Kanazawa University

Doctoral Thesis

A Numerical Scheme for the Hele-Shaw Problem in Oscillating Media

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Abstract

We study the Hele-Shaw problem in oscillating media, and mainly are interested in the averaging behavior of the free boundary velocity, that is a homogenization problem. Our focus is only on the one-phase Hele-Shaw problem neglecting the surface tension. A general formula for the homogenized velocity is unknown. Current results are known for the media periodic only in space or only in time. In this work, we develop an efficient numerical scheme to estimate the averaging velocity in two dimensions with periodic coefficients in both space and time. Also for comparison, we implement the regular finite difference to obtain a numerical solution and we describe how to get a free boundary position. We present several computation experiments to test the error of both for the numerical solution and the free boundary position.

1 Introduction

In many years, Hele-Shaw problem have become one of popular model in fluid mechanics. For two dimensions case, this is a popular model for pressure driven flow of an incompressible liquid between two parallel plates, names Hele-Shaw cell. And for three dimensions case, this is a model for pressure driven flow of an incompressible liquid in porous medium. In this work, we study the Hele-Shaw problem in oscillating media, and mainly are interested in the averaging behavior of free boundary velocity, i.e homogenization problem. Our focus is only on One-phase Hele-Shaw problem neglecting surface tension. One can simply represent this problem is flowing a liquid through the valleys area containing air only. For general periodic media, the homogenized solution is unknown. Current results have been revealed for the media that periodic only either in space or in time. We develop a numerical schemes to estimate the homogenized solution of normal velocity in Hele-Shaw problem with periodic coefficients in both space and time. Instead of directly solving the problem, we use the fact that One-phase Hele-Shaw problem is a zero heat specific limit of One-phase Stefan problem, then consider the equation in the enthalpy form. We implement the efficient numerical scheme, namely BBR method to solve the PDE, and use the idea in 1D case to estimate the average velocity $r(q)$ in two dimensions.

2 Problem Statement

Suppose the Hele-Shaw problem is in $\mathbb{R}^n$, $n = 2$. Given Ω_0 as an initial domain, closed subset $K \subset \Omega_0$, with $\Omega_0 \subset \Omega_t,\ t > 0$ is increasing. We want to find a pair of $u(x,t)$ and free boundary $\partial\Omega_t$ that satisfying (HS).

$$\begin{cases} \Delta u(x,t) = 0 & \text{for } (x,t) \in (\Omega_t \setminus K) \times [0,\infty) \\ u_t = g(x,t)|\nabla u|^2, & \text{for} (x,t) \in \partial\Omega_t \times (0,\infty) \\ u(x,t) = 1 & \text{for} (x,t) \in K \times (0,\infty) \\ u(x,t) = 0 & \text{for} (x,t) \in \overline{(\mathbb{R}^n \setminus \Omega_t)} \times (0,\infty) \end{cases} \tag{HS}$$

Solution $u(x,t)$ represents pressure of some viscous fluid that is injected into the air as in Figure HS. Neglecting surface tension, the area with air is assumed to have zero pressure on $\partial\Omega_t$, and function $\frac{1}{g(x,t)}$ represents the depth of hole at the free boundary that the liquid must fill while it is advancing. In fact, (HS) can be approximated by limiting specific heat coefficient $c \longrightarrow 0$ in the One-phase Stefan problem. The form of the One-phase Stefan problem is only distinguished by replacing the Laplace operator on (HS) with Heat operator with heat specific coefficient $c > 0$.

$$cu_t - \Delta u(x,t) = 0$$

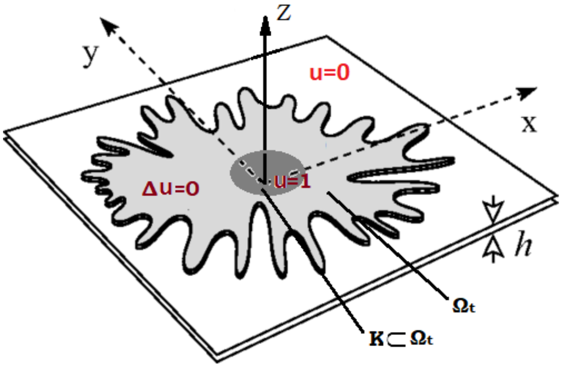

FIGURE 1: Hele-Shaw cell for problem (HS)

On the Stefan problem, the solution of u models the temperature diffusion in ice melting, where the temperature in the ice region is preserved to be 0^oC. Here, $\frac{-1}{g}$ is specified to be a latent energy in the phase transition solid to liquid. Then for numerical scheme, we rewrite the Stefan problem into the enthalpy formulation as written in (SF2).

$$\begin{cases} cu_t - \Delta h(u) = 0 & \text{in } (\mathbb{R}^n \setminus K) \times (0, \infty) \\ u = 1 & \text{in } K \times (0, \infty) \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^n \end{cases} \tag{SF2}$$

Operator $h(u) = u^+$, where implicitly revealing $(\Omega_t \setminus K) := \{x | u(x,t) > 0\}$.

2.1 Homogenization Problem

Suppose that the Hele-shaw cell in the periodic media is given by the problem to find $u = u(x,t)$, $\Omega \subset \mathbb{R}^n \times [0, \infty)$, $(\Omega(t) : \{x | (x,t) \in \Omega\})$ such that:

$$\begin{cases} \Delta u(\cdot, t) = 0 & \text{in } \Omega(t) \setminus K, t > 0 \\ V(x,t) = g(\frac{x}{\epsilon}, \frac{t}{\epsilon}) |\nabla u(x,t)| & x \in \partial\Omega(t), t > 0 \\ \Omega(0) = \Omega_0 & \\ u = h & \text{on } K \\ u = 0 & \text{on } \partial\Omega(t), t > 0 \end{cases} \tag{2.1}$$

for given $h > 0$, and $g > 0$ 1-periodic function $(g(x+k, t+l) = g(x,t), k \in \mathbb{Z}^N, l \in \mathbb{Z}^N)$.

The homogenized problem of (2.1) has almost the same form except on the formula for the normal velocity. Let u^ϵ denotes the solution of (2.1) for any specific ϵ. In Pozar (2015), there exist $r = r(q), r : \mathbb{R}^N \mapsto [0, \infty)$ depending only on g such that $u^\epsilon \to u, \Omega^\epsilon \to \Omega$ as $\epsilon \to 0$, where (u, Ω) is the solution of (2.1) with $V(x,t) = r(\nabla u(x,t))$. In general, $r(q)$ has no explicit form. However, in special cases we know the following formula for $r(q)$.

- If $g = g(t)$ depends only on t, then $r(q) = \langle g \rangle |q|$, where $\langle g \rangle = \int_0^1 g(\tau) d\tau$ is the average of g.
- If $g = g(x)$ depends only on x, then $r(q) = \frac{1}{\langle \frac{1}{g} \rangle} |q|$, where $\langle \frac{1}{g} \rangle = \int_0^1 \frac{1}{g(\tau)} d\tau$ is the average of $\frac{1}{g}$.

If g depends on both x and t, the explicit form of $r(q)$ is not known in general, and can appear to be complicated. The number of $r(q)$ is related to Poincare's rotation number. Nonethelesss, for particular case of q, it is interesting to observe the existence of the interval where velocity becoming constant, namely *pinning interval.*

Lemma 2.1. *Suppose that $g(x,t) = f(x+t)$ where $f = f(x)$ is a positive periodic continuous function. Then $r(q) = 1$ for $q \in [\frac{1}{\max f}, \frac{1}{\min f}]$.*

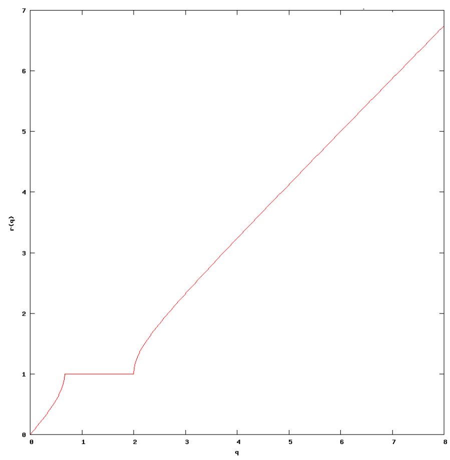

FIGURE 2: graph of $r(q)$ in 1D for $g(x,t) = \sin(2\pi(x+t))^2 + 1/2$ pinning interval is shown for constant $r(q) = 1$ in $[2/3, 2]$

However in the higher dimension, if function g is time-independent, homogenized solution convergence to $r(q) = \frac{1}{\langle \frac{1}{g} \rangle} |q|$, the same as in 1-D case. Also for simple scaling argument, it shows that solution homogenization with $g = g(t)$ converge to $r(q) = \langle g \rangle |q|$.

2.2 Estimating The Average Normal Velocity

To estimate the average normal velocity in 2D, we are motivated by the idea in 1D case, where the homogenized problem can be simplified to the ODE

$$\begin{cases} (y^\varepsilon)'(t) = g(\frac{y^\varepsilon(t)}{\varepsilon}, \frac{t}{\varepsilon})|q|, & t > 0, \\ y^\varepsilon(0) = y_0, \end{cases} \tag{ODE}$$

and the solution y^ε converge locally uniformly as $\varepsilon \to 0+$ to the solution of linear ODE.

$$\begin{cases} y'(t) = r(q), & t > 0, \\ y(0) = y_0, \end{cases}$$

Therefore, we can estimate $r(q)$ in 1D by solving (3.3) for a small $\varepsilon > 0$.

$$r(q) = y(1) - y_0 \approx y^{\varepsilon}(1) - y_0.$$

Or by scaling argument, fix $\varepsilon = 1$ and solve (3.3) for a large time $T \gg 1$.

$$r(q) = \frac{y(T) - y_0}{T} \approx \frac{y^1(T) - y_0}{T}.$$

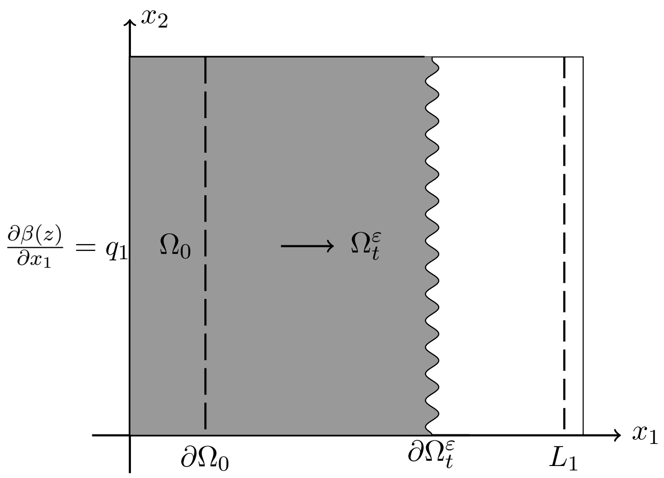

FIGURE 3: The setting of initial condition in 2D case to estimate the value of $r(q)$.

By the convergence in the Hausdorff distance, we see that $T_{\varepsilon} \to \frac{L_1 - L_0}{r(q)}$ as $\varepsilon \to 0$. This allows us to estimate $r(q)$ by choosing $0 < \varepsilon \ll 1$ and using

$$r(q) \approx \frac{L_1 - L_0}{T_{\varepsilon}}.$$

3 Numerical Method

3.1 Explicit Finite Difference Method

For two dimension case, the numerical scheme of explicit finite difference for problem (SF2) can be written as (3.1),

$$\begin{aligned} u_{i,j}^{k+1} = & \\ & u_{i,j}^{k} + \left(\frac{\Delta t}{c\Delta x^2}\right)\{h(u_{i-1,j}^{k}) \\ & + h(u_{i+1,j}^{k}) - 4h(u_{i,j}^{k}) + h(u_{i,j+1}^{k}) + h(u_{i,j-1}^{k})\} \end{aligned} \tag{3.1}$$

with stability condition as the follows.

$$\frac{\Delta t}{c\Delta x^2} < \frac{1}{2}. \tag{3.2}$$

The explicit finite difference method is implemented to obtain numeric solution of Hele-Shaw problem with radial source in periodic media, see Figure 1. The result is verified to the solution of isotropic case, which is given an initial $\Omega_0 = B(0, R_f(0)) \subset \mathbb{R}^2$ and $K = \overline{B(0, R_k)}$ on homogeneous media by setting $g = 1$. The exact solution appears to be solution of the ordinary differential equation (ODE), (3.3).

$$V_n = R_f'(t) = |\nabla u(x)| = \frac{1}{R_f(t) \log \frac{R_k}{R_f(t)}} \tag{3.3}$$

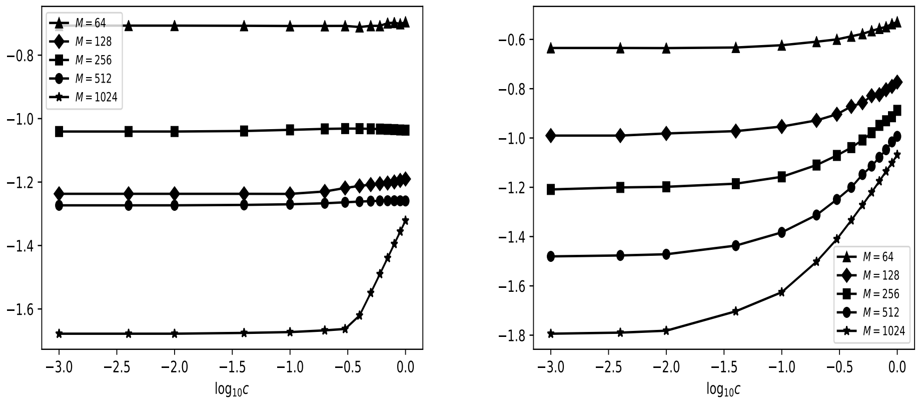

FIGURE 4: Maximum absolute error of solution with x-axis ϵ

Explicit finite different is very restricted to the stability condition. The reasonable small parameter of specific heat and acceptable mesh size compare to coefficient periodic, even lead to the computation much slower. Therefore, we do not recommend this method for numeric solution homogenization in 2D.

3.2 BBR Scheme as Implicit Method

Since the explicit method restrict the computation with the stability condition and the reasonable size of M for the coefficient period ϵ, we provide an efficient numerical scheme firstly introduced by Berger, Brézis and Rogers (1979) and the further studied by Murakawa(2011) for the the types of non-linear equation such a (SF2). We refer to this scheme as BBR scheme. Choosing a time step $\Delta t > 0$, we iteratively find the sequences

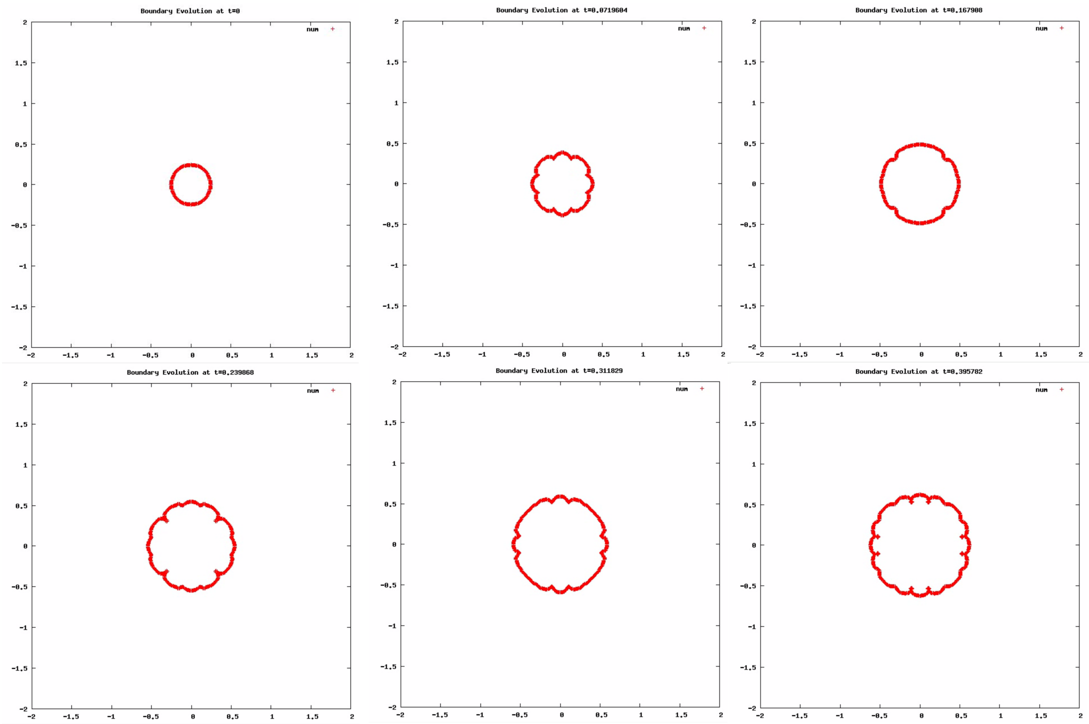

FIGURE 5: Numeric free boundary by explicit method for a given initial open ball domain with periodic function $g(x,t) > 0$

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$\{u^k\}_{k\geq 1}$, $\{z^k\}_{k\geq 0}$ of solutions of

$$\begin{cases} c\mu^{k-1}u^k - \tau\Delta u^k = c\mu^{k-1}\beta(z^{k-1}) & \text{in } U, \\ \dfrac{\partial u^k}{\partial x_1}(0,\cdot) = q_1, & \\ u^k(1,\cdot) = 0, & \\ u^k \quad \text{1-periodic in } x_2, & \end{cases} \tag{3.4a}$$

$$z^k = z^{k-1} + \mu^{k-1}(u - \beta(z^{k-1})) - \frac{\tau}{c}\left(\frac{\partial}{\partial t}\frac{1}{g^{\varepsilon}}\right)(\cdot, t_{k-\frac{1}{2}})\chi_{\text{int}\,\{z^{k-1}<0\}}, \tag{3.4b}$$

$$\mu^k = \frac{1}{\delta + \beta'(z^k)}, \tag{3.4c}$$

for $k = 1, 2, \ldots$, with $z^0 := z(\cdot, 0)$. Here $\delta > 0$ is a chosen regularization parameter that we discuss below, and we define

$$\beta'(s) := \begin{cases} 1, & s > 0, \\ 0, & s \leq 0. \end{cases}$$

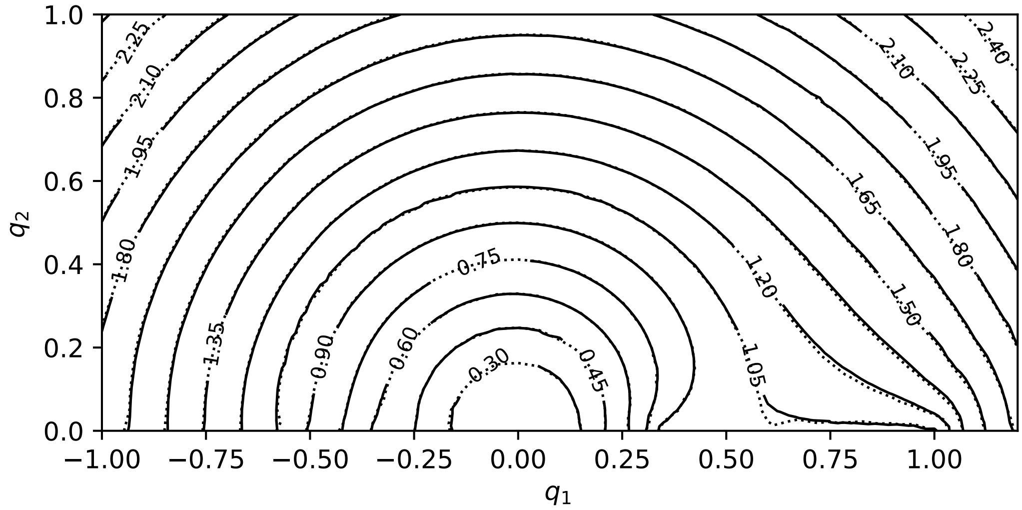

FIGURE 6: The contour plot of $r(q)$ with $g(x,t) = \sin(2\pi(x_1 + t)) + 2$. The pinning interval $[\frac{1}{3}, 1] \times \{0\}$ is apparent, where the average velocity is pinned to 1. The solid contours were obtained with $M = 256$, while the dotted contours were obtained with $M = 128$.

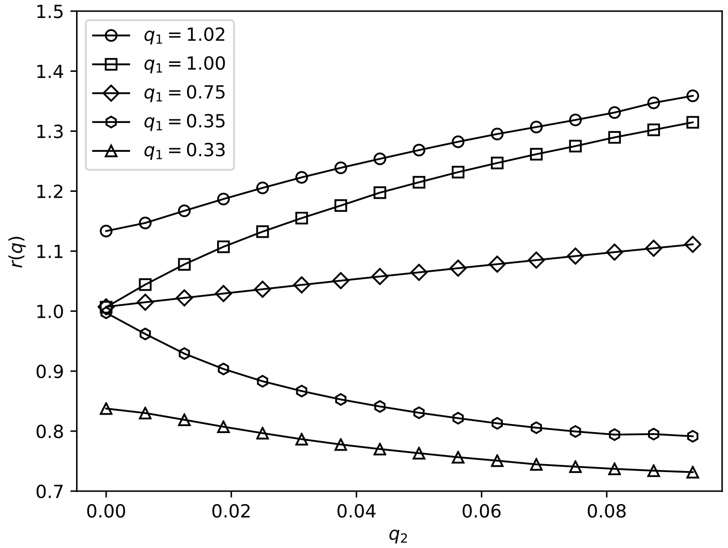

FIGURE 7: Values of $r(q)$, $q = (q_1, q_2)$, for $g(x,t) = \sin(2\pi(x + t)) + 2$ with $M = 1024$ as a function of q_2 for several chosen of q_1.

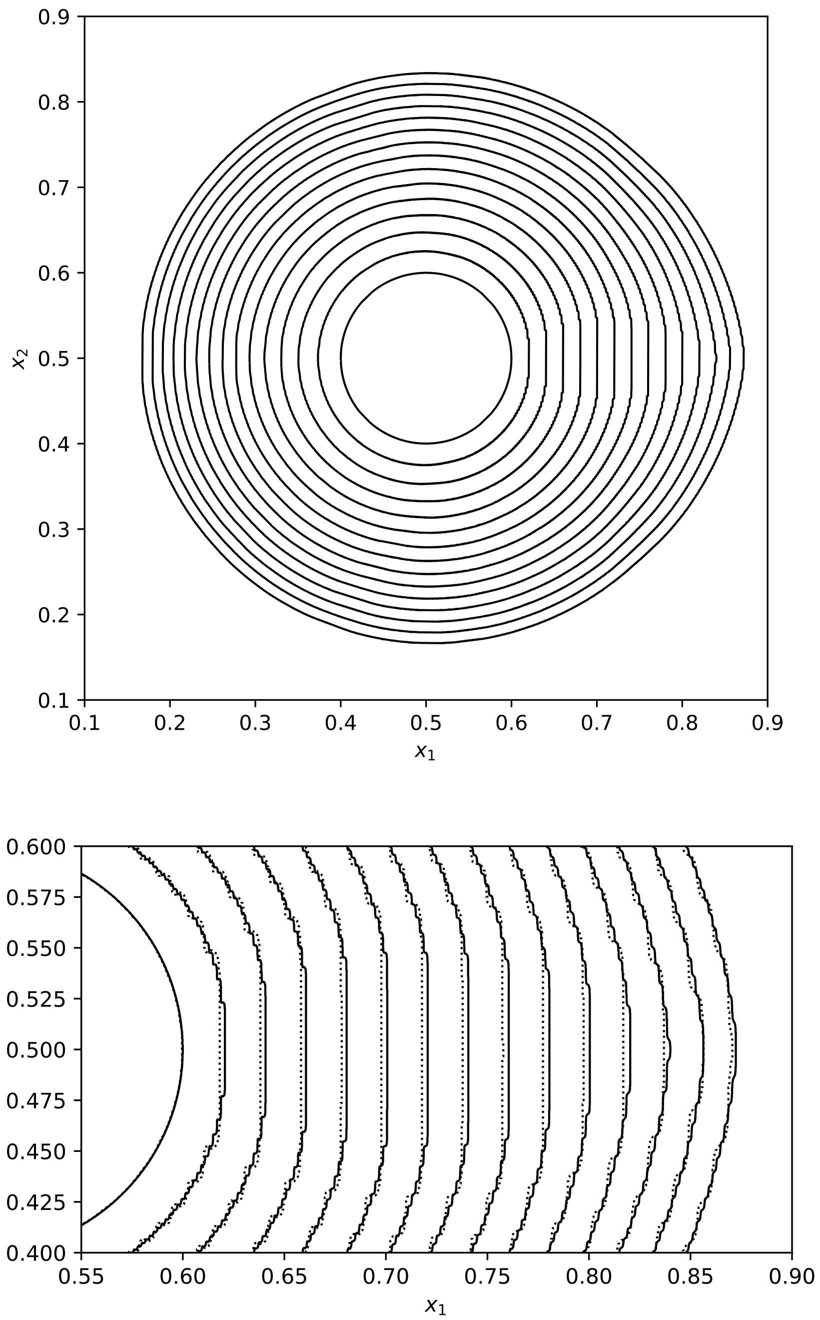

FIGURE 8: (Top) The free boundary of the numerical solution of the Hele-Shaw problem with a given source $f = 1500\max\{0.1 - |x - (\frac{1}{2}, \frac{1}{2})|, 0\}$ and a function $g(x,t) = \sin(2\pi(x_1 + t)) + 1.05$ with initial data $\Omega_0 = \{x : |x - (\frac{1}{2}, \frac{1}{2})| < 0.1\}$. The free boundary is plotted at times $t = 0.02m$, $m \in N$. A facet seems to appear in direction $(1, 0)$. It reaches its maximum length at $t \approx 0.12$. Solid line is the solution with $M = 8192$, $\varepsilon = \frac{1}{512}$, while the dotted line is with $M = 2048$, $\varepsilon = \frac{1}{128}$. We used 1 V-cycle. (Bottom) Detail of the region with facets.

平成 30 年 8 月 10 日

学位論文審査報告書（甲）

1．学位論文題目（外国語の場合は和訳を付けること。）

A Numerical Scheme for the Hele-Shaw Problem in Oscillating Media

摂動ムラのある媒体上の Hele-Shaw 問題の数値解析について

2．論文提出者 (1) 所　属　数物科学専攻　　　専攻

(2) ふりがな 氏　名　いるま ぱるぴ Irma Palupi

3．審査結果の要旨（600～650 字）

Irma Palupi さんは、2015 年 10 月に自然科学研究科数物科学専攻に入学した(数物科学グローバル人材育成コース給付生)。それ以降、自由境界を持つ流体の流れのモデルの数値解析に取り組んできた。特に、多孔質媒体や隘路において自由界面を有する非圧縮性流体をモデル化する Hele-Shaw 問題を研究してきた。Irma さんは、流路の表面の粗さに関する「高度に振動する媒質係数」の問題に焦点を当てた。この状況は高解像度のグリッドが必要と予見されるため、数値計算は非常に困難である。Irma さんは、振動する係数の性質により、どのように平均化していくかを判定する「均質化の理論」を構築し、数値計算と合わせて有用なソルバーを開発した。今までの理論では、時間と空間の両方に依存する係数を持つ Hele-Shaw 問題は取り扱われていない。Irma さんは、この難問に対して、数値計算と理論を駆使して自由境界の平均速度を求めるなど、先駆的な結果を得た。　一般に、Hele-Shaw 問題は、問題に合わせて数値アルゴリズムを開発する必要がある。 Irma さんはいくつかの方法をテストし、最終的に非常に効率的なマルチグリッドソルバーを使って時刻に対する陰的な離散化を実装した。このプログラムを使用して、いくつかの重要なケースで平均自由境界速度を推定し、その結果を分析した。特に、振動係数によって生じる流体界面の安定した平坦な部分の出現を観察した。これは、この問題に関する偏微分方程式理論および数値解析に対して、新たな興味深い問題をもたらす重要な結果である。この結果を原著論文 1 本にまとめた。以上により本論文は、博士(理学)を授与するに値すると判断した。

4．審査結果　(1) 判　定（いずれかに○印）　(合　格)・　不合格

(2) 授与学位　博　士（　理学　）