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Stability theorems for Γ-foliations associated with semi-simple flat homogeneous spaces

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1. Introduction.

An important problem in the foliation theory is to describe the influence exerted by a compact leaf upon the global structure of a foliation. For certain classes of foliations, this problem is reasonable. Stability theorems have been studied by G. Reeb [9], B.L. Reinhart[10], R.A. Blumenthal[1] and others:

THEOREM A (Reeb Stability[9]). *Let E be a foliation of codimension one on a compact connected manifold. If E has a compact leaf with finite fundamental group, then all the leaves of E are compact with finite fundamental group.*

THEOREM B (Reinhart Stability[10]). *Let E be a complete riemannian foliation of codimension $q \geqq 1$ on a connected manifold. Then all the leaves of E have the same universal cover. In paticular, if E has a compact leaf with finite fundamental group, then all the leaves of E are compact with finite fundamental group.*

THEOREM C (Blumenthal Stability[1]). *Let E be a complete conformal foliation of codimention $q \geqq 3$ on a connected manifold. Then all the leaves of E have the same universal cover. In particular, if E has a compact leaf with finite infinitesimal holonomy group of order 2, then all the leaves of E are compact with finite infinitesimal holonomy group of order 2.*

The aim of this paper is to prove the stability theorem for a foliation with the structure pseudogroup Γ of local automorphisms of a certain 2nd order G-structure which implies the some stability theorems.

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After the completion of this paper, the author recieved a preprinted form of Blumenthal's paper entitled "Cartan connections in foliated bundles". The author obtained the results in this paper independently.

THEOREM. *Let E be a complete Γ-foliation associated with a semi-simple flat homogeneous space L/L_0. Suppose that the Spencer cohomology $H^{2,1}(\mathfrak{l})$ of the graded Lie algebra $\mathfrak{l}$ of L vanishes. Then all the leaves of E have the same universal cover. In particular, if E has a compact leaf N_0 with finite infinitesimal holonomy group $H^2(N_0, x_0)$ of N_0 based at x_0, then all the leaves of E are compact and $H^2(N, x)$ is finite for all leaves N of E.*

We shall be in C^∞-category, and manifolds are supposed to be paracompact, Hausdorff spaces.

2. Semi-simple flat homogeneous spaces.

We shall review a brief survey of the basic materials on semi-simple flat homogeneous spaces. For details, see S. Kobayashi and T. Nagano[3], S. Kobayashi and T. Ochiai[4] and T. Ochiai[8].

A (transitive) semi-simple graded Lie algebra means a semi-simple Lie algebra $\mathfrak{l}=\mathfrak{g}_{-1}+\mathfrak{g}_0+\mathfrak{g}_1$, $\dim \mathfrak{g}_p<\infty$ $(p=-1, 0, 1)$ such that $[\mathfrak{g}_p, \mathfrak{g}_q]\subset\mathfrak{g}_{p+q}$ for all p, $q\geqq -1$ and $[X, \mathfrak{g}_{-1}]\neq 0$ for each non-zero $X\in\mathfrak{g}_p$, $p\geqq 0$. $\mathfrak{g}_{-1}$ is the dual vector space of $\mathfrak{g}_1$ by the nondegeneracy of the Killing form of $\mathfrak{l}$. Semi-simple graded Lie algebras have been classified in[3].

The Lie algebra cohomology $H(\mathfrak{l})=H(\mathfrak{g}_{-1}, \mathrm{ad}_{\mathfrak{l}}|\mathfrak{g}_{-1}, \mathfrak{l})$ of the abelian Lie algebra $\mathfrak{g}_{-1}$ with respect to its adjoint representation on $\mathfrak{l}$ is called the Spencer cohomology of a graded Lie algebra $\mathfrak{l}=\mathfrak{g}_{-1}+\mathfrak{g}_0+\mathfrak{g}_1$. More precisely, let $C^{p,q}=\mathfrak{g}_{p-1}\otimes\Lambda^q(\mathfrak{g}_{-1})^*$ be the vector space of all $\mathfrak{g}_{p-1}$-valued q-linear alternating map on $\mathfrak{g}_{-1}$. Define a coboundary operator $\partial: C^{p,q}\longrightarrow C^{p-1,q+1}$ by

$$(\partial c)(X_1, \dots, X_{q+1})=\sum_i(-1)^{i+1}[X_i, c(X_1, \dots, \hat{X}_i, \dots, X_{q+1})]$$

for $c\in C^{p,q}$ and $X_1, \dots, X_{q+1}\in\mathfrak{g}_{-1}$. Then $\partial^2=0$ and the Spencer cohomology $H(\mathfrak{l})=\sum H^{p,q}(\mathfrak{l})$ is defined by

$$H^{p,q}(\mathfrak{l})=\partial^{-1}(0)\cap C^{p,q}/\partial(C^{p+1,q-1}).$$

Let L/L_0 be a connected homogeneous space on which a (not necessarily connected) semi-simple Lie group L acts effectively and transitively. Since L_0 is the isotropy subgroup of L at the origin of L/L_0, there is a natural representation ρ of L, called the linear isotropy representation of L_0, on the tangent space of L/L_0 at the origin. ρ is a homomorphism from L_0 into $GL(\mathfrak{g}_{-1})=GL(q\,;\boldsymbol{R})$, $q=\dim L/L_0$. Let L_1 be the kernel of ρ. L/L_0 is called a semi-simple flat homogeneous space of order 2 if the Lie algebra $\mathfrak{l}$ of L has a semi-simple graded Lie algebra structure $\mathfrak{l}=\mathfrak{g}_{-1}+\mathfrak{g}_0+\mathfrak{g}_1$ such that $\mathfrak{g}_0+\mathfrak{g}_1$ is the Lie algebra of L_0. It is known that $\mathfrak{g}_0$ is the Lie algebra of the linear isotropy subgroup $G_0=L_0/L_1\subset GL(q\,;\boldsymbol{R})$ and L_0 is a semi-direct product $G_0\cdot G_1$ of G_0 and the vector group $G_1=\exp\mathfrak{g}_1$.

EXAMPLE 1. $\mathfrak{l}=\mathfrak{sl}(q+1\,;\boldsymbol{R})$ $(q\geqq 1)$.

$$\mathfrak{g}_{-1}=\left\{\begin{pmatrix}0 & 0\\ \xi & 0\end{pmatrix}\right\},\quad \mathfrak{g}_0=\left\{\begin{pmatrix}a & 0\\ 0 & A\end{pmatrix}\middle| a+\mathrm{Trace}\ A=0\right\},\quad \mathfrak{g}_1=\left\{\begin{pmatrix}0 & u\\ 0 & 0\end{pmatrix}\right\},$$

where ξ is a row q-vector, u is a column q-vector, $A\in\mathfrak{gl}(q\,;\boldsymbol{R})$ and $a\in\boldsymbol{R}$. The corresponding semi-simple flat homogeneous space L/L_0 of order 2 is a real projective space of dimension q, where $L=GL(q+1\,;\boldsymbol{R})/\boldsymbol{R}^*I_{q+1}$ and $L_0=\left\{\begin{pmatrix}b & v\\ 0 & B\end{pmatrix}\in GL(q+1\,;\boldsymbol{R})\right\}/\boldsymbol{R}^*I_{q+1}$.

EXAMPLE 2. $\mathfrak{l}=\mathfrak{o}(q+1,\ 1)=\{X\in\mathfrak{gl}(q+2\,;\boldsymbol{R})|{}^tXS+XS=0\}(q\geqq 3)$,

$$\text{where } S=\begin{bmatrix}0 & 0 & -1\\ 0 & I_q & 0\\ -1 & 0 & 0\end{bmatrix}$$

$$\mathfrak{g}_{-1}=\left\{\begin{pmatrix}0 & 0 & 0\\ \xi & 0 & 0\\ 0 & {}^t\xi & 0\end{pmatrix}\right\},\quad \mathfrak{g}_0=\left\{\begin{pmatrix}a & 0 & 0\\ 0 & A & 0\\ 0 & 0 & -a\end{pmatrix}\right\},\quad \mathfrak{g}_1=\left\{\begin{pmatrix}0 & u & 0\\ 0 & 0 & {}^tu\\ 0 & 0 & 0\end{pmatrix}\right\},$$

where ξ is a row q-vector, $A\in\mathfrak{o}(q)$, u is a column q-vector and $a\in\boldsymbol{R}$. The corresponding semi-simple flat homogeneous space L/L_0 of order 2 is a Möbius space of dimension q (q-sphere S^q), where $L=\{X\in GL(q+2\,;\boldsymbol{R})|{}^tXSX=S\}/\{\pm I_{q+2}\}$ and

$$L_0=\left\{X=\begin{pmatrix}* & * & *\\ 0 & * & *\\ 0 & 0 & *\end{pmatrix}\middle|{}^tXSX=S\right\}/\{\pm I_{q+2}\}.$$

EXAMPLE 3. $\mathfrak{l}=\{X\in\mathfrak{gl}(q+2\,;\boldsymbol{R})|{}^tXS+SX=0\}(q\geqq 3)$,

$$\text{where } S=\begin{pmatrix}0 & 0 & 1\\ 0 & S_{r,s} & 0\\ 1 & 0 & 0\end{pmatrix},\quad S_{r,s}=\begin{pmatrix}I_r & 0\\ 0 & -I_s\end{pmatrix},\quad r+s=q,\ r\geqq s\geqq 0.$$

$$\mathfrak{g}_{-1}=\left\{\begin{bmatrix}0 & 0 & 0 & 5\\ \xi' & 0 & 0 & 0\\ \xi'' & 0 & 0 & 0\\ 0 & {}^t\xi' & -{}^t\xi'' & 0\end{bmatrix}\right\},\quad \mathfrak{g}_0=\left\{\begin{pmatrix}a & 0 & 0\\ 0 & A & 0\\ 0 & 0 & -a\end{pmatrix}\right\},\quad \mathfrak{g}_1=\left\{\begin{bmatrix}0 & u' & u'' & 0\\ 0 & 0 & 0 & {}^tu'\\ 0 & 0 & 0 & -{}^tu''\\ 0 & 0 & 0 & 0\end{bmatrix}\right\},$$

where ξ' (ξ'') is a row r-vector (s-vector), u' (u'') is a column r-vector (s-vector), $A\in\mathfrak{o}(r,\ s)=\{B\in\mathfrak{gl}(q\,;\boldsymbol{R})|{}^tBS_{r,s}+S_{r,s}B=0\}$ and $a\in\boldsymbol{R}$. The corresponding semi-simple flat homogeneous space L/L_0 of order 2 is the quotient space $E_{r,s}=S^r\times S^s/\sim$ of $S^r\times S^s$ by the equivalence relation $\sim$ defined by $(x,\ y)\sim(-x,\ -y)$ for $(x,\ y)\in S^r\times S^s$([7]).

3. L_0-structure of order 2 associated with L/L_0.

We will refer to [7] and [11] in this and the following sections. Let B be a manifold of dimension q. Let $G^r(q) \longrightarrow p^r(B) \xrightarrow{\tilde{\pi}_r} B$ be the r-th frame bundle of B. $GL(q\,;\boldsymbol{R})=G^1(q)$ may be identified with a subgroup of $G^r(q)$ in canonical way. Then the natural projection $\tilde{\pi}^s_r: P^r(B) \longrightarrow P^s(B)$ for $r>s$ is $GL(q\,;\boldsymbol{R})$-equivariant and satisfies $\tilde{\pi}_s\circ\tilde{\pi}^s_r=\tilde{\pi}_r$. Let $\Gamma(B)$ be the pseudogroup of all local diffeomorphisms of B. The r-th prologation $\gamma^{(r)}$ of $\gamma\in\Gamma(B)$ is a local $G^r(q)$-bundle map of $P^r(B)$ such that $\tilde{\pi}_r\circ\gamma^{(r)}=\gamma\circ\tilde{\pi}_r$.

Now, we review the definition of the r-th canonical form $\tilde{\theta}^{(r)}$ on $P^r(B)$. We define the distinguished element $e^r\in P^r(\boldsymbol{R}^q)$ by $e^r=j^r_0(\text{identity})$ and set $\mathfrak{p}^r(q)= T_{e^r}(P^r(\boldsymbol{R}^q))$. The natural action of $G^r(q)$ on $\mathfrak{p}^{r-1}(q)$ is denoted by Ad. The map p^r_s: $\mathfrak{p}^r(q)\longrightarrow\mathfrak{p}^s(q)$ means the differential of $\tilde{\pi}^s_r: P^r(\boldsymbol{R}^q)\longrightarrow P^s(\boldsymbol{R}^q)$ for $r>s$ at e^r. In particular, $\mathfrak{p}^0(q)=\boldsymbol{R}^q$, $\mathfrak{p}^1(q)=\boldsymbol{R}^q+\mathfrak{gl}(q)$, which may be identified with the Lie algebra of the group of affine automorphisms of $\boldsymbol{R}^q$ and $p^0_1:\mathfrak{p}^1(q)\longrightarrow\mathfrak{p}^0(q)$ is the projection to the first factor. The r-th canonical form $\tilde{\theta}^{(r)}$ is a $\mathfrak{p}^{r-1}(q)$-valued 1-form on $P^r(B)$ defined as follows. Let $u=j^r_0(f)\in P^r(B)$ and $f:\boldsymbol{R}^q\longrightarrow B$ be a local diffeomorphism defined around the origin 0. Then the correspondence $j^{r-1}_0(\gamma)\longrightarrow j^{r-1}_0(f\cdot\gamma)$ defines a local diffeomorphism $\bar{f}: P^{r-1}(\boldsymbol{R}^q)\longrightarrow P^{r-1}(B)$ defined around e^{r-1} such that $\bar{f}(e^{r-1}) =u'=\tilde{\pi}^{r-1}_r(u)$, and the differential $\bar{u}:\mathfrak{p}^{r-1}(q)\longrightarrow T_u(P^{r-1}(B))$ of $\bar{f}$ at e^{r-1} is independent of choice of f. $\tilde{\theta}^{(r)}$ is defined by

$$\tilde{\theta}^{(r)}(X)=\bar{u}^{-1}(\tilde{\pi}^{r-1}_r)_*X \qquad \text{for } X\in T_u(P^r(B)).$$

It satisfies

$$R_a{}^*\tilde{\theta}^{(r)}=Ad(a^{-1})\tilde{\theta}^{(r)} \qquad \text{for } a\in G^r(q),$$
$$(\tilde{\pi}^s_r)^*\tilde{\theta}^{(s)}=p^{s-1}_{r-1}\tilde{\theta}^{(r)} \qquad \text{for } r>s,$$

where R_a means the right translation of $P^r(B)$ by $a\in G^r(q)$. In particular, let $\tilde{\theta}_{-1}$ and $\tilde{\theta}_0$ be the $\boldsymbol{R}^q$-component and $\mathfrak{gl}(q\,;\boldsymbol{R})$-component of the 2nd canonical form $\tilde{\theta}^{(2)}$ on $P^2(B)$ respectively, so that $\tilde{\theta}^{(2)}=\tilde{\theta}_{-1}+\tilde{\theta}_0$. Then we have

$$d\tilde{\theta}_{-1}+[\tilde{\theta}_0,\ \tilde{\theta}_{-1}]=0.$$

Let L/L_0 be a semi-simple flat homogeneous space of order 2. We set $\rho_0=\rho|G_0$. Then we have a commutative diagram;

$$\begin{array}{ccc} L_0 & \xrightarrow{\ \tau\ } & G^2(q) \\ \uparrow & \searrow^{\rho} & \downarrow \tilde{\pi}^1_2 \\ G_0 & \xrightarrow[\ \rho_0\ ]{} & GL(q\,;\boldsymbol{R}). \end{array}$$

ρ_0 is an injective homomorphism, which identifies G_0 with the Lie subgroup $\rho(L_0)$ of $GL(q;\boldsymbol{R})$. Let $L_0 \longrightarrow Q \longrightarrow B$ be a principal L_0-subbundle of $P^2(B)$, which is called a structure of order 2 associated with semi-simple flat homogeneous space L/L_0. For each Q, let Γ be the pseudogroup of local automorphisms of Q, that is,

$$\Gamma=\{\gamma\in\Gamma(B)|\gamma^{(2)}Q\subset Q\}.$$

We define a G_0-subbundle P of $P^1(B)$ by $P=\tilde{\pi}_2^1(Q)$, which is the G_0-structure associated with Q. It should be noted that for each $\gamma\in\Gamma$ the lst prologation $\gamma^{(1)}$ leaves P invariant, since $\gamma^{(1)}\cdot\tilde{\pi}_2^1=\tilde{\pi}_2^1\cdot\gamma^{(2)}$.

EXAMPLE 4. Let L/L_0 be as Example 1 in § 2. The linear isotropy subgroup G_0 coincides with $GL(q;\boldsymbol{R})$, and P is the bundle of linear frames on B. Γ is nothing but the pseudogroup of local projective transformations of a torsion-free linear connection on B.

EXAMPLE 5. Let L/L_0 be as Example 2 in § 2. The linear isotropy subgroup G_0 coincides with $CO(q)$, and P is a $CO(q)$-structure on B. Γ is nothing but the pseudogroup of local conformal transformations of a riemannian metric on B. It should be noted that P contains $O(q)$-structure (i.e. riemannial structure) as subbundle.

EXAMPLE 6. Let L/L_0 be as Example 3 in § 2. The linear isotropy subgroup G_0 coincides with $CO(r,s)$, and P is a $CO(r,s)$-structure on B. Γ is nothing but the pseudogroup of local conformal transformations of a pseudo-riemannian metric of signature (r,s) on B.

4. $\boldsymbol{\Gamma}$-foliations associated with $\boldsymbol{L/L_0}$.

Let M be a connected smooth manifold of dimension n and B be an another connected smooth manifold of dimension $q=\dim L/L_0$. Let $\Gamma(B)$ be the pseudogroup of local diffeomorphisms on B. Then a $\Gamma(B)$-foliation $\hat{E}$ may be defined by a $\Gamma(B)$ -cocycle $\hat{E}=\{(U_\alpha,\ f_\alpha,\ \gamma_{\alpha\beta})\}_{\alpha,\beta\in\Lambda}$ such that

(i) $\{U_\alpha\}$ is an open cover of M,

(ii) $f_\alpha: U_\alpha \longrightarrow B$ is a submersion,

(iii) for each $x\in U_\alpha\cap U_\beta$, there exists $\gamma_{\alpha\beta}^x\in\Gamma(B)$ such that $f_\beta=\gamma_{\alpha\beta}^x f_\alpha$ in some neighborhood of x.

In the other words, the fibers of each submersion f_α are pieced togather to define the leaves of the foliation $\hat{E}$. The kernel of the differentials $(f_\alpha)_*$ of submersions f_α constitute an integrable subbundle $\hat{E}$ of the tangent bundle TM. A pair (U_α,f_α) is called an adapted chart to $\hat{E}$.

In the same way, we may define a Γ-foliation E replacing $\Gamma(B)$ with Γ in the definition of the foliation $\hat{E}$. It should be noted that both $\hat{E}$ and E have the same structure of leaves. We shall define the r-th frame bundle $P^r(\hat{E})$ for $\hat{E}$ and the r-th canonical form $\theta^{(r)}$ on $P^r(\hat{E})$.

Take a point $o\in B$ and fix it once for all. Choose a local diffeomorphism $f: \boldsymbol{R}^q \longrightarrow B$ defined around 0 such that $f(0)=o$ and $j_0^2(f)\in Q$, and then identify a neighborhood of 0 in $\boldsymbol{R}^q$ with a neighborhood of o in B by means of f. We set

$$P^r(\hat{E})=\{j_x^r(f)|f\in\hat{E} \text{ defined around } x \text{ with } f(x)=0\},$$

and define the projection $\pi_r: P^r(\hat{E})\longrightarrow M$ by $\pi_r(j_x^r(f))=x$. The group $G^r(q)$ acts on $P^r(\hat{E})$ from right by

$$j_x^r(f)\cdot j_0^r(\varphi)=j_x^r(\varphi^{-1}\circ f) \qquad \text{for } j_0^r(\varphi)\in G^r(q).$$

Then we have a $G^r(q)$-bundle $G^r(q)\longrightarrow P^r(\hat{E})\xrightarrow{\pi_r} M$. Note that $P^r(\hat{E})$ may be identified with the r-th prologation of the frame bundle of the normal bundle $\nu(\hat{E})=TM/\hat{E}$. The natural projection $\pi_r^s: P^r(\hat{E})\longrightarrow P^s(\hat{E})$ for $r>s$ is also $GL(q;\boldsymbol{R})$-equivariant and satisfies $\pi_s\cdot\pi_r^s=\pi_r$. Let $f: V\longrightarrow B$ be a local submersion in $\hat{E}$. For each $j_x^r(\varphi)\in P^r(\hat{E})$ with $x\in V$, there exists a local diffeomorphism $\psi: \boldsymbol{R}^q\longrightarrow B$ defined around 0 such that $\psi(0)=f(x)$ and $\psi\circ\varphi=f$ around x. Then the correspondence $j_x^r(\varphi)\longrightarrow j_0^r(\psi)$ defines a $G^r(q)$-bundle map $f^{(r)}: P^r(\hat{E})|V\longrightarrow P^r(B)$. It satisfies $\tilde{\pi}_r\circ f^{(r)}=f\circ\pi_r$.

Let $v=j_x^r(f)\in P^r(B)$ and set $v'=\pi_r^{r-1}(v)$. For each $j_y^{r-1}(\varphi)\in P^{r-1}(\hat{E})$ near to v, there exists a local diffeomorphism $\psi: \boldsymbol{R}^q\longrightarrow\boldsymbol{R}^q$ defined around 0 such that $\psi(0)=f(y)$ and $\psi\circ\varphi=f$ around x. The correspondence $j_y^{r-1}(\varphi)\longrightarrow j_0^{r-1}(\psi)$ defines a local map $\bar{f}: P^{r-1}(\hat{E})\longrightarrow P^{r-1}(\boldsymbol{R}^q)$ defined around v' with $\bar{f}(v')=e^{r-1}$. The differential $\bar{v}: T_{v'}(P^{r-1}(\hat{E}))\longrightarrow \mathfrak{p}^{r-1}(q)$ of $\bar{f}$ at v' is independent of the choice of f. $\theta^{(r)}$ is defined by

$$\theta^{(r)}(x)=\bar{v}(\pi_r^{r-1})_*X \qquad \text{for } X\in T_v(P^r(\hat{E})).$$

Then the following relations hold;

$$R_a{}^*\theta^{(r)}=Ad(a^{-1})\theta^{(r)} \qquad \text{for } a\in G^r(q),$$

$$(\pi_r^s)^*\theta^{(s)}=p_{r-1}^{s-1}\theta^{(r)} \qquad \text{for } r>s,$$

$$\theta^{(r)}=f^{(r)*}\tilde{\theta}^{(r)} \text{ on } P^r(\hat{E})|V, \text{ for each local submersion } f: V\longrightarrow B \text{ in } \hat{E}.$$

LEMMA 1 ([7]). *Let $E=\{f_\alpha\}_{\alpha\in A}$ be a Γ-foliation associated with L/L_0 on M. Then*

(i) *There exists a unique L_0-subbundle $Q(E)$ of $P^2(\hat{E})$ such that $Q(E)|U_\alpha=(f_\alpha^{(2)})^{-1}Q$ for each $f_\alpha: U_\alpha\longrightarrow B$ in E.*

(ii) *There exists a unique G_0-subbundle $P(E)$ of $P^1(\hat{E})$ such that $P(E)|U_\alpha=(f_\alpha^{(1)})^{-1}P$ for each $f_\alpha: U_\alpha\longrightarrow B$ in E.*

(iii) $\pi_2^1 Q(E)=P(E)$.

LEMMA 2 ([11]). *There exists a G_0-equivariant section $s: P(E) \longrightarrow Q(E)$ of the bundle $\pi_2^1: Q(E) \longrightarrow P(E)$.*

We recall the existence theorem of Tanaka-Ochiai for a Cartan connection on the structure Q of order 2 associated with L/L_0.

THEOREM 3 ([8],[12]). *If the Spencer cohomology $H^{2,1}(\mathfrak{l})=0$, then there exists a unique normal Cartan connection of type L/L_0 on Q.*

We shall require following properties of a normal Cartan connection $\tilde{w}$:

(i) $\tilde{w}$ is a $\mathfrak{l}$-valued 1-form on Q.

(ii) $\tilde{w}$ is invariant under Γ.

(iii) Let $L \longrightarrow Q^L = Q \times_{L_0} L \longrightarrow B$ be the group extension of Q by L. Then $\tilde{w}$ is extended to a unique L-connection form on Q^L, which is also defined by $\tilde{w}$.

(iv) Let $\tilde{w}_i (i=-1, 0, 1)$ be the projection of $\tilde{w}$ with respect to the decomposition $\mathfrak{l} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$, so that $\tilde{w} = \tilde{w}_{-1} + \tilde{w}_0 + \tilde{w}_1$. We may regard $\mathfrak{g}_{-1} + \mathfrak{g}_0$ as a Lie subalgebra of $\mathfrak{p}^1(q) = \boldsymbol{R}^q + \mathfrak{gl}(q; \boldsymbol{R})$ by the map $id \oplus \rho_0$, where the differential of $\rho_0: G_0 \longrightarrow GL(q; \boldsymbol{R})$ is also denoted by ρ_0. Then $\tilde{\theta}^{(2)} = \tilde{w}_{-1} + \tilde{w}_0$ on Q, and hence $\tilde{w}_{-1} = p_1^0 \tilde{\theta}^{(2)}$ on Q.

Hereafter we suppose that $H^{2,1}(\mathfrak{l})=0$.

At each point $q \in Q$, we define a subspace H_q of $T_q(Q)$ by

$$H_q = \{X \in T_q(Q) | \langle X, \tilde{w}_0 \rangle = 0, \langle X, \tilde{w}_1 \rangle = 0\}.$$

Since $\langle X, \tilde{w} \rangle = 0$ for every non-zero vector X on Q, we have

$$H_q \cap (\mathfrak{g}_0 + \mathfrak{g}_1) = \{0\}.$$

And it is easily proved that $\tilde{w}_{-1}: \mathrm{H}_q \longrightarrow \boldsymbol{R}^q$ is an isomorphism. Then $\tilde{w}$ restricted to Q defines an absolute parallelism on Q.

Let $L \longrightarrow Q(E)^L = Q(E) \times_{L_0} L \longrightarrow M$ be the group extension of $Q(E)$ by L. For a submersion $f_\alpha: U_\alpha \longrightarrow B$ in E, the natural extension $Q(E)^L | U_\alpha \longrightarrow Q^L$ of the bundle map $f_\alpha^{(2)}: Q(E) | U_\alpha \longrightarrow Q$ is also denoted by $f_\alpha^{(2)}$. Now, the Γ-invariance of the normal Cartan connection $\tilde{w}$ implies the following Lemma.

LEMMA 4 ([7]). *There exists a unique L-connection form w on $Q(E)^L$ such that $f_\alpha^{(2)*} \tilde{w} = w$ on $Q(E)^L | U_\alpha$ for each $f_\alpha: U_\alpha \longrightarrow B$ in E.*

It should be noted that w respected to $Q(E)$ denotes an absolute parallelism on $Q(E)$.

Let $\Omega = dw + (1/2)[w,w]$ be the curvature of w and decompose w and Ω with respect to the decomposition $\mathfrak{l} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$,

$$w = w_{-1} + w_0 + w_1, \qquad \Omega = \Omega_{-1} + \Omega_0 + \Omega_1.$$

Then we have the following relations;

$$w_{-1} = p_1^0 \theta^{(2)},$$

$$\Omega_{-1}=dw_{-1}+[w_0,w_{-1}]=0,$$
$$[w_{-1},\ \Omega_0]=0.$$

Choose a G_0-equivariant section $s: P(E) \longrightarrow Q(E)$ by Lemma 2. Then the pull back $\bar{w}_0=s^*w_0$ of w_0 is a G_0-connection form on $P(E)$. The curvature of $\bar{w}_0$ is denoted by $\bar{\Omega}_0$.

PROPOSITION 5. *Define a subbundle $E^{(1)}$ of $T(P(E))$ with the same dimension as E by*

$$E^{(1)}=\{X \in T(P(E)) | (\pi_1)_*X \in E,\ \langle X,\ \bar{w}_0\rangle=0\}.$$

Then $E^{(1)}$ is an integrable subbundle of $T(P(E))$.

PROOF. Note that $\theta^{(1)}=s^*w_{-1}$ on $P(E)$ and $d\theta^{(1)}+[\bar{w}_0,\theta^{(1)}]=0$. Taking $\mathfrak{g}_0$-component of Ω, we have

$$\Omega_0=dw_0+(1/2)[w_0,w_0]+[w_{-1},w_1],$$

and hence

$$\bar{\Omega}_0=s^*\Omega_0-[s^*w_{-1},s^*w_1].$$

Let X and Y be in $E^{(1)}_v$ with $v \in P(E)$ and choose $f_\alpha: U_\alpha \longrightarrow B$ in E with $\pi_1(v) \in U_\alpha$. Then $\Omega_0=f^{(2)*}\tilde{\Omega}_0$ on $Q(E)|U_\alpha$ by Lemma 4, and hence $s^*\Omega_0=s^*f_\alpha^{(2)*}\tilde{\Omega}_0$ on $P(E)|U_\alpha$. Since $\tilde{\Omega}_0$ is holizontal and

$$\tilde{\pi}_{2*}(f_\alpha^{(2)}{}_*s_*X)=f_{\alpha*}\pi_{2*}s_*X=f_{\alpha*}\pi_{1*}X=0,$$

we have $(s^*\Omega_0)(X,Y)=0$. On the other hand, $\theta^{(1)}(X)=0$ with $X \in T(P(E))$ if and only if $(\pi_1)_*X \in E$. This fact implies that

$$[s^*w_{-1},\ s^*w_1](X,Y)=0 \qquad \text{for } X,\ Y \in E^{(1)}_v.$$

Then we have $\bar{\Omega}_0(X,Y)=0$ for X, $Y \in E^{(1)}_v$, which implies that $d\bar{w}_0=0$ on $E^{(1)}$. Therefore, $E^{(1)}$ is an integrable subbundle of $T(P(E))$ with the same dimension as E.

Let $N^{(1)}$ be a leaf of $E^{(1)}$. If $N=\pi_1(N^{(1)})$ is a leaf of E, then $\pi_1|N^{(1)}: N^{(1)} \longrightarrow N$ is a regular cover.

PROPOSITION 6. *Define a subbundle $E^{(2)}$ of $T(Q(E))$ with the same dimension as E by*

$$E^{(2)}=\{X \in T(Q(E)) | (\pi_2)_*X \in E,\ \langle X,\ w_0\rangle=0,\ \langle X,\ w_1\rangle=0\}.$$

Then $E^{(2)}$ is an integrable subbundle of $T(Q(E))$.

PROOF. Let X and Y be in $E^{(2)}_v$ with $v \in Q(E)$ and choose $f_\alpha: U_\alpha \longrightarrow B$ in E with $\pi_2(v) \in U_\alpha$. Then $\Omega=f_\alpha^{(2)*}\tilde{\Omega}$ on $Q(E)|U_\alpha$ by Lemma 4. Since $\tilde{\Omega}$ is horizontal and $\tilde{\pi}_{2*}(f_\alpha^{(2)}{}_*X)=f_{\alpha*}\pi_{2*}X=0$, we have $\Omega(X,Y)=0$, which implies that

$$d(w_0+w_1)=0 \qquad \text{on } E^{(2)}.$$

Therefore, $E^{(2)}$ is an integrable subbundle of $T(Q(E))$ with the same dimension as E.

Let $N^{(2)}$ be a leaf of $E^{(2)}$. If $N'=\pi_2(N^{(2)})$ is a leaf of E, then $\pi_2|N^{(2)}: N^{(2)} \longrightarrow N'$ is a regular cover.

5. Stability theorem I.

Let M be a connected manifold of dimension n and E be a foliation on M of dimension n-q. It is well-known that for $X \in TM$ the following conditions are equivalent:

(i) $[X, Y] \in E$ for any $Y \in E$.

(ii) In any neighborhood in M, the local one parameter group generated by X preserves the foliation E.

(iii) In any adapted chart (U_α, f_α), X may be written of the form

$$X=\sum_{i=1}^{n-q} X^i(x^1, \ldots, x^{n-q}, y^1, \ldots, y^q)\partial/\partial x^i + \sum_{a=1}^{q} X^a(y^1, \ldots, y^q)\partial/\partial y^a,$$

in terms of corresponding coordinates $(x^1, \ldots, x^{n-q}, y^1, \ldots, y^q)$.

$X \in TM$ is called a foliated vector field if X satisfies one of the above conditions. The image $\bar{X}$ of a foliated vector field X under the map: {foliated vector fields} $\longrightarrow$ {foliated vector fields}/E is called a transversal field associated with X. Let $\nu(E)= TM/E$ be the normal bundle of E. Then $\bar{X}$ is a section of $\nu(E)$. We may regard $\bar{X}$ as a transversal vector field (associated with a foliated vector field X) by identifying the normal bundle $\nu(E)$ of E with the complement $E^\perp$ of E. Since a foliated vector field X is written of the form (iii) in (U_α, f_α), by means of the above identification, we may write $\bar{X}$ of the form

$$\bar{X}=\sum_{a=1}^{q} X^a(y^1, \ldots, y^q)\bar{\partial}/\partial y^a,$$

where $\bar{\partial}/\partial y^a=\partial/\partial y^a+\sum_{i=1}^{n-q} A_a^i \partial/\partial x^i$ and A_a^i are functions on U_α.

A transversal parallelism with respect to E means that a family $\{\bar{X}_1, \ldots, \bar{X}_q\}$ of transversal vector fields associated foliated vector fields is linearly independent at each point of M. Then we also say that the normal bundle $\nu(E)$ of E has an absolute parallelism. Moreover, a transversal parallelism $\{\bar{X}_1, \ldots, \bar{X}_q\}$ with respect to E is complete if each $\bar{X}_a$ is a complete vector field.

LEMMA 7 ([1],[6]). *If M admits a complete transversal parallelism with respect to E, then the group* Aut (M, E) *of diffeomorphisms of M which preserve E acts transitively on M.*

Let E be a Γ-foliation of dimension n-q on M associated with L/L_0. Then $Q(E)$ admits a transversal parallelism with respect to $E^{(2)}$. In fact, for any $X \in E^{(2)}$ and any $Y \in \nu(E^{(2)})$, we have $\Omega(X, Y)=0$ by identifying the normal bundle $\nu(E^{(2)})$ with the complement $E^{(2)\perp}$ of $E^{(2)}$. Then, since $Q(E)$ has an absolute parallelism by means of the connection w, we have $[X, Y]=0$. Therefore, $\nu(E^{(2)})$ has an absolute parallelism.

A Γ-foliation E associated with L/L_0 is called to be complete if each vector field defining a transversal parallelism is a complete vector field.

Without loss of generality, we may suppose (by passing to a finite cover of M if necessarily) that $P^2(E)$, and hence $Q(E)$, is connected. If E is a complete Γ-foliation associated with L/L_0, then Aut $(Q(E), E^{(2)})$ acts transitively on $Q(E)$, and hence all the leaves of $E^{(2)}$ are diffeomorphic. Since the leaves of $E^{(2)}$ are covers of the leaves of E, all the leaves of E have the same universal cover.

Summing up, we have the following.

THEOREM 8 (Stability theorem I). *Let E be a Γ-foliation associated with a semi-simple flat homogeneous space L/L_0 of order 2. Suppose that the Spencer cohomology $H^{2,1}(\mathfrak{l})$ of the graded Lie algebra $\mathfrak{l}$ of L vanishes. If E is complete, then all the leaves of E have the same universal cover.*

REMARK 1. We have many examples of the graded Lie algebra $\mathfrak{l}=\mathfrak{g}_{-1}+\mathfrak{g}_0+\mathfrak{g}_1$ with $H^{2,1}(\mathfrak{l})=0$ ([8], Proposition 7-1, p. 177). In particular, the graded Lie algebras $\mathfrak{l}$ in Example 1 ($q\geqq 2$), Example 2 and Example 3 satisfy $H^{2,1}(\mathfrak{l})=0$.

Thus we have

COROLLARY 9. *If E is a complete conformal (projective) foliation, then all the leaves of E have the same unversal cover.*

REMARK 2. If E is a complete riemannian foliation, then $P(E)$ admits a complete transversal parallelism with respect to $E^{(1)}$, and hence Aut $(P(E), E^{(1)})$ acts transitively on $P(E)$. Thus we have that if E is a complete riemannian foliation, then all the leaves of E have the same universal cover (Theorem B).

5. Infinitesimal holonomy group and Stability theorem II.

Let E be a foliation of dimension n-q on a connected manifold M and N be a leaf of E. Take $x_0\in N$ and fix it once for all. Let $\sigma:[0, 1]\longrightarrow N$ be a loop at x_o. Choose a cover of M by a finite sequence of adapted charts (U_0, f_0), (U_1, f_1), ..., $(U_m, f_m)=(U_0, f_0)$ with $x_o\in U_0$, $f_0(x_o)=o$ (a fixed point of B) and $U_i\cap U_{i+1}\neq\phi$ for $i=0, 1, \dots, m-1$. For each $i(0\leqq i\leqq m-1)$, there exists a diffeomorphism $\gamma_{i+1,i}: f_i(U_i\cap U_{i+1})\longrightarrow f_{i+1}(U_i\cap U_{i+1})$ such that $f_{i+1}=\gamma_{i+1,i}\circ f_i$ on $U_i\cap U_{i+1}$. Then $\gamma_{m,m-1}\circ\cdots\circ\gamma_{1,0}$ is contained in the group $G(q)$ of all local diffeomorphisms in $\Gamma(B)$ fixing o defined around o (equivalently, the group of local diffeomorphisms in $\Gamma(\boldsymbol{R}^q)$ fixing 0 defined around 0). Let $H(\sigma)$ be its germ at 0. Since $H(\sigma)$ depends only on the homotopy class of σ, we may define a homomorphism $H:\pi_1(N, x_o)\longrightarrow G(q)$ up to conjugacy.

The image of H is called the holonomy group $H_1(N, x_o)$ of N based at x_o. The infinitesimal holonomy group $H^2(N, x_o)$ of order 2 of N at x_o is defined as the image of $\pi^2 \circ H$, where $\pi^2 : G(q) \longrightarrow G^2(q)$ is the canonical projection. The restriction $Q(E)|N$ has a canonical flat connection with the holonomy group $H^2(N, x_o)$. In fact, by definition, $j_0^2(\gamma_{m,m-1} \circ \cdots \circ \gamma_{1,0})$ is a 2-jet of an element of the holonomy group of the Bott connection on N. We have $(\pi_2^1)_* E^{(2)} = E^{(1)}$, and $Q(E)$ has an absolute parallelism. And $\pi_2|N^{(2)} : N^{(2)} \longrightarrow N$ is a regular cover. Therefore, $\pi_2 : N^{(2)} \longrightarrow N$ is a regular cover whose group of all deck transformations is isomorphic to $H^2(N, x_o)$.

In the same way, the holonomy group $H^1(N, x_o)$ is isomorphic to the group of Jacobians of elements of the holonomy group of the Bott connection on $P(E)$ ([2],[5]).

Then we have the following:

THEOREM 10 (Stability theorem II). *Let E be a complete Γ-foliation associated with a semi-simple flat homogeneous space L/L_0 of order 2. Suppose that the Spencer cohomology $H^{2,1}(\mathfrak{l})$ of the graded Lie algebra $\mathfrak{l}$ of L vanishes. If E has a compact leaf N_o with finite infinitesimal holonomy proup $H^2(N_o, x_o)$ of N_o based at x_o, then all the leaves of E are compact and $H^2(N, x)$ is finite for all leaves N of E.*

COROLLARY 11. *Let E be a complete conformal (projective) foliation. If E has a compact leaf N_o with finite infinitesimal holonomy group $H^2(N_o, x_o)$, then all the leaves of E are compact and $H^2(N, x)$ is finite for all leaves N of E.*

REMARK 3. For a riemannian foliation, we have Theorem B togather with Remark 2.

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