Dissertation

On a Coupled SPH-Rigid Body Method for the Surfing Problem

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Date of Submission : 29 June 2018

KANAZAWA UNIVERSITY

Abstract

Division of Mathematical and Physical Science
Graduate School of Natural Science and Technology

PhD

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by Reza Rendian Septiawan

In this work, we use the smoothed particle hydrodynamics (SPH) method coupled with a rigid body simulation to study the surfing problem. We simulate a surfing board on top of an ocean wave which moves at a constant velocity. A fluid-rigid body coupling is handled by using pure hydrodynamics-based force. External forces are applied to the board, representing a surfer trying to stabilize the board at a desired point along the uphill part of the ocean wave. An ordinary differential equation (ODE) control is used to manipulate the distribution of the external forces based on a position, velocity, and an inclination angle of the surfing board relative to the ocean wave. The control system successfully helps the surfing board to move and maintain its position at the desired point.

"So which of the favours of your Lord would you deny?"

(Quran 55:13)

Acknowledgements

First, the author would like to thank his Lord, Allah, for all of His blessings and favors.

The author would like to also thank all of his family members, for all of their love, support, and understanding. They help the author maintains his sanity all of the time.

The author would like to thank his Ph.D supervisor Professor Seiro Omata and Doctor Norbert Pozar for their invaluable supports, advices, and moral supports. This work cannot be done without their help.

Next, the author would also like to thank to all of his friends, in campus and also outside campus, for all of their support, in academic life and also regular daily life.

The author would like to thank to Ministry of Education, Culture, Sports, Science, and Technology (MEXT) Japan for their financial support through MEXT U2U Scholarship.

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Dedicated to my beloved Mom and Dad. And for my whole family and friends for their love and support and understanding. Because family and friends are always on the top priority.

Chapter 1

Introduction

Surfing on ocean waves poses some interesting mathematical and physics problems. One of them is a modeling of a surfer that controls the movement of the surfing board. The goal of what we called the surfing problem is to maintain the position of the surfing board on the upslope part of the ocean wave as long as possible. To reach the goal, the surfer maneuvers the surfing board by adjusting the distribution of forces given to the board via their feet in an attempt to modify the inclination angle of the surfing board.

To study the surfing problem, we can limit our domain frame to a small area of the uphill part of the ocean wave. This frame moves together with the wave, assuming the ocean wave moves with a constant velocity. In this dissertation we are developing an ordinary differential equation (ODE) control of the inclination angle of the surfing board to move the surfing board toward the desired point with respect to the wave. The ODE control that we propose is taking in the account of the position, velocity and the inclination angle of the surfing board. The output inclination angle from the ODE control acts as a "target angle" for the surfing board. To reach the target angle we mimic the surfer's attempt to control the board by giving two forces at the tips of the board. To find suitable parameters for which the control system is stable, we perform a stability analysis of a linearized simplified one-dimensional ODE model of the surfer on an ocean wave.

To verify the capabilities of the ODE control in controlling the movement of the rigid body under interaction with the fluid, we simulate the full system using the smoothed particle hydrodynamics (SPH) method. SPH is one of the most popular

Lagrangian solvers of the fluid equations. It was introduced by Lucy [1], and Gingold and Monaghan [2]. SPH method treats a system as a set of material points. Each material point brings physical informations and interacts with each others. In one of the original papers of SPH, Gingold and Monaghan [2] said that to recover the density from the known distribution of points, they need to recover a probability distribution from a sample, provided that the points are randomly distributed by their assumption. Because of its close relation with statistical ideas, therefore they described SPH as one of the Monte-Carlo methods. Later, in their next paper [3], they discussed that the motion of points is not sufficiently complex, therefore, the points cannot be considered as randomly distributed points through the whole domain.

Even though the SPH method is invented to approximate the numerical solution for the equations of fluid dynamics, at first the SPH method is developed to solve astronomical problems, such as a binary fission [1][3][4]. They modeled the motion of astronomical objects by the fluid dynamics equations with a dissipation term designed to stabilize the model before the system undergo the fission sequence. Since then, the SPH method has been widely used in solving fluid dynamics problems, such as shock wave problems. In [5], Gingold and Monaghan introduced a dissipative term designed to work with the SPH method since an artificial viscosity for finite difference scheme introduced in [6] produces excessive oscillation and smearing of the shock front due to the scale separation of points. Later they improved the artificial viscosity term to work with a subsonic flow [7]. Further, they use the solution from the Riemann problem to improve the artificial viscosity term for the SPH method in [8].

Another interesting fluid dynamics problems that can be solved by using the SPH method are heat transfer problems. In [9] they successfully designed the heat transfer model for the SPH method that can handle various problems of a heat conduction, such as the heat conduction between points with different properties (which implies points from different materials), a large discontinuity in a thermal conductivity, a temperature-dependent thermal conductivity, and the most important is the heat conduction with large deformation on material boundaries, which is the main advantage of the SPH method compared to other methods such as the finite difference method and the finite element method. Another heat transfer problem which is interesting is the solidification process [10]. In that work they improved the SPH method to deal with a phase-change phenomenon effectively, starting from the Stefan problem associated with a fluid with a single composition, up to a fluid with multiple compositions, in their case, a sodium nitrate solution, and compared it with an experimental data. The result from the SPH method is in a good agreement with the experimental result.

Modeling the surface tension and the contact angle in a multi-phase simulation is also an interesting problem to be solved by using the SPH method. In [11] they proposed a combination of point-point interactions which incorporates a short-range repulsive interaction and a long-range attractive interaction to model a surface tension into the SPH method.

Numerous technical improvements to make the computation more efficient for the SPH method have been developed as well. One of them is by using a parallel computation. Parallelization in a numerical scheme is usually done by assigning different computation workers to solve the problem in fixed spatial subdomains while minimizing communication between workers, which is usually called a static load-balancing method. But since the SPH method is a Lagrangian solver, SPH points can move freely through the whole domain, causing non-uniform computational load spatially, rendering a static load-balancing method inefficient. In [12], the dynamic load-balancing method for MPI protocol is used to improve the efficiency of parallel computing for the SPH method.

The computational capabilities of graphics processing unit (GPU) had a dramatic improvement in last several years, leads to a more common GPU utilization in many computational research fields, including fluid dynamics as well. Many trials have been done on implementing the SPH method into a GPU, but one of the difficulties comes from the neighbor searching algorithm which cannot be implemented in a trivial way on a GPU. One of the success studies of the SPH-GPU implementation is a massive parallelization with a GPU done in [13]. They succesfully run the SPH simulation fully in a GPU, efficiently prevent a data transfer bottlenecking problem when one part of the simulation needs to be done on a CPU and the other part is run on a GPU.

Another interesting topic of research that can be solved by the SPH method is the fluid-rigid body interaction. By the nature of the SPH method, boundary interaction between fluid and rigid body can be handled easily. Numerous studies on an implementation of the SPH method to model the fluid-rigid body interaction have been done by many researchers from different fields of study. One of the studies related to the simulation of the fluid-rigid body interaction by using the SPH method is [14]. They studied the impact of a rigid body when it hits the body of water, by using both laboratory experiments and numerical simulations. They modeled the interaction between the rigid body and the fluid by discretizing the rigid body into a set of boundary points that can interact with other SPH fluid points. The interaction between both types of points is done by exchanging momentum via a Lennard-Jones potential repulsive boundary force, which is called the Monaghan boundary force (MBF). The MBF is designed to prevent penetration of fluid points with a maximum velocity into a rigid body. However, the repulsive force from the MBF is thought to be too strong for a slow moving points. In [15] an impulse-based boundary force (IBF) was proposed. Recently, a more robust IBF is introduced in [16] by using a sequential impulse to solve the frictional contact problem with many contact points.

The IBF method relies heavily on the normal of the surface of rigid bodies. Hence, for a complex-shaped rigid body, the calculation is difficult. Moreover, IBF does not use pure hydrodynamics-based forces. A more versatile fluid-rigid body interaction was introduced in [17]. By the inclusion of boundary points in the density calculation, they also solved a neighbor deficiency problem near the boundary which is a common problem in the SPH method.

1.1 Problem Statement

In this dissertation we want to study a surfing problem. In what we call a surfing problem, the goal is to maintain the surfing board to stay on top of the upslope part of the ocean wave as long as possible. In an attempt to achieve this goal, the surfer needs to control the movement of the surfing board by utilizing his/her own body weight, balancing the net force between the drag force and the gravity. Here we propose an ODE control that can help the surfing board maintains its position at a desired point on the uphill part of the ocean wave.

The author uses SPH as his method of choice to simulate the surfing problem since the SPH method is capable to handle the free surface fluid motion. This problem is interesting since it combines two things at once; modeling of the ocean wave by a coupled fluid-rigid body simulation by using the SPH method and the ODE control to help the surfing board maintains its position.

Simulating the ocean wave with the surfing board is one of the main challenges in this study, since there is a large scale difference between both the ocean wave and the board. By choosing a right frame of reference, we can take just enough part of the ocean that we need to study our case. Another main challenge here is in the process of designing the ODE control that could incorporate all parameters that affect the motion of the surfing board, while still make it simple enough to be analysed easily. The result of the work in this dissertation is published in [18].

1.2 Goals

The goals of this study are:

1. Propose an ODE control that can control and maintain the position of the surfing board at a given desired point.
2. Design a coupled fluid-rigid body simulation using the SPH method that can model the movement of surfing board on top of an ocean wave.

1.3 Outline of Dissertation

This dissertation consists of 5 chapters that are briefed as follows.

Chapter 1 tells a description of the surfing problem and a brief history about the development of the SPH method over years since its invention. In this chapter the author cites several articles which give fundamental impacts to the SPH method overall and also related to this dissertation. The author also writes the main problem and the objectives of this study. This chapter is ended by an outline of the whole dissertation.

Chapter 2 explains about equations govern the problem stated in this dissertation. Starting from a set of governing equations for the hydrodynamics, then continued to the rigid body dynamics, and ended with a linear analysis of the ODE control.

Chapter 3 describes a fundamental theory of the SPH method. This chapter provides an SPH approximation for fluid dynamics, the implementation of rigid body simulation into SPH method, and the algorithm of the simulation.

Chapter 4 presents the setup of the simulation and the results of the simulation. In this chapter we will discuss about the results that we ge from the simulation as well.

Chapter 5 finally presents the conclusion of this work and a plan for the future works which have not yet been achieved through this work.

Chapter 2

Governing Equations

2.1 Fluid Motion

2.1.1 Substantial Derivative

Let us have $\Omega_0 \subset \mathbb{R}^3$ is a domain filled with a fluid at time $t = 0$. Next, consider that we have a material point of fluid is located at $x_0 \in \Omega_0$ at time $t = 0$. The position of that material point at a given time t is $X^{x_0}(t)$, with $X^{x_0}(0) = x_0$, and its velocity is given by $U^{x_0}(t) = \frac{d}{dt}X^{x_0}(t)$. Since material points move, the shape of the domain changes with time. Let Ω_t be the domain at a given time t. With a given $x \in \Omega_t$, the velocity of a material point located at $x = X^{x_0}(t)$ at a given time t is defined as $u(x,t) = \frac{d}{dt}X^{x_0}(t)$. $u(x,t)$ for $x \in \Omega_t$ is so-called velocity field of the fluid.

All the above physical quantities notated using small alphabets are in an Eulerian description, which is fixed in space, while physical quantities notated with capital alphabets are in a Lagrangian description, with their information is moving with the material point. The relationship between velocity u in an Eulerian description with a material point velocity U from a Lagrangian description at a given time t can be written as $U^{x_0}(t) = u(X^{x_0}(t), t)$. This relation also holds for other physical quantities that can be described in both descriptions, such as density.

To calculate the time derivative of a physical quantity $F^{x_0}(t)$ for a given material point x_0 (here, notation x_0 is also used to label the material point, which material point is located at $x_0 \in \Omega_0$ at time $t = 0$), we need its "counterpart" field function

$f(x,t)$ in an Eulerian description. Let $x = X^{x_0}(t) = (X_1^{x_0}(t), X_2^{x_0}(t), X_3^{x_0}(t)) \in \Omega_t$ for a given material point x_0 at time t, the velocity field is $u(x,t) = u(X^{x_0}(t), t) = u(X_1^{x_0}(t), X_2^{x_0}(t), X_3^{x_0}(t), t) = \frac{d}{dt}(X_1^{x_0}(t), X_2^{x_0}(t), X_3^{x_0}(t))$. The time derivative of $F^{x_0}(t)$ is

$$\begin{aligned}
\frac{d}{dt}F^{x_0}(t) &= \frac{df}{dt}(X^{x_0}(t), t) \\
&= \frac{df}{dt}(X_1^{x_0}(t), X_2^{x_0}(t), X_3^{x_0}(t), t) \\
&= \frac{\partial f}{\partial t}(X^{x_0}(t), t) + \sum_{i=1}^{3} \frac{\partial f}{\partial x_i}(X^{x_0}(t), t)\frac{dX_i^{x_0}(t)}{dt} \\
&= \frac{\partial f}{\partial t}(X^{x_0}(t), t) + u(X^{x_0}(t), t) \cdot \nabla f(X^{x_0}(t), t) \\
&= \left(\frac{\partial f}{\partial t} + u \cdot \nabla f\right)(X^{x_0}(t), t) \\
&=: \frac{Df}{Dt}(X^{x_0}(t), t).
\end{aligned}$$

The

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \cdot \nabla \tag{2.1}$$

operator is called the substantial derivative (or sometimes, material derivative) operator.

2.1.2 Transport Theorem

Before we move to conservation laws, it is useful to introduce density functions for any measurable physical quantities, which means we can assign a value $\psi(V_t, t)$ to any given sufficiently nice subdomain $V_t \subset \Omega_t$ at any given time $t \in [0, \infty)$, there exists a density function $f = f(x,t)$ such that

$$\psi(V_t, t) = \int_{V_t} f(x,t)\ dx\ .$$

Let us define a function $\phi(x_0, t) := X^{x_0}(t)$ as a function $\phi : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3$ is smooth with a non-zero Jacobian matrix [1] $D_{x_0}\phi(x_0, t) \neq 0, \forall x_0 \in \mathbb{R}^3, t \in [0, \infty)$ so

[1]Capital D here represents a Jacobian matrix operator; neither related with substantial derivative nor with any physical units in a Lagrangian description.

the material points will not have an infinite density; and a function $\phi_t(x_0) : \mathbb{R}^3 \to \mathbb{R}^3$ defined as $\phi_t(x_0) := \phi(x_0, t) = X^{x_0}(t)$ is invertible $\forall t \in [0, \infty)$ so that no more than one material point occupies the same position in space. A function ϕ is a flow map of the fluid which captures the kinematic evolution of the fluid. Above conditions guarantee that we have a nice regular flow, allow us to do the change of variables formula, differentiate, and integrate by parts.

Now let us consider a moving domain $\{V_t\}_{t\in[0,\infty)}$ which contains same material points for all time t, that is

$$X^{x_0}(\tau) \in V_\tau \text{ for some } x_0 \in \mathbb{R}^3, \tau \in [0, \infty) \Rightarrow X^{x_0}(t) \in V_t \text{ for all } t \in [0, \infty)\,,$$

which implies

$$V_t = \{X^{x_0}(t) : x_0 \in V_0\} = \{\phi_t(x_0) : x_0 \in V_0\} =: \phi_t(V_0)\ .$$

Next let us introduce the Jacobian $J(x_0, t)$ to be the determinant of the Jacobian matrix of the function ϕ_t at (x_0),

$$J(x_0, t) = \det\left(D\phi_t(x_0)\right) \quad x_0 \in \mathbb{R}^3, t \in [0, \infty)\,. \tag{2.2}$$

Since material points cannot overlap with each others, the "orientation" of the fluid remains the same and will not be flipped, hence, the Jacobian is non-negative. And since the Jacobian is also non-zero, so $J(x_0, t) = \det(D\phi_t(x_0)) > 0$.

From the formula for the derivative of a determinant of an invertible matrix,

$$\frac{d}{dt}\det(A(t)) = \det(A(t))\ \mathrm{tr}\left(A^{-1}(t)\frac{dA}{dt}(t)\right),$$

we get

$$\frac{\partial J}{\partial t}(x_0, t) = J(x_0, t)\ \mathrm{tr}\left((D\phi_t(x_0))^{-1}\ \frac{\partial D\phi_t}{\partial t}(x_0)\right). \tag{2.3}$$

Let us rewrite a velocity field $u(x, t)$ in term of $\phi(x_0, t)$ as

$$u(\phi(x_0, t), t) = \frac{\partial \phi}{\partial t}(x_0, t),$$

and differentiate the i-th component of both sides of the equation with respect to $x_{0,j}$ (let $x_0 = (x_{0,1}, x_{0,2}, x_{0,3})$ and $\phi = (\phi_1, \phi_2, \phi_3)$) as

$$\begin{aligned} \frac{\partial u_i}{\partial x_{0,j}}(\phi(x_0,t),t) &= \frac{\partial}{\partial t}\frac{\partial \phi_i}{\partial x_{0,j}}(x_0,t) \\ \sum_k \frac{\partial u_i}{\partial x_k}(\phi(x_0,t),t)\frac{\partial \phi_k}{\partial x_{0,j}}(x_0,t) &= \frac{\partial}{\partial t}\frac{\partial \phi_i}{\partial x_{0,j}}(x_0,t), \end{aligned}$$

which can be written as

$$D_x u(\phi(x_0,t),t) D\phi_t(x_0) = \frac{\partial}{\partial t} D\phi_t(x_0).$$

Since $D\phi_t(x_0)$ is invertible, we can multiply both sides of the equation by $(D\phi_t(x_0))^{-1}$, take the trace, and use the property $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, we get

$$\begin{aligned} \mathrm{tr}\left(D_x u(\phi(x_0,t),t)\right) &= \mathrm{tr}\left(\left(\frac{\partial}{\partial t} D\phi_t(x_0)\right)(D\phi_t(x_0))^{-1}\right) \\ &= \mathrm{tr}\left((D\phi_t(x_0))^{-1}\left(\frac{\partial}{\partial t} D\phi_t(x_0)\right)\right) . \end{aligned}$$

The left-hand side of the equation is $\mathrm{tr}\left(D_x u(\phi(x_0,t),t)\right) = \mathrm{div}(u(\phi(x_0,t),t))$. Substituting it back into equation (2.3) gives us

$$\frac{\partial J}{\partial t}(x_0,t) = J(x_0,t)\ \mathrm{div}(u(\phi(x_0,t),t)). \tag{2.4}$$

Now we want to express the time derivative of $\psi(V_t,t)$ in terms of the derivative of its density function f. By the change of variables, we get

$$\frac{d}{dt}\int_{V_t} f(x,t)\ dx = \frac{d}{dt}\int_{V_0} f(\phi(x_0,t),t)\ J(x_0,t)\ dx_0$$

Since now the right-hand side is an integral over a domain which is independent of t, we can exchange integration and differentiation and we get

$$\frac{d}{dt}\int_{V_t} f(x,t)\ dx = \int_{V_0} \frac{\partial}{\partial t}\left(f(\phi(x_0,t),t)\ J(x_0,t)\right)\ dx_0. \tag{2.5}$$

By the product rule, the chain rule, and substitution from (2.4), we have

$$\begin{aligned}\frac{\partial}{\partial t}\left(f(\phi(x_0,t),t)\ J(x_0,t)\right) &= f(\phi(x_0,t),t)\ \frac{\partial J}{\partial t}(x_0,t) + J(x_0,t)\left(\frac{\partial f}{\partial t} + u\cdot\nabla f\right)(\phi(x_0,t),t) \\ &= f(\phi(x_0,t),t)\left(J(x_0,t)\ \mathrm{div}(u(\phi(x,t),t))\right) \\ &\quad + J(x_0,t)\left(\frac{\partial f}{\partial t} + u\cdot\nabla f\right)(\phi_t(x_0,t),t) \\ &= J(x_0,t)\left(\frac{\partial f}{\partial t} + f\ \mathrm{div}(u) + u\cdot\nabla f\right)(\phi(x_0,t),t) \\ &= J(x_0,t)\left(\frac{\partial f}{\partial t} + \mathrm{div}(fu)\right)(\phi(x_0,t),t)\end{aligned}$$

. Finally, by substituting it back into (2.5) and by doing change of variables back into x and V_t, we get

$$\begin{aligned}\frac{d}{dt}\int_{V_t} f(x,t)\ dx &= \int_{V_0}\left(\frac{\partial f}{\partial t} + \mathrm{div}(fu)\right)(\phi(x_0,t),t)J(x_0,t)\ dx_0 \\ &= \int_{V_t}\left(\frac{\partial f}{\partial t} + \mathrm{div}(fu)\right)(x,t)\ dx. \qquad (2.6)\end{aligned}$$

Equation (2.6) is called The Transport Theorem.

2.1.3 Conservation of Mass

From continuum mechanics, we define density ρ in a given domain $\Omega \subset \mathbb{R}^3$ at a given time t as the function $\rho \in C(\Omega), \rho : \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}$ s.t. $m_\Omega(t) = \int_\Omega \rho(x,t)dx$, with $m_\Omega(t)$ is a mass of fluid inside domain Ω at time t.

Let u be a velocity field and V be a fixed subdomain of Ω does not change with time. By using the conservation of mass, the rate of change of mass in V equals to total mass flow rate through the boundary of V (with normal vector n points outward),

$$\frac{d}{dt}m_V(t) = -\int_{\partial V}\rho(x,t)u(x,t)\cdot n(x,t)\ dS.$$

By using the divergence theorem for the right-hand side term, we get

$$\frac{d}{dt}\int_V \rho(x,t)\ dx = -\int_V \operatorname{div}(\rho(x,t)u(x,t))\ dx$$

$$\int_V \frac{\partial \rho(x,t)}{\partial t} + \operatorname{div}(\rho(x,t)u(x,t))\ dx = 0. \tag{2.7}$$

Since (2.7) must be true for any $V \subset \Omega$,

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0. \tag{2.8}$$

Equation (2.8) is a continuity equation in an Eulerian description form.

By using a product rule for divergence, we get

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) &= 0 \\ \frac{\partial \rho}{\partial t} + \rho\ \operatorname{div}(u) + \nabla\rho \cdot u &= 0 \\ \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla\rho &= -\rho\ \operatorname{div}(u) \\ \frac{D\rho}{Dt} &= -\rho\ \operatorname{div}(u). \end{aligned} \tag{2.9}$$

Equation (2.9) is a continuity equation in a Lagrangian description form.

2.1.4 Conservation of Momentum

Forces acting on a material body can be classified into two types: stress forces act on the surface, and body (or external) forces act on the continuum itself.

The simplest type of stress forces is a pressure force, as stated from [19] pp 5, is defined as, for any motion of the fluid there is a function $p(x,t)$ called the **pressure** such that if S is a surface in the fluid with a chosen unit normal n, the force of stress exerted across the surface S per unit area at $x \in S$ at time t is $p(x,t)n$; i.e.,

$$\text{force across } S \text{ per unit area} = -p(x,t)n(x,t).$$

The pressure force is in the same direction with normal vector n, or in the other words, orthogonal to the surface and does not have a tangential part.

Let S_V be a total surface force acts on a given subdomain $V \subset \Omega$ at time t,

$$S_V = -\int_{\partial V} p(x,t)n(x,t)\ dS,$$

where p is a pressure force, n is a normal vector points outward, and ∂V is a surface of V. By multiplying it with any fixed vector $e \in \mathbb{R}^3$,

$$\begin{aligned} e \cdot S_V &= -\int_{\partial V} p(x,t)e \cdot n(x,t)\ dS \\ &= -\int_V \text{div}(p(x,t)e)\ dx \\ &= -\int_V p(x,t)\text{div}(e) + e \cdot \nabla p(x,t)\ dx \\ &= -\int_V e \cdot \nabla p(x,t)\ dx \\ S_V &= -\int_V \nabla p(x,t)\ dx. \end{aligned} \tag{2.10}$$

Now let b be a given body force per unit mass. The total body force B_V acts on V is

$$B_V = \int_V \rho(x,t)b(x,t)\ dx. \tag{2.11}$$

By definition, the linear momentum $M(V,t)$ has a density function $f(x,t) = \rho(x,t)u(x,t)$. Capital letter M here does not mean it is in a Lagrangian description. By the Newton's law, the rate of change of momentum equals to total force acts on the body. Using the similar argument with the conservation of mass, the rate of change of momentum equals to total momentum flow rate through the boundary plus rate of change of momentum by the source, which is in this case, total force acts on the body. Let $M(V,t) = (M_1(V,t), M_2(V,t), M_3(V,t))$ and $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$, as well with $S_V = (S_{V,1}, S_{V,2}, S_{V,3})$, $B_V = (B_{V,1}, B_{V,2}, B_{V,3})$ and $b = (b_1, b_2, b_3)$. For $i = 1, 2, 3$ we have

$$\begin{aligned} \frac{d}{dt}M_i(V,t) &= -\int_{\partial V} \rho(x,t)u_i(x,t)u \cdot n\ dS\ +\ S_{V,i}\ +\ B_{V,i} \\ \frac{d}{dt}\int_V \rho(x,t)u_i(x,t)\ dx &= -\int_V \text{div}(\rho(x,t)u_i(x,t)u(x,t))\ dx\ -\ \int_V \frac{\partial}{\partial x_i}p(x,t)\ dx \\ &\quad + \int_V \rho(x,t)b_i(x,t)\ dx \end{aligned}$$

$$\int_V \frac{\partial}{\partial t}\left(\rho(x,t)u_i(x,t)\right)\,dx = \int_V -\operatorname{div}(\rho(x,t)u_i(x,t)u(x,t)) \\ - \frac{\partial}{\partial x_i}p(x,t) + \rho(x,t)b_i(x,t)\,dx.$$

Since all ρ, p, b, and u (and, of course, u_i for $i = 1, 2, 3$) are functions of (x, t), let us drop it from the equation for simplicity. Since the equation must be true for any $V \subset \Omega$, the equation becomes

$$\frac{\partial(\rho u_i)}{\partial t} + \operatorname{div}(\rho u_i u) = -\frac{\partial p}{\partial x_i} + \rho b_i, \tag{2.12}$$

which is an Eulerian description for conservation of momentum. By doing product rules, we get

$$\begin{aligned} -\frac{\partial p}{\partial x_i} + \rho b_i &= \rho\frac{\partial u_i}{\partial t} + u_i\frac{\partial \rho}{\partial t} + \rho u_i \operatorname{div}(u) + \rho u \cdot \operatorname{div}(u_i) + u_i u \cdot \operatorname{div}(\rho) \\ &= \rho\left(\frac{\partial u_i}{\partial t} + u \cdot \operatorname{div}(u_i)\right) + u_i\left(\frac{\partial \rho}{\partial t} + \rho\operatorname{div}(u) + u \cdot \operatorname{div}(\rho)\right) \\ &= \rho\frac{Du_i}{Dt} + u_i\left(\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u)\right). \end{aligned}$$

By using (2.8), the second term of right-hand side of the equation became zero, leaving us with

$$\rho\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \rho b_i,$$

or

$$\rho\frac{Du}{Dt} = -\nabla p + \rho b, \tag{2.13}$$

which is a Lagrangian description for the conservation of momentum.

As mentioned before, (2.12) and (2.13) assume that the only force exerts on the surface comes from the pressure term which is in an orthogonal direction to the surface, which is the case of an inviscid flow of the fluid. Next we will try to take into account a tangential part of the force as well. Let us modify our previous definition into

$$\text{force across } S \text{ per unit area} = -p(x,t)n(x,t) + \sigma(x,t) \cdot n(x,t),$$

with σ is a matrix called stress tensor, fulfills some assumptions:

1. σ depends linearly to $\mathbf{D}u$ that is, the stress tensor σ is related to the gradient of the velocity field $\mathbf{D}u$ by some linear transfomations at each points.

2. σ is invariant under rigid body rotation, that is, let U is an orthogonal matrix,
$$\sigma(U \cdot \mathbf{D}u \cdot U^{-1}) = U \cdot \sigma(\mathbf{D}u) \cdot U^{-1}.$$

3. σ is symmetric.

$\mathbf{D}u$ consists of two parts, a symmetric part $\mathbf{D} = \frac{1}{2}(\mathbf{D}u + (\mathbf{D}u)^T)$ which represents the expansion or contraction of the fluid, hence denoted by $\mathbf{D}$ for "deformation", and a skew-symmetric $\mathbf{S} = \frac{1}{2}(\mathbf{D}u - (\mathbf{D}u)^T)$ represents the rotation of fluid. It is easily can be checked that $\mathbf{D}u = \mathbf{D} + \mathbf{S}$.

Since σ is a symmetric matrix, by properties 1 and 2, it can only depends on a symmetric part of $\mathbf{D}u$, that is the deformation matrix $\mathbf{D}$. Since σ is a linear function of $\mathbf{D}$, σ and $\mathbf{D}$ commute, and assuming both of them are diagonalizable, σ and $\mathbf{D}$ can be simultaneously diagonalized. Thus, the eigenvalues of σ are linear functions of the eigenvalues of $\mathbf{D}$. By the property 2, they must also be symmetric. The only linear function which fulfils all conditions is

$$\sigma_i = \lambda(d_1 + d_2 + d_3) + 2\mu d_i, \qquad i = 1, 2, 3, \tag{2.14}$$

where σ_i are the eigenvalues of σ, d_i are the eigenvalues of $\mathbf{D}$, λ and μ are constants. $d_1 + d_2 + d_3$ equals to $\mathrm{div}(u)$, and by transforming it back to usual basis, we get

$$\sigma = \lambda \ \mathrm{div}(u)\mathbf{I} + 2\mu\mathbf{D}, \tag{2.15}$$

where $\mathbf{I}$ is an identity matrix. By rearranging the equation, we get:

$$\sigma = 2\mu \left(\mathbf{D} - \frac{1}{3}\mathrm{div}(u)\mathbf{I} \right) + \zeta \ \mathrm{div}(u)\mathbf{I}, \tag{2.16}$$

where μ is the first coefficient of viscosity and $\zeta = \lambda + \frac{2}{3}\mu$ is the second coefficient of viscosity.

Now the total surface force S_V on a given subdomain $V \subset \Omega$ at a given time t is given by (drop all variables (x,t) for simplicity),

$$\begin{aligned}
S_V &= \int_{\partial V} -pn + \sigma \cdot n \; dS \\
&= \int_{\partial V} -pn + (\lambda \operatorname{div}(u)\mathbf{I} + 2\mu \mathbf{D}) \cdot n \; dS \\
&= -\int_{\partial V} pn \; dS \;+\; \lambda \int_{\partial V} \operatorname{div}(u) n \; dS \;+\; 2\mu \int_{\partial V} \frac{1}{2}((\mathbf{D}u) + (\mathbf{D}u)^T) \; dS \\
&= -\int_{\partial V} pn \; dS \;+\; \lambda \int_{\partial V} \operatorname{div}(u) n \; dS \;+\; \mu \int_{\partial V} ((\mathbf{D}u) + (\mathbf{D}u)^T) \; dS.
\end{aligned}$$

By multiplying it with a fixed vector $e \in \mathbb{R}^3$ we have

$$\begin{aligned}
e \cdot S_V &= -\int_{\partial V} pe \cdot n \; dS \;+\; \lambda \int_{\partial V} \operatorname{div}(u) e \cdot n \; dS \;+\; \mu e \cdot \int_{\partial V} (\mathbf{D}u + (\mathbf{D}u)^T) \; dS \\
&= -\int_V \operatorname{div}(pe) \; dx \;+\; \lambda \int_V \operatorname{div}(\operatorname{div}(u)e) \; dx \\
&\quad + \; \mu e \cdot \int_V (\operatorname{div}(\mathbf{D}u) + \operatorname{div}((\mathbf{D}u)^T)) \; dx.
\end{aligned}$$

Since e is fixed, the product rule of $\operatorname{div}(fe) = e \cdot \nabla f$ for any scalar function f. Also,

$$\begin{aligned}
(\operatorname{div}((\mathbf{D}u)^T))_i &= \sum_{j=1}^{3} \frac{\partial}{\partial x_j}((\mathbf{D}u)^T)_{ij} = \sum_{j=1}^{3} \frac{\partial}{\partial x_j}(\mathbf{D}u)_{ji} = \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\frac{\partial u_j}{\partial x_i} = \frac{\partial}{\partial x_i}\sum_{j=1}^{3}\frac{\partial u_j}{\partial x_j} \\
&= (\nabla(\operatorname{div}(u)))_i,
\end{aligned}$$

or

$$\operatorname{div}((\mathbf{D}u)^T) = \nabla(\operatorname{div}(u)).$$

Let us denote $\Delta u = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) u = \operatorname{div}(\mathbf{D}u)$ is a Laplacian of the velocity field u. Back to the total surface force, now we have

$$\begin{aligned}
e \cdot S_V &= -e \cdot \int_V \nabla p \; dx \;+\; \lambda e \cdot \int_V \nabla(\operatorname{div}(u)) \; dx \;+\; \mu e \cdot \int_V \Delta u \;+\; \nabla(\operatorname{div}(u)) \; dx \\
&= e \cdot \int_V -\nabla p \;+\; (\lambda + \mu)\nabla(\operatorname{div}(u)) \;+\; \mu \Delta u \; dx \\
S_V &= \int_V -\nabla p \;+\; (\lambda + \mu)\nabla(\operatorname{div}(u)) \;+\; \mu \Delta u \; dx.
\end{aligned}$$

By doing the similar things like before, by the conservation of momentum (and dropping all variables (x,t)), we have

$$\begin{aligned}\frac{d}{dt}M_i &= -\int_{\partial V} \rho u_i u \cdot n \; dS \;+\; S_{V,i} \;+\; B_{V,i} \\ \frac{d}{dt}\int_V \rho u_i dx &= -\int_V \operatorname{div}(\rho u_i u)\; dx \;+\; \int_V -\frac{\partial p}{\partial x_i} + \;(\lambda+\mu)\frac{\partial}{\partial x_i}(\operatorname{div}(u)) \\ &\quad + \;\mu \Delta u_i \; dx + \int_V \rho b_i \; dx \\ \int_V \frac{\partial}{\partial t}(\rho u_i)\, dx &= \int_V -\operatorname{div}(\rho u_i u) - \frac{\partial p}{\partial x_i} \;+\; (\lambda+\mu)\frac{\partial}{\partial x_i}(\operatorname{div}(u)) \;+\; \mu\Delta u_i \; + \rho b_i \; dx.\end{aligned}$$

Since it must be true for any $V \subset \Omega$,

$$\frac{\partial(\rho u_i)}{\partial t} + \operatorname{div}(\rho u_i u) = -\frac{\partial p}{\partial x_i} \;+\; (\lambda+\mu)\frac{\partial}{\partial x_i}(\operatorname{div}(u)) \;+\; \mu\Delta u_i \; + \rho b_i. \tag{2.17}$$

(2.17) is an Eulerian description of conservation of momentum for viscous fluid. We also have

$$\rho\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} \;+\; (\lambda+\mu)\frac{\partial}{\partial x_i}(\operatorname{div}(u)) \;+\; \mu\Delta u_i \; + \rho b_i,$$

or

$$\rho\frac{Du}{Dt} = -\nabla p \;+\; (\lambda+\mu)\nabla(\operatorname{div}(u)) \;+\; \mu\Delta u \; + \rho b, \tag{2.18}$$

which is a Lagrangian description of conservation of momentum for viscous fluid.

2.1.5 Conservation of Energy

The total energy $E(V_t,t)$ for the moving volume $V_t = V(t) \subset \Omega$ at a given time t is the summation of its kinetic energy $E_k(V_t,t)$ and the internal energy $E_i(V_t,t)$. The kinetic energy of V_t is defined as

$$E_k(V_t,t) = \frac{1}{2}\int_{V_t} \rho(x,t)\|u(x,t)\|^2 \; dx,$$

where $\rho(x,t)\|u(x,t)\|^2$ is the density function of the kinetic energy, and $\|u(x,t)\|^2 = u(x,t)\cdot u(x,t) = u_1^2(x,t) + u_2^2(x,t) + u_3^2(x,t)$. By using the transport theorem, the rate of change of the kinetic energy is written as

$$\begin{aligned}
\frac{d}{dt}E_k(V_t,t) &= \frac{1}{2}\frac{d}{dt}\int_{V_t}\rho(x,t)\|u(x,t)\|^2\ dx \\
&= \frac{1}{2}\int_{V_t}\left(\frac{\partial}{\partial t}(\rho\|u\|^2) + \operatorname{div}(\rho\|u\|^2 u)\right)(x,t)\ dx \\
&= \frac{1}{2}\int_{V_t}\left(\rho\frac{\partial\|u\|^2}{\partial t} + \|u\|^2\frac{\partial\rho}{\partial t} + \rho\ \operatorname{div}(\|u\|^2 u) + \|u\|^2 u\cdot\nabla\rho\right)(x,t)\ dx \\
&= \frac{1}{2}\int_{V_t}\left(\rho\frac{\partial\|u\|^2}{\partial t} + \|u\|^2\frac{\partial\rho}{\partial t} + \rho\|u\|^2\operatorname{div}(u) + \rho u\cdot\nabla(\|u\|^2)\right. \\
&\qquad\qquad \left. + \|u\|^2 u\cdot\nabla\rho\right)(x,t)\ dx \\
&= \frac{1}{2}\int_{V_t}\left(\|u\|^2\left(\frac{\partial\rho}{\partial t} + u\cdot\nabla\rho + \rho\ \operatorname{div}(u)\right)\right. \\
&\qquad\qquad \left. + \rho\left(\frac{\partial\|u\|^2}{\partial t} + u\cdot\nabla\left(\|u\|^2\right)\right)\right)(x,t)\ dx.
\end{aligned}$$

By using the conservation of mass, the first term is zero, and for the second term we will substitute:

$$\begin{aligned}
\frac{\partial\|u\|^2}{\partial t} + u\cdot\nabla(\|u\|^2) &= \frac{\partial(u\cdot u)}{\partial t} + u\cdot\nabla(u\cdot u) \\
&= 2u\cdot\frac{\partial u}{\partial t} + \sum_{i=1}^{3} u_i(\nabla(u\cdot u))_i \\
&= 2u\cdot\frac{\partial u}{\partial t} + \sum_{i=1}^{3} u_i\left(\frac{\partial}{\partial x_i}\left(\sum_{j=1}^{3} u_j u_j\right)\right) \\
&= 2u\cdot\frac{\partial u}{\partial t} + 2\sum_{i=1}^{3}\sum_{j=1}^{3} u_i u_j\frac{\partial u_j}{\partial x_i} \\
&= 2\left(u\cdot\frac{\partial u}{\partial t} + \sum_{j=1}^{3} u_j\sum_{i=1}^{3} u_i\frac{\partial u_j}{\partial x_i}\right) \\
&= 2\left(u\cdot\frac{\partial u}{\partial t} + u\cdot(u\cdot\nabla)u\right).
\end{aligned}$$

So now we have

$$\begin{aligned}
\frac{d}{dt}E_k(V_t,t) &= \frac{1}{2}\int_{V_t}\left(\rho\left(\frac{\partial\|u\|^2}{\partial t} + u\cdot\nabla\left(\|u\|^2\right)\right)\right)(x,t)\ dx \\
&= \int_{V_t}\left(\rho u\cdot\left(\frac{\partial u}{\partial t} + (u\cdot\nabla)u\right)\right)(x,t)\ dx, \qquad (2.19)
\end{aligned}$$

where

$$(u \cdot \nabla)u = \left\{ \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} \right\}_{i=1...3} .$$

In general, we will do a similar thing like before for the conservation of energy. The total time rate of change of the energy equals to the total heat flux into the system plus the total rate of works done by forces. Before we continue, we need to discuss about a case of an incompressible flow.

The incompressible fluid means that the volume of the fluid will not change with time. By the change of variables and (2.4), for a moving fluid element V_t at a given time t, we have

$$\begin{aligned}
\frac{d}{dt}\|V_t\| = \frac{d}{dt}\int_{V_t} dx = \frac{d}{dt}\int_{V_0} J(x_0,t)\ dx_0 &= \int_{V_0} \frac{\partial J}{\partial t}(x_0,t)\ dx_0 \\
&= \int_{V_0} \operatorname{div}(u(\phi(x_0,t),t))J(x_0,t)\ dx_0 \\
&= \int_{V_t} \operatorname{div}(u(x,t))\ dx = 0,
\end{aligned}$$

which implies that $\operatorname{div}(u) = 0$ for any V_t, with $\|V_t\|$ is the measure of the volume V_t. By assuming the kinetic energy is the only energy works on the system, and by the definition of works, for an inviscid incompressible fluid, the rate of change of the energy of V_t at time t is

$$\begin{aligned}
\frac{d}{dt}E_k(V_t,t) &= -\int_{\partial V_t} (pu\cdot n)(x,t)\ dS \\
&\quad + \int_{V_t} (\rho u \cdot b)(x,t)\ dx \\
\int_{V_t} \left(\rho u \cdot \left(\frac{\partial u}{\partial t} + (u\cdot\nabla)u\right)\right)(x,t)\ dx &= \int_{V} \left(-\operatorname{div}(pu) + \rho u\cdot b\right)(x,t)\ dx \\
&= \int_{V_t} \left(-u\cdot\nabla p + \rho u \cdot b\right)(x,t)\ dx,
\end{aligned}$$

since $\operatorname{div}(u) = 0$ for an incompressible case. So now we have

$$\int_{V_t} \left(u \cdot \left(\rho \frac{Du}{Dt}\right)\right)(x,t)\ dx = \int_{V_t} \left(u\cdot(-\nabla p + \rho b)\right)(x,t)\ dx.$$

This equation is also a consequence of the balance of momentum. This shows that if we only assume $E = E_k$, then the fluid must be an incompressible fluid (except if we have $p = 0$ for the whole domain).

Next, let $Q(V_t, t)$ be a total heat contained in a moving fluid element V_t at a given time t with its density function is represented by $\rho(x,t)\varrho(x,t)$, so that $Q(V_t,t) = \int_{V_t} \rho(x,t)\varrho(x,t)\, dx$. Let $E(V_t,t)$ be a total energy of V_t with $E(V_t,t) = E_i(V_t,t) + E_k(V_t,t)$, $E_i(V_t,t)$ is an internal energy of V_t and $E_k(V_t,t)$ is a kinetic energy of V_t at a given time t. The total heat flux flows into a system consists of two parts; the rate of the volumetric heating and the rate of the thermal conduction [20]. Let $H_v(V_t,t)$ be the rate of volumetric heating of V_t,

$$H_v(V_t,t) = \frac{d}{dt}\int_{V_t} (\rho\varrho)(x,t)\, dx.$$

By the transport theorem, we have

$$\begin{aligned}
H_v(V_t,t) &= \int_{V_t} \left(\frac{\partial}{\partial t}(\rho\varrho) + \operatorname{div}(\rho\varrho u)\right)(x,t)\, dx \\
&= \int_{V_t} \left(\rho\frac{\partial\varrho}{\partial t} + \varrho\frac{\partial\rho}{\partial t} + \rho\varrho\, \operatorname{div}(u) + u\cdot\nabla(\rho\varrho)\right)(x,t)\, dx \\
&= \int_{V_t} \left(\rho\frac{\partial\varrho}{\partial t} + \varrho\left(\frac{\partial\rho}{\partial t} + \rho\, \operatorname{div}(u)\right) + \rho u\cdot\nabla\varrho + \varrho u\cdot\nabla\rho\right)(x,t)\, dx \\
&= \int_{V_t} \left(\rho\left(\frac{\partial\varrho}{\partial t} + u\cdot\nabla\varrho\right) + \varrho\left(\frac{\partial\rho}{\partial t} + \rho\, \operatorname{div}(u) + u\cdot\nabla\rho\right)\right)(x,t)\, dx.
\end{aligned}$$

By the conservation of mass, the second term of right-hand side is equal to zero,

$$\begin{aligned}
H_v(V_t,t) &= \int_{V_t} \rho\left(\frac{\partial\varrho}{\partial t} + u\cdot\nabla\varrho\right)(x,t)\, dx \\
&= \int_{V_t} \left(\rho\frac{D\varrho}{Dt}\right)(x,t)\, dx. \qquad (2.20)
\end{aligned}$$

Now let $H_c(V_t,t)$ be the rate of thermal conduction through the surface of moving fluid element V_t at a given time t,

$$H_c(V_t,t) = -\int_{\partial V_t} (\rho\varrho u\cdot n)(x,t)\, dS.$$

By the divergence theorem, we have

$$H_c(V_t,t) = -\int_{V_t} (\operatorname{div}(\rho\varrho u))(x,t)\, dx.$$

By the definition of the heat-flux $q(x,t)$ as the flow of the energy per unit time per unit area, $q(x,t) = \varrho\rho u$, so now we have

$$H_c(V_t,t) = -\int_{V_t} \text{div}(q(x,t))\ dx. \tag{2.21}$$

By the Fourier's Law, the heat flux is $q(x,t) = -k\nabla\theta(x,t)$ where k is the thermal conductivity, and θ is a temperature, so

$$H_c(V_t,t) = \int_{V_t} \text{div}(k\nabla(\theta(x,t)))\ dx. \tag{2.22}$$

Let $W(V_t,t)$ be the rate of works done by forces act on V_t at a given time t,

$$\begin{aligned} W(V_t,t) &= -\int_{\partial V_t} (pu\cdot n)(x,t)\ dS \ + \ \int_{V_t} (\rho u\cdot b)(x,t)\ dx \\ &= \int_{V_t} (-\text{div}(pu) + \rho u\cdot b)(x,t)\ dx. \end{aligned}$$

Let $\rho(x,t)e(x,t)$ be a density function of internal energy, so for a moving fluid V_t at a given time t we have $E_i(V_t,t) = \int_{V_t} (\rho e)(x,t)\ dx$. Time rate of change of total energy of V_t is

$$\begin{aligned} \frac{d}{dt}E(V_t,t) &= H_v(V_t,t) + H_c(V_t,t) + W(V_t,t) \\ \frac{d}{dt}\left(E_i(V_t,t) + E_k(V_t,t)\right) &= \int_V \left(\rho\frac{D\varrho}{Dt} \ + \ \text{div}(k\nabla\theta) \ - \ \text{div}(pu) \ + \ \rho u\cdot b\right)(x,t)\ dx. \end{aligned}$$

Since last two integrands in the right-hand side of the equation equals to the rate of change of kinetic energy, conservation of energy equation is usually written in term of e. Rate of change of internal energy itself can be written as

$$\frac{d}{dt}(E_i(V_t,t)) = \frac{d}{dt}\int_{V_t} (\rho e)(x,t)\ dx.$$

By the transport theorem, we have

$$\begin{aligned} \frac{d}{dt}(E_i(V_t,t)) &= \int_{V_t} \left(\frac{\partial(\rho e)}{\partial t} + \text{div}(\rho e u)\right)(x,t)\ dx \\ &= \int_{V_t} \left(\rho\frac{\partial e}{\partial t} + e\frac{\partial \rho}{\partial t} + \rho e\ \text{div}(u) + \rho u\cdot\nabla e + e u\cdot\nabla\rho\right)(x,t)\ dx \\ &= \int_{V_t} \left(\rho\left(\frac{\partial e}{\partial t} + u\cdot\nabla e\right) + e\left(\frac{\partial\rho}{\partial t} + \rho\ \text{div}(u) + u\cdot\nabla\rho\right)\right)(x,t)\ dx. \end{aligned}$$

By the conservation of mass, now we have

$$\frac{d}{dt}(E_i(V_T,t)) = \int_{V_t} \left(\rho \frac{De}{Dt}\right)(x,t)\ dx. \tag{2.23}$$

Going back to our conservation of energy equation, now we have

$$\int_{V_t} \left(\rho \frac{De}{Dt}\right)(x,t)\ dx = \int_{V_t} \left(\rho \frac{D\varrho}{Dt} + \operatorname{div}(k\nabla\theta)\right)(x,t)\ dx.$$

By dropping all variables, in the end we have

$$\rho \frac{De}{Dt} = \rho \frac{D\varrho}{Dt} + \operatorname{div}(k\nabla\theta). \tag{2.24}$$

2.2 Rigid Body Dynamics

In this section we give a basic overview of the standard notions of the rigid body mechanics. Here we follow a standard textbook of [21]. Rigid body is defined as a collection of materials points whose mutual distance is independent of time, that is, if Ω_t is a rigid body in $\mathbb{R}^3$ at a given time t, by using the previous notation of the flow we have

$$|X^{x_0}(t) - X^{y_0}(t)| = |X^{x_0}(0) - X^{y_0}(0)|\,,\ \forall x_0, y_0 \in \mathcal{R},$$

where $\mathcal{R}$ is the reference configuration of the rigid body and we will assume that $|X^{x_0}(0) - X^{y_0}(0)| = |x_0 - y_0|$ for all $x_0, y_0 \in \mathcal{R}$. Therefore the motion must be given by a *rigid* transformation, that is, it is given by a translation and a rotation. If we pick a fixed point $x_0 \in \mathcal{R}$, there exists an orthogonal matrix $\mathbf{R}(t)$, called the *rotation matrix*, such that

$$X^x(t) = X^{x_0}(t) + \mathbf{R}(t)(x - x_0), \qquad \text{for all } x \in \mathcal{R}, t \in [0,\infty)\,.$$

The mass of the rigid body is constant. Specifically,

$$M = \int_{\mathcal{R}} \rho(x)\ dx.$$

A convenient choice of the point x_0 above is the center of mass,

$$x_0 = \frac{\int_{\mathcal{R}} \rho(x) x \; dx}{M}.$$

Let $X(t) = X^{x_0}(t)$, $U(t) = \dot{X}(t)$ and M be a position of the center of mass, velocity, and mass, respectively. The linear momentum $G(t)$ is defined as

$$G(t) = MU(t). \tag{2.25}$$

To describe the angular motion, it is useful to introduce the moment of inertia tensor of rigid body. Now let $\mathbf{J}$ be a moment of inertia tensor in the rigid body reference frame, which we understand to be the reference frame at time $t = 0$. $\mathbf{J}$ is defined as

$$\mathbf{J} = \int_{\mathcal{R}} \rho(x) \left(|x|^2 \mathbf{I} - x \otimes x \right) \; dx,$$

where $\mathbf{I}$ is an identity matrix. Matrix $\mathbf{J}$ can also be written components-wise,

$$J_{ij} = \int_{\mathcal{R}} \rho(x) \left(\delta_{ij} \left(\sum_{k=1}^{3} x_k^2 \right) - x_i x_j \right) \; dx,$$

where δ_{ij} is a Kronecker delta. We can see that $\mathbf{J}$ is a symmetric tensor. To be more explicit, components of $\mathbf{J}$ are

$$\begin{aligned} J_{11} &= \int_{\mathcal{R}} \rho(x) \left(x_2^2 + x_3^2 \right) \; dx & J_{12} = J_{21} &= -\int_{\mathcal{R}} \rho(x) x_1 x_2 \; dx \\ J_{22} &= \int_{\mathcal{R}} \rho(x) \left(x_1^2 + x_3^2 \right) \; dx & J_{13} = J_{31} &= -\int_{\mathcal{R}} \rho(x) x_1 x_3 \; dx \\ J_{33} &= \int_{\mathcal{R}} \rho(x) \left(x_1^2 + x_2^2 \right) \; dx & J_{23} = J_{32} &= -\int_{\mathcal{R}} \rho(x) x_2 x_3 \; dx. \end{aligned}$$

J_{11}, J_{22}, and J_{33} are called the moments of inertia, while J_{12}, J_{13}, and J_{23} are called the products of inertia.

We define $\omega(t)$ as the angular velocity vector of the rigid body in the following way: Let $\omega(t)$ be the vector such that for any fixed vector $v \in \mathbb{R}^3$ we have

$$\frac{d}{dt}(\mathbf{R}(t) v) = \omega(t) \times \mathbf{R}(t) v. \tag{2.26}$$

Let us explain why such a vector $\omega(t)$ exists. Differentiation yields

$$\begin{aligned}\frac{d}{dt}(\mathbf{R}(t)v) &= \frac{d\mathbf{R}(t)}{dt}v \\ &= \frac{d\mathbf{R}(t)}{dt}\mathbf{R}^{-1}(t)\mathbf{R}(t)v, \\ &= \mathbf{W}(t)\mathbf{R}(t)v,\end{aligned}$$

where $\mathbf{W}(t) = \frac{d\mathbf{R}(t)}{dt}\mathbf{R}^{-1}(t)$. Since $\mathbf{R}(t)$ is an orthogonal matrix, we have $\mathbf{R}^{-1}(t) = \mathbf{R}^T(t)$ and $\mathbf{R}(t)\mathbf{R}^T(t) = \mathbf{I}$. Hence

$$0 = \frac{d}{dt}\left(\mathbf{R}(t)\mathbf{R}^T(t)\right) = \frac{d\mathbf{R}(t)}{dt}\mathbf{R}^T(t) + \mathbf{R}(t)\frac{d\mathbf{R}^T(t)}{dt} = \mathbf{W}(t) + \mathbf{W}^T(t).$$

Therefore $\mathbf{W}(t)$ is a skew-symmetric matrix and can be written as

$$\mathbf{W}(t) = \begin{bmatrix} 0 & -\omega_3(t) & \omega_2(t) \\ \omega_3(t) & 0 & -\omega_1(t) \\ -\omega_2(t) & \omega_1(t) & 0 \end{bmatrix}. \tag{2.27}$$

The vector $\omega(t) = (\omega_1(t), \omega_2(t), \omega_3(t))$ satisfies (2.26) since direct computation yields $\omega(t) \times w = \mathbf{W}(t)w$ for any vector $w \in \mathbb{R}^3$.

The angular momentum of the rigid body relative to its center of mass is defined as

$$H(t) = \mathbf{R}(t)\mathbf{J}\mathbf{R}^T(t)\omega(t).$$

It is convenient for the matrix $\mathbf{J}$ to be as simple as possible, that is, to be a diagonal matrix. We can achieve this by choosing a reference frame for the rigid body such that $\mathbf{J}$ is diagonal by rotating the coordinate system. As $\mathbf{J}$ is a real symmetric matrix, there exist an orthogonal matrix $\mathbf{P}$ such that $\mathbf{PJP}^T$ is diagonal. We can therefore make $\mathbf{J}$ diagonal by using $\mathbf{P}\mathcal{R}$ as the reference frame instead of $\mathcal{R}$. The columns of $\mathbf{P}$ are three orthogonal eigenvectors of matrix $\mathbf{J}$ and the diagonal components of $\mathbf{PJP}^T$ are the corresponding eigenvalues. In the rest of this thesis, we assume that $\mathbf{J}$ is diagonal.

The total moment acts on the rigid body equals to the rate of change of its angular momentum. Let $K(t) = \sum_i (x^i - X(t)) \times F(x^i, t)$ be a total moment acts on a rigid body with center of mass $X(t)$ at a given time t, where $F(x^i, t)$ is a total force acts at a point x^i. The rate of change of the angular momentum is

$$\begin{aligned} K(t) &= \frac{d}{dt} H(t) \\ &= \frac{d}{dt}\left(\mathbf{R}(t)\mathbf{J}\mathbf{R}^T(t)\omega(t)\right) \\ &= \dot{\mathbf{R}}(t)\mathbf{J}\mathbf{R}^T(t)\omega(t) + \mathbf{R}(t)\mathbf{J}\dot{\mathbf{R}}^T(t)\omega(t) + \mathbf{R}(t)\mathbf{J}\mathbf{R}^T(t)\dot{\omega}(t), \end{aligned}$$

where $\dot{\mathbf{R}} = \frac{d}{dt}\mathbf{R}(t)$ and $\dot{\omega}(t) = \frac{d}{dt}\omega(t)$ is an angular acceleration at a given time t. Since $\mathbf{W}(t)v = \dot{\mathbf{R}}(t)\mathbf{R}^T(t)v = \omega(t) \times v$ for any vector $v \in \mathbb{R}^3$, we can write the first term on the righthand side as

$$\begin{aligned} \dot{\mathbf{R}}(t)\mathbf{J}\mathbf{R}^T(t)\omega(t) &= \dot{\mathbf{R}}(t)\mathbf{R}^T(t)\mathbf{R}(t)\mathbf{J}\mathbf{R}^T(t)\omega(t) \\ &= \omega(t) \times \left(\mathbf{R}(t)\mathbf{J}\mathbf{R}^T(t)\omega(t)\right), \end{aligned}$$

and by the property of a matrix transpose $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ and skew-symmetricity of $\mathbf{W}(t)$, so that $\mathbf{W}^T(t) = -\mathbf{W}(t)$, the second term becomes

$$\begin{aligned} \mathbf{R}(t)\mathbf{J}\dot{\mathbf{R}}^T(t)\omega(t) &= \mathbf{R}(t)\mathbf{J}\mathbf{R}^T\mathbf{R}(t)\dot{\mathbf{R}}^T(t)\omega(t) \\ &= \mathbf{R}(t)\mathbf{J}\mathbf{R}^T\left(\dot{\mathbf{R}}(t)\mathbf{R}^T(t)\right)^T\omega(t) \\ &= \mathbf{R}(t)\mathbf{J}\mathbf{R}^T(t)\mathbf{W}^T(t)\omega(t) \\ &= \mathbf{R}(t)\mathbf{J}\mathbf{R}^T(t)\left(-\omega(t) \times \omega(t)\right) \\ &= 0, \end{aligned}$$

leaving us with

$$K(t) = \omega(t) \times \left(\mathbf{R}(t)\mathbf{J}\mathbf{R}^T(t)\omega(t)\right) + \mathbf{R}(t)\mathbf{J}\mathbf{R}^T(t)\dot{\omega}(t).$$

Now let us operate the above equation with matrix $\mathbf{R}^T(t)$ from the left, and by the properties of cross product $\mathbf{A}(b \times c) = \mathbf{A}b \times \mathbf{A}c$, provided $\mathbf{A}$ is an orthogonal matrix, we have

$$\begin{aligned} \mathbf{R}^T(t)K(t) &= \mathbf{R}^T(t)\left(\omega(t) \times \left(\mathbf{R}(t)\mathbf{J}\mathbf{R}^T(t)\omega(t)\right)\right) + \mathbf{R}^T(t)\mathbf{R}(t)\mathbf{J}\mathbf{R}^T(t)\dot{\omega}(t) \\ &= \mathbf{R}^T(t)\omega(t) \times \mathbf{J}\mathbf{R}^T(t)\omega(t) + \mathbf{J}\mathbf{R}^T(t)\dot{\omega}(t). \end{aligned} \tag{2.28}$$

In general, the relation between the transpose rotation matrix $\mathbf{R}^T(t)$ with the rate of change of any vector function $v(t) \in \mathbb{R}^3$ is

$$\begin{aligned}
\mathbf{R}^T(t)\frac{dv(t)}{dt} &= \frac{d}{dt}\left(\mathbf{R}^T(t)v(t)\right) - \frac{d\mathbf{R}^T(t)}{dt}v(t) \\
&= \frac{d}{dt}\left(\mathbf{R}^T(t)v(t)\right) - \mathbf{R}^T(t)\mathbf{R}(t)\frac{d\mathbf{R}^T(t)}{dt}v(t) \\
&= \frac{d}{dt}\left(\mathbf{R}^T(t)v(t)\right) - \mathbf{R}^T(t)\mathbf{W}^T(t)v(t) \\
&= \frac{d}{dt}\left(\mathbf{R}^T(t)v(t)\right) + \mathbf{R}^T(t)\left(\omega(t) \times v(t)\right).
\end{aligned}$$

Let $\hat{\omega}(t) = \mathbf{R}^t(t)\omega(t)$ be an angular velocity transformed into a body reference frame, we have

$$\begin{aligned}
\mathbf{R}^T(t)\dot{\omega}(t) &= \mathbf{R}^T(t)\frac{d}{dt}\omega(t) \\
&= \frac{d}{dt}\left(\mathbf{R}^T(t)\omega(t)\right) + \mathbf{R}^T(t)(\omega(t) \times \omega(t)) \\
&= \frac{d}{dt}\left(\mathbf{R}^T(t)\omega(t)\right) = \frac{d}{dt}\hat{\omega}(t) =: \dot{\hat{\omega}}(t),
\end{aligned}$$

which means, the transformation of the derivative equals to the derivative of the transformation. Let $\hat{K}(t) = \mathbf{R}^T(t)K(t)$ be a transformed total moment, from (2.28) now we have

$$\hat{\omega}(t) \times \mathbf{J}\hat{\omega}(t) + \mathbf{J}\dot{\hat{\omega}}(t) = \hat{K}(t), \tag{2.29}$$

which is the Euler's equation of the rigid body dynamics.

2.3 Linear analysis of the ODE control

In our system, we choose a reference frame which moves together with the wave, and assuming that the wave has a constant velocity. The illustration of the frame can be seen in Figure 2.1. The surfing board also moves together with the ocean wave, hence, the velocity of the surfing board at the stable condition equals to zero relative to the frame of reference. In the surfing problem, the only parameter that can be controled is the inclination angle of the board. In this work we simplify the goal to be to design an ODE control of the inclination angle that capable to control the position of the surfing board to be at a given desired point in 1-dimensional.

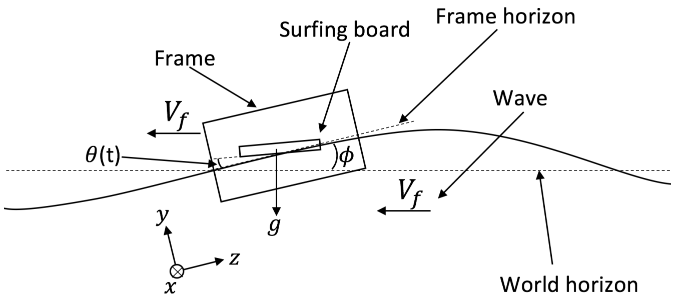


Figure 2.1: The illustration of the frame of system's domain.

Assume that we are interested to control the board in the third axis of the system. We introduce the following ODE control for the inclination angle,

$$\dot{\theta}(t) = a(Z(t) - \tilde{Z}) + b(V(t) - \tilde{V}) + c(\theta(t) - \tilde{\theta}), \tag{2.30}$$

where $Z(t)$ and $V(t)$ are the third components of position $X(t)$ and linear velocity of the surfing board $U(t)$, respectively, and $\theta(t)$ is the inclination angle of a surfing board. $\tilde{Z}$, $\tilde{V}$, and $\tilde{\theta}$ are the desired position, desired velocity, and the desired angle, respectively. a, b, and c are constants to be fixed below. The desired position $\tilde{Z}$ is given. Since in a stable condition, the velocity of the surfing board is zero relative to both the reference frame and the ocean wave, we take $\tilde{V} = 0$. $\tilde{\theta}$ is a desired inclination angle that helps stabilize the board. Up to now we do not have any information about $\tilde{\theta}$.

To find suitable parameters a, b, and c, we consider a simplified linearized ODE model with the ODE control (2.30). Then we do a stability analysis of this ODE model and choose parameters a, b, and c that can stabilize the system.

From the definition, we have $\dot{Z}(t) = V(t)$. We assume that the acceleration of the rigid body depends on an external body force (in this case, gravity) and a drag force. Since we do not know the drag force in our system, by assuming that the system is close to the stationary point, we can linearize the acceleration of the system as a function of the position, velocity, and the inclination angle of the board. Let $\xi(t) = (Z(t), V(t), \theta(t))$ be the unknown of the system. We consider a simplified linearized ODE model in the following form:

$$\begin{cases} \dot{Z}(t) = V(t), \\ \dot{V}(t) = -\mu\theta(t) - \mu_v V(t) - \mu_z Z(t) - \mu_0, \\ \dot{\theta}(t) = a(Z(t) - \tilde{Z}) + bV(t) + c(\theta(t) - \tilde{\theta}), \end{cases} \tag{2.31}$$

where μ, μ_v, μ_z, and μ_0 here are constants related to the drag and gravity forces. For now, let us assume $\mu_v = \mu_z = 0$ for simplicity. Later in the results section we will see that this assumption is not correct, but it does not seem to influence the stability.

We can write (2.31) in a matrix form as

$$\begin{aligned} \dot{\xi}(t) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -\mu \\ a & b & c \end{bmatrix} \xi(t) + \begin{bmatrix} 0 \\ -\mu_0 \\ 0 \end{bmatrix} \\ &=: \mathbf{A}\xi(t) + \zeta. \end{aligned} \tag{2.32}$$

The stationary point of (2.32) is

$$\xi^* = -\mathbf{A}^{-1}\zeta, \tag{2.33}$$

and the solution of (2.32) is in a form of

$$\xi(t) = \xi^* + \nu(t), \tag{2.34}$$

where ν is the solution of

$$\dot{\nu}(t) = \mathbf{A}\nu(t), \tag{2.35}$$

which is

$$\nu(t) = e^{\mathbf{A}t}\nu_0, \tag{2.36}$$

where $\nu_0 = \nu(0)$. By plugging in (2.36) into (2.34), we have

$$\xi(t) = \xi^* + e^{\mathbf{A}t}\nu_0, \tag{2.37}$$

and for $t = 0$ we have

$$\nu_0 = \xi_0 - \xi^*, \tag{2.38}$$

with $\xi_0 = \xi(0)$. Now we have the solution $\xi(t)$ which is

$$\xi(t) = \xi^* + e^{\mathbf{A}t} \left(\xi_0 - \xi^*\right). \tag{2.39}$$

Since we want $\xi(t) = \xi^*$, $e^{\mathbf{A}t}$ must go to zero.

Let us decompose matrix $\mathbf{A}$ into $\mathbf{A} = \mathbf{PDP}^{-1}$ with $\mathbf{P}$ is a matrix composed by eigenvectors of $\mathbf{A}$, and $\mathbf{D}$ is a diagonal matrix constructed by its corresponding eigenvalues. The exponent of matrix could be represented by a power series

$$e^{\mathbf{A}t} = \sum_{i=0}^{\infty} \frac{\mathbf{A}^i t^i}{i!} = \sum_{i=0}^{\infty} \frac{\left(\mathbf{PDP}^{-1}\right)^i t^i}{i!} = \mathbf{P} \sum_{i=0}^{\infty} \left(\frac{\mathbf{D}^i t^i}{i!}\right) \mathbf{P}^{-1} = \mathbf{P} e^{\mathbf{D}t} \mathbf{P}^{-1}, \tag{2.40}$$

with $\mathbf{A}^0 = \mathbf{D}^0 = \mathbf{I}$ is an identity matrix. Now we can make $e^{\mathbf{A}t}$ goes to zero by making $e^{\mathbf{D}t}$ goes to zero, which can be done by choosing such a, b, and c that make eigenvalues (or real part of eigenvalues, if they are complex numbers) of $\mathbf{A}$ to be negative.

The characteristic equation of $\mathbf{A}$ is

$$\det(\mathbf{A} - \lambda \mathbf{I}) = -\lambda^3 + c\lambda^2 - b\mu\lambda - a\mu = 0 \tag{2.41}$$

By using Vieta's formulas, we see that for all roots of (2.41) to be negative, the value of a and b must be positive, and c must be negative.

Let us set the roots of (2.41) to be $\lambda_1 = -3$, $\lambda_2 = -4$, and $\lambda_3 = -5$. This yields the equation

$$-\lambda^3 - 12\lambda^2 - 47\lambda - 60 = 0. \tag{2.42}$$

Comparing this with (2.41), we get $a = \frac{60}{\mu}$, $b = \frac{47}{\mu}$, and $c = -12$.

The inclination angle from the ODE control is passed to the simulation as a target inclination angle for controlling the movement of the rigid body. We model the controlling action of the surfer on a surfing board by using a two-points model, imitating the locations where the surfer places their feet. Both points are located

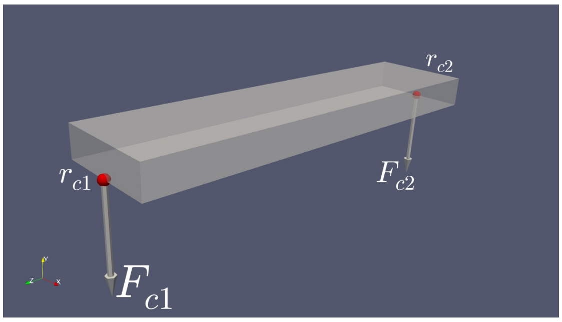


FIGURE 2.2: The locations of contact points.

at the tips of the elongated axis of the rigid body and can be seen in Figure 2.2. The total force is constant, but the individual forces are controlled by a ratio $T(t)$ as

$$F_{c1}(t) = T(t)W \tag{2.43}$$

$$F_{c2}(t) = (1 - T(t))W, \tag{2.44}$$

where W is the weight of the surfer, $F_{c1}(t)$ and $F_{c2}(t)$ are the forces given at each point at a given time t, with the total force is constant as they represent the weight of the surfer. The ratio is controlled by the linear function

$$T(t) = 0.5 - \sigma(\hat{\theta}(t) - \min(\theta(t), \theta_m)), \tag{2.45}$$

where $\theta(t)$ is the angle from the ODE control in Section 2.3, θ_m is an allowed maximum inclination angle, and $\hat{\theta}(t)$ is an observed angle at a given time t. σ is a given constant. Here we use a clipped function for the inclination angle since the surfer does not want the front part of their surfing board to be immersed into the water. Note that we choose the inclination angle to be negative when the board is inclined upward, and vice versa. We do not impose $T(t) \in [0, 1]$, which is reasonable if the surfer is using straps on their feet.

Chapter 3

Smoothed Particle Hydrodynamics

3.1 Basic Idea of the SPH Method

The basic idea of the SPH method comes from a convolution. Convolution between any two measurable functions u and v in $\mathbb{R}^n$ is defined as

$$(u * v)(x) := \int_{\mathbb{R}^n} u(y)v(x-y)\ dy, \quad x \in \mathbb{R}^n. \tag{3.1}$$

Some basic properties of convolution are

1. $u * v = v * u$
2. $u * (v * w) = (u * v) * w$

where u, v, and w are measurable functions in $\mathbb{R}^n$, and assuming all integrals exist. Another property of convolution is, if $u \in L^1(\mathbb{R}^n)$ and $v \in L^p(\mathbb{R}^n)$, then $u * v \in L^p(\mathbb{R}^n)$ and $||u * v||_p \leq ||u||_1||v||_p$, see [22] for more details. Because of this property, convolution is useful for constructing approximations of a function by using a function that are more regular. In other words, we can construct arbitrarily smooth functions by convolving a function with a smooth function. The smooth function used in convolution is usually called the kernel or the mollifier function.

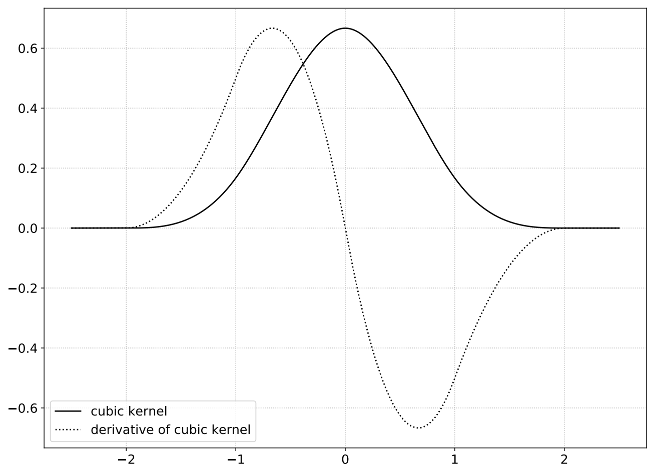


FIGURE 3.1: The graph of cubic kernel and its first derivative in 1D.

In the SPH method we choose a compact-supported sufficiently smooth mollifier function $\psi \in C_c^k(\mathbb{R}^n)$. Let us define

$$\psi_h(x) := \frac{1}{h^n}\psi\left(\frac{x}{h}\right), \quad x \in \mathbb{R}^n, \tag{3.2}$$

with $h > 0$. We got that

$$\int_{\mathbb{R}^n} \psi_h(x)\ dx = \int_{\mathbb{R}^n} \psi(x)\ dx, \tag{3.3}$$

for all $h > 0$, but the "mass" of ψ_h is more concentrated to the origin point as $h \to 0$. If we have $\int_{\mathbb{R}^n} \psi(x)\ dx = 1$, the family of functions $\{\psi_h\}_{h>0}$ is called an approximate identity. For any field function $f \in C^k(\mathbb{R}^n)$ for some $1 \leq k < \infty$, we have $f * \psi_h \to f$ uniformly as $h \to 0$,

$$f(x) \approx \int_{\mathbb{R}^n} f(y)\psi_h(x-y)\ dy, \tag{3.4}$$

and for the derivative of f is [22]

$$\partial^\alpha f(x) \approx \int_{\mathbb{R}^n} f(y)\partial^\alpha \psi_h(x-y)\ dy, \qquad |\alpha| \leq k, \tag{3.5}$$

for $\psi \in C_c^k(\mathbb{R}^n)$. Both (3.4) and (3.5) are SPH approximations for a field function and its derivative in an integral form.

We choose a smoothing function ψ. One of the most popular kernel function for SPH method is a piecewise cubic kernel function [23]

$$\psi(x) = \frac{\alpha_n}{6} \begin{cases} (2-|x|)^3 - 4(1-|x|)^3, & 0 \leq |x| < 1 \\ (2-|x|)^3, & 1 \leq |x| < 2 \\ 0, & 2 \leq |x| \end{cases} \tag{3.6}$$

where α_n is equal to 1, $\frac{15}{7\pi}$, or $\frac{3}{2\pi}$ for $n = 1, 2, 3$ respectively. Note that here $\psi \in C_c^2(\mathbb{R}^n)$.

3.2 SPH Approximation for Fluid Dynamics

To discretize the fluid, let us have N material points of the fluid which bring their own physical quantities with the position of a point i at a given time t is $X_i(t) = X^{x_0^i}(t)$ with x_0^i is a position of point i at $t = 0$. Notice that now we are "tagging" points using an index i. Let all N points are reasonably uniformly distributed. Now we approximate (3.4) and (3.5) by discrete integrations,

$$f(x) \approx \int_{\mathbb{R}^n} f(y)\psi_h(x-y)\, dy \approx \sum_{i=1}^{N} f(X_i)\psi_h(x-X_i)V(E_i), \tag{3.7}$$

and

$$\partial^\alpha f(x) \approx \int_{\mathbb{R}^n} f(y)\partial^\alpha\psi_h(x-y)\, dy \approx \sum_{i=1}^{N} f(X_i)\partial^\alpha\psi_h(x-X_i)V(E_i), \tag{3.8}$$

where $V(E_i)$ is a volume of the set E_i around a point X_i which we assume that the integrand to be a constant or close to being linear if E_i is symmetric and X_i is at its center of mass. Take a note that we drop a time variable t from equations to simplify the derivation. We need an efficient way to approximate the volume $V(E_i)$. One way is by using the Voronoi tesselation, by defining E_i as the closest point to X_i,

$$E_i = \{x : |x - X_i| < |x - X_j|,\ i \neq j,\ i, j \in \{1, 2, \ldots, N\}\}. \tag{3.9}$$

Another common method to estimate the volume $V(E_i)$ is by using a mass and density at a point X_i,

$$V(E_i) \approx \frac{m_i}{\rho_i}, \tag{3.10}$$

where m_i and ρ_i are a mass and density at a point X_i, respectively. By using (3.7) for density field function and by (3.10), we have

$$\rho(x) \approx \sum_{i=1}^{N} \rho_i \psi_h(x - X_i) V(E_i) \approx \sum_{i=1}^{N} m_i \psi_h(x - X_i). \tag{3.11}$$

Then we have an approximation for $V(E_i)$ as

$$V_i(E_i) \approx \frac{m_i}{\sum_{j=1}^{N} m_j \psi_h(X_i - X_j)}. \tag{3.12}$$

We can use (3.11) to approximate a density of a point i,

$$\rho(X_i) = \sum_{j=1}^{N} m_j \psi_h(X_i - X_j). \tag{3.13}$$

(3.13) is one of the most common method to update density in SPH, such as in [24][5].

Another popular method to update the density is by using the SPH approximation for div(u) in (2.9). Let $u_i(t) = u(X_i(t), t) = (u_{i,1}(X_i(t), t), u_{i,2}(X_i(t), t), u_{i,3}(X_i(t), t))$, and let us drop the time variable t,

$$\begin{aligned}
\operatorname{div}(u_i) &= \sum_{\alpha=1}^{3} \partial^\alpha u_{i,\alpha} \\
&\approx \sum_{\alpha=1}^{3} \sum_{j=1}^{N} \frac{m_j}{\rho_j} u_{j,\alpha} \partial^\alpha \psi_h(X_i - X_j) \\
&= \sum_{j=1}^{N} \frac{m_j}{\rho_j} u_j \cdot \nabla \psi_h(X_i - X_j),
\end{aligned} \tag{3.14}$$

where m_i, ρ_i and u_i are a mass, density, and velocity at a point X_i, respectively. Now, the approximation for (2.9) is

$$\frac{d\rho_i}{dt} = -\rho_i \sum_{j=1}^{N} \frac{m_j}{\rho_j} u_j \cdot \nabla\psi_h(X_i - X_j). \tag{3.15}$$

Considering that the SPH approximation of the derivative of a constant $C = 1$ is

$$0 = \partial^\alpha C = \sum_{j=1}^{N} \frac{m_j}{\rho_j} \partial^\alpha \psi_h(X_i - X_j). \tag{3.16}$$

We can write (3.16) as

$$\rho_i u_i \sum_{j=1}^{N} \frac{m_j}{\rho_j} \partial^\alpha \psi_h(X_i - X_j) = \rho_i \sum_{j=1}^{N} \frac{m_j}{\rho_j} u_i \partial^\alpha \psi_h(X_i - X_j). \tag{3.17}$$

By adding (3.17) to (3.15), we have [23]

$$\frac{d\rho_i}{dt} = \rho_i \sum_{j=1}^{N} \frac{m_j}{\rho_j} (u_i - u_j) \cdot \nabla\psi_h(X_i - X_j). \tag{3.18}$$

(3.18) is the anti-symmetrized form of an SPH approximation for (2.9). This form reduces errors arising from the points inconsistency problem [24][23][7]. Another common anti-symmetrized form of continuity equation is achieved by modifying the righthand side of (2.9) into

$$-\rho_i \mathrm{div}\,(u_i) = -\left(\mathrm{div}\,(\rho_i u_i) - u_i \cdot \nabla\rho_i\right), \tag{3.19}$$

so the approximation of the continuity equation becomes

$$\begin{aligned} \frac{d\rho_i}{dt} &= -\left(\sum_{j=1}^{N} \left(\frac{m_j}{\rho_j} \rho_j u_j \cdot \nabla\psi_h(X_i - X_j)\right) - u_i \cdot \sum_{j=1}^{N} \left(\frac{m_j}{\rho_j} \rho_j \nabla\psi_h(X_i - X_j)\right)\right) \\ &= \sum_{j=1}^{N} m_j (u_i - u_j) \cdot \nabla\psi_h(X_i - X_j). \end{aligned} \tag{3.20}$$

Both density summation approximation (3.15) and density continuity approximations (3.18) and (3.20) have their own advantages and disadvantages. While the summation approximation conserves total mass of the system, the continuity approximation does not conserve total mass. However, the summation approximation gives the "edge" effect while some points lack of neighboring points, i.e. near the boundaries, near material interfaces for multi-phase system when points from different phases are not allowed to interact, or when they are located far from the rest of points. The edge effect could lead to the clumping of points, smoothen out the density of points near the edge, and make the distribution of points irregular between the edge with the interior points.

By applying (3.8) directly to the SPH approximation of momentum equation for inviscid fluid (2.18), we have

$$\begin{aligned}\frac{du_i}{dt} &= -\frac{1}{\rho_i}\nabla p_i + b_i \\ &= -\frac{1}{\rho_i}\sum_{j=1}^{N}\left(\frac{m_j}{\rho_j}p_j\nabla\psi_h(X_i - X_j)\right) + b_i,\end{aligned} \tag{3.21}$$

where p_i is a pressure at point X_i and b_i is an external (body) force acts on X_i. Similar with (3.17), we can rewrite (3.16) into

$$-\frac{p_i}{\rho_i}\sum_{j=1}^{N}\frac{m_j}{\rho_j}\partial^\alpha\psi_h(X_i - X_j) = -\frac{1}{\rho_i}\sum_{j=1}^{N}\frac{m_j}{\rho_j}p_i\partial^\alpha\psi_h(X_i - X_j) = 0, \tag{3.22}$$

and adding (3.22) into (3.21) yields

$$\frac{du_i}{dt} = -\sum_{j=1}^{N}\left(m_j\frac{p_i + p_j}{\rho_i\rho_j}\nabla\psi_h(X_i - X_j)\right) + b_i, \tag{3.23}$$

which is one of the anti-symmetrized form of SPH approximations for the momentum equation. We can get another popular anti-symmetrized form by considering

$$-\frac{1}{\rho_i}\nabla p_i = -\left(\nabla\left(\frac{p_i}{\rho_i}\right) - p_i\nabla\left(\frac{1}{\rho_i}\right)\right) = -\left(\nabla\left(\frac{p_i}{\rho_i}\right) + \frac{p_i}{\rho_i^2}\nabla\rho_i\right), \tag{3.24}$$

and modify the first term on the righthand side of (2.18) into (3.24), the SPH approximation for momentum equation becomes

$$\begin{aligned}\frac{du_i}{dt} &= -\left(\sum_{j=1}^{N}\left(m_j\frac{p_j}{\rho_j^2}\nabla\psi_h(X_i - X_j)\right) + \frac{p_i}{\rho_i^2}\sum_{j=1}^{N}\left(\frac{m_j}{\rho_j}\rho_j\nabla\psi_h(X_i - X_j)\right)\right) + b_i \\ &= -\sum_{j=1}^{N}\left(m_j\left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2}\right)\nabla\psi_h(X_i - X_j)\right) + b_i.\end{aligned} \tag{3.25}$$

Now we approximate the incompressibility of the fluid by a "slightly compressible" condition. That is, it allows the density of points to deviate from the reference density. Then we choose the pressure to be a function of a density

$$p_i = B\left(\left(\frac{\rho_i}{\rho_0}\right)^{\gamma} - 1\right), \tag{3.26}$$

where ρ_0 is a reference density, $\gamma = 7$ for water-like fluid, and B is a problem dependent parameter. B can be chosen to be equal to the reference density ρ_0 [25], or set it to be [26]

$$B = \frac{\rho_0 c_s^2}{\gamma}, \tag{3.27}$$

where c_s is a speed of sound in a material, or in this case, speed of sound in a fluid.

Up to now we consider the fluid as an inviscid fluid. If we want to implement the SPH approximation to a viscous fluid, we need to do the approximation for the momentum equation for a viscous fluid. Another approach to do the SPH simulation for a viscous fluid is by adding an artificial viscosity term to the approximation of momentum equation (3.25),

$$\frac{du_i}{dt} = -\sum_{j=1}^{N}\left(m_j\left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} + \Pi_{ij}\right)\nabla\psi_h(X_i - X_j)\right) + b_i. \tag{3.28}$$

where Π_{ij} is the artificial viscosity between points X_i and X_j. Π_{ij} is defined as [23]

$$\Pi_{ij} = \frac{-\alpha_\Pi c_s \nu_{ij} + \beta_\Pi \nu_{ij}^2}{\overline{\rho_{ij}}}, \tag{3.29}$$

where α_Π and β_Π are artificial viscosity constants which values are around magnitude of 1 [23], c_s is a speed of sound in fluid, $\overline{\rho_{ij}} = \frac{\rho_i+\rho_j}{2}$ is an average density betwen points X_i and X_j, and

$$\nu_{ij} = \frac{h\,\overline{u_{ij}}\cdot\overline{X_{ij}}}{|\overline{X_{ij}}|^2 + 0.01h^2}, \tag{3.30}$$

where $\overline{u_{ij}} = \frac{u_i+u_j}{2}$, $\overline{X_{ij}} = \frac{X_i+X_j}{2}$, and h is a kernel function parameter.

One choice to do a time integration for the SPH method is by using the second-order accurate leapfrog method. The leapfrog method is a symplectic solver and suits well to be used with the SPH simulation. The leapfrog method also gives better accuracy while using almost the same computational cost compared to Euler integrator. Here we use leapfrog scheme as our time integrator. Leapfrog integrator scheme can be written as

$$u_i\left(t+\frac{\tau}{2}\right) = u_i\left(t-\frac{\tau}{2}\right) + a_i\left(t\right)\tau \tag{3.31}$$

$$X_i\left(t+\tau\right) = X_i\left(t\right) + u_i\left(t+\frac{\tau}{2}\right)\tau, \tag{3.32}$$

where $a_i(t) = \frac{du_i}{dt}(t)$ is an acceleration of a material point X_i at a given time t, and τ is a timestep. To approximate the velocity at an even timestep, we use

$$u_i(t+\tau) = u_i\left(t+\frac{\tau}{2}\right) + a_i(t)\frac{\tau}{2}. \tag{3.33}$$

At the very first timestep, we need the value $u_i\left(-\frac{\tau}{2}\right)$ which can be approximated by

$$u_i\left(-\frac{\tau}{2}\right) = u_i(0) - a_i(0)\frac{\tau}{2}. \tag{3.34}$$

3.3 Rigid Body Dynamics

From (2.29), we can rewrite it as

$$\mathbf{J}\dot{\hat{\omega}}(t) + \hat{\omega}(t) \times \mathbf{J}\hat{\omega}(t) = \hat{K}(t). \tag{3.35}$$

Since $\mathbf{J}$ is a diagonal matrix, we can write (3.35) component-wise as

$$J_\alpha\dot{\hat{\omega}}_\alpha(t) = \hat{K}(t)_\alpha + \hat{\omega}_\beta(t)\hat{\omega}_\gamma(t)\left(J_\beta - J_\gamma\right), \tag{3.36}$$

where $(\alpha,\beta,\gamma) = (1,2,3), (2,3,1),$ and $(3,1,2)$. Following [27], in the spirit of the leapfrog time integrator, the angular velocity vector $\hat{\omega}(t)$ can be updated by using the fixed point iteration

$$\hat{\omega}_\alpha^{(m+1)}\left(t+\frac{\tau}{2}\right) = \hat{\omega}_\alpha\left(t-\frac{\tau}{2}\right) + \frac{\tau}{J_\alpha}\left(\hat{K}_\alpha(t) + \hat{\omega}_\beta^{(m)}(t)\hat{\omega}_\gamma^{(m)}(t)\left(J_\beta - J_\gamma\right)\right), \tag{3.37}$$

where τ is a timestep, and (m) is an iteration step. Initially, we can set

$$\hat{\omega}_\alpha^{(0)}\left(t+\frac{\tau}{2}\right)=\hat{\omega}_\alpha\left(t-\frac{\tau}{2}\right).$$

We also choose

$$\hat{\omega}_\beta^{(m)}(t)\hat{\omega}_\gamma^{(m)}(t)=\frac{1}{2}\left(\hat{\omega}_\beta\left(t-\frac{\tau}{2}\right)\hat{\omega}_\gamma\left(t-\frac{\tau}{2}\right)\ +\ \hat{\omega}_\beta^{(m)}\left(t+\frac{\tau}{2}\right)\hat{\omega}_\gamma^{(m)}\left(t+\frac{\tau}{2}\right)\right).$$

Later, we update the rotation matrix by using (2.26) and in the sense of leapfrog integrator as

$$\begin{aligned}\mathbf{R}(t+\tau)&=\mathbf{R}(t)+\tau\frac{d}{dt}\mathbf{R}\left(t+\frac{\tau}{2}\right)\\&=\mathbf{R}(t)+\tau\mathbf{W}\left(t+\frac{\tau}{2}\right)\mathbf{R}\left(t+\frac{\tau}{2}\right),\end{aligned}$$

where $\mathbf{W}$ is a matrix from (2.27) and $\mathbf{R}\left(t+\frac{\tau}{2}\right)=\frac{(\mathbf{R}(t)+\mathbf{R}(t+\tau))}{2}$. By some algebraic operations, now we have

$$\begin{aligned}\mathbf{R}(t+\tau)&=\left(\mathbf{I}-\frac{\tau}{2}\mathbf{W}\left(t+\frac{\tau}{2}\right)\right)^{-1}\left(\mathbf{I}+\frac{\tau}{2}\mathbf{W}\left(t+\frac{\tau}{2}\right)\right)\mathbf{R}(t)\\&=\Theta\left(t+\frac{\tau}{2}\right)\mathbf{R}(t),\end{aligned}\tag{3.38}$$

where $\mathbf{I}$ is an identity matrix, and

$$\Theta\left(t+\frac{\tau}{2}\right):=\left(\mathbf{I}-\frac{\tau}{2}\mathbf{W}\left(t+\frac{\tau}{2}\right)\right)^{-1}\left(\mathbf{I}+\frac{\tau}{2}\mathbf{W}\left(t+\frac{\tau}{2}\right)\right),$$

or more explicitly

$$\Theta\left(t+\frac{\tau}{2}\right)=\frac{\mathbf{I}\left(1-\frac{\tau^2}{4}\omega^2\left(t+\frac{\tau}{2}\right)\right)-\tau\mathbf{W}\left(t+\frac{\tau}{2}\right)+\frac{\tau^2}{2}\left(\omega\otimes\omega\right)\left(t+\frac{\tau}{2}\right)}{1+\frac{\tau^2}{4}\omega^2\left(t+\frac{\tau}{2}\right)},$$

where $\omega^2\left(t+\frac{\tau}{2}\right)=(\omega\cdot\omega)\left(t+\frac{\tau}{2}\right)$.

3.4 Rigid body discretization and coupling with SPH

The rigid body is discretized into N_b points with a regular distance between them. In this work we set the distance between rigid body points to be equal to h. A position of each point is denoted by $X_i(t) = (X_{i,1}(t), X_{i,2}(t), X_{i,3}(t))$ at a given time t. To make the rigid body dynamics calculation easier, we configure the calculation of the rigid body to be in a principal frame. Let $\check{\mathbf{J}}$ be the moment of inertia tensor of the rigid body. By assuming the density of the rigid body is uniform and equals to ρ_b, components of $\check{\mathbf{J}}$ are

$$\begin{aligned}
\check{J}_{11} &= \rho_b \sum_{i=1}^{N_b} \left(X_{i,2}^2 + X_{i,3}^2 \right)\ dx & \check{J}_{12} &= \check{J}_{21} = -\rho_b \sum_{i=1}^{N_b} X_{i,1} X_{i,2}\ dx \\
\check{J}_{22} &= \rho_b \sum_{i=1}^{N_b} \left(X_{i,1}^2 + X_{i,3}^2 \right)\ dx & \check{J}_{13} &= \check{J}_{31} = -\rho_b \sum_{i=1}^{N_b} X_{i,1} X_{i,3}\ dx \\
\check{J}_{33} &= \rho_b \sum_{i=1}^{N_b} \left(X_{i,1}^2 + X_{i,2}^2 \right)\ dx & \check{J}_{23} &= \check{J}_{32} = -\rho_b \sum_{i=1}^{N_b} X_{i,2} X_{i,3}\ dx.
\end{aligned} \tag{3.39}$$

We calculate the *principal* moment of inertia tensor $\mathbf{J}$ of the rigid body by doing an eigendecomposition of the moment of inertia tensor $\check{\mathbf{J}}$. Eigenvalues of $\check{\mathbf{J}}$ are diagonal components of $\mathbf{J}$, while their corresponding eigenvectors construct an orthogonal rotation matrix $\mathbf{R}(0)$.

$$\check{\mathbf{J}} = \Lambda \mathbf{J} \Lambda^{-1}, \qquad \mathbf{R}(0) = \Lambda. \tag{3.40}$$

Similar with fluid points, we keep track of the physical quantities with rigid body points as well. Rigid body points also have a density function which is set to be equal with reference density ρ_0. The mass and volume of rigid body points is also set to match with our SPH configuration described in (3.10) and (3.12). Take a note that the density of rigid body points is a completely different quantity with the density of rigid body ρ_b.

After the discretization process, interactions between fluid points and rigid body points are handled in exact same ways with interactions between fluid points. Fluid points see rigid body points as exactly *same objects with other fluid points* *and do not discriminate interactions between all of them*, on both density update step and acceleration update step.

As mentioned before in the introduction part, in this work we consider a purely hydrodynamics force for interactions between rigid body points and fluid points. Based on (3.25), since the density of rigid body point is equal to the density reference yielding zero pressure for rigid body points, total forces applied on a rigid body points X_i is

$$f_i = -m_i c_V \left(\left(\sum_{j=1}^{N} m_j \frac{p_j}{\rho_j^2} \nabla \psi_h (X_i - X_j) \right) + b_i \right), \tag{3.41}$$

where $c_V = \frac{h^3}{V(E_i)}$. c_V is needed since there is a discrepancy on an SPH volume calculation $V(E_i)$ compared to the volume of the cube with length of the edge h occupied by each rigid body point with. Take a note that (3.41) is calculated only for points r_j are fluid points. We do not consider an interaction between rigid-rigid points.

To calculate a linear movement of the rigid body, from (2.25), we have

$$A(t) = \frac{1}{M} \left(\left(\sum_{i=1}^{N_b} f_i(t) \right) + F_{c1} + F_{c2} \right), \tag{3.42}$$

where M is the mass of the rigid body, $A(t)$ is a linear acceleration of the rigid body, $f_i(t)$ is a total forces applied on a rigid point X_i, F_{c1} and F_{c2} are forces from the ODE control given at a contact point 1 and contact point 2, respectively. Updating a linear velocity and a position of the center of mass of the rigid body can be done using the leapfrog time integrator scheme in a similar fashion described in (3.31)–(3.34).

Next, to calculate a rotation movement of the rigid body, we need to sum total moments of force applied on the rigid body,

$$K(t) = \left(\sum_{i=1}^{N_b} (r_i(t) - X(t)) \times f_i(t) \right) + (r_{c1}(t) - X(t)) \times F_{c1} + (r_{c2}(t) - X(t)) \times F_{c2}, \tag{3.43}$$

where $K(t)$ and $X(t)$ are a total moments of force applied on the rigid body and a position of the rigid body at a given time t, respectively, $f_i(t)$ and $r_i(t)$ are a total forces applied on a rigid body point X_i and a position of a rigid body point at a given time t, r_{c1} and r_{c2} are a position of a contact point 1 and contact point 2, respectively. We use the Euler's equation of rigid body dynamics (2.29) and the iterative scheme (3.37) to update the angular velocity of the rigid body. Then we update the rotation matrix of the rigid body by using (3.38). The position of a rigid body point is updated based on its relative position to the center of mass of the rigid body,

$$X_i(t+\tau) = X(t+\tau) + \mathbf{R}(t+\tau)\mathbf{R}^T(t)\left(X_i(t) - X(t)\right), \tag{3.44}$$

where τ is the timestep. The velocity of a rigid body point is updated based on a linear velocity and an angular velocity of the rigid body

$$U_i(t+\tau) = U\left(t+\frac{\tau}{2}\right) + \frac{\tau}{2}A(t) + \omega(t+\tau) \times \left(X_i(t+\tau) - X(t+\tau)\right), \tag{3.45}$$

where $U_i(t)$ is a velocity of a rigid body point r_i at a given time t, $U(t)$ and $\omega(t)$ are a linear velocity and angular velocity of the rigid body at a given time t.

3.5 Algorithm of the simulation

The simulation is started by an initialization of all parameters needed for the simulation and initial configuration for all SPH points as discretized representations of fluid and rigid body. The main loop of the simulation is started with updating the density, pressure, and hydrodynamics forces-based acceleration of all SPH points. The forces applied on rigid body points are used to update a linear movement of the rigid body. Continued with solving the ODE to update the contact forces needed to control the rigid body. Then, all forces including the contact forces are used to calculate the rotational movement of the rigid body. Next, we update the velocity and position of the rigid body points based on the evolution of the rigid body. Finally, we update the velocity and position of the fluid points, and going back to the starting point of the main loop, until the final timestep is reached. The full algorithm of the simulation can be seen in Figure 3.2.

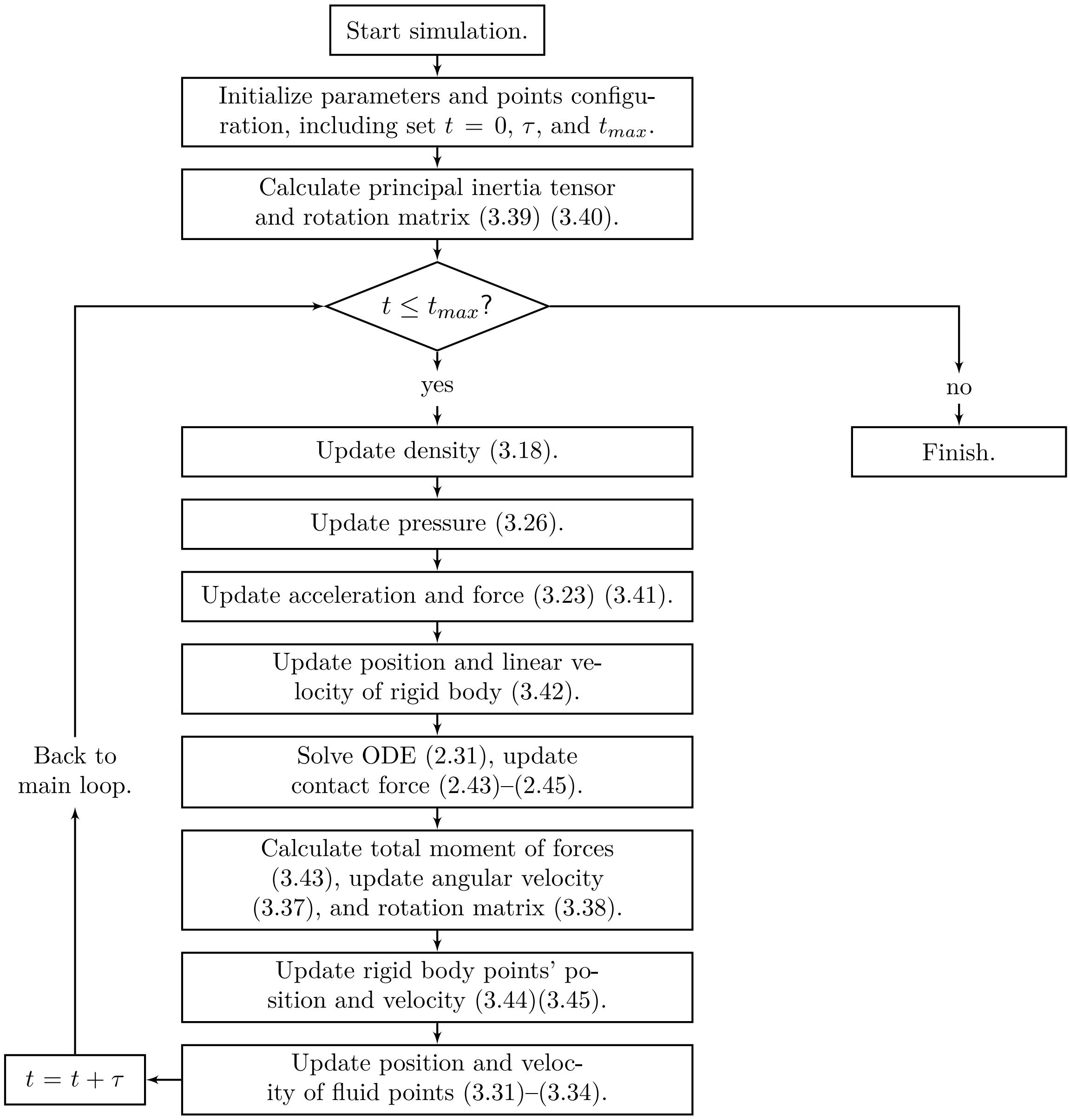


FIGURE 3.2: Algorithm of the simulation.

Chapter 4

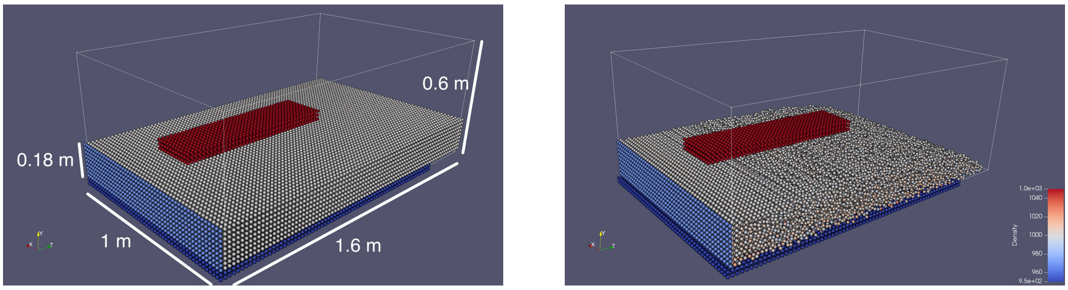
Results and Discussions

4.1 Set-up of the Simulation

In this work we choose the frame of reference which moves together with the ocean wave and inclined, so it is parallel with the slope part of the ocean wave. We assume the ocean wave moves with a constant speed $V_f = 2.5$ and the inclination angle (ϕ in Figure 2.1) to be $\phi = \frac{\pi}{18}$ degree. Since now the frame of reference is rotated, the direction of the gravity is slanted to $-z$-axis in a frame coordinate. Gravity is set to be $g = (0,\ -9.81\ \cos\phi,\ -9.81\ \sin\phi)$. The size of the domain is set to be $1 \times 0.6 \times 1.6$ centered at the origin point.

The system has a periodic boundary condition in x-axis, non-zero Dirichlet boundary on some parts of left boundary by using ghost points (rendered with light blue color in Figure 4.1a) for $-0.3 \leq y < -0.12$, bottom boundary is also set to be a non-zero Dirichlet boundary by using a non-moving boundary points (rendered with dark blue color in Figure 4.1a) for $-0.8 \leq z < 0.56$, and free boundary in right boundary, top boundary, left boundary for $-0.12 \leq y < 0.3$, and bottom boundary for $0.56 \leq z \leq 0.8$. Take a note that the origin point is located at the center of the domain. The depth of the fluid is 0.18. The size of the rigid body is $0.2 \times 0.06 \times 0.8$. The rigid body is represented by red points in Figure 4.1a. Initially, the center of mass of rigid body is positioned at $(0, -0.04, -0.27)$.

The density reference ρ_0 is set to be $\rho_0 = 1000$, while the density of the rigid body is $\rho_b = 100$. The fluid is initialized to have an initial velocity $u_i(0) = V_f = 2.5$ toward $+z$-axis, and initial density to be $\rho_i(0) = \rho_0$ for all fluid points X_i. By



(A) (B)

FIGURE 4.1: (a) The initial configuration of the system, and (b) the initial condition after relaxation process.

our choice of piecewise cubic kernel (3.6) as a mollifier function and choose the points to be initialized in a regular grid with a distance h in each axis, $V(E_i)$ and m_i for all points are $V(E_i) = 8 \times 10^{-6}$ and $m_i = 0.008$. We set the parameter of kernel function $h = 0.02$. Time step size is set to be $\tau = 0.0005$ with speed of sound is chosen to be $c = 20$. W from (2.43) and (2.44) is set to $W = 10$. We choose σ and θ_m of (2.45) to be $\sigma = 10$ and $\theta_m = -0.05$. The positions of contact point 1 and contact point 2 are $r_{c1} = (0,\ -0.03,\ -0.4)$ and $r_{c2} = (0,\ -0.03,\ 0.4)$, respectively, relative to the position of center of mass of the rigid body.

In this work we neglect the viscosity term. By looking at the Reynold's number of this setting,

$$Re = \frac{\rho_0 V_f L}{\nu} = \frac{1000 \times 2.5 \times 0.2}{0.001} \approx 5 \times 10^5,$$

where ν is the real dynamic viscosity of the water at 20 degree Celsius, and we take the characteristic length L to be the width of the rigid body, it is clear that the viscosity force is negligible compared to the inertial force of the fluid.

Free boundary condition is implemented by changing the type of any fluid points leaving the domain into a ghost point which its velocity does not change with time and its density always equal to the reference density ρ_0, but still interacts with other points. If a ghost point enters the domain, it will be marked as a normal fluid point again. But if it leaves the domain farther than h, the point will be removed from the simulation.

Before we run the actual simulation, we run the "relaxation" process to stabilize the flow of the water up to $t = 1.5$. The initial condition after relaxation can be seen on Figure 4.1b.

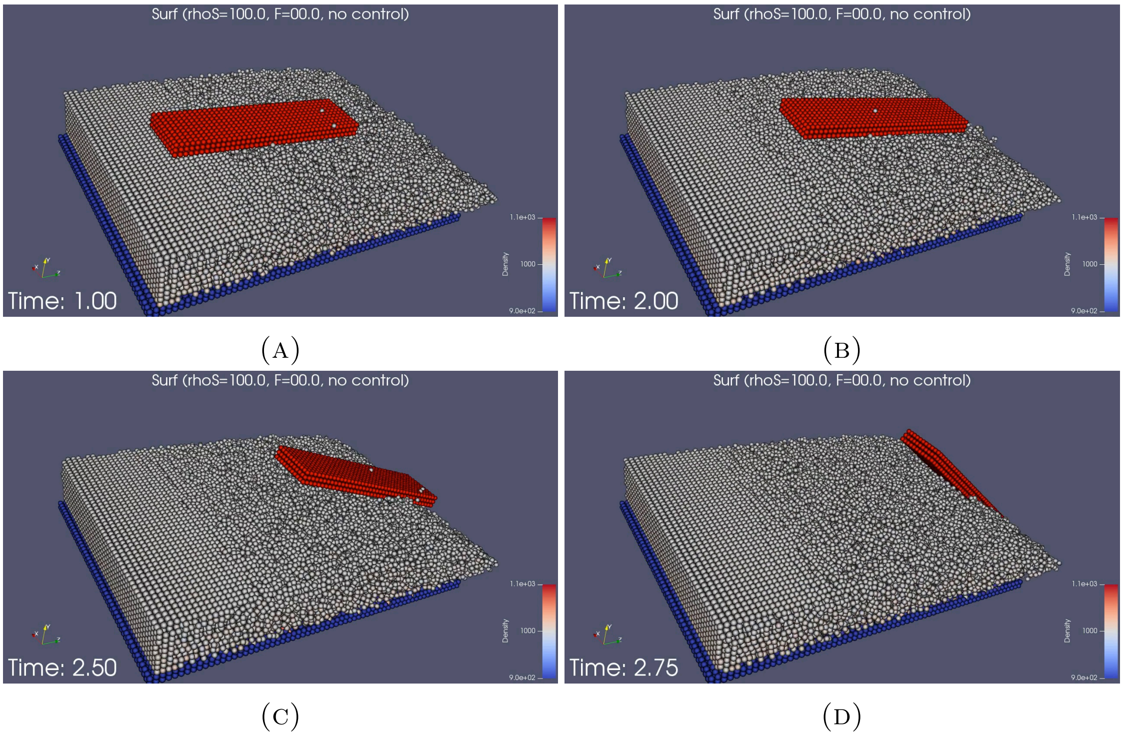


FIGURE 4.2: (A) The curve of positions in z-axis and inclination angles of the board through time without an ODE controller; snapshots of the surfing board simulation without an ODE controller at (B) $t = 1.00$ s, (C) $t = 2.00$ s, (D) $t = 2.50$ s, (E) $t = 2.75$ s.

4.2 Results and Discussions

First, we try to run a simulation without an ODE control. We can see in Figure 4.2 that the board cannot maintain its position and drifts away with the flow of the fluid.

To find the best $\tilde{\theta}$ and μ, for each $\tilde{Z} \in \{-0.6, -0.5, -0.4, -0.3, -0.2\}$ we try each combination of $\tilde{\theta} \in \{-0.05, -0.06, -0.07, -0.08, -0.09, -0.1\}$ and $\mu \in \{1, 2, 5, 10, 20, 50\}$. For each simulation, we take the data of the position of the center of mass of the rigid body for each time step to assess the quality of the corresponding parameters configuration. Let $Z(\mu, \tilde{\theta}, \tilde{Z}, m\tau)$ be the third component of the position of the center of mass of the rigid body at a time step $m\tau$ for a simulation with a parameters configuration μ, $\tilde{\theta}$, and $\tilde{Z}$, with $m \in \{0, 1, \dots, M\}$. To assess the chosen parameters configuration, we take an average of the difference between $Z(\mu, \tilde{\theta}, \tilde{Z}, m\tau)$ with a corresponding $\tilde{Z}$ for the whole simulation time,

$$\overline{\epsilon_Z}(\mu, \tilde{\theta}, \tilde{Z}) := \frac{\sum_{m=0}^{M} \left| Z(\mu, \tilde{\theta}, \tilde{Z}, m\tau) - \tilde{Z} \right|}{M}. \tag{4.1}$$

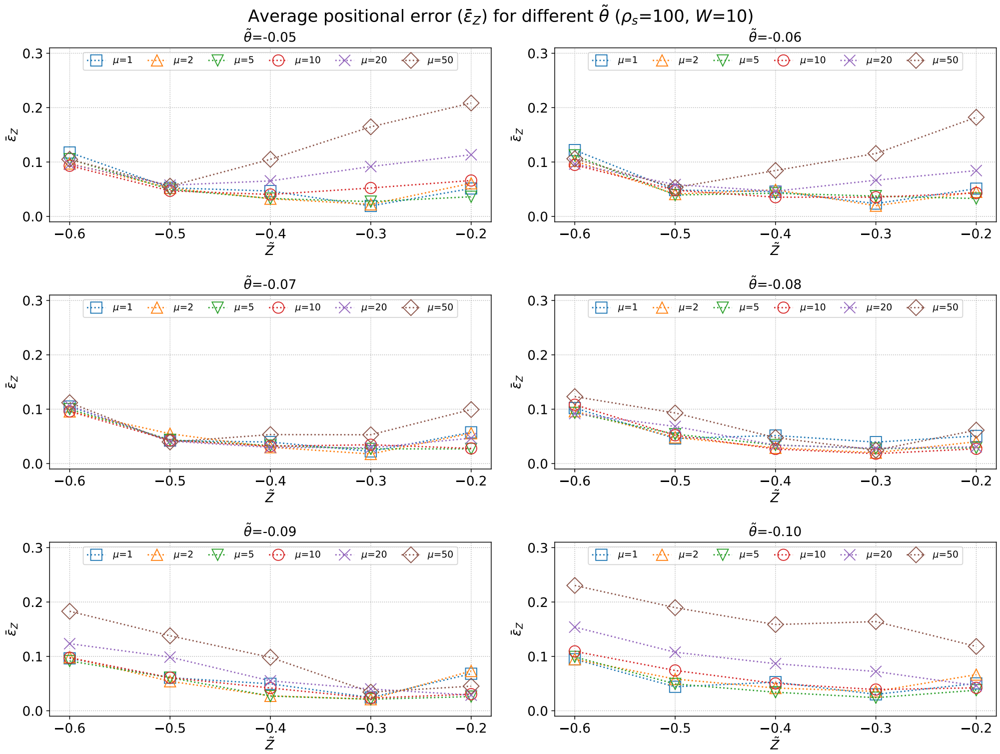


FIGURE 4.3: The graphs of the average positional error for different parameters $\tilde{\theta}$, μ, and $\tilde{Z}$.

Let us call $\bar{\epsilon_Z}$ as the average positional error. The graphs of $\bar{\epsilon_Z}$ can be seen on Figure 4.3. Then we take an average of the average positional error for each $\tilde{Z}$,

$$\overline{\overline{\epsilon_Z}}(\mu, \tilde{\theta}) := \frac{\sum_{\tilde{Z} \in \tilde{Z}_{\text{set}}} \overline{\epsilon_Z}(\mu, \tilde{\theta}, \tilde{Z})}{\# \tilde{Z}_{\text{set}}}, \qquad \tilde{Z}_{\text{set}} = \{-0.6, -0.5, -0.4, -0.3, -0.2\}, \quad (4.2)$$

and find cumulative errors for each $\tilde{\theta}$ and μ. The average of average positional error and its cumulative errors for each $\tilde{\theta}$ and μ can be seen on the following table:

TABLE 4.1: Table of the average of average positional error and its cumulative errors for each $\tilde{\theta}$ and μ.

$\mu \backslash \tilde{\theta}$	-0.05	-0.06	-0.07	-0.08	-0.09	-0.10	C.E.
1	5.74e-02	5.81e-02	5.33e-02	5.80e-02	5.98e-02	5.44e-02	3.41e-01
2	5.30e-02	5.13e-02	5.08e-02	4.67e-02	5.48e-02	5.94e-02	3.16e-01
5	5.02e-02	5.29e-02	4.62e-02	4.76e-02	4.53e-02	4.89e-02	2.91e-01
10	5.97e-02	5.13e-02	4.66e-02	4.65e-02	5.08e-02	6.29e-02	3.18e-01
20	8.47e-02	7.04e-02	5.07e-02	5.07e-02	6.92e-02	9.33e-02	4.19e-01
50	1.28e-01	1.08e-01	7.15e-02	7.00e-02	9.97e-02	1.72e-01	6.49e-01
C.E.	4.33e-01	3.92e-01	3.19e-01	3.20e-01	3.80e-01	4.91e-01	

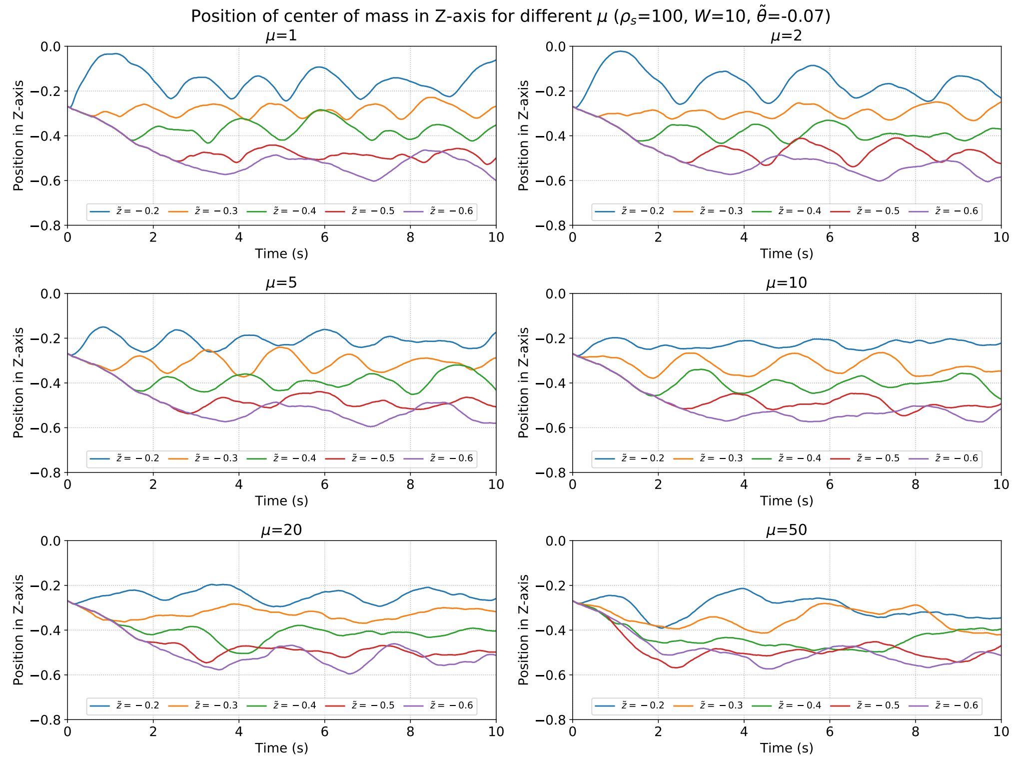


FIGURE 4.4: The z-axis-component position of surfing board for $\tilde{\theta} = -0.07$.

From Table 4.1 we can see that $\tilde{\theta} = -0.07$ and $\mu = 5$ give the smallest cumulative errors for $\tilde{\theta}$ and μ, respectively. Now let us see more in detail the simulation results for $\tilde{\theta} = -0.07$ in Figure 4.4.

As we can see in Figure 4.4, smaller values of μ give more oscillations to the position compared to larger μ values. But as the larger value of μ dampens the amplitude and frequency of oscillation to the position, it also shifts the stable position of surfing board. That problem also occurs when we solve the ODE (2.31) directly by using an ODE solver which we can see in Figure 4.5.

The direct ODE solver is used to observe the behavior of the solution under different parameters choices. The parameters a and b of the ODE control in (2.31) depend on μ. Note again that we do not know the actual value μ of our system. Hence, we do trials and errors by varying the value of μ in our simulation, yields a and b that differ from the correct ones. By using the direct ODE solver we want to see the effect of the choice of μ in simulation that mismatch with the actual μ of the system. In the direct ODE solver we choose $\mu_{\text{actual}} = 20$ and use various μ, effectively varying the values of a and b. We choose $\tilde{Z} = -0.5$. From Figure 4.5 we can see that as the value of μ increases, the oscillation is decreasing

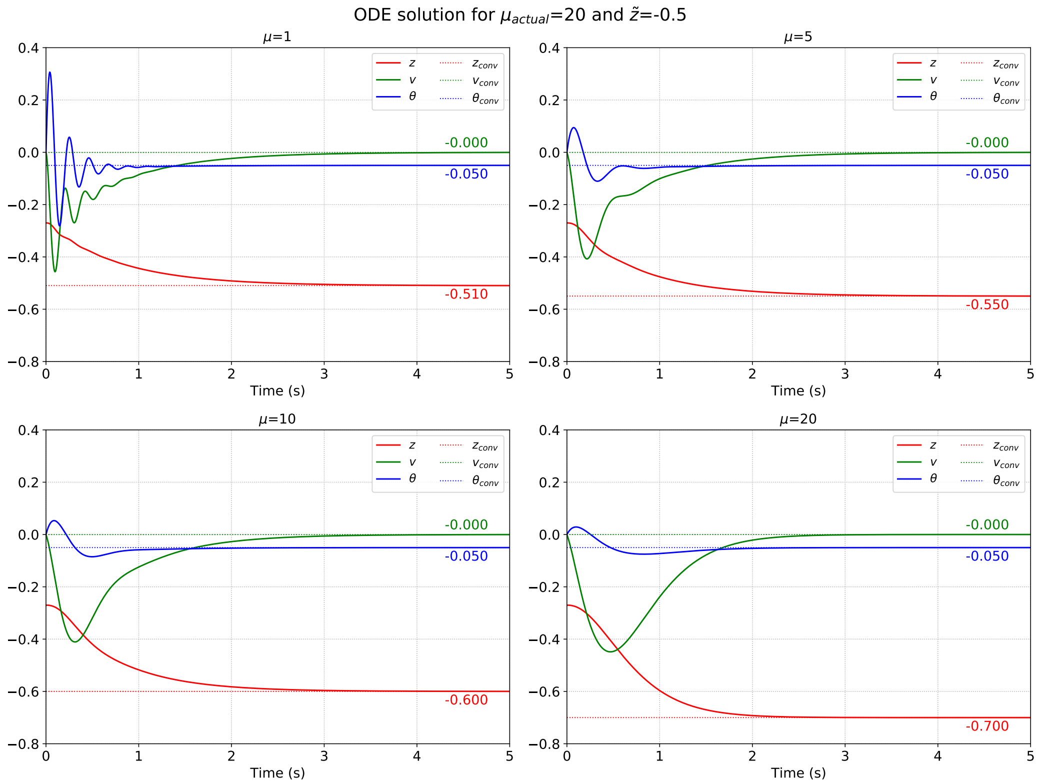
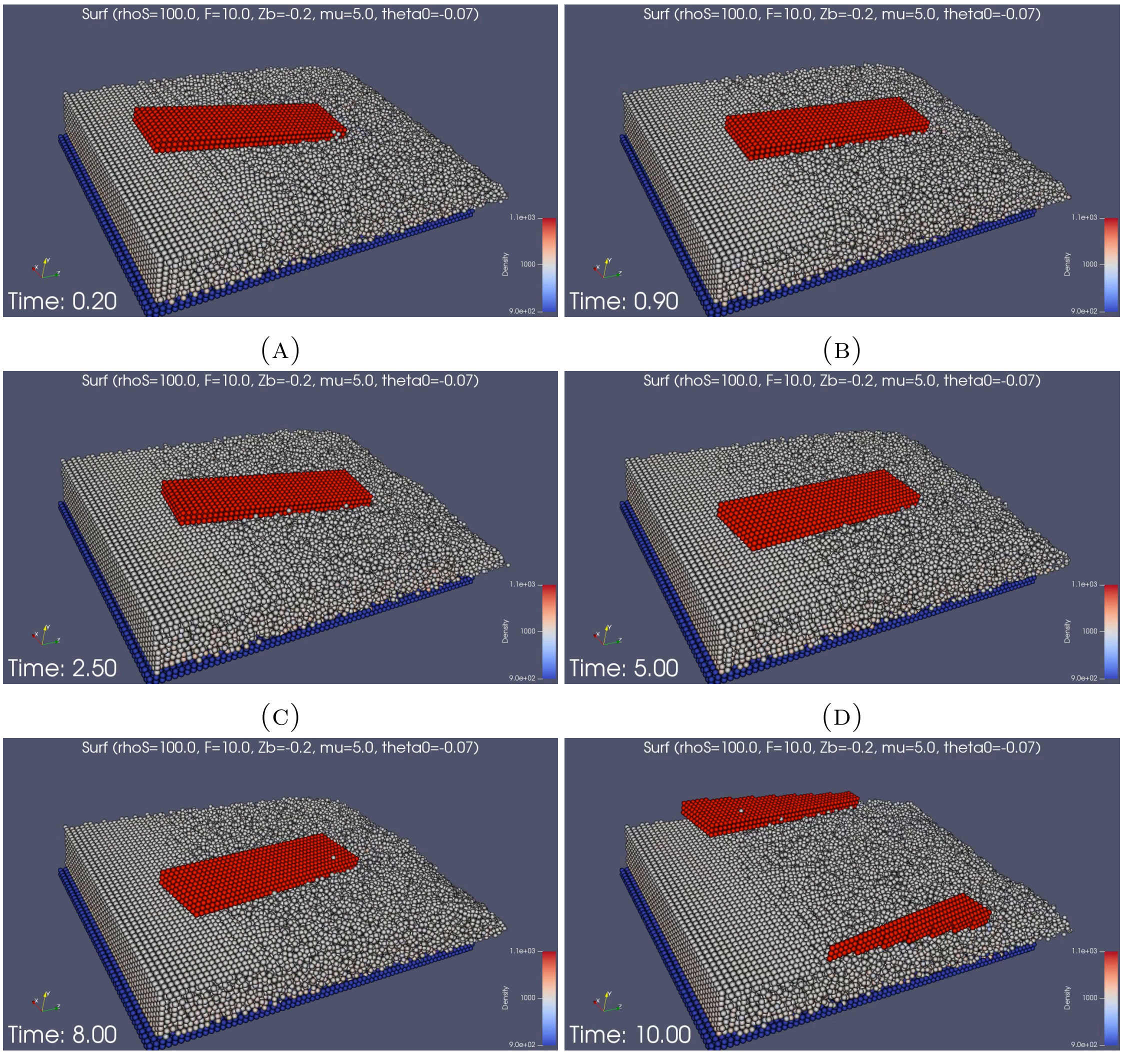


FIGURE 4.5: Solution of the ODE system for different μ.

in both amplitude and frequency. But the increasing of μ also shifts the stable position farther from the desired position $\tilde{Z} = -0.5$, justifying the same behavior that occurs on our simulation. Adding an additional ODE control for $\tilde{\theta}$ might be the solution for this problem. $\tilde{\theta}$ acts as a "target" inclination angle in our current ODE control. By controlling $\tilde{\theta}$, it is possible to disturb the stability of the system when the board is not located at the desired position, forcing the board to nudge slowly to the desired position.

Notice that in the direct ODE solver, the value of μ affects the oscillation of θ and V instead of θ and Z in our SPH simulations. This happens because our SPH simulation cannot translate the change of the inclination angle into the change of the velocity fast enough. The delayed response in velocity propagates the oscillation in the inclination angle to the position in SPH simulations. By tweaking σ from (2.45) it is possible to transfer the change of the inclination angle θ from the ODE control to the inclination angle of the board in the SPH simulation $\hat{\theta}$ faster, which makes a faster change in the velocity and helps the surfing board stabilizes faster.



(A) (B) (C) (D) (E) (F)

FIGURE 4.6: Snapshots of the surfing board simulation with an ODE controller with parameters $\tilde{Z} = -0.2$, $\mu = 5$, and $\tilde{\theta} = -0.07$ at (a) $t = 0.20$ s, (b) $t = 0.90$ s, (c) $t = 2.50$ s, (d) $t = 5.00$ s, (e) $t = 8.00$ s, and (f) $t = 10.00$ s.

The oscillation in Figure 4.4 also occurs since we set μ_v and μ_z from (2.31) to be zero, ignoring the dependency of acceleration to the position and velocity. In fact, this is not correct since the drag force depends not only to the shape of the interface between the fluid and the rigid body, but also depends on the relative velocity between both of them. Initially we assume that the flow has a constant velocity through the whole domain on z-axis of our frame. But, in reality, the velocity of the flow depends on the position, as the gravity slows the velocity of the flow as a consequence of our choice of the frame.

Snapshots of a simulation with the ODE control can be seen on Figure 4.6.

Chapter 5

Conclusion

Now we arrived at the last chapter of this work. Here we wll conclude and summarize our work, and also discuss about some future works that can be done in regard of improving the result of this work and deepen our understanding about the surfing problem.

5.1 Conclusions of The Work

By observing the result that we got from this work, we can conclude that we successfully designed an ODE control for the surfing board that can control the position of the board to be at the desired position relative to the ocean wave. With several assumptions, we can do a linear analysis to the ODE control easily and get some parameters needed to stabilize the surfing board. We successfully verify the capabilities of the ODE control by implementing it into a coupled fluid-rigid body simulation by the SPH method. The coupling between the fluid and the rigid body is done by discretizing the rigid body into a set of SPH points which interact with other fluid points by exchanging momentum with pure hydrodynamics-based forces. The implementation of the ODE control into the SPH simulation is handled by giving two additional external forces to the rigid body, representing the surfer maneuvering the board by adjusting the distribution of their weight on the board via their feet.

In our case, we get the best parameters combination is $\tilde{\theta} = -0.07$ and $\mu = 5$. We still face several problems, including the shifting problem caused by the choice of a parameter μ and the oscillation problem came from our assumption of μ_z and μ_v are zero.

5.2 Future Work

For future work, we have several plans that might improve the results from current work. As we already mentioned before, here we made several assumptions to simplify the calculation, both numerically and analytically. We made an assumption $\mu_z = \mu_v = 0$, ignoring the drag's dependency to the position and velocity, which is not correct in the real situation. Also, an additional ODE control for the $\tilde{\theta}$ is needed to automatically choose such $\tilde{\theta}$ that can nudge the board toward a desired position without disturbing the stability too much. In the future, the author wants to study further the parameters of the ODE control and add an additional ODE control for the $\tilde{\theta}$ in an attempt to perfecting the ODE control for the surfing problem. Furthermore, the effect of inclination angle clipping needs to be studied more since it might affect the ODE control as well.

Another possible future work related with current work is to modify the transfer function between the ODE control and the SPH simulation into a more sophisticated mapping since current linear function cannot translate the change of an inclination angle from ODE into an observed angle of the SPH system swift enough. In this work we observed a delayed response which affects the performance of the ODE control, although we did not measure the effect quantitatively.

Last but not least, the boundary between a fluid part with a rigid body part of SPH points might need a better coupling scheme. In this work we are using a simple momentum transfer scheme that works, but does not have any justification in the background. Further study in a coupling SPH-rigid body simulation is also an interesting challenge to be done.

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