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Notes on space-like foliations of codimension one in Lorentz manifolds

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1. Introduction.

This note is divided into three parts. In the first part, we will give spherically symmetric maximal foliations of codimension one in a Lorentz 4-manifold with a Lorentz metric:

$$ds^2=-g(r)dt^2+f(r)dr^2+r^2d\theta^2+r^2\sin^2\theta d\varphi^2.$$

The result given by Reinhart[7] is a special case. If $g(r)=f(r)=1$ for $r>0$, then the metric is called the Lorentz-Minkowski metric. In the second part, we will prove that a spherically symmetric maximal foliation of codimension one in the Lorentz-Minkowski $(n+1)$-space is totally geodesic. This is essentially Cheng and Yau's Theorem[1] (foliation version). Because we consider only spherically symmetric foliations, our method is different from one in[1] and is elementary. In the third part, we will discuss the geometric properties of space-like foliations of codimension one in connected Lie groups with left inveriant Lorentz metrics, and several examples will be shown.

The discussion in this note is motivated by Reinhart's paper[7] and extension of Oshikiri's Theorem ([8]) to non-compact case.

All the objects in this note are of class C^∞ unless otherwise stated.

2. Foliation of codimension one.

A foliation of codimension one in a manifold M is a family of hypersurfaces filling M. Each hypersurface is called a leaf of the foliation. The exact definition of foliation of codimension one is as follows. If a 1-form ω on a manifold M satisfies the integrability condition, i.e. $d\omega=\eta\wedge\omega$, then ω is called to define a foliation $\mathscr{F}$ of codimension one in M ([6]). And the foliation $\mathscr{F}$ is called to be defined by $\omega=0$. In other words, if an integrable subbundle E of rank dim $M-1$ of the tangent bundle TM over M annihilates ω, then E is called to define $\mathscr{F}$ ([8]). Here, E is integrable if $[X, Y]\in\Gamma(E)$ for any X, $Y\in\Gamma(E)$, and E annihilates w if $w(X)=0$ for any $X\in\Gamma(E)$ ($\Gamma(E)$ denotes the space of all the sections of the bundle E). Each maximal connected integral manifold of E is called a leaf of $\mathscr{F}$. For details, see Reinhart[6].

A foliation $\mathscr{F}$ of codimension one in a Lorentz manifold M is space-like if each leaf of $\mathscr{F}$ is a space-like hypersurface of M ([4], [7]).

3. Spherically symmetric maximal foliations and special Lorentz metrics.

Let $N(a)=\{(x^1, x^2, x^3)\in \boldsymbol{R}^3 | r=((x^1)^2+(x^2)^2+(x^3)^2)^{1/2}>a\}$ where $a\geqq 0$. We consider a Lorentz manifold $M=\boldsymbol{R}\times N(a)$ with a Lorentz metric

$$ds^2=-g(r)dt^2+f(r)dr^2+r^2d\theta^2+r^2\sin^2\theta d\varphi^2 \tag{1}$$

by means of the spherical coordinate (r, θ, φ) on $N(a)$, where g and f are positive valued functions of r on an open interval (a, ∞). Let $\mathscr{F}$ be a space-like foliation of codimension one in M defined by $\omega=0$. The foliation $\mathscr{F}$ is spherically symmetric if ω can be expressed by

$$\omega=dt+h(r)dr \tag{2}$$

where h is a function of r on (a, ∞) ([7]). The foliation $\mathscr{F}$ is maximal if the trace of the second fundamental form (by means of the Levi-Civita connection ∇ with respect to the metric (1)) of each leaf of $\mathscr{F}$ is zero ([7], [8]). We will seek a closed 1-form $\omega=dt+h(r)dr$ on M such that the space-like foliation $\mathscr{F}$ defined by $\omega=0$ is maximal.

The unit normal vector field T to the leaves of $\mathscr{F}$ is given by

$$T=(g(r)^{-1}-f(r)^{-1}h(r)^2)^{-1/2}\left(-g(r)^{-1}\frac{\partial}{\partial t}+f(r)^{-1}h(r)\frac{\partial}{\partial r}\right).$$

We notice that $g(r)^{-1}-f(r)^{-1}h(r)^2>0$ because $\mathscr{F}$ is space-like. An orthonormal frame field $\{X_1, X_2, X_3\}$ tangent to the leaves of $\mathscr{F}$ is given by

$$X_1=(g(r)^{-1}-f(r)^{-1}h(r)^2)^{-1/2}\left(-f(r)^{-1/2}g(r)^{-1/2}h(r)\frac{\partial}{\partial t}+f(r)^{-1/2}g(r)^{-1/2}\frac{\partial}{\partial r}\right)$$

$$X_2 = r^{-1}\frac{\partial}{\partial\theta}$$

$$X_3 = r^{-1}(\sin\theta)^{-1}\frac{\partial}{\partial\varphi}.$$

Then we have

$$\begin{aligned}
&\textstyle\sum_{i=1}^{3} < \nabla_{X_i} X_i,\ T > \\
&= (g(r)^{-1} - f(r)^{-1}h(r)^2)^{-3/2} r^{-1}\Big\{-2f(r)^{-1}g(r)^{-1}h(r) \\
&\quad + 2f(r)^{-2}h(r)^3 - rf(r)^{-1}g(r)^{-1}h'(r) \\
&\quad - rf(r)^{-1}g(r)^{-2}g'(r)h(r) + \frac{1}{2}rf(r)^{-2}f'(r)g(r)^{-1}h(r) \\
&\quad + \frac{1}{2}rf(r)^{-2}g(r)^{-1}g'(r)h(r)^3\Big\},
\end{aligned}$$

where $< \ , \ >$ denotes the inner product with respect to the metric (1), and $'$ denotes the derivative with respect to r. The trace of the second fundamental form vanishes if and only if

$$\begin{aligned}
(3) \qquad h'(r) &= \left(\frac{1}{2}f(r)^{-1}f'(r) - g(r)^{-1}g'(r) - 2r^{-1}\right)h(r) \\
&\quad + \left(\frac{1}{2}f(r)^{-1}g'(r) + 2f(r)^{-1}g(r)r^{-1}\right)h(r)^3.
\end{aligned}$$

The differential equation (3) is easily solved by means of an auxiliary variable : $k(r) = g(r)^{1/2}f(r)^{-1/2}h(r)$. Then (3) can be rewritten in the form

$$(4) \qquad k'(r) = \left(\frac{1}{2}g(r)^{-1}g'(r) + 2r^{-1}\right)(-k(r) + k(r)^3)$$

Then, the equation (4) has the solutions :

$$k(r) = \pm(1 + Cg(r)r^4)^{-1/2}, \text{ or } = 0,$$

where C is a constant.

Thus we have

THEOREM 1. *Let $M = \mathbf{R}\times N(a)$ be a Lorentz 4-manifold with a Lorentz metric* (1), *and let $\mathcal{F}$ be a spherically symmetric space-like foliation of codimension one in M*

defined by $\omega=0$, *where* $\omega=dt+h(r)dr$. *Then* $\mathscr{F}$ *is maximal if and only if*

$$h(r)=\pm g(r)^{-1/2}f(r)^{1/2}(1+Cg(r)r^4)^{-1/2},\ or\ =0$$

on (a, ∞) *for some constant* C.

The case $g(r)=f(r)=1$ for $r\in(0, \infty)$, that is, the case of the Lorentz-Minkowski space, is discussed in the next section. The case of $g(r)=1-2m/r$ and $f(r)=g(r)^{-1}$ for $r\in(2m, \infty)$ was discussed by Reinhart[7], and then (M, ds^2) is the Schwarzschild exterior space-time ([4]). We must remark that Reinhart has found all spherically symmetric maximal foliations in extended Schwarzschild space ([7]).

Now we may consider the case $g(r)=1-4m^2/r^2$ and $f(r)=r^{-2}g(r)^{-1}$ for $r\in(2m, \infty)$. Then we have

$$ds^2=-(1-4m^2/r^2)dt^2+r^{-2}(1-4m^2/r^2)^{-1}dr^2+r^2d\theta^2+r^2\sin^2\theta d\varphi^2. \tag{5}$$

If we set

$$\hat{t}=t+\log(r^2-4m^2)^{1/2} \tag{6}$$

then the metric (5) can be rewritten in the form

$$ds^2=-(1-4m^2/r^2)d\hat{t}^2+2r^{-1}d\hat{t}dr+r^2d\theta^2+r^2\sin^2\theta d\varphi^2. \tag{7}$$

The metric (7) has been introduced by Otsuki[5], who has discussed the "black holes" by means of smooth "general connection" on the space-time with this metric (7).

If we set

$$\begin{aligned} v&=(r^2-4m^2)^{1/2}\sinh t\\ u&=(r^2-4m^2)^{1/2}\cosh t \end{aligned} \tag{8}$$

then the metric (5) can be also rewritten in the form

$$ds^2=r^{-2}(-dv^2+du^2)+r^2d\theta^2+r^2\sin^2\theta d\varphi^2. \tag{9}$$

Thus we may have analogous discussion on the space $\boldsymbol{R}\times N(0)$ with the metric (5) as Reinhart's discussion ([7]) on extended Schwarzschild space.

4. Spherically symmetric maximal foliations in the Lorentz-Minkowski space.

Let M be the Lorentz-Minkowski $(n+1)$-space with the global coordinate $(t, x^1, \cdots, x^n)$ and the Lorentz metric

(10) $$ds^2 = -dt^2 + (dx^1)^2 + \cdots + (dx^n)^2.$$

Let $r = ((x^1)^2 + \cdots + (x^n)^2)^{1/2}$, and we set that $M^+ = \{(t, x^1, \cdots, x^n) \in M \mid r > 0\}$. Then M^+ is an open Lorentz submanifold of M ([4]). By means of the spherical coordinate $(r, \theta_1, \cdots, \theta_{n-1})$, the induced metric on M^+ from (10) can be rewritten in the form

(11) $$ds^2 = -dt^2 + dr^2 + r^2(d\theta_1)^2 + \sum_{j=2}^{n-1} r^2 \sin^2\theta_1 \cdots \sin^2\theta_{j-1}(d\theta_j)^2.$$

We consider a 1-form ω on M defined by

(12) $$\omega = dt + \sum_{i=1}^{n} f_i dx^i$$

where f_i are functions on M, and we suppose that ω defines a foliation $\mathscr{F}$ of codimension one in M. Then the foliation $\mathscr{F}$ is spherically symmetric if the 1-form ω is written as follows on M^+;

(13) $$\omega = dt + h(r)dr$$

where h is a function of $r > 0$.

Under the assumption that the foliation $\mathscr{F}$ in M is space-like, $\mathscr{F}$ is totally geodesic if the second fundamental form (by means of the Levi-Civita connection with respect to the metric (10)) of each leaf of $\mathscr{F}$ vanishes identically ([4], [8]).

Hereafter we suppose that the foliation $\mathscr{F}$ of codimension one in M defined by $\omega = 0$ is spherically symmetric and space-like. Let $\mathscr{F}|M^+$ be the restricted foliation of $\mathscr{F}$ in M^+. Since $\mathscr{F}|M^+$ is space-like, we have that $1 - h(r)^2 > 0$ for $r > 0$.

By means of the coordinate $(t, r, \theta_1, \cdots, \theta_{n-1})$ in M^+, the unit normal vector field T to the leaves of $\mathscr{F}|M^+$ is given by

$$T = (1 - h(r)^2)^{-1/2}\left(-\frac{\partial}{\partial t} + h(r)\frac{\partial}{\partial r}\right)$$

and an orthonormal frame field tangent to the leaves of $\mathscr{F}|M^+$ is given by

$$X_1=(1-h(r)^2)^{-1/2}\left(-h(r)\frac{\partial}{\partial t}+\frac{\partial}{\partial r}\right)$$
$$X_2=r^{-1}\frac{\partial}{\partial\theta_1}$$
$$X_k=(r\sin\theta_1\ \cdots\ \sin\theta_{k-2})^{-1}\frac{\partial}{\partial\theta_{k-1}}\qquad k=3,\ 4,\ \cdots,\ n.$$

Then we have

$$\sum_{i=1}^{n}<\nabla_{X_i}X_i,\ T>=-(1-h(r)^2)^{1/2}\{h'(r)+(n-1)r^{-1}h(r)(1-h(r)^2)\}.$$

Thus the trace of the second fundamental form of each leaf of $\mathcal{F}|M^+$ is zero if and only if

$$h'(r)=-(n-1)r^{-1}h(r)(1-h(r)^2)\qquad (r>0). \tag{14}$$

The differential equation (14) has the solutions

$$(15_a)\qquad h(r)=\pm(1+Cr^{2(n-1)})^{-1/2}\qquad (r>0)$$
$$(15_b)\qquad h(r)=0\qquad (r>0)$$

where C is a positive constant.

REMARK. A function $f(r)=(1+Cr^{2(n-1)})^{-1/2}$ is well-defined on $[0,\ \infty)$ for any positive constant C. As r goes to infinity, $f(r)$ approaches 0. And we have that $f(0)=1$.

Then we have the following theorem :

THEOREM 2. *Let M be the Lorentz-Minkowski $(n+1)$-space and $\mathcal{F}$ a spherically symmetric and space-like foliation of codimension one in M defined by $\omega=0$, where $\omega=dt+\sum_{i=1}^{n}f_i dx^i$ and f_i are functions on M. If $\mathcal{F}$ is maximal, then $\mathcal{F}$ is totally geodesic.*

We will give a proof of Theorem 2 for $n=3$. By our assumption, the function h in (13) is given by (15_a) or (15_b). We may set

$$x^1=r\cos\theta_1$$
$$x^2=r\sin\theta_1\cos\theta_2$$

$x^3 = r\sin\theta_1\sin\theta_2.$

By (12) for $n=3$ and (13), we have

(16)
$$\begin{aligned} &f_1\cos\theta_1 + f_2\sin\theta_1\cos\theta_2 + f_3\sin\theta_1\sin\theta_2 = h(r)\\ &-f_1\sin\theta_1 + f_2\cos\theta_1\cos\theta_2 + f_3\cos\theta_1\sin\theta_2 = 0\\ &-f_2\sin\theta_1\sin\theta_2 + f_3\sin\theta_1\cos\theta_2 = 0. \end{aligned}$$

Let N be the vector field dual to ω, that is,

(17)
$$N = -\frac{\partial}{\partial t} + \sum_{i=1}^{3} f_i \frac{\partial}{\partial x^i}.$$

Since $\mathscr{F}$ is space-like, N is a time-like vector field on M ([4]).

For any fixed $t^* \in \boldsymbol{R}$, we consider a curve $c : \boldsymbol{R} \longrightarrow M$ given by $c(s) = (t^*, 2^{-1/2}s, 2^{-1}s, 2^{-1}s)$. By (16), we have, for $s \neq 0$,

(18)
$$\begin{aligned} &2^{1/2}f_1(c(s)) + f_2(c(s)) + f_3(c(s)) = 2h(|s|)\\ &-2^{1/2}f_1(c(s)) + f_2(c(s)) + f_3(c(s)) = 0\\ &-f_2(c(s)) + f_3(c(s)) = 0. \end{aligned}$$

We suppose that $h(r) = \pm(1+Cr^4)^{-1/2}$ (i.e. (15_a) for $n=3$). As r approaches 0, $h(r)$ approaches ± 1. Thus, as s approaches 0, (18) implies that

$$\begin{aligned} &f_1(c(0)) = \pm 2^{-1/2}\\ &f_2(c(0)) = f_3(c(0)) = \pm 2^{-1}. \end{aligned}$$

By (17), we have

$$N_{c(0)} = -\frac{\partial}{\partial t}\bigg|_{c(0)} + \left(2^{-1/2}\frac{\partial}{\partial x^1}\bigg|_{c(0)} + 2^{-1}\frac{\partial}{\partial x^2}\bigg|_{c(0)} + 2^{-1}\frac{\partial}{\partial x^3}\bigg|_{c(0)}\right),$$

which is a light-like vector ([4]). This contradicts the fact that N is a time-like vector field on M.

Therefore, we have that $h(r)=0$ $(r>0)$. Then, by (16), we have

$$f_i = 0 \qquad i = 1, 2, 3$$

on M. Thus we have that $\omega=dt$ on M, which means that $\mathscr{F}$ is totally geodesic.

By the same way as above, we can prove Theorem 2 for $n>3$.

5. Space-like foliations in Lie groups.

Let G be an $n+1$ dimensional connected Lie group and $\mathfrak{g}$ be the associated Lie algebra of all vector fields on G that are invariant under left translations. If we take a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, then we have a foliation $\mathscr{F}(\mathfrak{h})$ in G ([8]). In fact, for each point $x\in G$, a submanifold $L_x(H)$ of G is the leaf through x of the foliation $\mathscr{F}(\mathfrak{h})$ in G, where L_x denotes the left translation of G by $x\in G$, and H is a connected subgroup of G whose Lie algebra is $\mathfrak{h}$. Let $\{e_0, e_1, \cdots, e_n\}$ be a basis of $\mathfrak{g}$. Then we denote by C^J_{IK} the structure constants of $\mathfrak{g}$ with respect to $\{e_0, e_1, \cdots, e_n\}$. Here and hereafter, unless otherwise stated, $I, J, K=0, 1, \cdots, n$ and $i, j=1, \cdots, n$.

In this section, we consider G whose Lie algebra $\mathfrak{g}$ is given by

(19) $$\mathfrak{g}=\{e_0\}+\mathfrak{h}$$

where $\mathfrak{h}$ is an ideal of codimension one of $\mathfrak{g}$, and $\{e_1, \cdots, e_n\}$is a basis of $\mathfrak{h}$. Now we can take a Lorentz inner product $<\ ,\ >$ on $\mathfrak{g}$ such that

$$<e_0, e_0>=-1 \qquad <e_0, e_i>=0 \qquad <e_i, e_j>=\delta_{ij}.$$

Then we have a left invariant Lorentz metric $<\ ,\ >$ on G induced from the Lorentz inner product $<\ ,\ >$ on $\mathfrak{g}$ ([2], [3]). Let ∇ be the Levi-Civita connection on G with respect to the left invariant Lorentz metric $<\ ,\ >$ ([2], [3]).

Thus we have a space-like foliation $\mathscr{F}(\mathfrak{h})$ in G with the left invariant Lorentz metric $<\ ,\ >$. We set

$$h^0_{ij}=<\nabla_{e_i}e_j,\ e_0>=(C^i_{0j}+C^j_{0i})/2.$$

Then we have the following definitions ([8]) : $\mathscr{F}(\mathfrak{h})$ is maximal (resp. totally geodesic) if

$$\textstyle\sum^n_{i=1}h^0_{ii}=0 \qquad (\text{resp. } h^0_{ij}=0 \text{ for all } i,\ j).$$

And $\mathscr{F}(\mathfrak{h})$ is of constant mean curvature if $\sum^n_{i=1}h^0_{ii}$ is a constant. Next, the curvature tensor R of ∇ defined by

$$R(e_I, e_J)e_K = \nabla_{[e_I, e_J]}e_K - \nabla_{e_I}\nabla_{e_J}e_K + \nabla_{e_J}\nabla_{e_I}e_K,$$

and the Ricci curvature $Ric(e_0)$ in the direction of e_0 is defined by

$$Ric(e_0) = \sum_{i=1}^{n} < R(e_0, e_i)e_0, e_i >$$

([2], [4]). By the direct calculation, we have

$$Ric(e_0) = -\sum_{i=1}^{n}(C_{0i}^{i})^2 - \frac{1}{2}\sum_{i<j}(C_{0j}^{i} + C_{0i}^{j})^2.$$

Thus we have the following Proposition :

PROPOSITION 3. *Let $\mathcal{F}(\mathfrak{h})$ be a space-like foliation of codimension one in the above Lie group G with the above left invariant Lorentz metric. Then the Ricci curvature $Ric(e_0)$ in the direction of e_0 is non-positive. Moreover $Ric(e_0)=0$ if and only if $\mathcal{F}(\mathfrak{h})$ is totally geodesic. In particular, if the left invariant Lorentz metric is flat then $\mathcal{F}(\mathfrak{h})$ is totally geodesic.*

EXAMPLE 1.

$$G = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \mid a, b, c \in \boldsymbol{R} \right\} \quad : \text{Heisenberg group}$$

$$\mathfrak{g} = \left\{ \begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} \mid x, y, z \in \boldsymbol{R} \right\}$$

$$e_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad e_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$[e_0, e_1] = e_2 \qquad [e_0, e_2] = 0 \qquad [e_1, e_2] = 0$

$\mathfrak{h} = \{e_1, e_2\}$

$< \ , \ >$: a left invariant Lorentz metric on G such that

$$< e_0, e_0 > = -1 \qquad < e_0, e_i > = 0$$

$$< e_i, e_j > = \delta_{ij} \qquad (i, j = 1, 2)$$

$Ric(e_0) = -1/2$

Then the space-like foliation $\mathcal{F}(\mathfrak{h})$ of codimension one in G is maximal and not totally geodesic.

EXAMPLE 2.

$$G=\left\{\begin{bmatrix} a & 0 & b \\ 0 & a & c \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; a>0,\ b,\ c \in \boldsymbol{R}\right\}$$

$$\mathfrak{g}=\left\{\begin{bmatrix} x & 0 & y \\ 0 & x & z \\ 0 & 0 & 0 \end{bmatrix} \;\middle|\; x,\ y,\ z \in \boldsymbol{R}\right\}$$

$$e_0=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad e_1=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad e_2=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$[e_0, e_1]=e_1 \quad [e_0, e_2]=e_2 \quad [e_1, e_2]=0$

$\mathfrak{h}=\{e_1, e_2\}$

$<\ ,\ >$: a left invariant Lorentz metric on G such that

$$<e_0, e_0>=-1 \qquad <e_0, e_i>=0$$

$$<e_i, e_j>=\delta_{ij} \qquad (i, j=1, 2)$$

$Ric(e_0)=-2$

Then the Lorentz manifold $(G, <\ ,\ >)$ is of constant curvature -1, and the space-like foliation $\mathscr{F}(\mathfrak{h})$ of codimension one in G is of constant mean curvature (not maximal).

EXAMPLE 3.

$$G=\left\{\begin{bmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; \theta,\ a,\ b \in \boldsymbol{R}\right\}$$: group of rigid motions of Euclidean 2-space

$$\mathfrak{g}=\left\{\begin{bmatrix} 0 & -x & y \\ x & 0 & z \\ 0 & 0 & 0 \end{bmatrix} \;\middle|\; x,\ y,\ z \in \boldsymbol{R}\right\}$$

$$e_0=\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad e_1=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad e_2=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$[e_0, e_1]=e_2 \quad [e_0, e_2]=-e_1 \quad [e_1, e_2]=0$

$\mathfrak{h}=\{e_1, e_2\}$

$<\ ,\ >$: a left invariant Lorentz metric on G such that

$$<e_0, e_0>=-1 \qquad <e_0, e_i>=0$$
$$<e_i, e_j>=\delta_{ij} \qquad (i, j=1, 2)$$

$Ric(e_0)=0$

Then $(G, < \ , \ >)$ is a flat Lorentz manifold, and the space-like foliation $\mathscr{F}(\mathfrak{h})$ of codimension one in G is totally geodesic.

EXAMPLE 4.

$$G=\left\{\begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} \mid a>0,\ b,\ c,\ d\in \boldsymbol{R}\right\}$$

$$\mathfrak{g}=\left\{\begin{bmatrix} x & y & z \\ 0 & x & u \\ 0 & 0 & x \end{bmatrix} \mid x,\ y,\ z,\ u\in \boldsymbol{R}\right\}$$

$$e_0=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad e_1=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad e_2=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad e_3=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$[e_0, e_i]=0 \quad (i=1, 2, 3) \quad [e_1, e_2]=0 \quad [e_1, e_3]=e_2 \quad [e_2, e_3]=0$

$\mathfrak{h}=\{e_1, e_2, e_3\}$

$< \ , \ >$: a left invariant Lorentz metric on G such that

$$<e_0, e_0>=-1 \qquad <e_0, e_i>=0$$
$$<e_i, e_j>=\delta_{ij} \qquad (i, j=1, 2, 3)$$

$Ric(e_0)=0 \qquad Ric(e_1)=-1/2$

Then the space-like foliation $\mathscr{F}(\mathfrak{h})$ of codimension one in G is totally geodesic.

References

[1] S.-Y. Cheng and S.-T. Yau, Maximal space like hypersurfaces in the Lorentz-Minkowski spaces, Ann. of Math. 104 (1976), 407-419.

[2] J. Milnor, Curvatures of left invariant metrics on Lie groups, Advances in Math. 21 (1976), 293 -329.

[3] K. Nomizu, Left invariant Lorentz metrics on Lie groups, Osaka Math. J. 16 (1979), 143-150.

[4] B. O'Neill, Semi-Riemannian Geometry with Applications to Relativity, Academic Press, New York, 1983.

[5] T. Otsuki, A certain space-time metric and smooth general connections, Kodai Math. J. 8 (1985), 307-316.

[6] B.L. Reinhart, Foliated manifolds with bundle-like metrics, Ann. of Math. 69 (1959), 119-132.

[7] B.L. Reinhart, Maximal foliations of extended Schwarzschild space, J. Math Phys. 14 (1973), 719.

[8] R. Takagi and S. Yorozu, Minimal foliations on Lie groups, Tohoku Math. J. 36 (1984), 541-554.

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