Singularities of an Involutive Differential System and Its Applications to a Certain Submantifold of a Riemannian Manifold

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In this note we will first give a property of singularities of involutive linear differential systems and then prove the theorems of a certain submanifold of a riemannian manifold similar to ones owing to W. S. Massey in a surface of Gaussian curvature 0 in 3-euclidean space. [3]

We shall always be in the C^∞-category. Let $L(M)$ be the Lie algebra of the vector fields on an n-dimensional manifold M.

Definition. An involutive differential system is a Lie subalgebra L of $L(M)$.

Given such an involutive system L and a point p in M, we put $L(p)=\{u(p) \mid u \in L\}$. $L(p)$ is a real subspace of the tangent space $T_p(M)$ to M at p. An integral manifold of L is a connected submanifold N of M such that $T_p(N) = L(p)$ at every point p in N.

1. We choose a neighbourhood U of p in M and denote by $O(U)$ the oscillation of $dim.\ L(p)$; $O(U) = \max_{x,y \in U} (dim.\ L(x) - dim.\ L(y))$. The oscillation $O(p)$ at p in M is defined by $O(p) = \min_{U \ni p} O(U)$. If $V \subset U$ we have $O(V) \leqq O(U)$. As $O(U)$ is an integer-valued function, there exists a neighbourhood U' of p in M such that $O(U') = O(p)$. A point p for which $O(p)=0 (O(p) \neq 0)$ will be called L-regular (L-singular). For a L-regular point p, there exists a neighbourhood U such that $O(U)=O(p)=0$, i. e. $dim.\ L(x)=$constant for all x in U.

(1.1) The set Ω of L-singular points is a closed set without interior points.

Let p be a limit point of Ω and let U be a neighbourhood of p. Then there exists at least one point x in U such that $O(x) \neq 0$, i. e. x has a neighbourhood V such that $V \subset U$, $O(V)=O(x) \neq 0$. This implies that $O(U) \neq 0$, and since U was arbitrary, we have $O(p) \neq 0$, i. e. $p \in \Omega$. Ω is closed.

Let p be an interior point of Ω, and let U be a set such that $p \in U \subset \Omega$, $O(U) = O(p)$. Then there exists a point y in U such that $O(p)=dim.\ L(p) - dim.\ L(y)$, and there exists a set V such that $y \in V \subset U$, $O(y)=O(V)$. Then there exists z in V such that $O(y)=dim.\ L(y)-dim.\ L(z)$. Since $O(y) \neq 0$, we have $dim.\ L(p) - dim.\ L(z)=O(p)+O(y)>O(p)$, $z \in U$. This implies $O(U)>O(p)$, contradictory to that $O(U)=O(p)$. Hence p cannot be an interior point of Ω.

(1.2) The set Ω is nowhere dense in M.

It follows from that a locally compact Hausdorff space is a Baire space.

(1.3) For each connected component M_0 of the set of L-regular points, $dim.\ L(p)=$constant for all p in M_0.

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(1.4) The boundary of each connected component M_0 of L-regular points consists of L-singular points.

Applying the Frobenius integrability theorem to each connected component of L-regular points, we have,

(1.5) Let M_0 be a connected component of the set of L-regular points in M. Then through each p in M_0 there exists an unique maximal connected submanifold Q such that $T_q(Q)=L(q)$ *for* each q in Q.

Summarlizing the aboves, we have,

Theorem 1.

Let L be an involutive linear differential system on M. Then the set Ω of L-singular points in M is nowhere dense in M, and through each p in each connected component in the set of L-regular points in M, there exists an integral manifold of L.

2. Let M be an n-dimensional riemannian manifold. Each point p in M has a positive definite product $<,>$ by the metric. M carries the riemannian connection ∇. Let N be a submanifold of codimension k. Since each tangent space $T_p(N)$ to N at p in N is identified with a subspace of the tangent space $T_p(M)$ to M at p in M, the given inner product $<,>$ on $T_p(M)$ can be restricted to $T_p(N)$ to define a positive definite inner product there also. Thus N inherits a riemannian metric. (so-called, induced metric). For p in N let $T_p^{\perp}(N)$ be the orthogonal complement of $T_p(N)$ in $T_p(M)$ with respect to $<,>$. For $u \in T_p^{\perp}(N)$, $v, w \in T_p(N)$ we choose the vector fields X, Y, Z such that X and Y are tangent to N, Z normal to N, so that $X(p)=v$, $Y(p)=w$, $Z(p)=u$ and define the second fundamental form S by $S_u(v, w)=<\nabla_X Y, Z>(p)=<\nabla_X Y(p), Z(p)>$.

Definition. A vector $v \in T_p(N)$ is a characterisitic vector of $S_u(v, T_p(N))=0$ for all $u \in T_p^{\perp}(N)$. C_p denotes the subspace of $T_p(N)$ of these characterisitic vectors at p in N. Let C'_p be the following subspace of C_p;

$C'_p=\{v \in C_p \mid R(v, T_p(N))(T_p(N)) \subset T_p(N)\}$, where R denotes the curvature tensor on M.

Let C be the set of vector fields X on M such that $X(p) \in C'_p$ for each p in N. Then C defines a linear differential system on N. We define the oscillations O corresponding to ones of L. A point p for which $O(p)=0$ ($\neq 0$) will be called C'-regular (C'-singular).

Theorem 2. The set of C-singular points is nowhere dense in N.

Theorem 3.. Suppose that $R(C'_p; C'_p, T_p(N))(T_p(N))=0$, for all $p \in N$, where ; denotes the covariant derivatives of the curvature tensor R on M. Let N_0 be a connected component of the set of C'-regular points in N. Then through each p in N_0 there is an unique maximal connected submanifold Q such that $T_q(Q)=C'_q$ for each q in Q. Further, Q is a totally geodesic submanifold of N such that tangent spaces are self-parallel along Q.

Proof. If we show that $\nabla_X Y \in C'$ for X, $Y \in C$, we have $[X, Y] \in C'$. Then

C' is involutive. Let X and Y be any vector fields in C' and Z any tangent vector field on N.

$$\nabla_{\nabla_X Y} Z = \nabla_Z \nabla_X Y + [\nabla_X Y, Z]$$
$$= R(Z, X)(Y) + \nabla_X(\nabla_Z Y) + \nabla_{[Z, X]} Y + [\nabla_X Y, Z]$$

But $\nabla_Z X$, $\nabla_Y Z$ and $\nabla_{[Z, X]} Y$ are tangent to N, since $\langle \nabla_Z X, u \rangle (p) = S_u(X, Z)_p = 0$, $\langle \nabla_Y Z, u \rangle (p) = S_u(Y, Z)_p = 0$. for all u normal to N. and $[Z, X]$ is tangent to N. Then $\nabla_{\nabla_X Y} Z$ is tangent to N for any tangent vector field Z on N, which proves that $(\nabla_X Y)_p \in C_p$ for all p in N. Now, for Z, W tangent to N, $R(\nabla_X Y, Z)(W) = \nabla_X (R(Y, Z)(W)) - R(Y, \nabla_X Z)(W) - R(Y, Z)(\nabla_X W)$ which is tangent to N. Therefore $\nabla_X Y \in C'$. The second statement is evident. q. e. d.

Reference

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