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A Convergence Condition for Stochastic Processes Associated with Lévy Generating Operators

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§1. Introduction

In our previous papers [3], [4], we studied certain convergence conditions for stochastic processes associated with time-homogeneous Lévy generating operators. In this note, we consider such a condition in time-inhomogeneous cases; besides, we make a detailed explanation and a slight improvement to the results in [3], [4].

By a d-dimensional Lévy generating operator, say shortly *Lévy operator*, we mean the following type integro-differential operator:

$$Lf(s, x) \equiv L(s)f(s, x)$$

$$= \frac{1}{2}\sum_{j,k=1}^{d} a^{jk}(s, x)\partial_j\partial_k f(s, x) + \sum_{j=1}^{d} b^j(s, x)\partial_j f(s, x)$$

$$+ \int_{R^d\setminus\{0\}}\{f(s, x+y) - f(s, x) - \frac{1}{1+|y|^2}\sum_{j=1}^{d}\partial_j f(s, x)\, y_j\}\nu(s, x\,; dy)$$

($s \geqq 0$, $x \in R^d$), where $\partial_j = \partial/\partial x_j$, $a = (a^{jk}(s, x))$ is a nonnegative definite symmetric $d\times d$-matrix, $b = (b^j(s, x))$ is a d-vector, and $\nu = \nu(s, x\,; dy)$ is a σ-finite measure on $R^d\setminus\{0\}$ for each $s \geqq 0$ and $x \in R^d$ such that

$$\int_{R^d\setminus\{0\}} |y|^2/(1+|y|^2)\ \nu(s, x\,; dy) < \infty \qquad (s \geqq 0,\ x \in R^d).$$

The data a, b, and ν of the operator L are called the diffusion matrix, the drift vector, and the Lévy measure of L, respectively. Then L is said to be the Lévy operator made of data $[a, b, \nu]$. In particular, the Lévy operator made of constant data defines the infinitesimal generator of a d-dimensional Lévy process.

The convergence conditions are described in terms of the data of Lévy operators, and they ensure the weak convergence of corresponding processes under a certain uniqueness condition on the limit operator. We will treat the weak convergence of stochastic processes associated with Lévy operators in the scheme of the martingale formulation

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introduced by Stroock and Varadhan (i.e. the *martingale problem*).

Such convergence conditions for Lévy operators were first studied by Skorokhod [7], and recently by Jacod-Shiryaev [2], Chap. IX, §4a and by the authors. Their conditions somewhat differ from each other. Our condition is given in the form of a natural extension of the convergence condition for infinitely divisible distributions. Those convergence conditions are expected to have several applications. For example, in [5], we used it to show the uniqueness of solutions to the martingale problem for certain Lévy operators. There, we have to show the convergence of the semigroups associated with approximate Lévy operators of the original one. This is done by combining an analytical method with a convergence property of stochastic processes associated with the approximate operators.

§2. Notations and preliminaries

First we introduce some notations. For the d-dimensional Euclidean space R^d, we set $Q=[0, \infty)\times R^d$ and further, for T, $R>0$, $Q_{T;R}=[0, T]\times\{|x|\leqq R\}$, where $\{|x|\leqq R\}=\{x\in R^d : |x|\leqq R\}$. Then we denote the supremum norms of functions defined on R^d and $Q_{T;R}$ by $\|\cdot\|$ and $\|\cdot\|_{T;R}$, respectively. When G is any subspace of a Euclidean space, we denote by $C_b(G)$ the space of bounded continuous functions on G. For $n=1, 2,\cdots, \infty$, $C_b^n(R^d)$ stands for the subspace of $C_b(R^d)$ consisting of those functions which have bounded continuous derivatives up to order n; $C_0^n(R^d)$ stands for the subspace of $C_b^n(R^d)$ consisting of those functions whose derivatives up to order n vanish at infinity; $C_c^n(R^d)$ stands for the subspace of $C_b^n(R^d)$ consisting of those functions which have compact support. $C_b^{1,2}(Q)$ denotes the subspace of $C_b(Q)$ consisting of those functions which have bounded continuous time derivatives of the first order and bounded continuous spatial derivatives up to order 2.

Next, following Stroock [8], we state the martingale problem for Lévy operators. Let Ω be the Skorokhod space (with the Skorokhod topology) of R^d-valued càdlàg functions on $[0, \infty)$. Given $\omega\in\Omega$, we denote by $x(t, \omega)$ the position of ω at time t. Let $\mathscr{M}_t^s$ and $\mathscr{M}^s$ be the σ-algebras of subsets of Ω generated by $\{x(u) : s\leqq u\leqq t\}$ and $\{x(u) : s\leqq u\}$, respectively. A probability measure P on $(\Omega, \mathscr{M}^s)$ is called a *solution to the martingale problem* for a Lévy operator L starting at $(s, x)\in Q$, if $P[x(u)=x,\ 0\leqq u\leqq s]=1$ and for every $f\in C_b^{1,2}(Q)$

$$M_f(t)\equiv f(t, x(t))-f(s, x(s))-\int_s^t\{\partial/\partial u+L(u)\}f(u, x(u))du$$

is a P-martingale with respect to the filtration $\{\mathscr{M}_t^s\}$. It is known that a probability measure P is a solution of the martingale problem for L if and only if $M_f(t)$ is a P-martingale with respect to the filtration $\{\mathscr{M}_t^s\}$ for every $f\in C_c^\infty(R^d)$ (see [8], Theorem (1.1)). For $f\in C_b^2(R^d)$, $M_f(t)$ is given in the form:

$$M_f(t)=f(x(t))-f(x(s))-L_f(t),$$

where

$$L_f(t)=\int_s^t L(u)f(x(u))du.$$

We denote by $\langle M_f \rangle$ the predictable quadratic variation process for the martingale M_f.

For the Lēvy measure ν of L, we set

$$\tilde{\nu}(s,\ x\ ;\ dy)=\frac{|y|^2}{1+|y|^2}\ \nu(s,\ x\ ;\ dy)$$

and $\tilde{\nu}(s,\ x\ ;\ \{0\})=0$. Then $\tilde{\nu}$ can be regarded as a function from Q to $C_b(R^d)'$, the dual space (with the weak*-topology) of the Banach space $C_b(R^d)$ (with the norn$\|\cdot\|$). Hence we define continuity and boundedness of $\tilde{\nu}$ by the corresponding properties of the function : $(s,\ x)\in Q\longrightarrow\tilde{\nu}(s,\ x\ ;\ \cdot)\in C_b(R^d)'$. For example, the continuity of $\tilde{\nu}$ means that $\int f(y)\tilde{\nu}(s,\ x\ ;\ dy)$ is continuous for every $f\in C_b(R^d)$.

In the following, for simplicity, we assume that the starting time s is equal to 0.

PROPOSITION 2.1.(cf. [3], Prop. 2.1) *Assume that a, b, and $\tilde{\nu}$ of L are bounded. Then, for any solution P to the martingale problem for L, we have*

(i) *for each* $f\in C_0^2(R^d)$

$$\langle M_f\rangle(t)=L_{f^2}(t)-2\int_0^t f(x(u))dL_f(u)\quad (P\text{-}a.s.),$$

(ii) $P[x(t)\neq x(t-)]=0$ *for each* $t\geqq 0$.

PROOF. (i) By the definition of $M_f(t)$ and $M_{f^2}(t)$,

$$\{M_f(t)\}^2=L_{f^2}(t)-\{L_f(t)\}^2-2f(x(0))\ L_f(t)-2M_f(t)\ L_f(t)+M_{f^2}(t)-2f(x(0))\ M_f(t).$$

From [1], p. 343, Theorem 18, it follows that

$$M_f(t)\ L_f(t)=\int_0^t M_f(u)dL_f(u)+\int_0^t L_f(u-)dM_f(u),$$

where $\int_0^t L_f(u-)dM_f(u)=\int_0^t L_f(u)dM_f(u)$ is a stochastic integral with respect to the martingale $M_f(t)$. Using the above equality and the equality $2\int_0^t L_f(u)dL_f(u)=\{L_f(t)\}^2$, we have

$$\{M_f(t)\}^2=L_{f^2}(t)-2\int_0^t f(x(u))dL_f(u)+M_{f^2}(t)-2f(x(0))M_f(t)-2\int_0^t L_f(u)dM_f(u).$$

This implies the conclusion, because $L_{f^2}(t)-2\int_0^t f(x(u))dL_f(u)$ is a continuous process with paths of finite variation and $M_{f^2}(t)-2f(x(0))M_f(t)-2\int_0^t L_f(u)dM_f(u)$ is a P-martingale.
(ii) Since $E^P[\{M_f(t)-M_f(s)\}^2]=E^P[\langle M_f\rangle(t)-\langle M_f\rangle(s)]$ $(t>s)$, $P[M_f(t)\neq M_f(t-)]=0$. Hence, using the fact that $C_0^2(R^d)$ separates points of R^d, we have the conclusion.

REMARK. By Proposition 2.1 (ii), we see that if $P_n\longrightarrow P$ weakly as $n\to\infty$ and P is a solution to the martingale problem for L, then any finite dimensional distribution of P_n converges to that of P.

§3. The convergence condition

In this section, L and L_n ($n=1, 2, \cdots$) denote the Lévy operators made of data $[a, b, \nu]$ and $[a_n, b_n, \nu_n]$ ($n=1, 2, \cdots$), respectively. Denote by $\sigma\equiv\sigma(s, x ; \theta)$ the symbol of L (as a pseudo-differential operator):

$$\sigma(s, x ; \theta)=e^{-ix\cdot\theta}L(s)e^{ix\cdot\theta}$$

$$=-\frac{1}{2}\sum_{j,k=1}^{d} a^{jk}(s, x)\theta_j\theta_k+i\sum_{j=1}^{d} b^j(s, x)\theta_j$$

$$+\int_{R^d\setminus\{0\}} \{e^{i\theta\cdot y}-1-\frac{i\theta\cdot y}{1+|y|^2}\}\nu(s, x ;dy),$$

where $i=\sqrt{-1}$, $\theta=(\theta_1, \theta_2,\cdots, \theta_d)$, and the symbol $\cdot$ means the inner product in R^d. For each $(s, x)\in R^d$, $\sigma(s, x ; \theta)$ defines the logarithmic characteristic function of the infinitely divisible distribution having $a(s, x)$, $b(s, x)$, and $\nu(s, x ; dy)$ as its diffusion matrix, drift vector, and Lévy measure, respectively. Therefore the data of L are uniquely determined. Furthermore, if $L_nf(s, x) \longrightarrow Lf(s, x)$ as $n\to\infty$ for every $(s, x)\in Q$ and $f\in C_b^2(R^d)$, then it holds that for each $(s, x)\in Q$

(i) $$\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty} \{\int_{|y|<\varepsilon} y_jy_k\nu_n(s, x ; dy)+a_n^{jk}(s, x)\}$$

$$=\lim_{\varepsilon\downarrow 0}\liminf_{n\to\infty} \{\int_{|y|<\varepsilon} y_jy_k\nu_n(s, x ; dy)+a_n^{jk}(s, x)\}$$

$$=a^{jk}(s, x) \qquad (j, k=1, 2,\cdots, d);$$

(ii) $$\lim_{n\to\infty} | b_n^j(s, x)-b^j(s, x)|=0 \qquad (j=1, 2, \cdots, d);$$

(iii) $$\lim_{n\to\infty}\int_{R^d} g(y)\nu_n(s, x ; dy)=\int_{R^d} g(y)\nu(s, x ; dy)$$

for every bounded continuous function g vanishing in some neighborhood of 0 (cf. [6]).

Note that (i) and (ii) are equivalent to the following (i′) and (iii′), respectively:

(i′) $$\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}|\int_{|y|<\varepsilon} y_jy_k\nu_n(s, x ; dy)+a_n^{jk}(s, x)-a^{jk}(s, x)|=0$$

$$(j, k=1, 2, \cdots, d);$$

(iii′) there exists a sequence $\{\varepsilon_k\}$ such that $\varepsilon_k\downarrow 0$ and, for every $g\in C_b(R^d)$,

$$\lim_{n\to\infty}\int_{|y|\geqq\varepsilon_k} g(y)\nu_n(s, x; dy)=\int_{|y|\geqq\varepsilon_k} g(y)\nu(s, x; dy),$$

because the equivalence of (i) and (i′) follows from a general property of sequences and that of (iii) and (iii′) is a known result on the weak convergence (cf.[6]).

Motivated by the above fact, we introduce the following convergence condition:

(C) (1) $\{[a_n, b_n, \tilde{\nu}_n]\}_{n=1}^{\infty}$ is uniformly bounded and $[a, b, \tilde{\nu}]$ are bounded.

(2) For every $T, R>0$,

(i) $\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}\|\int_{|y|<\varepsilon} y_j y_k \nu_n(\cdot, \cdot; dy)+a_n^{jk}-a^{jk}\|_{T;R}=0$ $(j, k=1, 2, \cdots, d)$;

(ii) $\lim_{n\to\infty}\|b_n^j-b^j\|_{T;R}=0$ $(j=1, 2, \cdots, d)$;

(iii) there exists a sequence $\{\varepsilon_k\}$ such that $\varepsilon_k\downarrow 0$ and, for every $g\in C_b(R^d)$,

$$\lim_{n\to\infty}\|\int_{|y|\geqq\varepsilon_k} g(y)\{\nu_n(\cdot, \cdot; dy)-\nu(\cdot, \cdot; dy)\}\|_{T;R}=0.$$

REMARK. In the above condition, if we replace $\|\cdot\|_{T;R}$ by the supremum norm over Q, then the uniform boundedness of $\{[a_n, b_n, \tilde{\nu}_n]\}_{n=1}^{\infty}$ follows from the boundedness of $[a, b, \tilde{\nu}]$.

In the case where the data $[a, b, \tilde{\nu}]$ of L are bounded continuous, Stroock [8] proved the existence of solutions to the martingale problem for L, and moreover we see that if $f\in C_b^2(R^d)$ (resp. $f\in C_0^2(R^d)$), then $Lf\in C_b(R^d)$ (resp. $Lf\in C_0(R^d)$).

THEOREM 3.1. *Assume that the data* $[a_n, b_n, \tilde{\nu}_n]$ *of* L_n $(n=1, 2, \cdots)$ *are bounded and that the data* $[a, b, \tilde{\nu}]$ *of* L *are bounded continuous. Then, under the convergence condition* (C), *it holds that, for* $f\in C_b^2(R^d)$,

$$L_n f(s, x)\longrightarrow Lf(s, x) \quad \text{as } n\to\infty$$

locally uniformly and boundedly in $(s, x)\in Q$.

To prove the theorem, we need the following lemma.

LEMMA 3.2. *Under the same assumption as in Theorem* 3.1, *it holds that if* ε *belongs to the sequence* $\{\varepsilon_k\}$ *in the condition* 2 (iii) *of* (C), *then*

$$\lim_{n\to\infty}\|\int_{|y|\geqq\varepsilon} f(\cdot+y)\{\nu_n(\cdot, \cdot; dy)-\nu(\cdot, \cdot; dy)\}\|_{T;R}=0$$

for every $f\in C_b(R^d)$.

PROOF. Let

$$\varphi_l(y)=\begin{cases}0 & (|y|\leqq l \\ |y|-l & (l\leqq|y|\leqq l+1) \\ 1 & (|y|\geqq l+1).\end{cases}$$

Since $\int\varphi_l(y)\tilde{\nu}(s,\ x\,;dy)$ is bounded continuous and $\int\varphi_l(y)\tilde{\nu}(s,\ x\,;dy)\downarrow 0$ as $l\to\infty$, using Dini's theorem, we see that

$$\lim_{l\to\infty}\|\int\varphi_l(y)\tilde{\nu}(\cdot,\ \cdot\,;dy)\|_{T;R}=0.$$

This fact implies that

$$\lim_{l\to\infty}\|\int_{|y|>l}\nu(\cdot,\ \cdot\,;dy)\|_{T;R}=0\,;$$

hence, by using 2 (iii) of (C), we see that for any $\eta>0$ there exists a positive number l_0 such that

$$\|\int_{|y|>l_0}\nu(\cdot,\ \cdot\,;dy)\|_{T;R}<\eta$$

and

$$\limsup_{n\to\infty}\|\int_{|y|>l_0}\nu_n(\cdot,\ \cdot\,;dy)\|_{T;R}<2\eta.$$

Next we divide the integral into two parts:

$$\int_{|y|\geqq\varepsilon}f(x+y)\{\nu_n(s,\ x\,;dy)-\nu(s,\ x\,;dy)\}$$

$$=\int_{|y|\geqq\varepsilon}f(x+y)\{1-\varphi_{l_0}(y)\}\{\nu_n(s,\ x\,;dy)-\nu(s,\ x\,;dy)\}$$

$$+\int_{|y|\geqq\varepsilon}f(x+y)\varphi_{l_0}(y)\{\nu_n(s,\ x\,;dy)-\nu(s,\ x\,;dy)\}.$$

The function $f(x+y)$ on $\{|x|\leqq R\}\times\{\varepsilon\leqq|y|\leqq l_0+1\}$ can be uniformly approximated by functions of the form $\sum_{j=1}^{m}g_j(x)h_j(y)$, where $g_j(x)$ and $h_j(y)$ $(j=1,2,\cdots,m)$ are confinuous on $\{|x|\leqq R\}$ and $\{\varepsilon\leqq|y|\leqq l_0+1\}$, respectively. Therefore

$$\|\int_{|y|\geqq\varepsilon}f(\cdot+y)\{\nu_n(\cdot,\ \cdot\,;dy)-\nu(\cdot,\ \cdot\,;dy)\}\|_{T;R}$$

$$\leqq\sum_j\|g_j\|\,\|\int_{|y|\geqq\varepsilon}h_j(y)\{1-\varphi_{l_0}(y)\}\{\nu_n(\cdot,\ \cdot\,;dy)-\nu(\cdot,\ \cdot\,;dy)\}\|_{T;R}$$

$$+\sup_{|x|\leqq R,\varepsilon\leqq|y|\leqq l_0+1}|\{f(x+y)-\sum_{j=1}^{m}g_j(x)h_j(y)\}\frac{1+|y|^2}{|y|^2}|$$

$$\times \{\sup_n \| \tilde{\nu}_n(\cdot, \cdot\,; R^d)\|_{T;R} + \| \tilde{\nu}(\cdot, \cdot\,; R^d)\|_{T;R}\}$$

$$+\|f\|\{\| \nu_n(\cdot, \cdot\,; \{|y|>l_0\})\|_{T;R} + \| \nu(\cdot, \cdot\,; \{|y|>l_0\})\|_{T;R}\}.$$

Consequently, the above estimates yield the conclusion.

Proof of Theorem 3.1. It is enough to show the locally uniform convergence. For simplicity, we write $\partial_j f(x)$ and $\partial_j \partial_k f(x)$ as $f_{,j}(x)$ and $f_{,jk}(x)$, respectively. Then

$$\begin{aligned}
&(L_n - L)f(s, x)\\
&= \frac{1}{2}\sum_{j,k} \{\textstyle\int_{|y|<\varepsilon} y_j y_k \nu_n(s, x\,; dy) + a_n^{jk}(s, x) - a^{jk}(s, x)\} f_{,jk}(x)\\
&+ \sum_j \{b_n^j(s, x) - b^j(s, x)\} f_{,j}(x)\\
&+ \sum_{j,k} \textstyle\int_{|y|<\varepsilon} [\int_0^1 (1-\theta)\{f_{,jk}(x+\theta y) - f_{,jk}(x)\} d\theta]\, y_i y_k \nu_n(s, x\,; dy)\\
&+ \sum_j f_{,j}(x) \textstyle\int_{|y|<\varepsilon} y_j \tilde{\nu}_n(s, x\,; dy)\\
&- \textstyle\int_{|y|<\varepsilon} \{f(x+y) - f(x) - \dfrac{1}{1+|y|^2}\sum_j f_{,j}(x) y_j\} \nu(s, x\,; dy)\\
&+ \textstyle\int_{|y|\geqq\varepsilon} \{f(x+y) - f(x) - \dfrac{1}{1+|y|^2}\sum_j f_{,j}(x) y_j\}\{\nu_n(s, x\,; dy) - \nu(s, x\,; dy)\}\\
&= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V} + \mathrm{VI}.
\end{aligned}$$

Then

$$\|\mathrm{I}\|_{T;R} \leqq \frac{1}{2}\sum_{j,k} \| \textstyle\int_{|y|<\varepsilon} y_j y_k \nu_n(\cdot, \cdot\,; dy) + a_n^{jk} - a^{jk}\|_{T;R} \|f_{,jk}\|,$$

$$\|\mathrm{II}\|_{T;R} \leqq \sum_j \|b_n^j - b^j\|_{T;R} \|f_{,j}\|.$$

Denote the modulus of continuity of a function g on $\{|x|\leqq R\}$ by $\omega_R(g\,; \delta)$:

$$\omega_R(g\,; \delta) = \sup_{|x|\leqq R, |y|\leqq\delta} |g(x+y) - g(x)| \quad (\delta>0).$$

Then

$$\|\mathrm{III}\|_{T;R} \leqq \sum_{j,k} \omega_R(f_{,jk}\,; \varepsilon) \sup_n \| \tilde{\nu}_n(\cdot, \cdot\,; R^d)\|_{T;R}.$$

Moreover

$$||\mathrm{IV}||_{T;R} \leqq \varepsilon \sum_j ||f_{,j}|| \sup_n || \widetilde{\nu}_n(\cdot, \cdot; R^d)||_{T;R}.$$

$$||V||_{T;R} \leqq \{\sum_{j,k} ||f_{,jk}|| + \sum_j ||f_{,j}||\} ||\widetilde{\nu}(\cdot, \cdot; \{|y|<\varepsilon\})||_{T;R}.$$

Finally we get

$$||\mathrm{VI}||_{T;R} \leqq ||\int_{|y|\geqq\varepsilon} f(\cdot+y)\{\nu_n(\cdot, \cdot; dy) - \nu(\cdot, \cdot; dy)\}||_{T;R}$$

$$+ ||f|| \, ||\int_{|y|\geqq\varepsilon} \{\nu_n(\cdot, \cdot; dy) - \nu(\cdot, \cdot; dy)\}||_{T;R}$$

$$+ \sum_j ||f_{,j}|| \, ||\int_{|y|\geqq\varepsilon} \frac{y_j}{1+|y|^2} \{\nu_n(\cdot, \cdot; dy) - \nu(\cdot, \cdot; dy)\}||_{T;R}.$$

Hence, for ε belonging to the sequence $\{\varepsilon_k\}$ of 2 (iii) in (C), using Lemma 3.2, we have

$$\lim_{n\to\infty} ||\mathrm{VI}||_{T;R} = 0.$$

Therefore, using the above estimates and the convergence condition (C), we have the conclusion.

THEOREM 3.3. *Assume that the data* $[a_n, b_n, \widetilde{\nu}_n]$ *of* L_n $(n=1, 2, \cdots)$ *are bounded and that the data* $[a, b, \nu]$ *of* L *are bounded continuous. For* $n=1, 2, \cdots$, *let* P_n *be a solution to the martingale problem for* L_n *starting at* $(s, x(s))$ *such that*

$$\lim_{l\to\infty} \sup_n P_n[|x(s)|>l] = 0.$$

If

$$L_n f(u, x) \longrightarrow Lf(u, x) \quad \text{as } n\to\infty$$

locally uniformly and boundedly in $(u, x) \in Q$ *for every* $f \in C_c^\infty(R^d)$, *then* $\{P_n\}_{n=1}^\infty$ *is tight and any limit point* P_∞ of $\{P_n\}_{n=1}^\infty$ *is a solution to the martingale problem for* L.

The theorem is proved in exactly the same way as in Appendix II (1°) (ii) of [5]; hence the proof is omitted.

Suppose that the uniqueness to the martingale problem holds for L_n $(n=1, 2, \cdots)$ and L, respectively. Then denote by $P_{s,x}^{(n)}$ and $P_{s,x}$ the solutions to the martingale problems for L_n and L starting at $(s, x) \in Q$, respectively. Let $\{T_{s,t}^{(n)}\}$ and $\{T_{s,t}\}$ be the time-inhomogeneous semigroups on $C_b(R^d)$ associated with L_n and L, respectively, that is,

$$T_{s,t}^{(n)} f(x) = E^{P_{s,x}^{(n)}}[f(x(t))],$$

$$T_{s,t} f(x) = E^{P_{s,x}}[f(x(t))],$$

where $f \in C_b(R^d)$, $0 \leqq s \leqq t$, and $x \in R^d$. Then, noting Remark to Proposition 2.1, we have

COROLLARY 3.4. *Under the uniqueness assumption to the martingale problems for* L_n *and* L *and under the assumption of Theorem* 3.3, *it holds that*

$$\lim_{n \to \infty} T_{s,t}^{(n)} f(x) = T_{s,t} f(x)$$

for $0 \leqq s \leqq t$, $f \in C_b(R^d)$, *and* $x \in R^d$.

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