parallel^2V_{AN}^2)=0. \quad (86)$$

Therefore, unlike Eq. (54) the whistler mode dispersion (56) is not included in Eq. (86) owing to the neglect of terms of order ϵ^2 and ϵ^3 in Eq. (62).

VII. SUFFICIENT CONDITION FOR STABILITY

From Eq. (85) one can derive a simple quadratic form to study stability following the same procedure as used previously.[8] Multiplying Eq. (85) by h^*, where the asterisk denotes the complex conjugate, and then by operating $\int_{-\infty}^{\infty}dx$, one obtains

$$\int_{-\infty}^{\infty}\rho_0\left[\Omega^2-\frac{(\mathbf{k}\cdot\mathbf{B}_0)^2}{\rho_0\mu_0}\right]\left|\frac{dh}{dx}\right|^2dx+\int_{-\infty}^{\infty}\rho_0k^2\left[\Omega^2-\frac{(\mathbf{k}\cdot\mathbf{B}_0)^2}{\rho_0\mu_0}\right]|h|^2dx=0, \quad (87)$$

where one assumed that $h(\pm\infty)=0$. This equation can be reduced to

$$A\omega^2+2B\omega+C=0, \quad (88)$$

where

$$A=\int_{-\infty}^{\infty}\rho_0|\phi|^2dx>0, \quad (89)$$

$$B=-\int_{-\infty}^{\infty}\rho_0(\mathbf{k}\cdot\mathbf{V}_0)|\phi|^2dx, \quad (90)$$

$$C=\int_{-\infty}^{\infty}\left[\rho_0(\mathbf{k}\cdot\mathbf{V}_0)^2-\frac{(\mathbf{k}\cdot\mathbf{B}_0)^2}{\mu_0}\right]|\phi|^2dx, \quad (91)$$

and

$$|\phi|^2\equiv\left|\frac{dh}{dx}\right|^2+k^2|h|^2. \quad (92)$$

From Eq. (88) one obtains

$$\omega=\frac{-B\pm\sqrt{B^2-AC}}{A}. \quad (93)$$

Since B^2 and A are positive definite, a sufficient condition for stability becomes

$$C\leq 0. \quad (94)$$

This condition is derived under the neglect of $O(\epsilon^2)$ and $O(\epsilon^3)$ terms in Eq. (62) and is the same as the sufficient stability condition for ideal incompressible MHD (Ref. 8) except for the fact that $\mathbf{V}_0$ in Eq. (91) includes the ion diamagnetic drift velocity, which is a finite ion inertial length correction to the ideal MHD flow velocity [see Eq. (63)].

The sufficient stability condition (94) was not obtained from a variational principle such as the energy principle,[1] which states that a plasma equilibrium is stable if and only if the minimum of the potential energy of ideal MHD is positive or equal to zero. Therefore, Eq. (94) is different from stability criteria used in fusion[1,2] and magnetospheric[3,4] plasmas, which are obtained from the energy principle and expressed by using unperturbed quantities. That is, the coefficients A, B, and C in Eq. (88) contain an unknown function $\phi(x)$. Thus, a sufficient condition for stability (94) cannot be evaluated until the unknown function $\phi(x)$ is determined. However, in a special case condition (94) provides a useful sufficient condition for stability, which is expressed by using unperturbed quantities only. That is, when

$$[\rho_0(\mathbf{k}\cdot\mathbf{V}_0)^2]_{\max} \le \left[\frac{(\mathbf{k}\cdot\mathbf{B}_0)^2}{\mu_0}\right]_{\min}, \tag{95}$$

where the subscripts max and min denote the maximum and minimum values of quantities in the brackets in the region for integration, respectively, condition (94) is satisfied because of $[\rho_0(\mathbf{k}\cdot\mathbf{V}_0)^2-\mu_0^{-1}(\mathbf{k}\cdot\mathbf{B}_0)^2]\le 0$ for all x. Since Eq. (95) contains only unperturbed quantities, it is not necessary to know the unknown function $\phi(x)$ in evaluating Eq. (95). Thus, Eq. (95) is a useful sufficient condition for stability.

As can be seen from Eq. (95), $\mathbf{k}\cdot\mathbf{B}_0$ must be nonzero and large in order to satisfy this condition. This suggests that a sufficient amount of magnetic shear, which makes $(\mathbf{k}\cdot\mathbf{B}_0)^2$ large, can suppress a magnetic fluctuation excited by Kelvin–Helmholtz instability driven by shear in the ion diamagnetic drift velocity. However, since the $(\mathbf{k}\cdot\mathbf{B}_0)^2$ term in Eq. (91) represents a line bending term, the field line bending, which occurs when $\mathbf{k}\cdot\mathbf{B}_0\neq 0$, contributes to stabilization with the stabilizing magnetic tension force and the magnetic shear itself is not responsible for the stabilization. Therefore, a large magnetic field without shear may well also stabilize this fluid instability.

In a real plasma, the compressibility may also contribute to stabilization. However, since incompressibility is assumed in the present analysis, there is no condition related to the compressibility. Even if the compressibility is taken into account, the above sufficient condition for stability (95) would remain important since tokamaks have a typical β value of several percent and the magnetic energy dominates the internal energy.

The lowest order approximation used in deriving Eq. (94) means that although higher-order corrections involving whistler mode components are neglected, the effect of the finite ion inertial length scale is accurately taken into account in calculating $\mathbf{V}_0$ [see Eq. (63)] and hence in obtaining a sufficient condition for stability (94).

VIII. DISCUSSION AND SUMMARY

For a flute mode satisfying $\mathbf{k}\cdot\mathbf{B}_0=0$, the existence of the Kelvin–Helmholtz instability driven by shear in the ion diamagnetic drift velocity has been proven when there is no unperturbed electric field[14] and when there is an unperturbed electric field yielding an unperturbed $\mathbf{E}\times\mathbf{B}$ drift velocity.[20] An unstable flute mode $(\mathbf{k}\cdot\mathbf{B}_0=0)$ found by solving compressible Hall MHD equations for a one-dimensional current sheet configuration with $\mathbf{E}_0=0$ (Ref. 21) is also essentially driven by shear in the ion diamagnetic drift velocity, although no mention of the shear in the ion diamagnetic drift velocity was made. Since a shear in the ion diamagnetic drift velocity in a nonuniform pressure plasma is essential in this instability, this instability cannot be found by a stability analysis for a uniform plasma,[22] even if ion inertial effects are taken into account.

For a flute mode satisfying $\mathbf{k}\cdot\mathbf{B}_0=0$, the stability of finite Larmor radius hydrodynamics was also studied.[23] However, since an electrostatic perturbation was assumed, instability driven by shear in the ion diamagnetic drift velocity[14] could not be obtained. This instability driven by shear in the ion diamagnetic drift velocity is essentially fluidlike and not driven by an inverse Landau damping. Therefore, this instability is different from drift instability and is considered to be more universal.

In tokamaks, there is a magnetic shear and therefore, it is impossible to have $\mathbf{k}\cdot\mathbf{B}_0=0$ everywhere. Thus, the flute mode stability is not relevant to tokamaks and the stability condition (95) is necessary to seek tokamak parameters ensuing stability. Let us assume that there is no parallel flow and $\mathbf{V}_0=\mathbf{V}_{0\perp}$. Since $\mathbf{k}_\perp\cdot\mathbf{V}_{0\perp}$ becomes a maximum when $\mathbf{k}_\perp$ is parallel to $\mathbf{V}_{0\perp}$, the sufficient condition for stability (95) becomes

$$[\rho_0(k_\perp V_{0\perp})^2]_{\max} \le [\mu_0^{-1}(k_\parallel B_0)^2]_{\min}. \tag{96}$$

Therefore, one obtains for stability

$$\rho_{0\,\max}(k_{\perp\max}V_{0\perp\max})^2 \le \mu_0^{-1}(k_{\parallel\min}B_{0\,\min})^2, \tag{97}$$

where subscripts max and min in this equation and following equations represent those values at radial positions, where $\rho_0(k_\perp V_{0\perp})^2$ is maximized and $(k_\parallel B_0)^2$ is minimized, respectively.

From Eq. (97) one obtains for stability

$$M_A \equiv \frac{V_{0\perp\max}}{V_{A\,\min}} \le \sqrt{\frac{\rho_{0\,\min}}{\rho_{0\,\max}}}\frac{k_{\parallel\min}}{k_{\perp\max}}, \tag{98}$$

where $V_{A\,\min}=B_{0\,\min}/\sqrt{\mu_0\rho_{0\,\min}}$.

Inequality (97) gives a sufficient condition for stability. By using ion beta $\beta_{i\,\max}$ defined by $2\mu_0 p_{i0}/B_{0\,\max}^2$, the scale length of the pressure gradient L_p, and the ion inertial scale length $\lambda_{i\,\max}$ defined by $\sqrt{m_i/(n_{0\,\max}\mu_0 e^2)}$, Eq. (97) can be expressed as

$$\beta_{i\,\max} \le \frac{2L_p}{\lambda_{i\,\max}}\frac{B_{0\,\min}}{B_{0\,\max}}\frac{L_p k_{\parallel\min}}{L_p k_{\perp\max}}, \tag{99}$$

where one assumed that the unperturbed electric field is zero and used $|\mathbf{V}_{0\perp}|=|\nabla p_{i0}\times\mathbf{B}_0/(n_0 e B_0^2)|\approx p_{i0}/(L_p n_0 e B_0)$. The stability analysis of the Kelvin–Helmholtz instability driven by shear in the ion diamagnetic drift velocity for $\mathbf{k}\cdot\mathbf{B}_0=0$ shows that $2ak_{\perp\max}\lesssim 1$ for instability, where $2a$ is the width of the shear of the ion diamagnetic drift velocity. Therefore, by assuming $L_p\approx 2a$ the most pessimistic form of Eq. (99) becomes

$$\beta_{i\,\max} \le \frac{2L_p}{\lambda_{i\,\max}} \frac{B_{0\,\min}}{B_{0\,\max}} L_p k_{\parallel\min} \tag{100}$$

for stability. Here, $k_{\parallel\min}=2\pi/\lambda_{\parallel\min}$ and $\lambda_{\parallel\min}$ is the parallel wavelength. Since $\lambda_{\parallel\min}$ is the periodicity length in the parallel direction, the most pessimistic limit of Eq. (100) can be obtained by assuming $\lambda_{\parallel\min}=\ell_{\parallel\min}$, where $\ell_{\parallel\min}$ is the length of the field line passing through the point in the radial direction, where $(k_\parallel B_0)^2$ is minimum. That is,

$$\beta_{i\,\max} \le \frac{4\pi L_p^2}{\lambda_{i\,\max}\ell_{\parallel\min}} \frac{B_{0\,\min}}{B_{0\,\max}} \tag{101}$$

for stability.

Although this is a very crude evaluation of the sufficient condition for stability, Eq. (101) may give a reasonable parametric dependence of a maximum achievable $\beta_{i\,\max}$ in order to satisfy the stability condition for the Kelvin–Helmholtz instability driven by shear in the ion diamagnetic drift velocity in tokamaks. Condition (101) contains an ion inertial scale length $\lambda_{i\,\max}$, which is zero in the ideal MHD limit. Condition (101) shows that when $\ell_{\parallel\min}$ is smaller, the sheared ion diamagnetic flow is more stable since the stabilizing magnetic tension force is larger. In tokamaks, when toroidal magnetic geometries have rational surfaces, the field line is closed and the field line length is a function of the plasma safety factor q. When toroidal magnetic geometries have ergodic surfaces, $\ell_{\parallel\min}$ is considered to be infinite. When q is an integer, $\ell_{\parallel\min}$ is nearly proportional to q. Therefore, in such a case, the sheared ion diamagnetic flow is more unstable for larger q. Although this seems somewhat unphysical, this is reasonable on physical grounds since if the field line length is larger, the stabilizing magnetic tension force is smaller. For erdgodic surfaces, $\ell_{\parallel\min}$ is considered to be infinite and therefore, there is no stabilizing tension force and the instability cannot be stabilized by the tension force. Condition (101) also shows a strong dependence of the stability on L_p and that for larger L_p the sheared ion diamagnetic drift is more stable.

The above observation suggests that even for ideal MHD stable tokamaks there is a small scale magnetic fluctuation in the vicinity of the ion pressure gradient, which is destabilized by the Kelvin–Helmholtz instability driven by shear in the ion diamagnetic drift velocity.[14] Since the ion pressure gradient is mainly in the radial direction, this magnetic fluctuation has an eigenmode structure, which is peaked near the ion pressure gradient and decays radially on both sides. The magnetic fluctuation is elongated along the field line. When the toroidal magnetic field is much stronger than the poloidal field, this is elongated almost in the toroidal direction. For such a configuration the ion diamagnetic drift is mainly a poloidal flow and the magnetic fluctuations are periodic in the poloidal direction with $L_p k_\perp \approx 1$ according to the stability analysis for $\mathbf{k}\cdot\mathbf{B}_0=0$.[14] Therefore, the periodic length in the poloidal direction is nearly equal to $\lambda \approx 2\pi L_p$. This magnetic fluctuation is of fluid origin (not destabilized by an inverse Landau damping) and can be completely stabilized by stabilizing magnetic tension force when the stability condition (101) is satisfied.

In summary, a fourth-order general eigenmode equation for a magnetic field perturbation has been derived for an arbitrary plasma equilibrium in a slab geometry in the framework of incompressible single-fluid equations with a simplified generalized Ohm's law. For a uniform plasma the eigenmode equation for $\mathbf{k}\cdot\mathbf{B_0}\neq 0$ gives dispersion relations for Alfvén and whistler modes. For a special case of $\mathbf{k}\cdot\mathbf{B}_0=0$, the eigenmode equation is reduced to a second-order differential equation, which was derived previously.[14] For a general case of $\mathbf{k}\cdot\mathbf{B_0}\neq 0$, the ratio of the characteristic ion inertia length to the shortest characteristic scale length of the inhomogeneity is chosen as a small parameter for expansion. By retaining the finite ion inertial length correction in the unperturbed flow velocity but neglecting higher-order whistler mode components, a simple quadratic form is obtained to investigate the general stability. A sufficient condition for stability against the Kelvin–Helmholtz instability driven by shear in the ion diamagnetic drift velocity is obtained from this quadratic form. Although the application of the present results is restricted to an equilibrium in a slab geometry, the obtained sufficient condition for stability provides a useful method to investigate the single-fluid stability of a general stationary plasma equilibrium with velocity shear and magnetic shear.

ACKNOWLEDGMENTS

This work was supported by Grant-in-Aid for Scientific Research under Grant No. 20540436.

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