A Theorem Concerning with a Transversal Map to the Foliation.

Haruo KITAHARA*

(Received Jan. 5, 1970)

In this note, we will prove a theorem concerning with a foliation induced by a transversal map to the given foliation on a manifold. We shall be in C^∞-category and follow R. S. Palais for the terminology for foliation.

Let M be an m-dimensional Hausdorff manifold with a regular foliation L of codimension k, and A an n-dimensional Hausdorff manifold. If M is a manifold, $T_x(M)$ denotes the tangent space to M at x. If f is a map of A to $M, f_*: T_a(A) \longrightarrow T_{f(a)}(M)$ denotes the linear map induced on tangent vectors.

Definition. A map f of A into M is said to be transversal to L at a point a in A if $f_{*,a}(T_a(A)) + L_{f(a)} = T_{f(a)}(M)$ (direct sum). f is said to be transversal to L if it is transversal to L at every point of A.

(1) f is transversal to L if and only if there is a subspace E of $T_a(A)$ of dimension k such that $f_{*,a}E$ is injective and $f_{*,a}(E) \cap L_{f(a)} = \{0\}$

If the above condition is satisfied, since $dim.\ f_{*,a}(E) + dim.\ L_{f(a)} = dim.\ T_{f(a)}(M)$, we have $f_{*,a}(E) + L_{f(a)} = T_{f(a)}(M)$. Hence f is transversal to L at a. Conversely, if f is transversal to L at a, let E' be a subspace of $f_{*,a}(T_a(A))$ of dimension k such that $E' \cap L_{f(a)} = \{0\}$. There is a subspace E of $T_a(A)$ of dimenison k such that $f_{*,a}(E) = E'$ and the result follows.

(2) For each $a \in A$, let L'_a be the set of vectors in $T_a(A)$ mapped by f_* into $L_{f(a)}$. Then $L'; p \longrightarrow L'_p$ is a regular foliation of codimension k in A.

If $a \in A$ then we can find a cubical coordinate system $(y_1, \ldots, y_m; Q)$ in M centered at f(a) flat with respect to L such that for a leaf $\Sigma_{f(a)}$ of L through f(a), $Q \cap \Sigma_{f(a)} = \{p \in Q \mid y_i(p) = 0, i = 1, \ldots, k\}$.

Since $(dy_i)_{f(a)} \mid L_{f(a)} = 0$, it follows that $(dy_i) \mid E'$ are linearly independent.

Since $f_{*,a}: E \longrightarrow E'$ is an isomorphism, $d(y_i \circ f)_a \mid E$ are linearly independent, in particular, $d(y_i \circ f)_a$ are linearly independent.

Since $f^{-1}(Q) \cap f^{-1}(\Sigma_{f(a)}) = \{p \in f^{-1}(Q) \mid (y_i \circ f)(p) = 0\}$, $f^{-1}(\Sigma_{f(a)})$ is a regularly imbedded submanifold in A of codimension k. Then we can find a coordinate system $(x_1, \ldots, x_n; Q')$ in A such that $x_i = y_i \circ f$. We can assume that $(x_1, \ldots, x_n; Q')$ is cubical, centered at a and $f(Q') \subseteqq Q$. The relation $dx_i = d(y_i \circ f) = f_* dy_i$ together with the fact that $dy_i\ (1, \ldots, k)$ are a base for the annihilator of L implies that dx_i are a base for the annihilator of $L' = f_*^{-1}(L)$. Then $(x_1, \ldots, x_n : Q')$ is flat with respect to L'. therefore L' is involutive and regular.

Since $f_{*,a}(L'_a) \subseteq L_{f(a)}$ for all $a \in A$, there exists a differentiable map h of A/L' to

* *Department of Mathematics*

M/L satisfying $\pi_L \cdot f = h \cdot \pi_{L'}$ where $\pi_L(\pi_L) : M \longrightarrow M/L\,(A \longrightarrow A/L')$ is an natural projection.

(3) h is a local diffeomorphism of A/L' into M/L.
π_L and π_L' are projection-like maps and $dim.\ (A/L') = dim.\ (M/L) = k$. Then the fact that rank $(f_*) = k$ implies that rank$(h_*) = k$.

(4) For each $\Sigma' \in A/L'$, $f_* \mid \Sigma'$ is a differentiable map of Σ to $h\ (\Sigma')$.
Let Σ'_a be a leaf of L containing $a \in A$. Since f (Σ'_a) is in a regularly imbedded submanifold $\Sigma_{f(a)}$, f $\mid \Sigma'_a$ is differentiable.

(5) Let A/L' be compact, connected. and M/L connected. If h is an open map, then $(A/L'$, h) is a covering space for M/L.
h (A/L') is an open and compact, and hence open and closed subset of M/L, so h$(A/L') = M/L$. If $\Sigma \in M/L$, then, since h is a local diffeomorphism, $h^{-1}\ (\Sigma)$ is a discrete subset of A/L'. hence (A/L', h) is a covering space for M/L. Summerizing the aboves, we have,

Theorem. Let M be an m-dimensional Hausdorff manifold with a regular foliation L of codimension k, and A an n-dimensional Hausdorff manifold. Let f be a transversal map to L. For each $a \in A$, let L'_a be the set of vectors in $T_a(A)$ mapped by f_* into $L_{f(a)}$. Then $L' : a \longrightarrow L'_a$ is a regular foliation of codimension k in A. There is a local diffeomorphism h of A/L' into M/L with following properties.
1) For each $\Sigma' \in A/L'$, $f_* \mid \Sigma$ is a differentiable map of Σ' to h (Σ').
2) Let A/L' be compact, connected and M/L connected. If h is an open map, then $(A/L'$, h) is a covering space for M/L.

Reference

[1] R. S. Palais A global formulation of the Lie theory of transformation groups. Memoirs of A. M. S. No. 22. 1957.