Dissertation Abstract

Long-time behavior of the one-phase Stefan problem in periodic media and random media

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1. Abstract

The Stefan problem is a particular kind of partial differential equations (PDEs) with a moving boundary which are the so-called *free boundary problems.* We study the long-time behavior of solutions of the one-phase Stefan problem in inhomogeneous media in dimensions $n \geq 2$.

We first consider the isotropic diffusion in periodic or stationary ergodic random media. Using the technique of rescaling, which is consistent with the evolution of the free boundary, we are able to show the homogenization of the free boundary velocity as well as the locally uniform convergence of the rescaled solution to a self-similar solution of the homogeneous Hele-Shaw problem with a point source. Moreover, by viscosity solution methods, we also deduce that the rescaled free boundary uniformly approaches a sphere with respect to the Hausdorff distance.

Similar results apply for the anisotropic case when we restrict our consideration to the periodic media with the space dimension $n \geq 3$. We first show that the rescaling in the isotropic case is also compatible with the anisotropic case, therefore the same rescaling is used. In this generalization, beside the homogenization of the free boundary velocity as in the previous case, we also obtain the homogenization of the elliptic operator. More precisely, we prove the locally uniform convergence of a rescaled solution to the solution of the homogenized Hele-Shaw-type problem with a point source. Analogously to the isotropic case, we show the convergence of the rescaled free boundary to a self-similar profile with respect to the Hausdorff distance using viscosity arguments.

2. Introduction

The *one-phase Stefan problem* is a specific type of free boundary problems, which typically models the melting of an ice body in contact with a water region due to heat conduction and an exchange of latent heat energy. Here we assume that the ice is maintained at temperature 0. The one-phase Stefan problem includes the heat equation in the liquid phase and an additional condition at the free boundary, which is the so-called *Stefan condition*, that expresses the local velocity of a moving boundary.

More precisely, let $n \geq 2$ and $K \subset \mathbb{R}^n$ be a compact set with sufficiently regular boundary, for instance, $\partial K \in C^{1,1}$, and assume that $0 \in \operatorname{int} K$. The one-phase Stefan problem (on an exterior domain) with inhomogeneous latent heat of phase transition is to find a function $v(x,t) : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ that satisfies the free boundary problem

$$\begin{cases} v_t - \Delta v = 0 & \text{in } \{v > 0\} \backslash K, \\ v = 1 & \text{on } K, \\ V_\nu = g(x)|Dv| & \text{on } \partial\{v > 0\}, \\ v(x,0) = v_0 & \text{on } \mathbb{R}^n, \end{cases} \tag{2.1}$$

where D and Δ are respectively the spatial gradient and the Laplacian, v_t is the partial derivative of v with respect to time variable t, V_ν is the normal velocity of the *free boundary* $\partial\{v > 0\}$. v_0 and g are given functions, see below. Note that the results in this chapter can be trivially extended to general time-independent positive continuous boundary data, 1 is taken only to simplify the exposition. The related *Hele-Shaw problem* is usually referred to in the literature as the quasi-stationary limit of the one-phase Stefan problem when the heat operator is replaced by the Laplace operator. This problem typically describes the flow of an injected viscous fluid between two parallel plates which form the so-called Hele-Shaw cell, or the flow in porous media.

In this work, we assume that the function g satisfies the following two conditions, which guarantee respectively the well-posedness of (2.1) and averaging behavior as $t \to \infty$:

(1) g is a Lipschitz function in $\mathbb{R}^n$, $m \leq g \leq M$ for some positive constants m and M.
(2) $g(x)$ has some averaging properties so that 2.5 applies, for instance, one of the following holds:
  (a) g is a $\mathbb{Z}^n$-periodic function,
  (b) $g(x, \omega) : \mathbb{R}^n \times A \to [m, M]$ is a stationary ergodic random variable over a probability space $(A, \mathcal{F}, P)$.

For a detailed definition and overview of stationary ergodic media, we refer to [**5**, **4**] and the references therein.

Throughout most of the work, we will assume that the initial data v_0 satisfies

$$\begin{aligned} &v_0 \in C^2(\overline{\Omega_0 \backslash K}), v_0 > 0 \text{ in } \Omega_0, v_0 = 0, \text{ on } \Omega_0^c := \mathbb{R}^n \setminus \Omega_0, \text{ and } v_0 = 1 \text{ on } K, \\ &|Dv_0| \neq 0 \text{ on } \partial\Omega_0, \text{ for some bounded domain } \Omega_0 \supset K. \end{aligned} \tag{2.2}$$

This will guarantee the existence of both the weak and viscosity solutions below and their coincidence, as well as the weak monotonicity (3.5). However, the asymptotic limit, Theorem 3.4, is independent of the initial data, and therefore the result applies to arbitrary initial data as long as the (weak) solution exists, satisfies the comparison principle, and the initial data can be approximated from below and from above by data satisfying (2.2). For instance, $v_0 \in C(\mathbb{R}^n)$, $v_0 = 1$ on K, $v_0 \geq 0$, $\operatorname{supp} v_0$ compact is sufficient.

Due to the singularities of the moving boundary, the classical solution of the Stefan problem in dimension $n \geq 2$ is not expected to exist for all time. Thus, it is natural to generalize the notion of solutions. The classical approach to define a generalized solution is to integrate v in time and introduce $u(x,t) := \int_0^t v(x,s)ds$ [**1**, **11**]. If v is sufficiently regular, then u solves the variation inequality

$$\begin{cases} u(\cdot, t) \in \mathcal{K}(t), \\ (u_t - \Delta u)(\varphi - u) \geq f(\varphi - u) \text{ a.e } (x,t) \text{ for any } \varphi \in \mathcal{K}(t), \end{cases} \tag{2.3}$$

where $\mathcal{K}(t) = \{\varphi \in H^1(D), \varphi \geq 0, \varphi = 0 \text{ on } \partial B, \varphi = t \text{ on } K\}$ with B is some large ball, $D = B \backslash K$ and f is

$$f(x) = \begin{cases} v_0(x), & v_0(x) > 0, \\ -\dfrac{1}{g(x)}, & v_0(x) = 0. \end{cases} \tag{2.4}$$

We also use the notion of viscosity solutions introduced by Kim [**2**] in our work. The problem is well-posed in both weak sense and viscosity sense (see [**1**, **2**]) and the coincidence of two notions of solutions was obtained by Kim and Mellet [**3**, **4**]. The regularity of the one-phase Stefan problem was studied many authors. Furthermore, the asymptotic behavior of solutions is one of the concerned problems in the literature. The asymptotic homogenization of the Hele-Shaw and the one-phase Stefan problem was given in [**9**, **3**, **4**]. The convergence of the Stefan problem to Hele-Shaw as $t \to \infty$ in homogeneous media was observed in [**8**]. Moreover, the long-time behavior of the related Hele-Shaw problem was studied in details in [**5**]. In our recent work, we focus on the long-time behavior of the one-phase Stefan and Stefan-type problems in some periodic or random media in dimension $n \geq 2$.

The first main part of our work is the investigation of the behavior of the solution of (2.1) and its free boundary when $t \to \infty$. Following [**8**, **5**] we use the natural rescaling of solutions of the form

$$v^\lambda(x,t) := \lambda^{(n-2)/n} v(\lambda^{1/n} x, \lambda t) \qquad \text{if } n \geq 3,$$

(see Section 3.1 for the corresponding rescaling for variational solutions and rescaling for $n = 2$). Then the rescaled viscosity solution satisfies the free boundary velocity law

$$V_\nu^\lambda = g(\lambda^{1/n} x)|Dv^\lambda|.$$

Heuristically, if g has some averaging properties, such as in condition (2), the free boundary velocity law should homogenize as $\lambda \to \infty$. Since the latent heat of phase transition $1/g$ should average out, the homogenized velocity law will be

$$V_\nu = \frac{1}{\langle 1/g \rangle} |Dv|,$$

where $\langle 1/g \rangle$ represents the "average" of $1/g$. More precisely, the quantity $\langle 1/g \rangle$ is the constant in the subadditive ergodic theorem such that

$$\int_{\mathbb{R}^n} \frac{1}{g(\lambda^{1/n} x, \omega)} u(x)dx \to \int_{\mathbb{R}^n} \left\langle \frac{1}{g} \right\rangle u(x)dx \text{ for all } u \in L^2(\mathbb{R}^n), \text{ for a.e. } \omega \in A. \tag{2.5}$$

In the periodic case, it is just the average of $1/g$ over one period. Since we always work with $\omega \in A$ for which the convergence above holds, we omit it from the notation in the rest of the work. Moreover, as $t \to \infty$ the diffusion in the process usually reaches to the *steady-state* and the heat equation in the Stefan problem loses the first term v_t. Thus, we can expect that the limit function of v^λ, if exists, will satisfies the following homogeneous Hele-Shaw problem with a point source

$$
\begin{cases}
-\Delta v = C\delta & \text{in } \{v > 0\},\\
v_t = \dfrac{1}{\langle 1/g\rangle}|Dv|^2 & \text{on } \partial\{v > 0\},\\
v(\cdot, 0) = 0,
\end{cases}
\tag{2.6}
$$

where δ is the Dirac δ-function, C is a constant depending on K and n, the quantity $\langle 1/g\rangle$ was defined by (2.5).

The subadditive ergodic theorem yields the first result on the homogenization of the variational inequality. Using barrier arguments we can precise the singularity of the limit function at the origin and then prove the locally uniformly convergence of the rescaled variational solution to the solution of the limit obstacle problem corresponding to (2.6). Next, we will use the coincidence of the weak and viscosity solution of the one-phase Stefan problem and the viscosity arguments to obtain the locally uniform convergence of the rescaled viscosity solution to the solution of (2.6) and show that the free boundary approaches a sphere with respect to the Hausdorff distance. This part in a generalization of the results in [**5**] for the Hele-Shaw problem. However, solutions of the Hele-Shaw problem have a very useful monotonicity in time, which is missing in the Stefan problem. This makes some steps more difficult. We instead take advantage of a weak monotonicity property (3.5), which holds for regular initial data satisfying (2.2) and then show the convergence result for general initial data using the uniqueness of the limit and the comparison principle. Moreover, the heat operator is not invariant under the rescaling, unlike the Laplace operator. The rescaled parabolic equation becomes elliptic when $\lambda \to \infty$, which causes some issues when applying parabolic Harnack's inequality, for instance.

The second main part of our work is the extension of the isotropic case to the anisotropic case, where the heat operator is replaced by more general linear parabolic operators of divergence form. Now, instead of (2.1), we observe the behavior of the solution v of the problem

$$
\begin{cases}
v_t - D_i(a_{ij}D_j v) = 0 & \text{in } \{v > 0\}\backslash K,\\
v = 1 & \text{on } K,\\
\dfrac{v_t}{|Dv|} = g a_{ij} D_j v \nu_i & \text{on } \partial\{v > 0\},\\
v(x, 0) = v_0 & \text{on } \mathbb{R}^n,
\end{cases}
\tag{2.7}
$$

where D is the gradient, v_t is the partial derivative of v with respect to time variable t and $\nu = \nu(x, t)$ is inward spatial unit normal vector of $\partial\{v > 0\}$ at point (x, t). Here we use Einstein summation convention. v_0 is a given function satisfying (2.2). We also assume that $A(x) = (a_{ij}(x))$ is symmetric, bounded, and uniformly elliptic, i.e, there exits some positive constants α and β such that

$$\alpha|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \beta|\xi|^2 \text{ for all } x \in \mathbb{R}^n \text{ and } \xi \in \mathbb{R}^n. \tag{2.8}$$

And moreover, a_{ij} and g are

(1) Lipschitz functions in $\mathbb{R}^n$, $m \le g \le M$ for some positive constants m and M,
(2) $\mathbb{Z}^n$-periodic functions.

In this consideration, besides the homogenization of the normal velocity as before, the coefficients of the elliptic operator should also homogenize in the limit. In fact, we will show that the limit function satisfies the homogenized Hele-Shaw-type problem with a point source

$$\begin{cases} -q_{ij}D_{ij}v = C\delta & \text{in } \{v > 0\}, \\ \dfrac{v_t}{|Dv|} = \dfrac{1}{\langle 1/g\rangle} q_{ij}D_i v \nu_j & \text{on } \partial\{v > 0\}, \\ v(\cdot, 0) = 0, \end{cases} \tag{2.9}$$

where δ is the Dirac δ-function, q_{ij} are constants satisfying a uniform ellipticity with some elliptic coefficients, C is a constant depending on K, n, q_{ij} and the boundary data 1, and the constant $\langle 1/g\rangle$ is the average quantity of the latent heat $L(x) = \frac{1}{g(x)}$.

In this setting, the variational structure is preserved, thus we are still able to use the notions of the weak solution as well as the viscosity solution and their coincidence. However, the main difficulties come from the loss of radially symmetric solutions which were used as barriers in the isotropic case and the homogenization problems appear not only for velocity law but also for elliptic operators. To overcome the first difficulty, we will construct some barriers for our problem from the fundamental solution of the corresponding elliptic equation of divergence form. Unfortunately, even though the unique fundamental solution of this elliptic equation exists for $n \ge 2$, its behavior in the case dimension $n = 2$ and dimension $n \ge 3$ are significantly different. Moreover, we need to make use of a very useful gradient estimate for the fundamental solution, which only holds for the periodic structure. Therefore, we will restrict our problem into the problem in periodic media and dimension $n \ge 3$. From the construction of the barriers, we also obtain the growth rate of the free boundary, more precisely, the free boundary expands with the rate of $t^{1/n}$ when t is large enough, which is the same with the isotropic case. Thus we use the same rescaling as before and obtain the locally uniform convergence of the rescaled variational solution to the solution of the limit obstacle problem corresponding to (2.9). Using the constructed barriers, we are able to prove the correct singularity of the limit function as $|x| \to 0$. The aim is then to prove the homogenization effects of the rescaling to our problem, which will be done with the help of Γ-convergence techniques. As the last step, we also use the viscosity method to prove the locally uniform convergence of the rescaled viscosity solution and its free boundary to the asymptotic profile.

3. Long-time behavior of the one-phase Stefan problem in periodic and random media

3.1. Rescaling. We will use the following rescaling of solutions as in [**5**].

3.1.1. *For* $n \geq 3$. For $\lambda > 0$ we use the rescaling

$$v^\lambda(x,t) = \lambda^{\frac{n-2}{n}} v(\lambda^{\frac{1}{n}} x, \lambda t), \qquad u^\lambda(x,t) = \lambda^{-\frac{2}{n}} u(\lambda^{\frac{1}{n}} x, \lambda t).$$

If we define $K^\lambda := K/\lambda^{\frac{1}{n}}$ and $\Omega_0^\lambda := \Omega_0/\lambda^{\frac{1}{n}}$ then v^λ satisfies the problem

$$\begin{cases} \lambda^{\frac{2-n}{n}} v_t^\lambda - \Delta v^\lambda = 0 & \text{in } \Omega(v^\lambda)\backslash K^\lambda, \\ v^\lambda = \lambda^{\frac{n-2}{n}} & \text{on } K^\lambda, \\ v_t^\lambda = g^\lambda(x)|Dv^\lambda|^2 & \text{on } \Gamma(v^\lambda), \\ v^\lambda(\cdot,0) = v_0^\lambda, \end{cases} \tag{3.1}$$

where $g^\lambda(x) = g(\lambda^{\frac{1}{n}} x)$. And the rescaled u^λ satisfies the obstacle problem

$$\begin{cases} u^\lambda(\cdot,t) \in \mathcal{K}^\lambda(t), \\ (\lambda^{\frac{2-n}{n}} u_t^\lambda - \Delta u^\lambda)(\varphi - u^\lambda) \geq f(\lambda^{\frac{1}{n}} x)(\varphi - u^\lambda) & \text{a.e } (x,t) \in \mathbb{R}^n \times (0,\infty) \\ & \text{for any } \varphi \in \mathcal{K}^\lambda(t), \\ u^\lambda(x,0) = 0, \end{cases} \tag{3.2}$$

where $\mathcal{K}^\lambda(t) = \{\varphi \in H^1(\mathbb{R}^n), \varphi \geq 0, \varphi = \lambda^{\frac{n-2}{n}} t \text{ on } K^\lambda\}$.

Remark 3.1. We can take the admissible set $K^\lambda(t)$ as above due to the continuity with respect to the H^1 norm of all terms in the variational inequality and the fact that the variational solution u has a compact support in space at every time.

3.1.2. *For n=2.* For dimension $n = 2$, we use a different rescaling that preserves the singularity of logarithm, namely

$$v^\lambda(x,t) = \log \mathcal{R}(\lambda) v(\mathcal{R}(\lambda)x, \lambda t), \qquad u^\lambda(x,t) = \frac{\log \mathcal{R}(\lambda)}{\lambda} u(\mathcal{R}(\lambda)x, \lambda t), \tag{3.3}$$

where $\mathcal{R}(\lambda)$ is the unique solution of $\mathcal{R}^2 \log \mathcal{R} = \lambda$, $\lim_{\lambda\to\infty} \mathcal{R}(\lambda) \to \infty$ (see [**5**] for more details). v^λ and u^λ satisfy rescaled problems analogous to (3.1) and (3.2). In particular, the term $\lambda^{(2-n)/n}$ in front of the time derivatives is replaced by $1/\log(\mathcal{R}(\lambda)) \to 0$ as $\lambda \to \infty$.

3.2. Convergence results. Let $V_{C,L}$ be the classical solution of (2.6), here C is a constant taken from the asymptotic behavior of the near field limit in [**8**] and $L = \langle 1/g \rangle$ is the constant defined by (2.5). $V_{C,L}$ has an explicit form as in [**5**]. Let $U_{C,L}(x,t) := \int_0^t V_{C,L}(x,s)\, ds$ then as shown in [**5**], $U_{C,L}$ is the unique solution of the limit obstacle problem

$$
\begin{cases} w \in \mathcal{K}_t, \\ a(w,\phi) \geq \langle -L, \phi \rangle, & \text{for all } \phi \in V, \\ a(w,\psi w) = \langle -L, \psi w\rangle & \text{for all } \psi \in W, \end{cases} \tag{3.4}
$$

where $\mathcal{K}_t = \left\{\varphi \in \bigcap_{\varepsilon>0} H^1(\mathbb{R}^n \backslash B_\varepsilon) \cap C(\mathbb{R}^n\backslash B_\varepsilon) : \varphi \geq 0, \lim_{|x|\to 0} \frac{\varphi(x)}{U_{C,L}(x,t)} = 1\right\}$,

$$
\begin{aligned}
V &= \left\{\phi \in H^1(\mathbb{R}^n) : \phi \geq 0, \phi = 0 \text{ on } B_\varepsilon \text{ for some } \varepsilon > 0\right\}, \\
W &= V \cap C^1(\mathbb{R}^n),
\end{aligned}
$$

and

$$
a(u,v) := \int_{\mathbb{R}^n} Du \cdot Dv dx, \quad \langle u, v\rangle := \int_{\mathbb{R}^n} uv dx.
$$

Lemma 3.2 (Convergence for radial case). *Let $\theta(x,t)$ be a radial solution of the Stefan problem (2.1) in exterior domain $\mathbb{R}^n\backslash B(0,a)$ satisfying an initial condition $\theta(x,0) = \theta_0(|x|)$ if $|x| \geq a$, a fixed boundary condition $\theta(x,t) = Ca^{2-n}$ on $\{|x| = a\}$ and the Stefan condition on the free boundary. Then θ^λ converges locally uniformly to $V_{A,L}$ in the set $(\mathbb{R}^n\backslash\{0\}) \times [0,\infty)$.*

Using the radially symmetric solutions as the barriers for the Stefan problem and the subadditive ergodic theorem, we are able to prove the convergence for the rescaled variational solution.

Theorem 3.3. *Let u be the unique solution of variational problem (2.3) and u^λ be its rescaling. Then the functions u^λ converges locally uniformly to $U_{A,L}$ as $\lambda \to \infty$ on $(\mathbb{R}^n\backslash\{0\}) \times [0,\infty)$.*

The convergence for viscosity solution and its free boundary will be proved by pointwise viscosity arguments. Let v be the solution of the Stefan problem (2.1). We define the half-relaxed limits in $\{|x| \neq 0, t \geq 0\}$:

$$
v^*(x,t) = \limsup_{(y,s),\lambda\to(x,t),\infty} v^\lambda(y,s), \qquad v_*(x,t) = \liminf_{(y,s),\lambda\to(x,t),\infty} v^\lambda(y,s),
$$

Theorem 3.4. *The rescaled viscosity solution v^λ of the Stefan problem (2.1) converges locally uniformly to $V = V_{C,L}$ in $(\mathbb{R}^n\backslash\{0\}) \times [0,\infty)$ as $\lambda \to \infty$ and*

$$
v_* = v^* = V.
$$

Moreover, the rescaled free boundary $\{\Gamma(v^\lambda)\}_\lambda$ converges to $\Gamma(V)$ locally uniformly with respect to the Hausdorff distance.

We first prove Theorem 3.4 under the assumption (2.2) with the help of the following monotonicity.

Lemma 3.5. *Suppose that v_0 satisfies (2.2). Then there exist $C \geq 1$ independent of x and t such that*

$$v_0(x) \leq Cv(x,t) \ \text{in } \mathbb{R}^n \backslash K \times [0,\infty). \tag{3.5}$$

Then we deduce the convergence result for general initial data by the uniqueness of the limit and the comparison principle.

4. Long-time behavior of one-phase Stefan-type problems with anisotropic diffusion in periodic media

We will use the same rescaling for solutions as in Section 3.1.

4.1. Construction of a sub-solution and a super-solution from fundamental solution. Since we do not have the radially symmetric solution as the isotropic case, we instead construct a sub-solution θ_1 of the form

$$\theta_1(x,t) := \left[c_1 F(x) + \frac{c_2 h(x)}{t} - c_3 t^{(2-n)/n}\right]_+ \chi_{E(t)},$$

where $h(x)$ is a function having quadratic growth and satisfying $\mathcal{L}h = n, Dh(x) = (A(x))^{-1}x$,

$$E(t) := \{x : F_b'(|x|,t) < 0\}, \quad F_b(r,t) := Cc_1 r^{2-n} + \frac{c_2\tilde{c}r^2}{t} - c_3 t^{(2-n)/n},$$

$C, \tilde{c}, c_1, c_2, c_3$ are constants and $F_b'(r,t)$ is the derivative of $F_b(r,t)$ with respect to r.

We also use a super-solution θ_2 of the form

$$\theta_2(x,t) := [C_1 F(x) - C_2 t^{(2-n)/n}]_+,$$

where C_1, C_2 are constants.

Using the estimates for the fundamental solution of an elliptic equation of divergence form and its gradient, we can choose some appropriate constants c_1, c_2, c_3, C_1, C_2 such that θ_1 is a sub-solution of (2.7) and θ_2 is a super-solution of (2.7) and then we will use θ_1, θ_2 as the barriers in our analysis. From the construction of barriers, we deduce that the free boundary expands with the rate of $t^{1/n}$ when t large enough, which is the same with the rate in the isotropic case obtained in [**8**]. Moreover, the homogenization of the fundamental solution in the classical theory automatically yields the convergence of the rescaled barriers.

4.2. Convergence results. Let $V_{C,L}$ be the classical solution of (2.9), here C is a constant taken from the asymptotic behavior of the near field limit similar to [**8**] and $L = \langle 1/g \rangle$ is the constant defined by (2.5). $V_{C,L}$ is obtained by the solution of (2.6) after changing the coordinate. Let $U_{C,L}(x,t) := \int_0^t V_{C,L}(x,s)\,ds$ then $U_{C,L}$ is the unique solution of the corresponding obstacle problem of (2.9) similar to that in Section 3.

Using the barriers constructed in Section 4.1 and the Γ-convergence techniques, we are able to prove a convergence result for the rescaled variational solution analogously to the isotropic case in Section 3.

Theorem 4.1. *Let u be the unique solution of variational problem corresponding to* (2.7) *and u^λ be its rescaling. Then the functions u^λ converge locally uniformly to $U_{A,L}$ as $\lambda \to \infty$ on $(\mathbb{R}^n \backslash \{0\}) \times [0, \infty)$.*

Finally, we obtain the convergence of the rescaled viscosity solution by the same approach with the first case in Section 3.

Theorem 4.2. *Let $n \geq 3$. The rescaled viscosity solution v^λ of the Stefan problem (2.7) converges locally uniformly to $V = V_{C,L}$ in $(\mathbb{R}^n \backslash \{0\}) \times [0, \infty)$ as $\lambda \to \infty$ and*

$$v_* = v^* = V.$$

Moreover, the rescaled free boundary $\{\Gamma(v^\lambda)\}_\lambda$ converges to $\Gamma(V)$ locally uniformly with respect to the Hausdorff distance.

References

[1] A. Friedman and D. Kinderlehrer, *A one phase Stefan problem*, Indiana Univ. Math. J. **24** (1974/75), no. 11, 1005–1035, DOI 10.1512/iumj.1975.24.24086. MR0385326

[2] I. C. Kim, *Uniqueness and existence results on the Hele-Shaw and the Stefan problems*, Arch. Ration. Mech. Anal. **168** (2003), no. 4, 299–328, DOI 10.1007/s00205-003-0251-z. MR1994745

[3] I. C. Kim and A. Mellet, *Homogenization of a Hele-Shaw problem in periodic and random media*, Arch. Ration. Mech. Anal. **194** (2009), no. 2, 507–530, DOI 10.1007/s00205-008-0161-1. MR2563637

[4] ———, *Homogenization of one-phase Stefan-type problems in periodic and random media*, Trans. Amer. Math. Soc. **362** (2010), no. 8, 4161–4190, DOI 10.1090/S0002-9947-10-04945-7. MR2608400

[5] N. Požár, *Long-time behavior of a Hele-Shaw type problem in random media*, Interfaces Free Bound. **13** (2011), no. 3, 373–395, DOI 10.4171/IFB/263. MR2846016

[6] N. Požár and G. T. T. Vu, *Long-time behavior of the one-phase Stefan problem in periodic and random media*, To appear in DCDS-S (2017). Available at https://arxiv.org/abs/1702.07119.

[7] ———, *Long-time behavior of one-phase Stefan-type problems with anisotropic diffusion in periodic media*, Preprint (2017).

[8] F. Quirós and J. L. Vázquez, *Asymptotic convergence of the Stefan problem to Hele-Shaw*, Trans. Amer. Math. Soc. **353** (2001), no. 2, 609–634, DOI 10.1090/S0002-9947-00-02739-2. MR1804510

[9] J.-F. Rodrigues, *Free boundary convergence in the homogenization of the one-phase Stefan problem*, Trans. Amer. Math. Soc. **274** (1982), no. 1, 297–305, DOI 10.2307/1999510. MR670933

[10] ———, *The Stefan problem revisited*, Mathematical models for phase change problems (Óbidos, 1988), Internat. Ser. Numer. Math., vol. 88, Birkhäuser, Basel, 1989, pp. 129–190. MR1038069

[11] ———, *Variational methods in the Stefan problem*, Phase transitions and hysteresis (Montecatini Terme, 1993), Lecture Notes in Math., vol. 1584, Springer, Berlin, 1994, pp. 147–212, DOI 10.1007/BFb0073397. MR1321833

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学位論文審査報告書（甲）

１．学位論文題目（外国語の場合は和訳を付けること。）

Long-time behavior of the one-phase Stefan problem in periodic media and random media

(周期的な媒体やランダムな媒体上での一相 Stefan 問題の時間大域的な解の挙動について)

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(2) 氏　　名（ふりがな）　Vu Thi Thu Giang（ぶ てぃ とぅ ざん）

３．審査結果の要旨（600～650 字）

Vu Thi Thu Giang さんは、２０１５年４月に自然科学研究科数物科学専攻に入学した（ベトナム政府奨学金給付生)。それ以降、二種類の媒体上での Stefan 問題という放物型偏微分方程式の自由境界問題の時間大域的な解の挙動の解析を行ってきた。Giang さんはある種の scaling を用いて、Stefan 問題（放物型）の解が、Hele-Shaw タイプの楕円型自由境界問題の解に収束することを示した。また、収束先は熱源が特異点となる Green 関数になることも示した。この収束問題の難点は、Stefan 問題の強い解の存在が期待できないこと、その弱解の時間に対する単調性も期待できないこと、時刻無限大での極限問題が楕円型自由境界問題になることにある。彼女はこの問題に対して、変分法の手法を用いて、Stefan 問題の Rescaled 変分解が Point source を持つ Hele-Shaw 問題の解に収束することを証明した。さらに、時間に対する弱単調性という概念を導入する事で、放物型の Harnack 不等式を使うことを可能にし、これらと比較原理に基づく粘性解の手法で Rescaled 自由境界が極限問題の自由境界に Hausdorff 距離で局所一様収束することを示した。これより周期的な媒体上での異方拡散をもつ Stefan 問題にも応用可能となった。ここでは、楕円型問題の基本解を用いた Barrier 関数を作り、極限関数が正しい境界条件（原点での特異点を含む）を満たすことを空間３次元の場合に示す事が重要なテクニックである。

Giang さんは、この結果を原著論文１本と Preprint１本にまとめた。以上により本論文は、博士（理学）を授与するに値すると判断した。

４．審査結果　(1) 判　　定（いずれかに○印）　(合　格)・　不合格

(2) 授与学位　博　士（　理学　）