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On a Counter-Example of the Conjecture of Dodziuk-Singer by M. T. Anderson.

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Let M be a complete simply-connected riemannian manifold with non-positive sectional curvature of even dimension m. Dodziuk-Singer conjecture [D] means that $H_2^p(M)=0$ if $p\neq m/2$ and $\dim H_2^{m/2}(M)=\infty$. Here, $H_2^p(M)$ is the space of smooth L^2-harmonic forms. The affirmative solution of Dodziuk-Singer conjecture implies, by means of the L_2-index theorem for a regular cover of M. F. Atiyah [A], the positive solution of the well-known E. Hopf Conjecture; If $(M,\langle\ ,\ \rangle)$ is a compact riemannian manifold of even dimension m with negative sectional curvature, then

$$(-1)^{m/2}\chi(M)>0.$$

Recently, M. T. Anderson [An] has announced that the conjecture is false, that is,

Theorem An. [An] *For any $m\geqq 2$, $0<p<m$ and $a>|m-2p|$, with $a\geqq 1$, there are complete simply-connected riemannian manifolds M with sectional curvature $-a^2\leqq K\leqq -1$ such that $\dim H_2^p(M)=\infty$.*

The manifold M in his proof is a warped product $H^{2p}(-a^2)\times_f S^{m-2p}(1)$ where $H^{2p}(-a^2)$ (resp. $S^{m-2p}(1)$) is the hyperbolic (resp. spherical) space form of constant curvature $-a^2$ (resp.1), and $f(x):=\sinh s(x)$, s is the distance from x to a fixed totally geodesic hyperplane H^{2p-1} in $H^{2p}(-a^2)$.

We shall point out his important mistakes from the following point of views. First, the following theorem on the warped products is well-known;

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THEOREM B-O. (R.L. Bishop-B. O'Neill [B-O]) *Let M and N be riemannian manifolds, and let* $f>0$ *be a differentiable function on M. Then the warped product* $M \times_f N$ *with metric* $ds^2=ds_M{}^2+f^2ds_N{}^2$ *has curvarture* $K<0$ *if the following conditions hold :*
(1) *dim* $M=1$*, or the sectional curvature of M is negative.*
(2) f *is strictly convex.*
(3) (a) *dim* $N=1$*, or* (b) *the sectional curvature* $K_N<0$ *if* f *has a minimum ;* $K_N \leqq 0$ *if* f *does not have a minimum.*
Conversely, if M is complete, and $M \times_f N$ *has negative curvature, then conditions* (1), (2) *and* (3) *hold.*

Then, theorem B-O implies that his manifold $H^{2p}(-a^2) \times_f S^{m-2p}(1)$ can not have negative curvature for all warping functions. Therefore, his statement is false.

Second, his warping function f is not strictly positive. In fact, $f(x)=0$ for all $x \in H^{2p-1}$. Then, his manifold is not riemannian. If his warping function f is miss-printed for cosh s(x), then $-a^2 \leqq K \leqq 1$. Therefore, his statement is also false.

It should seem for authors that he constructed the manifold M as followings ; First, he noted $H^{2p}=R \times_{\cosh t} H^{2p-1}$ for $t \in R$ and $\{0\} \times H^{2p-1}$ is a totally geodesic hyperplane in $H^{2p}(-a^2)$. Second, he considered $H^{2p}(-a^2)$ as $(0, \infty) \times_{\cosh t} H^{2p-1} \cup (-\infty, 0) \times_{\cosh t} H^{2p-1} \cup \{0\} \times H^{2p-1}$, and then set $H^{2p}(-a^2)=(0, \infty) \times_{\cosh s(x)} H^{2p-1}$, s(x) is the distance from x to H^{2p-1}. In fact, every differential form in his proof is invariant under reflection through H^{2p-1}. But he forgot the $\{0\} \times H^{2p-1}$ -factor.

If we replace his warping function f by cosh r(x), r(x) is the distance from x to a fixed point $0 \in H^{2p}(-a^2)$, then we have

THEOREM.1. For any $m \geqq 2$, $0<p<m$ *and* $a>|m-2p|$ *with* $a \geqq 1$*, there are complete, simply-connected riemannian manifold with sectional curvature* $-a^2 \leqq K \leqq 1$ *such that dim* $H_2^p(M)=\infty$.

REMARK. H. Donnelly and F. Xavier[D-X] have proved that, for any $p<(m-1)/2$, $0 \leqq \varepsilon < 1-4p^2/(m-1)^2$, a complete, simply-connected riemannian manifold with sectional curvature $-1 \leqq K \leqq -1+\varepsilon$ of dimension m has neither L_2-harmonic p-forms nor L_2-harmonic (m-p)-forms.

Now we return Theorem B-O, and want to have a counter example of the conjecture in the same way as M. T. Anderson. Unfortunately, authors don't know examples of a complete, simply-connected manifold of negative curvature and of finite volume. In fact, there does not exist a complete, simply-connected riemannian manifolds of constant negative curvature -1 and of finite volume.

Then if we lack the assumption of simply-connectedness of M, we have Theorem 2, 3 with a slight reversions.

THEOREM 2. For any $m \geqq 2$, $0<p<m$ and $a>|m-2p|$, with $a \geqq 1$, there are complete non-simply connected riemannian manifolds with sectional curvature $-a^2 \leqq K \leqq -1$ such that dim $H_2^p(M)=\infty$.

THEOREM 3. For any $m \geqq 2$, $0<p<m$ and $a>|m-2p|$, with $a \geqq 1$, there are incomplete simply-connected riemannian manifolds with sectional curvature $-a^2 \leqq K \leqq -1$ such that dim $H_2^p(M)=\infty$.

From the above theorems, it seems to authors that the Conjecture of Dodziuk-Singer is unsolved yet.

1. L_2-cohomology spaces.

Let (M, <, >) be a riemannian manifold of dimension m. Let $\Lambda^*(M) := \Sigma \Lambda^p(M)$ be the space of differential forms on M. Let

$$0 \to R \to \Lambda^0(M) \xrightarrow{d} \Lambda^1(M) \xrightarrow{d} \Lambda^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^p(M) \xrightarrow{d} \Lambda^{p+1}(M) \xrightarrow{d} \cdots$$

be the de Rham complex on M.

Let $\Lambda_0^p(M)$ be the space of smooth p-forms on M with compact support. We define the global inner product $\ll \ , \ \gg$ on $\Lambda_0^p(M)$:

$$\ll \omega, \eta \gg := \int_M < \omega, \eta > dvol_M = \int_M \omega \Lambda * \eta,$$

where $*$ is the Hodge star operator.

Let δ be the formal adjoint of d :

$$\ll d\omega, \eta \gg = \ll \omega, \delta\eta \gg \qquad \text{for } \omega \text{ or } \eta \text{ in } \Lambda_0^*(M).$$

Let $\Delta := -(d\delta + \delta d)$ be the Lalacian acting on $\Lambda^*(M)$. $H^p(M) := \{\omega \in \Lambda^p(M) | \Delta\omega = 0\}$ is called the space of harmonic p-forms. Let $L_2^p(M)$ be the completion of $\Lambda_0^p(M)$ with respect to $\ll \ , \ \gg$. Then it is a Hilbert space. $H_2^p(M) := \{\omega \in L_2^p(M)$

$|\Delta\omega=0\}$ is a closed subspace in $L_2^p(M)$. In fact, by the ellipticity of Δ, every weak solution ω is smooth. Then $H_2^p(M)=\Lambda^p(M)\cap L_2^p(M)$. And if we take any sequence ω_k in $H_2^p(M)$ converging to $\omega \in L_2^p(M)$, then we have

$$\langle\langle \omega, \Delta\eta \rangle\rangle = 0 \qquad \text{for any } \eta \in \Lambda_0^p(M),$$

for $0=\langle\langle \Delta\omega_k, \eta\rangle\rangle=\langle\langle \omega_k, \Delta\eta\rangle\rangle \to \langle\langle \omega, \Delta\eta\rangle\rangle$ $(k\to\infty)$. By the ellipticity of Δ, $\omega \in \Lambda^p(M)$. Therefore, $H_2^p(M)$ is a closed subspace. $H_2^p(M)$ is called the space of L_2-harmonic p-forms.

Since $\Lambda_0^p(M)$ is dense in $L_2^p(M)$, d has the strong closure $\bar{d}$. Hereafter, we use the same letters for every operator and it's closure. Let D_d be the domain of d. If there is an $\eta \in L_2^{p+1}(M)$ and $\omega_k \to \omega$, $d\omega_k \to \eta$, then $d\omega=\eta$ and $\omega \in D_d$. It is well known ([de R], [G. 2. 3] [Che]) that $d\omega=\eta$ in the sense of distributions, *i. e.*

$$\langle\langle \eta, \phi\rangle\rangle = \langle\langle \omega, \delta\phi\rangle\rangle \qquad \text{for all } \phi \in \Lambda_0^p(M)$$

if and only if $\omega \in D_d$, $d\omega=\eta$. $H_{2,\#}^p(M):=Z_2^p(M)/\overline{B_2^p(M)}$ is called the reduced L_2-cohomology space, where

$$Z_2^p(M):=\{\omega \in D_d \mid d\omega=0\}$$

and

$$B_2^p(M):=\{\omega \in L_2^p(M) \mid \omega=d\eta,\ \eta \in D_d\}.$$

Let $(M, <\ ,\ >)$ be a complete riemannian manifold of dimension m. We define a smooth function $\mu: R\to R$ by

$$0\leqq \mu(t)\leqq 1 \ (t\in R),\ \mu(t)=1 \ (t\leqq 1),\ \mu(t)=0 \ (t\geqq 2),$$

and define the sequence of functions w_k by

$$w_k(x):=\mu(\rho(x)/k) \ (k\in N)$$

where $\rho(x)$ is the geodesic distance from x to some fixed $x_0\in M$.

LEMMA (1.1). *For* $\omega\in\Lambda^p(M)$,

$$\|dw_k \Lambda \omega\| \leqq (c/k)\|\omega\|,$$
$$\|\delta w_k \Lambda * \omega\| \leqq (c/k)\|\omega\|.$$

Proof. $\rho(x)$ is a locally Lipschiz function, and at points which its derivatives exist, $<\partial_i \rho, \partial_j \rho> \leqq m$. Then we have

$$|dw_k|^2 \leqq (c/k)^2.$$

THEOREM (1.2). *If M is complete, then $H_2^p(M) = \{\omega \in L_2^p(M) \mid d\omega = \delta\omega = 0\}$.*

Proof. Note that $H_2^p(M) = \Lambda^p(M) \cap L_2^p(M)$. If $d\omega = \delta\omega = 0$, then $\Delta\omega = 0$. Conversely, if $\omega \in H_2^p(M)$, $w_k\omega \in \Lambda_0^p(M)$ and so

$$0 = \ll \Delta\omega, w_k\omega \gg = -\ll d\omega, dw_k \Lambda\omega \gg - \ll d\omega, w_k d\omega \gg$$
$$- \ll \delta\omega, w_k\delta\omega \gg + \ll \delta\omega, *(dw_k \Lambda * \omega) \gg.$$

Letting $k \to \infty$, by means of Lemma (1.1), we have

$$\|d\omega\|^2 + \|\delta\omega\|^2 = 0.$$

Therefore, we have $d\omega = \delta\omega = 0$.

We consider Δ as an unbounded operator on $L_2^p(M)$ with the domain $\Lambda_0^p(M)$. Since $\Lambda_0^p(M)$ is dense in $L_2^p(M)$, there is the adjoint Δ^* and since Δ is a negative-definite symmetric operator, there is the closure Δ. Let $D_{\Delta*}^p$ (resp. D_Δ^p) be the domain of Δ^* (resp. Δ). Then we have

$$D_\Delta^p * = D_\Delta^p \oplus D_\lambda^p,$$

where $D_\lambda := \{\omega \in D_{\Delta*} \mid \Delta^*\omega = \lambda\omega, \lambda > 0\}$.

LEMMA (1.3). ([R-S], pp.136-137) *Let A be any closed, negative-definite, symmetric, densely defined operator on a Hilbert space. Then $A = A^*$ if and only if there are no eigenvectors with positive eigenvalues in D_{A*}.*

LEMMA (1.4). (Yau [Y]) *Suppose that M is complete. If ω is a L_2-p-form such that $\Delta^*\omega = \lambda\omega$ for some $\lambda > 0$, then ω is identically zero.*

Proof. Since $\langle\langle \Delta\phi, \omega\rangle\rangle = \langle\langle \phi, \Delta^*\omega\rangle\rangle = \lambda\langle\langle \phi, \omega\rangle\rangle$ for all $\phi \in \Lambda_0^p(M)$, ω is a weak solution of an elliptic equation and so ω is smooth.

$$\begin{aligned}&\lambda\langle\langle w_k{}^2\omega, \omega\rangle\rangle\\ &\quad=\langle\langle w_k{}^2\omega, \Delta^*\omega\rangle\rangle\\ &\quad=-\langle\langle d(w_k{}^2\omega), d\omega\rangle\rangle-\langle\langle \delta(w_k{}^2\omega), \delta\omega\rangle\rangle\\ &\quad=-\langle\langle w_k{}^2 d\omega, d\omega\rangle\rangle-\langle\langle w_k{}^2\delta\omega, \delta\omega\rangle\rangle\\ &\qquad-2\langle\langle w_k dw_k\Lambda\omega, d\omega\rangle\rangle+2\langle\langle \omega, w_k dw_k\Lambda\delta\omega\rangle\rangle.\end{aligned}$$

Then, we have

$$\begin{aligned}&\|w_k d\omega\|^2+\|w_k\delta\omega\|^2\\ &=-\lambda\langle\langle w_k{}^2\omega, \omega\rangle\rangle-2\langle\langle w_k dw_k\Lambda\omega, d\omega\rangle\rangle+2\langle\langle \omega, w_k dw_k\Lambda\delta\omega\rangle\rangle\\ &\leqq 2|\langle\langle w_k dw_k\Lambda\omega, d\omega\rangle\rangle|+2|\langle\langle \omega, w_k dw_k\Lambda\delta\omega\rangle\rangle|\\ &\leqq 2\|dw_k\|_\infty\|\omega\|(\|w_k d\omega\|+\|w_k\delta\omega\|),\end{aligned}$$

and so

$$\|w_k d\omega\|+\|w_k\delta\omega\|\leqq 4\|dw_k\|_\infty\|\omega\|.$$

(Here, we use the inequality : $x^2+y^2\leqq c(|x|+|y|)$ implies that $|x|+|y|\leqq 2c$.) Letting $k\to\infty$, we have $d\omega=\delta\omega=0$. Therefore, we have

$$\omega=\lambda^{-1}\Delta^*\omega=0.$$

Lemma (1.4) implies that $D^p_{\Delta*}=D^p_\Delta$, *i. e.* $\Delta^*=\Delta$.

THEOREM (1.5). *Δ is an essentialy self-adjoint operator on a complete riemannian manifold.*

COROLLARY (1.6). *If ω and $\Delta\omega$ are in $L_2^p(M)$, then $d\omega$ and $\delta\omega$ are in $L_2^p(M)$, and there is a sequence ω_k in $\Lambda_0^p(M)$ such that $\omega_k\to\omega$, $\Delta\omega_k\to\Delta\omega$, $d\omega_k\to d\omega$, $\delta\omega_k\to\delta\omega$ in L_2-sense.*

Proof. For $\omega\in\Lambda^p(M)$, $\|d\omega\|+\|\delta\omega\|=-\langle\langle\omega, \Delta\omega\rangle\rangle$, and so $\|d\omega\|^2+\|\delta\omega\|^2\leqq\|\omega\|\,\|\Delta\omega\|$. By the continuity, their estimates hold also on $D^p_{\Delta*}=D^p_\Delta$. The existence of such a sequence ω_k in $\Lambda_0^p(M)$ follows from the definition of $D_\Delta{}^p$ and the above estimates.

Theorem (1.7). (The general Hodge Theorem of Kodaira) ([de R])
The orthogonal direct sum decomposition holds on any riemannian manifold :

$$L_2^p(M)=\overline{\delta\Lambda_0^{p+1}(M)}\oplus\overline{d\Lambda_0^{p-1}(M)}\oplus H_2^p(M).$$

This decomposition holds also for $\Lambda^p(M)\cap L_2^p(M)$.

Corollary (1.8). (Gaffney [G1]) *On a complete riemannian manifold,*

$$H_2^p(M)=H_{2,\#}^p(M)\quad (\textit{as Hilbert spaces}).$$

Proof. Note that $H_2^p(M)=(d\Lambda_0^{p-1}(M))^{\perp}\cap(\delta\Lambda_0^{p+1}(M))^{\perp}$, where $(\ )^{\perp}$ is the orthogonal complement of ( ) in $L_2^p(M)$. By Theorem (1.7), the natural map $\iota: H_2^p(M)\to H_{2,\#}^p(M)$ is surjective and $\iota|(d\Lambda_0^{p-1}(M))^{\perp}$ is an isometric injection. Moreover, d is a closed operator, ker d is a closed subspace in $L_2^p(M)$. By Theorem (1.5), we have

$$\ll d\omega,\ \eta\gg=\ll\omega,\ \delta\eta\gg \text{ for } \omega\in D_d,\ \eta\in D_\delta.$$

Therefore, ι is injective.

2. The Laplacians of Warped Product.

Let (M,ds_M^2) and (N,ds_N^2) be a riemannian manifold of dimension m and n respectively. We define a warped product $(M\times_f N,\ ds^2)$ as a riemannian manifold with a metric $ds^2:=ds_M^2+f^2(x)ds_N^2$ where f is a positive function on M, called a warping function. Let $M\times_f N$ be a warped product. We shall discuss the influence of f on the de Rham complex, in particular, the Laplacian.**) Let $L_2^{p,q}(M\times_f N)$ be the completion of

$$\Lambda_0^{p,q}(M\times N):=\Lambda_0^p(M)\wedge\Lambda_0^q(N)$$

in $L_2^{p+q}(M\times_f N)$.

Then, $L_2^r(M\times_f N)$ can be generated by

$$\oplus_{p+q=r}L_2^{p,q}(M\times_f N).$$

**) S.Zuker [Z] computed the Laplacians on a warped product. But his computations contain errors.

We note that the volume element of the warped product is ,in terms of $dvol_M$ and $dvol_N$,

$$dvol=f^n dvol_M dvol_N.$$

Then, we have

Lemma (2.1). *For $\phi\Lambda\eta\in\Lambda_0^p(M)\Lambda\Lambda_0^q(N)$, the L_2-norm $\|\phi\Lambda\eta\|$ is given by*

$$\|\phi\Lambda\eta\|^2=\int\|\phi\|_M^2\|\eta\|_N^2 f^{n-2q}\ dvol_M dvol_N.$$

The exterior derivative d associated to the product stucture may be written

$$d=d_M\otimes 1_N+(-1)^p 1_M\otimes d_N$$

on $\Lambda_0^{p,q}(M\times N)$.

Since the following calculations are local in nature, we may suppose that M and N are orientable. The relation between the Hodge star operators$*,*_M$, and $*_N$ is as follows ;

(2.2) $\quad *=(-1)^{q(m-p)}*_f*_N \qquad$ on $\Lambda_0^{p,q}(M\times N)$,

where $*_f:=F_q*_M$ and $F_q:=f^{n-2q}$. Let d^* and $d_N{}^*$ be the formal adjoint of d and d_N respectively. Then we have

(2.3) $\quad d^*(\phi\Lambda\eta)=(-1)^p(*_f^{-1}d*_f\phi)\Lambda\eta+(-1)^p f^{-2}\phi\Lambda d_N{}^*\eta$

for $\phi\Lambda\eta\in\Lambda_0^{p,q}(M\times N)$.

Lemma (2.4). *For $\phi\Lambda\eta\in\Lambda_0^{p,q}(M\times N)$,*

$$d_f{}^*(\phi\Lambda\eta)=d_M{}^*(\phi\Lambda\eta)-(n-2q)(\iota_{d(\log f)}\phi)\Lambda\eta,$$

where

(2.5) $\quad d_f{}^*:=(-1)^p*_f^{-1}d*_f.$

Proof. The equalities

$$d_f{}^*(\phi \Lambda \eta)=(-1)^p F_q{}^{-1} *_M{}^{-1} d(F_q *_M \phi)\Lambda \eta$$

and

$$d(F_q *_M \phi)=dF_q \Lambda(*_M \phi)+F_q d(*_M \phi)$$

imply that

$$(2.6)\quad d_f{}^*(\phi \Lambda \eta)=(-1)^p F_q{}^{-1} *_M{}^{-1}(dF_q \Lambda *_M \phi)\Lambda \eta+(d_M^* \phi)\Lambda \eta.$$

And $F_q=f^{n-2q}$ implies that

$$dF_q \Lambda *_M \phi=(n-2q)f^{n-2q-1} df \Lambda *_M \phi,$$

and the first term of the right hand side (2.6) is equal to

$$(-1)^p(n-2q) *_M{}^{-1}(d(\log f)\Lambda *_M \phi)\Lambda \eta.$$

Setting

$$d(\log f):=\sum f_j \omega^j \qquad \text{and} \qquad \phi:=\phi_{i_1 \cdots i_p} \omega^{i_1} \Lambda \cdots \Lambda \omega^{i_p}$$

in terms of a local orthonormal coframing $\{\omega^i\}$ on M, we have

$$\begin{aligned}
&*_M(d(\log f)\Lambda *_M \phi)\\
&=(-1)^{(m-p)(p-1)} f_j \phi_{i_1 \cdots i_p}\, sgn\begin{pmatrix} i_1 \cdots\cdots i_p \\ j k_1 \cdots\cdots k_{p-1}\end{pmatrix} \omega^{k_1}\Lambda \cdots \Lambda \omega^{k_{p-1}}\\
&=(-1)^{(m-p)(p-1)} \iota_{d(\log f)} \phi,
\end{aligned}$$

where $\iota_{(.)}$ means the interior multiplication by $(.)$. Then, we have

$$*_M^{-1}(d(\log f) *_M \phi)=(-1)^{p-1} \iota_{d(\log f)} \phi,$$

which implies that the first term of the right hand side of (2.6) is equal to

$$-(n-2q)(\iota_{d(\log f)} \phi)\Lambda \eta.$$

Therefore, we may compute the Laplacian $\Delta:=d^*d+dd^*$ acting on $\Lambda_0^r(M \times N)$.

THEOREM (2.7).

$$\Delta=\Delta_f+f^{-2}\Delta_N+2(-1)^p\iota_{d(\log f)}d_N+2(-1)^{p+1}f^{-2}\varepsilon_N d_N{}^*,$$

where $\Delta_f:=d_M d_f{}^*+d_f{}^* d_M,$

$d_f{}^*:=(-1)^p *_f{}^{-1} d_M *_f,$

$*_f:=f^{n-2q} *_M,$

and ε_N *is the left exterior multiplication by* $d_M(\log f)$.

Moreover, we have,

THEORM (2.8).

$$\Delta=\Delta_M+(-1)^p(d\iota_F-\iota_F d) \qquad on \qquad \Lambda^p(M),$$

where $F:=n\ d(\log f)$.

REMARK (2.9). $F:=$n d$(\log f)=-$(the mean curvature of N in $M\times_f N$).

We consider the complex on $M\times_f N$:

(2.10) $$0\to R\to\Lambda^0(M\times_f N)\xrightarrow{d}\Lambda^1(M\times_f N)\xrightarrow{d}\Lambda^2(M\times_f N)\xrightarrow{d}\cdots\xrightarrow{d}\Lambda^p(M\times_f N)\xrightarrow{d}\cdots$$

which is sometimes called the "basic" de Rham complex ([K.1,2]). If $M\times_f N$ is complete and N is compact, then we have

COROLLARY (2.11). *The complex* (2.10) *satisfies the Poincaré duality in the* L_2- *"basic" cohomology if* $M\times_f N$ *is a product manifold, i.e,* f *is constant.*

In fact, the Pioncaré duality holds in the "basic" cohomology if N is minimal, df$=$0 ([K-T]. [K2]).

3. The sectional curvatures of a warped product.

Let $(M\times_f N, ds^2)$ be a warped product. We have the orthogonal decomposition of the tangent bundle $T(M\times_f N, ds^2)$;

$$T(M \times_f N,\ ds^2)=D_M \oplus D_N,$$

where D_M and D_N are integrable distributions on $M \times_f N$. The leaves of D_M (resp. D_N) are isometric to $(M,\ ds_M^2)$
(resp.$(N,\ f^2 ds_N^2)$).

Proposition (3.1). ([B-O]) *The sectional curvature $K(\Pi)$ of $(M \times_f N,\ ds^2)$ is given by*

$$\begin{aligned}K(\Pi)=&K_M(X,\ Y)\|X\Lambda Y\|_M^2-f(x)\{<W,\ W>_N((\nabla_M)^2 f)(X,\ X)\\&-2<V,\ W>_N((\nabla_M)^2 f)(X,\ Y)+<V,\ V>_N((\nabla_M)^2 f)(Y,\ Y)\}\\&+f^2(x)\{K_N(V,\ W)-\|grad\ f\|_M^2\}<V\Lambda W,\ V\Lambda W>,\end{aligned}$$

where $\nabla_{(.)}$ (resp. $K_{(.)}$) is the covariant derivative (resp. the sectional curvature of $(\,.\,)$), and $(\nabla_M)^2 f$ is the Hessian of f.

Hereafter, let $H^m(-a^2)$ and $N^n(-1)$ be the hyperbolic space form of dimension m and a compact (or finite volume) hyperbolic space of dimension n. And let $s(x)$ be the geodesic distance from x to a fixed point 0 in $H^{2p}(-a^2)$, and set $f(x) := \cosh s(x)$.

Example 1. $H^{2p}(-a^2) \times_f N^{m-2p}(-1)$. Then we have,

$$-a^2 \leqq K(\Pi) \leqq -1.$$

Proof. Let Π be generated by orthonormal framing $\{X,\ V\}$. Then we have

$$K(\Pi)=-a\,\frac{\cosh\ as}{\sinh\ as}\,\frac{\sinh\ s}{\cosh\ s},$$

which implies that if $s\to\infty$, $K(\Pi)\to -a$ and if $s\to 0$, $K(\Pi)\to -1$. On the other hand, if $\Pi \subset D_N$(resp. D_H), $K(\Pi)=-1$ (resp. $-a^2$). Therefore, we have the the above estimate.

Example 2. $H^{2p}(-a^2) \times_f S^{m-2p}(1)$. Then we have

$$-a^2 \leqq K(\Pi) \leqq 1.$$

4. Proofs of Theorems.

We shall prove Theorem 2. Our proof is a slight reversion of one of M.T.Anderson [An]. $M := H^{2p}(-a^2) \times_f N^{m-2p}(-1)$ is a complete, not simply-connected riemannian manifold. Let $\{X_i\}$ be a local orthonormal framing on $H^{2p}(-a^2)$ of eigenvectors of $\nabla^2 f$ and $\{V_i\}$ a local orthonormal framing on $N^{m-2p}(-1)$. Then the 2-framings $\{X_i \Lambda X_j\}$, $\{X_i \Lambda V_j\}$ and $\{V_i \Lambda V_j\}$ diagonalize the curvature tensor $R : \Lambda^2(M) \to \Lambda^2(M)$ with corresponding sectional curvature $-a^2$, $-a(\coth as)(\tanh s)$, -1.

For $\omega \in \Lambda^p(H^{2p}(-a^2))$,

$$\Delta_M \omega = \Delta_{H^{2p}} \omega + (-1)^p [d\iota_F - \iota_F d]\omega.$$

where $F = (m-2p)\ d\ (\log f)$. And it holds that

$$H^{2p}(-a^2) = H^2(-a^2) \times_g H^{2p-2}(-a^2)$$

where $g : H^2(-a^2) \to R$, $g(x) := \cosh a\rho(x)$, $\rho(x)$ is the geodesic distance from a fixed point $0 \in H^2(-a^2)$ to x. By this decomposition, F is tangent to the $H_2(-a^2)$-factor. Let

$$\omega := \phi \Lambda \eta,\ \phi \in \Lambda^1(H^2(-a^2)),\ \eta \in \Lambda^{p-1}(H^{2p-2}(-a^2)).$$

If η is any harmonic $(p-1)$-form on $H^{2p-2}(-a^2)$, then $\Delta\omega = 0$ if and only if

$$(4.1) \qquad \Delta\phi - [d\iota_F - \iota_F d]\phi = 0 \qquad \text{on } \Lambda^1(H^2(-a^2)).$$

Setting $\phi := du$ and using the conformal equivalence of $H^2(-a^2)$ with $\Omega := \{(x, \theta) | x \in R,\ \theta \in (-\pi/2, \pi/2)\}$, we have that (4.1) is equivalent to

$$(4.2) \qquad \partial^2 u / \partial x^2 + \partial^2 u / \partial \theta^2 + h(\theta) \partial u / \partial \theta = 0,$$

$$h(\theta) := (1/f_1)(\partial f_1 / \partial \theta),$$
$$f_1 := f|_{H^2(-a^2)} = 1/2\{[\alpha^{1/a} - \beta^{1/a}] / \cos^{1/a}\theta\},$$
$$\alpha := 1 + \sin\theta,\ \beta := 1 - \sin\theta.$$

Note that $h=0$ on $\partial\Omega$. We may suppose, without loss of generaliry, that $(m-2p)>0$ and so $h>0$. (4.2) has solutions, smooth up to $\partial\Omega$. If we conformally identify $H^2(-a^2)$ with $B^2(1)$ with the flat metric, we may obtain an infinite dimensional space of solutions of (4.2).

$\| \|$ is a conformal invariant on forms in the middle dimension. For ω as above, we have

$$\int_M |\omega|^2 = \int_{H^{2p}\times N^{m-2p}} |\omega|^2 f^{m-2p}\ dvol_H dvol_N$$
$$= vol\ (N^{m-2p}) \int_{H^2\times H^{2p-2}} |\phi|^2 |\eta|^2 f^{m-2p} dovol_{H^2} dvol_{H^{2p-2}}$$
$$\leqq vol(N^{m-2p})\ vol(B^{2p-2}(1)) \int_{B^2(1)} f^{m-2p}\ dvol_B.$$

Here, we have used the conformal equivalence of $H^k(-a^2)$ with $B^k(1)$, $k=2$, $2p-2$ and have supposed that η is a harmonic $(p-1)$-form with $|\eta|_\infty \leqq 1$ with respect to the flat metric on $B^{2p-2}(1)$, e.g.

$$\eta = (1/(p-1)!) dx^1 \wedge \cdots \wedge dx^{p-1}.$$

Then we have

$$\int_{B^2(1)} f^{m-2p}\ dvol < c \int_0^{\pi/2} cos^{-(m-2p)/a}\theta\ d\theta,$$

and so if $(m-2p)/a < 1$, we have $\int_M |\omega|^2 < \infty$.

Theorem 1 can be also obtained by replacing $N^{m-2p}(-1)$ by $S^{m-2p}(1)$.

SO(m,1)/SO(m) is a complete, simply-connected riemannian manifold with a compact quotient $\Gamma\backslash SO(m,1)/SO(m)$, where Γ is a group acting differentiably and properly discontinuously on SO(m,1)/SO(m).

LEMMA (4.3). *There is a relatively compact open set C in SO(m,1)/SO(m) with* $\Gamma C =$ *SO (m,1)/ SO(m).*

If we replace $N^{m-2p}(-1)$ by the above C, $M := H^{2p}(-a^2) \times_f C$ is an incomplete, simply-connected riemannian manifold. $vol(C) \leqq vol(\bar{C})$ implies Theorem 3.

COMMENT. If we may find a complete, simply-connected riemannian manifold with non-positive curvature, and finite volume, the conjecture of Dodziuk-Singer is false by M. T. Anderson's construction.

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