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L^2-vector fields on a non-compact Riemannian manifold with boundary

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1. The results of the study of conformal and Killing vector fields on a compact Riemannian manifold with boundary had been listed in Yano's book [3]. It has been tried by the third named author to generalize to the non-compact case, that is, in [5], the non-existence of L^2-Killing vector fields on a non-compact Riemannian manifold with boundary was discussed.

The purpose of the present paper is to investigate the properties and the non-existences of L^2-conformal and Killing vector fields on a non-compact Riemannian manifold with boundary.

We shall be in C^∞-category. Latin indices run from 1 to n+1 and Greek ones from 1 to n. The Einstein summation convention will be used.

2. Let $\mathscr{M}$ be a complete, non-compact, connected and orientable Riemannian manifold of dimension n+1 with a Riemannian metric g. Let ∇ be the Riemannian connection on $\mathscr{M}$ associated with g. We take a non-compact manifold $\bar{M}=\partial M\cup M$ such that M is a non-compact, connected and open submanifold of $\mathscr{M}$ and $\partial M=\bar{M}-M$ is an n dimensional, compact, connected submanifold of $\mathscr{M}$, where $\bar{M}$ denotes the closure of M in $\mathscr{M}$. Then $\bar{M}$ is an orientable Riemannian manifold with boundary ∂M and the Riemannian metric induced from g on $\mathscr{M}$. We denote by g the induced Riemannian metric on $\bar{M}$ and by ∇ the Riemannian connection on $\bar{M}$ associated with g. It is trivial that $\bar{M}$ is complete as a metric space with the distance determined by g.

At each point p of ∂M, there exists a coordinate neighborhood system $\{U, (x^i)\}$ of p in $\mathscr{M}$ such that $U\cap\bar{M}$ is represented by $x^{n+1}\geqq 0$ and $U\cap\partial M$ is represented by $x^{n+1}=0$. Such a coordinate neighborhood system is called a boundary coordinated system. And let $\{U\cap\partial M, (x^\alpha)\}$ be the induced coordinate neighborhood system on ∂M. If $\{U, (x^i)\}$ and $\{V,(y^i)\}$ are boundary coordinate systems satisfying $U\cap V\neq\phi$, then we have

(2.1) $\frac{\partial y^{n+1}}{\partial x^{n+1}}>0$ and $\frac{\partial y^{n+1}}{\partial x^{\alpha}}=0$ (for any α)

on $\partial M\cap U\cap V$. Since the Jacobian of the coordinate transformation of $\{U,(x^i)\}$ and $\{V,(y^i)\}$ is positive, the Jacobian of the coordinate transformation of $\{U\cap\partial M,(x^{\alpha})\}$ and $\{V\cap\partial M,(y)^{\alpha}\}$ is positive. Thus ∂M is orientable.

Let $i:\partial M\longrightarrow\bar{M}$ be the inclusion. Let $\{U,(x^i)\}$ be a boundary coordinate system of $p\in\partial M$ in $\mathscr{M}$ and $\{U',(u^{\alpha})\}$ be a coordinate neighborhood system of p in ∂M such that $U'\subset U\cap\partial M$. Then the inclusion i may be represented locally by

(2.2) $x^i=x^i(u^{\alpha})$.

We denote by B the differential of the inclusion i, that is,

(2.3) $B=(B^i_{\alpha})=(\frac{\partial x^i}{\partial u^{\alpha}})$.

The induced metric $\tilde{g}=(\tilde{g}_{\alpha\beta})$ on ∂M is given by

(2.4) $\tilde{g}_{\alpha\beta}=B^i_{\alpha}B^j_{\beta}g_{ij}$,

where $g=(g_{ij})$. We may choose the unit outer normal vector field N on ∂M. Let $\tilde{\nabla}$ be the Riemannian connection on ∂M associated with $\tilde{g}$. The equations of Gauss and Weingarten are stated as follows:

(2.5) $\nabla_{i_*\tilde{X}}i_*\tilde{Y}=i_*(\tilde{\nabla}_{\tilde{X}}\tilde{Y})+h(\tilde{X},\tilde{Y})\cdot N$

(2.6) $\nabla_{i_*\tilde{X}}N=i_*(-A\tilde{X})$

for any vector fields $\tilde{X}$ and $\tilde{Y}$ on ∂M, where h denotes the second fundamental form of ∂M with respect to N and A is defined by $h(\tilde{X},\tilde{Y})=\tilde{g}(A\tilde{X},\tilde{Y})$.

3. Let $\Lambda^s(\bar{M})$ (resp. $\Lambda^s(\partial M)$) be the space of all s-forms on $\bar{M}$ (resp. ∂M). Let d (resp. $\tilde{d}$) denote the exterior derivative on $\Lambda^s(\bar{M})$ (resp. $\Lambda^s(\partial M)$). An operator δ (resp. $\tilde{\delta}$) is defined by

(3.1) $\delta=(-1)^m*d*$ (resp. $\tilde{\delta}=(-1)^m*\tilde{d}*$)

on $\Lambda^s(\bar{M})$ (resp. $\Lambda^s(\partial M)$), where $m=sn+s+n$ (resp. $m=sn+n+1$) and $*$ denotes the star operator.

The local scalar product on $\Lambda^s(\bar{M})$ (or $\Lambda^s(\partial M)$) is denoted by $<,>$, and let $\ll, \gg$ be the global scalar product on $\Lambda^s_0(\bar{M})$ (or $\Lambda^s_0(\partial M)$). Here $\Lambda^s_0(\bar{M})$ (resp. $\Lambda^s_0(\partial M)$) denotes the subspace of $\Lambda^s(\bar{M})$ (resp. $\Lambda^s(\partial M)$) composed of forms with compact supports. For example, we have

$$\ll \xi, \eta \gg = \int_{\bar{M}} \langle \xi, \eta \rangle * 1 = \int_{\bar{M}} \xi \Lambda * \eta \tag{3.2}$$

for any $\xi, \eta \in \Lambda^s_0(\bar{M})$. Then we have

$$\ll d\xi, \zeta \gg = \ll \xi, \delta\zeta \gg \tag{3.3}$$

for any $\xi \in \Lambda^s_0(\bar{M})$ and $\zeta \in \Lambda^{s+1}_0(\bar{M})$. Let $L^{2,s}(\bar{M})$ be the completion of $\Lambda^s_0(\bar{M})$ with respect to the scalar product $\ll, \gg$. We set $\| \ \| = \ll, \gg^{1/2}$. We remark that $\ll, \gg$ is extended to the space of (u, v)-tensor fields on $\bar{M}$. For any (0, 2)-tensor fields S and T on $\bar{M}$ with compact supports, we have that $\ll S, T \gg = \int_{\bar{M}} <S, T> * 1$ $= \int_{\bar{M}} \frac{1}{2!} S_{ij} T^{ij} * 1$ ([5]). We remark that the local scalar product in [3] is given by $<S, T> = S_{ij} T^{ij}$.

For any $\eta \in \Lambda^1(\bar{M})$ and $\tilde{\zeta} \in \Lambda^1(\partial M)$, we define $t\eta \in \Lambda^1(\partial M)$, $n\eta \in \Lambda^0(\partial M)$ and $C\tilde{\zeta} \in \Lambda^1(\partial M)$ by

$$(t\eta)(\tilde{Y}) = \eta(i_* \tilde{Y})$$

$$n\eta = \eta(N) \tag{3.4}$$

$$(C\tilde{\zeta})(\tilde{Y}) = \tilde{\zeta}(A\tilde{Y})$$

for any vector field $\tilde{Y}$ on ∂M.

For a vector field $X = \xi^i \, \partial/\partial x^i$ on $\bar{M}$, the 1-form ξ dual to X is defined by $\xi = \xi_i \, dx^i = g_{ij} \, \xi^j \, dx^i$. We set

$$\nabla_i = \nabla_{\partial/\partial x^i} \quad \text{and} \quad \nabla^j = g^{ij} \nabla_j,$$

where (g^{ij}) denotes the inverse matrix of (g_{ij}).Then we have following expression:

$$(d\xi)_{ij} = \nabla_i \xi_j - \nabla_j \xi_i$$
$$\delta\xi = -\nabla^i \xi_i$$
$$(t\xi)_\alpha = B^{\ i}_{\alpha} \xi_i \tag{3.5}$$

$$n\xi = \xi_i N^i$$
$$C\,(t\xi)_\alpha = A^\beta_\alpha B^i_\beta \xi_i$$
$$\nabla_i \nabla_j\, \xi^i - \nabla_j \nabla_i \xi^i = R_{ji} \xi^i$$

where R_{ji} denote the components of the Ricci tensor field of ∇ ([3], [5]). Moreover, we define $D(\delta\xi)$, $\Delta\xi$ and $\Box X$ as follows:

$$(D(\delta\xi))^i = \nabla^i\,(\delta\xi) = -\nabla^i \nabla^j \xi_j$$

$$(3.6) \qquad (\triangle\xi)_i = (d\delta\xi + \delta d\xi)_i = -\nabla^j \nabla_j \xi_i + R^j_i \xi_j$$

$$(\Box X)^i = \nabla^j \nabla_j \xi^i + R^i_j \xi^j$$

where $R^i_j = g^{ik} R_{kj}$([3], [5]).

Let X be a vector field on $\bar{M}$ and ξ the dual 1-form. Then $X \mid_{\partial M}$ has the following expression:

$$(3.7) \qquad X = i_* \tilde{X} + (n\xi)\cdot N \quad \text{on } \partial M,$$

for some vector field $\tilde{X}$ on ∂M.

DEFINITION 3.1 ([3]). A vector field X on $\bar{M}$ is called *tangential* (resp. *normal*) to ∂M if $n\xi = 0$ (resp. $t\xi = 0$) for the dual 1-form ξ to X.

By (3.7), we remark that $t\xi = 0$ is equivalent to $i_* \tilde{X} = 0$.

DEFINITION 3.2. A vector field X on $\bar{M}$ is called a *conformal vector field* if $L_X g = 2\lambda \cdot g$, where L denotes the Lie derivative operator and λ is a function on $\bar{M}$.

DEFINITION 3.3. A vector field X on $\bar{M}$ is called a *Killing vector field* if $L_X g = 0$.

By local expression, a vector field X on $\bar{M}$ is a conformal (resp. Killing) vector field if

$$(3.8) \qquad \nabla_i \xi_j + \nabla_j \xi_i = 2\lambda\; g_{ij} \quad (\text{resp. } \nabla_i \xi_j + \nabla_j \xi_i = 0),$$

where ξ is the dual 1-form to X.

DEFINITION 3.4 ([4]). A vector field X on $\bar{M}$ has *finite L^2-norm* if $\xi \in L^{2,1}(\bar{M}) \cap \Lambda^1(\bar{M})$ for the dual 1-form ξ to X. A vector field X on $\bar{M}$ with finite L^2-norm is called an *L^2-vector field*.

We remark that, in [4], an L^2-vector field is called a vector field with finite global norm. An L^2-vector field X on $\bar{M}$ satisfies $\| X \| < \infty$.

4. For each point p of $\bar{M}$, we denote by $\rho(p)$ the distance from p to ∂M. Since ∂M is compact and connected, ρ is well-defined. We set

$$B(2r) = \{ p \in \bar{M} \mid \rho(p) \leqq 2r \} \tag{4.1}$$

for any $r>0$. A function μ on R satisfies the following properties:

$$\begin{array}{ll} 0 \leqq \mu(t) \leqq 1 & \text{on } R \\ \mu(t) = 1 & \text{for } t \leqq 1 \\ \mu(t) = 0 & \text{for } t \geqq 2. \end{array} \tag{4.2}$$

Then we define locally Lipschitz continuous functions ω_r on $\bar{M}$ by

$$\omega_r(p) = \mu(\rho(p)/r) \qquad r = 1, 2, \cdots \tag{4.3}$$

for any $p \in \bar{M}$.

LEMMA 4.1 ([1], [4]). *There exists a positive number C, depending only on μ, such that*

$$\| d\omega_r \Lambda \eta \|^2_{B(2r)} \leqq (n+1)\, C\, r^{-2} \| \eta \|^2_{B(2r)}$$

$$\| d\omega_r \Lambda * \eta \|^2_{B(2r)} \leqq (n+1)\, C\, r^{-2} \| \eta \|^2_{B(2r)}$$

$$\| d\omega_r \otimes \eta \|^2_{B(2r)} \leqq (n+1)\, C\, r^{-2} \| \eta \|^2_{B(2r)}$$

for any $\eta \in \Lambda^s(\bar{M})$, where $\| \cdot \|^2_{B(2r)} = \langle\!\langle \cdot, \cdot \rangle\!\rangle_{B(2r)} = \int_{B(2r)} \langle \cdot, \cdot \rangle * 1$.

We remark that, for any $\eta \in L^{2,s}(\bar{M}) \cap \Lambda(\bar{M})$, $\omega_r \eta \in \Lambda_0^s(\bar{M})$ and $\omega_r \eta \longrightarrow \eta$ as $r \longrightarrow \infty$ in the strong sense.

LEMMA 4.2. *For any* $\eta \in \Lambda^1(\bar{M})$, *it holds that*

$$d(\omega_r^2 \eta) = \omega_r^2 d\eta + 2\omega_r\, d\omega_r \wedge \eta \qquad a.\ e.\ on\ M$$

$$\delta(\omega_r^2 \eta) = \omega_r^2 \delta\eta - *(2\omega_r\, d\omega_r \wedge *\eta) \quad a.\ e.\ on\ M,$$

where "a. e." means "almost everywhere".

5. By the direct calculation, we have

PROPOSITION 5.1. *Let* X *be a vector field on* $\bar{M}$ *and* ξ *the dual* 1*-form to* X. *Then it holds that*

$$\begin{aligned}&<\Box X - (1-\frac{2}{n+1})\, D(\delta\xi),\ \omega_r^2 X>\\ &+ <L_X g + \frac{2}{n+1}(\delta\xi)\cdot g,\ L_{(\omega_r^2 X)} g + \frac{2}{n+1}(\delta(\omega_r^2\xi))\cdot g>\\ &= -\delta\,(i(\omega_r^2 X)(L_X g) + \frac{2}{n+1}(\omega_r^2 \delta\xi)\cdot\xi).\end{aligned}$$

By Stokes' theorem, we have

$$(5.1) \qquad \int_{B(2r)} d(*(\omega_r^2\eta)) = \int_{\partial B(2r)} \omega_r^2 \eta(N) * 1.$$

Since $d(*(\omega_r^2\eta)) = -*\delta(\omega_r^2\eta)$ and $\partial B(2r) = \partial M \cup \{p \in \bar{M} \mid \rho(p) = 2r\}$, (5.1) implies

$$(5.2) \qquad -\int_{B(2r)} \delta(\omega_r^2\eta) * 1 = \int_{\partial M} \eta(N) * 1.$$

By Proposition 5.1 and (5.2), we have

PROPOSITION 5.2. *Let* X *be a vector field on* $\bar{M}$ *and* ξ *the dual* 1*-form to* X. *Then it holds that*

$$\ll \Box X - (1-\frac{2}{n+1})\, D(\delta\xi),\ \omega_r^2 X \gg_{B(2r)}$$

$$+\ll L_X g+\frac{2}{n+1}(\delta\xi)\cdot g,\ L_{(\omega_r^2 X)}g+\frac{2}{n+1}(\delta(\omega_r^2\xi))\cdot g\gg_{B(2r)}$$

$$=\int_{\partial M}(L_X g+\frac{2}{n+1}(\delta\xi)\cdot g)(X, N)*1.$$

Thus we have

THEOREM 5.3. *Let* X *be an* L^2*-vector field on* $\bar{M}$ *and* ξ *the dual* 1*-form to* X. *Then* X *is an* L^2*-conformal vector field on* $\bar{M}$ *if and only if*

$$(\#)\qquad \begin{cases} \Box X-(1-\frac{2}{n+1})D(\delta\xi)=0 & on\ \bar{M} \\ (L_X g+\frac{2}{n+1}(\delta\xi)\cdot g)(X,N)=0 & on\ \partial M \end{cases}$$

PROOF. If X satisfies (#), then, by Proposition 5.2, we have

$$0=\ll L_X g+\frac{2}{n+1}(\delta\xi)\cdot g,\ L_{(\omega_r^2 X)}g+\frac{2}{n+1}(\delta(\omega_r^2\xi))\cdot g\gg_{B(2r)}$$

$$=||\omega_r(L_X g+\frac{2}{n+1}(\delta\xi)\cdot g)||^2_{B(2r)}+4\ll\omega_r(L_X g+\frac{2}{n+1}(\delta\xi)\cdot g),\ d\omega_r\otimes\xi\gg_{B(2r)}.$$

Thus we have, by Lemma 4.1,

$$||\omega_r(L_X g+\frac{2}{n+1}(\delta\xi)\cdot g)||^2_{B(2r)}$$

$$=-4\ll\omega_r(L_X g+\frac{2}{n+1}(\delta\xi)\cdot g),\ d\omega_r\otimes\xi\gg_{B(2r)}$$

$$\leqq\frac{1}{2}||\omega_r(L_X g+\frac{2}{n+1}(\delta\xi)\cdot g)||^2_{B(2r)}+8(n+1)Cr^{-2}||\xi||^2_{B(2r)}.$$

Thus we have

$$\frac{1}{2}||\omega_r(L_X g+\frac{2}{n+1}(\delta\xi)\cdot g)||^2_{B(2r)}\leqq 8(n+1)Cr^{-2}||\xi||^2_{B(2r)}.$$

As $r\longrightarrow\infty$, we have

$$||L_X g+\frac{2}{n+1}(\delta\xi)\cdot g||^2=0.$$

Therefore X is an L^2-conformal vector field on $\bar{M}$. The converse is trivial.

The following theorem is given as corollary of Theorem 5.3.

THEOREM 5.4: *Let* X *be an* L^2*-vector field on* $\bar{M}$ *and* ξ *the dual* 1*-form to* X. *Then* X *is an* L^2*-Killing vector field on* $\bar{M}$ *if and only if*

$$\square X=0 \quad and \quad \delta\xi=0 \qquad on\ \bar{M}$$
$$(L_X g)(X, N)=0 \qquad on\ \partial M$$

Let τ be the mean curvature of the boundary ∂M defined by

$$\tau=\Sigma_\alpha h(\tilde{X}_\alpha, \tilde{X}_\alpha), \tag{5.3}$$

where $\{\tilde{X}_\alpha\}$ denotes the orthonormal local frame on ∂M with respect to $\tilde{g}$.

LEMMA 5.5. *Let* X *be a vector field on* $\bar{M}$ *and* ξ *the dual* 1*-form to* X. *Then, on* ∂M, *it holds that*

$$(\nabla_N\xi)(N)=\tilde{\delta}\tilde{\xi}+\tau(n\xi)-\delta\xi$$

$$(L_X g)(X, N)=(\nabla_N\xi)(X)+h(\tilde{X}, \tilde{X})+\tau(n\xi)^2$$
$$+2\cdot(n\xi)\cdot\tilde{\delta}\tilde{\xi}-\tilde{\delta}((n\xi)\tilde{\xi})-(n\xi)\cdot\delta\xi,$$

where $X=i_*\tilde{X}+(n\xi)\cdot N$ *and* $\tilde{\xi}$ *denotes the dual* 1*-form to* $\tilde{X}$.

By Theorems 5.3 and 5.4 and Lemma 5.5, we have

COROLLARY 5.6. *Let* X *be an* L^2*-vector field on* $\bar{M}$ *tangential* (*resp. normal*) *to the boundary* ∂M *and* ξ *the dual* 1*-form to* X. *Then* X *is an* L^2*-conformal vector field on* $\bar{M}$ *if and only if*

$$\square X-\left(1-\frac{2}{n+1}\right)D(\delta\xi)=0 \qquad on\ \bar{M}$$

$$(\nabla_N\xi)(X)+h(\tilde{X}, \tilde{X})=0 \qquad on\ \partial M$$

$$(resp.\ \tau\cdot(n\xi)^2-\frac{n-1}{n+1}(n\xi)\cdot\delta\xi=0 \qquad on\ \partial M).$$

COROLLARY 5.7. *Let* X *be an* L^2*-vector field on* $\bar{M}$ *tangential (resp. normal) to the boundary* ∂M *and* ξ *the dual* 1*-form to* X. *Then* X *is an* L^2*-Killing vector field on* $\bar{M}$ *if and only if*

$$\Box X=0, \quad \delta\xi=0 \qquad on\ \bar{M}$$

$$(\nabla_N\xi)(X)+h(\tilde{X}, \tilde{X})=0 \qquad on\ \partial M$$

$$(\text{resp.}\quad \tau\cdot(n\xi)^2=0 \qquad on\ \partial M).$$

6. Let X be a conformal vector field on $\bar{M}$, that is, $L_X\, g=2\lambda\cdot g$, and let ξ be the dual 1-form to X. We define a 1-form η by

(6.1) $\quad \eta=(\nabla_j\xi_i)\,\xi^j\, dx^i.$

Then we have

LEMMA 6.1. *It holds that*

$$-\delta\eta=<\mathscr{R}\,\xi, \xi>-2<\nabla\xi, \nabla\xi>+(n+1)<d\lambda, \xi>+2(n+1)<\lambda, \lambda>$$

$$-\delta\xi=(n+1)\,\lambda$$

$$<N, \eta>=<\tilde{d}\,(n\xi), t\xi>+<C\,(t\xi), t\xi>$$

$$*\,(2\omega_r d\omega_r\Lambda * \eta)=-2<\omega_r\, d\omega_r\otimes\xi, \nabla\xi>,$$

where $(\quad\xi)_i=R_i^j\xi_j$ *and* $(\nabla\xi)_{ij}=\nabla_i\,\xi_j$

PROPOSITION 6.2. *Let* X *be a conformal vector field on* $\bar{M}$ *and* ξ *the dual* 1*-form to* X. *Then it holds that*

$$\ll\omega_r\mathscr{R}\,\xi, \omega_r\,\xi\gg_{B(2r)}-2\ll\omega_r\nabla\xi, \omega_r\nabla\xi\gg_{B(2r)}$$

$$+(n+1)\ll\omega_r\, d\lambda, \omega_r\,\xi\gg_{B(2r)}+2(n+1)\ll\omega_r\lambda, \omega_r\lambda\gg_{B(2r)}$$

$$-2\ll d\omega_r\otimes\xi, \omega_r\nabla\xi\gg_{B(2r)}$$

$$=\int_{\partial M}<\tilde{d}\,(n\xi),\ t\xi>+<C\,(t\xi),\ t\xi>*1.$$

PROOF. For a 1-form $\eta=(\nabla_j\xi_i)\ \xi^j\ dx^i$ on $\bar{M}$, we have

$$(6.2)\qquad d\,(*(\omega_r^2\eta))=-*\delta\,(\omega_r^2\eta)$$
$$=*\ (\omega_r^2(-\delta\eta)+*\,(2\omega_r d\omega_r\Lambda*\eta)).$$

By Stokes' theorem, we have

$$(6.3)\qquad \int_{B(2r)}d(*(\omega_r^2\eta))=\int_{\partial B(2r)}<N,\ \omega_r^2\eta>*1.$$

By Lemma 6.1, (6.2) and (6.3), we have

$$(6.4)\qquad \ll\omega_r\mathscr{R}\ \xi,\ \omega_r\xi\gg_{B(2r)}-2\ll\omega_r\nabla\xi,\ \omega_r\nabla\xi\gg_{B(2r)}$$
$$+(n+1)\ll\omega_r d\lambda,\ \omega_r\xi\gg_{B(2r)}+2(n+1)\ll\omega_r\lambda,\ \omega_r\lambda\gg_{B(2r)}$$
$$-2\ll d\omega_r\otimes\xi,\ \omega_r\nabla\xi\gg_{B(2r)}$$
$$=\int_{\partial B(2r)}\omega_r^2\ \{<\tilde{d}(n\xi),\ t\xi>+<C\,(t\xi),\ t\xi>\}\ *\ 1.$$

Since $\partial B(2r)=\partial M\cup\ \{p\in\bar{M}\mid\rho(p)=2r\}$, $\omega_r=1$ on ∂M and $\omega_r=0$ on $\{p\in\bar{M}\mid\rho\ (p)=2r\}$, the right hand side of (6.4) is equal to $\int_{\partial M}<\tilde{d}\ (n\xi),\ t\xi>+<C\,(t\xi),\ t\xi>*1$.

LEMMA 6.3. *It holds that*

$$\ll\omega_r d\lambda,\ \omega_r\xi\gg_{B(2r)}=-(n+1)\ll\omega_r\lambda,\ \omega_r\lambda\gg_{B(2r)}-\ll\omega_r\lambda,\ *\,(2\ d\omega_r\Lambda*\xi)\gg_{B(2r)}.$$

Then Proposition 6.2 and Lemma 6.3 imply

$$\ll\omega_r\mathscr{R}\ \xi,\ \omega_r\xi\gg_{B(2r)}$$
$$=2\|\omega_r\nabla\xi\|^2_{B(2r)}+(n+1)\ (n-1)\|\omega_r\lambda\|^2_{B(2r)}+(n+1)\ll\omega_r\lambda,\ *\,(2\ d\omega_r\Lambda*\xi)\gg_{B(2r)}$$
$$+2\ll d\omega_r\otimes\xi,\ \omega_r\nabla\xi\gg_{B(2r)}+\int_{\partial M}<\tilde{d}(n\xi),\ t\xi>+<C\,(t\xi),\ t\xi>*1$$

By the Schwarz inequality and Lemma 4.1, we have

$$2\,|\ll \omega_r\lambda,\ *(d\omega_r\Lambda * \xi)\gg_{B(2r)}|$$

$$\leqq 2\|\omega_r\lambda\|_{B(2r)}\|d\omega_r\Lambda * \xi\|_{B(2r)}$$

$$\leqq 2\|\omega_r\lambda\|^2_{B(2r)}+(n+1)C\cdot 2^{-1}r^{-2}\|\xi\|^2_{B(2r)}$$

and

$$2\,|\ll d\omega_r\otimes\xi,\ \omega_r\nabla\xi\gg_{B(2r)}|$$

$$\leqq\|\omega_r\nabla\xi\|^2_{B(2r)}+(n+1)\,Cr^{-2}\|\xi\|^2_{B(2r)}.$$

Thus we have

$$\ll\omega_r\mathscr{R}\ \xi,\ \omega_r\xi\gg_{B(2r)}$$

$$\geqq 2\|\omega_r\nabla\xi\|^2_{B(2r)}+(n+1)\,(n-1)\|\omega_r\lambda\|^2_{B(2r)}$$

$$-2\,(n+1)\|\omega_r\lambda\|^2_{B(2r)}-(n+1)^2C\cdot 2^{-1}r^{-2}\|\xi\|^2_{B(2r)}.$$

$$-\|\omega_r\nabla\xi\|^2_{B(2r)}-(n+1)\,Cr^{-2}\|\xi\|^2_{B(2r)}$$

$$+\int_{\partial M}<\tilde{d}\ (n\xi),\ t\xi>+<C\ (t\xi),\ t\xi> * 1$$

$$=\|\omega_r\nabla\xi\|^2_{B(2r)}+(n+1)\,(n-3)\|\omega_r\lambda\|^2_{B(2r)}$$

$$+\int_{\partial M}<\tilde{d}\ (n\xi),\ t\xi>+<C\ (t\xi),\ t\xi> * 1$$

$$-(n+1)\,(n+3)\,C\cdot 2^{-1}r^{-2}\|\xi\|^2_{B(2r)}.$$

PROPOSITION 6.4. *Let* X *be an* L^2-*conformal vector field on* $\bar{M}$ *and* ξ *the dual* 1-*form to* X. *If* $\dim\bar{M}=n+1\geqq 4$ *and* $\limsup_{r\to\infty}\ll\omega_r\mathscr{R}\ \xi,\ \omega_r\xi\gg_{B(2r)}<\infty$, *then*

$$\limsup_{r\to\infty}\ll\omega_r\mathscr{R}\ \xi,\ \omega_r\xi\gg_{B(2r)}$$

$$\geqq \|\nabla\xi\|^2+(n+1)(n-3)\|\lambda\|^2+\int_{\partial M}<\tilde{d}\,(n\xi),\, t\xi>+<C\,(t\xi),\, t\xi> * 1.$$

COROLLARY 6.5 ([5]). *Let* X *be an* L^2*-Killing vector field on* $\bar{M}$ *and* ξ *the dual* 1*-form to* X. *If* $\limsup_{r\to\infty} \ll \omega_r \mathscr{R}\ \xi,\ \omega_r\xi \gg_{B(2r)} < \infty$, *then*

$$\limsup_{r\to\infty} \ll \omega_r \mathscr{R}\ \xi,\ \omega_r\xi \gg_{B(2r)}$$

$$\geqq \|\nabla\xi\|^2+\int_{\partial M}<\tilde{d}\,(n\xi),\, t\xi>+<C\,(t\xi),\, t\xi> * 1.$$

If the second fundamental form of ∂M with respect to N is non-negative, then we have

$$\int_{\partial M}<C\,(t\xi),\, t\xi> * 1 \geqq 0.$$

Thus, by Proposition 6.4 and Corollary 6.5, we have

THEOREM 6.6. (i) *Suppose that* $\limsup_{r\to\infty} \ll \omega_r \mathscr{R}\ \xi,\ \omega_r\xi \gg_{B(2r)} \leqq 0$ *for any* $\xi \in L^{2,1}(\bar{M}) \cap \Lambda^1(\bar{M})$, *the second fundamental form on* ∂M *with respect to the unit outer normal vector field is non-negative, and* $\dim \bar{M} = n+1 \geqq 4$. *Then every* L^2*-conformal vector field on* $\bar{M}$ *tangential to* ∂M *is parallel.* (ii) *Suppose that* $\limsup_{r\to\infty} \ll \omega_r \mathscr{R}\ \xi,\ \omega_r\xi \gg_{B(2r)} \leqq 0$ *for any* $\xi \in L^{2,1}(\bar{M}) \cap \Lambda^1(\bar{M})$ *and* $\dim \bar{M} = n+1 \geqq 4$. *Then every* L^2*-conformal vector field on* $\bar{M}$ *normal to* ∂M *is parallel.*

THEOREM 6.7 ([5]). (i) *Suppose that* $\limsup_{r\to\infty} \ll \omega_r \mathscr{R}\ \xi,\ \omega_r\xi \gg_{B(2r)} \leqq 0$ *for any* $\xi \in L^{2,1}(\bar{M}) \cap \Lambda^1(\bar{M})$ *and the second fundamental form of* ∂M *with respect to the unit outer normal vector field is non-negative. Then every* L^2*-Killing vector field on* $\bar{M}$ *tangential to* ∂M *is parallel.* (ii) *Suppose that* $\limsup_{r\to\infty} \ll \omega_r \mathscr{R}\ \xi,\ \omega_r\xi \gg_{B(2r)} \leqq 0$ *for any* $\xi \in L^{2,1}(\bar{M}) \cap \Lambda^1(\bar{M})$. *Then every* L^2*-Killing vector field on* $\bar{M}$ *normal to* ∂M *is parallel.*

THEOREM 6.8. (i) *Suppose that* $\bar{M}$ *is of negative Ricci curvature,* $\dim \bar{M} = n+1 \geqq 4$, *and the second fundamental form of* ∂M *with rerpect to the unit outer normal vector field is non-negative. Then there are no non-zero* L^2*-conformal vector fields on* $\bar{M}$ *tangential to* ∂M. (ii) *Suppose that* $\bar{M}$ *is of negative Ricci curvature and* $\dim \bar{M} = n+1 \geqq 4$. *Then there are no non-zero* L^2*-conformal vector fields on* $\bar{M}$ *normal to* ∂M.

THEOREM 6.9 ([5]). (i) *Suppose that* $\bar{M}$ *is of negative Ricci curvature and the second fundamental form of* ∂M *with respect to the unit outer normal vector field is non-negative. Then there are no non-zero* L^2*-Killing vector fields on* $\bar{M}$ *tangential to* ∂M. (ii) *Suppose that* $\bar{M}$ *is of negative Ricci curvature. Then there are no non-zero* L^2*-Killing vector fields on* $\bar{M}$ *normal to* ∂M.

EXAMPLE 6.10. We set $r=(x^2+y^2+z^2)^{1/2}$ for any point $(x, y, z) \in R^3$ and

$$x = r\cos\theta_1 \qquad y = r\sin\theta_1\cos\theta_2 \qquad z = r\sin\theta_1\sin\theta_2,$$

that is, (θ_1, θ_2, r) is the spherical coordinates in R^3. For two positive constant numbers a_1 and a_2 $(a_1 < a_2)$, we consider a metric ds^2 on R^3 such that

$$ds^2 = r^2 ((d\theta_1)^2 + \sin^2\theta_1 (d\theta_2)^2) + (dr)^2 \qquad r \leqq a_1$$

$$ds^2 = r^{-2/3}((d\theta_1)^2 + \sin^2\theta_1 (d\theta_2)^2) + (dr)^2 \qquad r \geqq (a_1 + a_2)/2.$$

Then $(\mathscr{M}, ds^2) = (R^3, ds^2)$ is a complete, non-compact, connected and orientable Riemannian manifold. We set $\bar{M} = \{(\theta_1, \theta_2, r) \in \mathscr{M} \mid r \geqq a_2\}$, then $\bar{M}$ is a non-compact, connected and orientable Riemannian manifold with a compact and connected boundary $\partial M = \{(\theta_1, \theta_2, r) \in \mathscr{M} \mid r = a_2\}$. Then $X = r^{-1/3}\, \partial/\partial r$ is an L^2-conformal vector field on $\bar{M}$ normal to ∂M.

LEMMA 6.11. *Let* W *be a Riemannian manifold with a Riemmannian metric* g_w *and* X *a Killing vector field on* W *with respect to the metric* g_w. *If* f *be a positive function on* W *such that* $X(f) \neq 0$ *on* W, *then* X *is a conformal vector field on* W *with respcet to the metric* $f \cdot g_W$.

In Lemma 6.11, if g_W is a complete metric on W and f is a bounded function on W, then $f \cdot g_W$ is a complete metric on W ([2]). Thus L^2-conformal vector fields on $\bar{M}$ tangential to ∂M are given by examples in [5].

REMARK 6.12. If, in Theorems 6.8 and 6.9, we take that $\bar{M}$ is of non-positive Ricci curvature and of infinite volume, then we also have the same conclusion.

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