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On the Criterion of C. Carathéodory.

Haruo KITAHARA.* and Nobukazu MATSUOKA.**

In his axiomatic foundation of thermodynamics, C. Carathéodory had the following theorem about a Pfaffian form defined on the 3-dimensional Euclidean space.

Theorem.***

If a linear differential form H is such that in every neighbourhood of an arbitrary point P there exist points which cannot be attained from P by any integral curve of H, then H is completely integrable.

In this note, we shall generalize the above theorem to one on a differentiable manifold in C^∞-category.

Let M be an n-dimensional manifold. The Lie algebra $V(M)$ of vector fields on M is a module over the ring $F(M)$ of functions on M. A linear differential system H on M is an $F(M)$-submodule of $V(M)$. For $p \in M$, H_p is a linear subspace $\{X(p) \mid X \in H\}$ of the tangent space M_p to M at p. The dimension of H_p is called the rank $r_H(p)$ of H at p. We shall suppose that H is of rank n-1, i. e. $r_H(p) = n-1$ for all $p \in M$.

Definition. $D(H)$ is the smallest Lie subalgebra of $V(M)$ containing H. Clearly, $n-1 = r_H(p) \leqq r_{D(H)}(p) \leqq n$ for all p.

(1) *If* $r_{D(H)}(H)(p) = n$ *for some* p, *then* $r_{D(H)}(q) = n$ *for all points* q *sufficiently close to* p.

Proof. Let X_i, $i, j = 1, \cdots\cdots, n$, be the elements of $D(H)$ which form a basis of $D(H)_p$. We put $X_i = \sum_{\gamma=1}^{n} A_{ij} \frac{\partial}{\partial x^j}$. Since the vector fields $\frac{\partial}{\partial x^j}$ form a basis of the tangent space at each point, the dimension of $D(H)_q$ is equal to the rank of $n \times n$ matrix $(A_{ij}(q))$. Hence we have $r_{D(H)}(q) = rank\ (A_{ij}(q)) = rank\ (A_{ij}(p)) = n$ for all q sufficiently close to p.

Definition. A curve $\sigma(t)$, $0 \leqq t \leqq 1$. is called an H-($D(H)$-) integral curve if $\sigma'(t) \in H_{\sigma(t)}$ ($\sigma'(t) \in D(H)_{\sigma(t)}$), $0 \leqq t \leqq 1$.

Definition. A linear differential system H is said to satisfy the criterion of Carathéodory, if the following geometrical property (C) holds;

(C). *For each point* $p \in M$, *there exist points* q *arbitrary close to* p *that are inaccessible from* p *along* H*-integral curves.*

An integral curve $exp\ tY$, $|t| < \epsilon$, of the vector field Y can be considered

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*Department of mathematics.

**Kanazawa Institute of Technology.

***The geometrical proof of its generalization has been showed by S. KASHIWABARA.

as an orbit of a local one-parameter group of local diffeomorphisms generated by Y. Since a transformed vector field $(exp\ tY)_*(X)$ of a vector field X is defined by $(exp\ tY)_*(X(q)) = (exp\ tY)_*(X)\ (exp\ tY\cdot q)$, $exp\ tY$ maps an integral curve of X to an integral curve of $(exp\ tY)_*(X)$. Then $s \to (exptY)\cdot(exp\ sX)\cdot p$ is an integral curve of $(exp\ tY)_*(X)$.

Hence we have the following;

(2) $(exp\ tY)\cdot(exp\ sX)\cdot p = exp\ (s\,(exp\ tY)_*(X))\cdot exp\ tY\cdot p$.

(3) *If H satisfies the property* (C), *then* $H_p = D(H)_p$.

Proof. Suppose that $H_p \neq D(H)_p$, i. e. $r_{D(H)}\ (p) = n$. Let X_a, $a, b = 1, \cdots\cdots, n\text{-}1$, be the elements of H which form a basis of H_p. Then there exists a pair X_a, X_b such that $[X_a, X_b] \bar{\in} H$ in a neighbourhood of p. Since, by definition,

$$[X_a, X_b]_p = \lim_{t \to 0} \frac{((exp\ tX_a)_*(X_b))_p - (X_b)_p}{t}$$

where the limit on the right hand side is taken with respect to the natural topology of the tangent space M_p, there exists some t such that $(exp\ tX_a)_*(X_b) \bar{\in}$ H, $|t| < \varepsilon$. Then we can suppose that $(exp\ X_a)_*(X_b) \bar{\in} H$. By (2), every point that can be reached along a $D(H)$-integral curves can be reached along H-integral curves. Since $D(H)_q = M_q$ for all points q sufficiently close to p, those points q can be reached along H-integral curves.
This cont-radicts the property (C).

Hence we have the following;

Theorem.

If a linear differential system H of rank n-1 satisfies the Criterion of Carathéodory, then H is completely integrable. The converse also holds.

Reference

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