Tempearture Dispersion Relation for Impedance

Hisanobu Ishibashi and Syurei Iwao

Abstract

An impedance for the substance is considered to be a function of both temperature and frequency of the applied alternating current. We have specifically investigated the temperature dependence of reactance at the fixed frequency from the available data.

1. Introduction

The dispersion relation for the impedance with respect to the frequency at a fixed temperature is a famous example on the subject.[1] We shall first present it as a warming -up and discuss the once-subtracted dispersion relation. Unfortunately we do not have the experimental data on this case, so we will point out only the specific properties involved in the subtraction technique.

From the summary of the experimental data[2] on the resistance one finds that it is a function of temperature. Thus it is suitable to consider that the impedance is a simultaneous function of both temperature and frequency. Instead of considering the double dispersion relation for these two parameters we shall investigate the temperature dependence of dispersion relation at the fixed frequency. There is no suggestion on the symmetry properties of resistance and reactance under the temperature reversal, so we shall consider all the possibilities.

Section 2 is devoted to the fixed T dispersion relation, so one can skip this section if he is familiar with the subject. In Section 3, the dispersion relation for T at the fixed frequency is discussed. Its numerical study is presented in Section 4. The last section summarizes our result.

2. Frequency Dispersion Relation

This is a well known dispersion theory on impedance with respect to the frequency. It seems, however, natural to explain shortly an essential aspect of the theory in order to develop that for the temperature.

Department of Physics, College of Liberal Arts, Kanazawa University, Kanazawa 920, Japan.

Let us extend our concept of impedance to that which is a complex function of the complex variable of the angular frequency ω. However, the observed impedance is a function of the real value ω_0, which then appears on the real axis on the contour of upper complex plane for the Cauchy integral representation of the assumed analytic function $Z(\omega)$. In this case one has to take the principal value of the integral, viz.,

$$Z(\omega_0) = \frac{1}{\pi i} P \oint \frac{Z(\omega)}{\omega - \omega_0} \, d\omega, \tag{1}$$

where one decomposes $Z(\omega)$ into

$$Z(\omega) = R(\omega) + i X(\omega). \tag{2}$$

Here R and X are the resistance and reactance, respectively. Since

$$P \int_{-\infty}^{\infty} \frac{1}{\omega - \omega_0} \, d\omega = 0, \tag{3}$$

one can subtract the corresponding null contributions from the real and the imaginary parts of Eq. (1) and find

$$R(\omega_0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{X(\omega) - X(\omega_0)}{\omega - \omega_0} \, d\omega \tag{4}$$

and

$$X(\omega_0) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{R(\omega) - R(\omega_0)}{\omega - \omega_0} \, d\omega. \tag{5}$$

In writing out Eqs. (4) and (5), we assumed Eq. (3) together with the null contribution of the integrand from the upper large semicircle of the contour. This point will be recovered further by taking into account the so-called subtracted dispersion relation (see, later discussion). In addition to this their representations, however, contain the contribution from an unobservable negative frequency effect. This uncomfortable situation can be remedied if we take into account the symmetry properties of the resistance and the reactance under the frequency reversal. As is known from an elementary text book it is natural to assume

$$R(-\omega) = R(\omega) \tag{6}$$

and

$$X(-\omega) = -X(\omega). \tag{7}$$

Substituting these relations into Eqs. (4) and (5) under an elementary rule known in the real integral, one finds immediately as

$$R(\omega_0) = \frac{2}{\pi} \int_0^{\infty} \frac{\omega X(\omega) - \omega_0 X(\omega_0)}{\omega^2 - \omega_0^2} \, d\omega \tag{8}$$

and

$$X(\omega_0) = -\frac{2\omega_0}{\pi}\int_0^\infty \frac{R(\omega)-R(\omega_0)}{\omega^2-\omega_0^2}\,d\omega \tag{9}$$

One sees from these expressions that the convergence property of the integral at infinite frequency is better in the latter than the former, if $X(\omega)$ and $R(\omega)$ take the similar behavior at $\omega=\infty$.

Let us turn our attention to a once subtracted dispersion relation which is obtained from Eq. (8) or (9) by changing the argument from ω_0 to another one, say, ω_1 and taking the difference of two relations. By an algebraic calculation one finds

$$R(\omega_1)-R(\omega_0) = \frac{2}{\pi}\int_0^\infty$$

$$\frac{(\omega_1^2-\omega_0^2)\omega X(\omega)-\omega^2(\omega_1 X(\omega_1)-\omega_0 X(\omega_0))+\omega_0\omega_1(\omega_0 X(\omega_1)-\omega_1 X(\omega_0))}{(\omega^2-\omega_0^2)\,(\omega^2-\omega_1^2)}\,d\omega \tag{10}$$

and

$$X(\omega_1)-X(\omega_0) = -\frac{2}{\pi}\int_0^\infty \left(\frac{(\omega_1-\omega_0)\omega^2 R(\omega)-\omega^2(\omega_1 R(\omega_1)-\omega_0 R(\omega_0))+\omega_0\omega_1(\omega_1-\omega_0)R(\omega)}{(\omega^2-\omega_0^2)\,(\omega^2-\omega_1^2)}\right.$$

$$\left.\frac{+\omega_0\omega_1(\omega_0 R(\omega_1)-\omega_1 R(\omega_0))}{}\right)\,d\omega. \tag{11}$$

Here $R(\omega_0)$ and $X(\omega_0)$ are so-called subtraction constants which are assumed to be known if we wish to estimate $R(\omega_1)$ and $X(\omega_1)$ from the measured quantities appearing in each expression. Comparing Eqs. (8) and (10) one sees that the convergence property of the integrand at infinite frequency becomes better in the latter, while it is not improved in case of reactance as seen by the comparison of Eqs. (9) and (11). One may proceed to work out a relation with further subtraction if necessary.

3. Temperature Dispersion Relation

As we discussed the impedance is considered to be a function of the frequency and the terperature, $Z(\omega, T)$, where T is the absolute terperature. If we consider the fixed frequency dispersion relation the derivation of the terperature dispersion relation proceeds completely parallel to the one discussed in the previous section, viz., one finds

$$Z(T_0) = \frac{1}{\pi i}\,P\oint \frac{Z(T)}{T-T_0}\,dT. \tag{12}$$

Here one should take again the principal value of the Cauchy integral on the upper half plane of the complex T plane, owing to the reality of T_0 on the contour assumed.

In order to perform our program we have to assume the symmetry properties of the resistance as well as the reactance under the temperature reversal operation. We do not know apriori which symmetry we have to choose unless the time and temperature corre-

spondence holds in our example, so we shall consider the following four possibilities.

(i) $R(-T) = R(T), \quad X(-T) = -X(T),$ (13)

(ii) $R(-T) = R(T), \quad X(-T) = X(T),$ (14)

(iii) $R(-T) = -R(T), X(-T) = X(T)$ (15)

and

(iv) $R(-T) = -R(T), X(-T) = -X(T).$ (16)

Here the dispersion relations for the first case are written down immediately by replacing simply ω by T in Eqs. (8) and (9). For the sake of completeness we shall spell out all of them explicitly.

They are summarized as follows.

Case (i)

$$R(T_0) = \frac{2}{\pi}\int_0^\infty \frac{TX(T) - T_0 X(T_0)}{T^2 - T_0^2}\, dT,$$ (17a)

$$X(T_0) = -\frac{2T_0}{\pi}\int_0^\infty \frac{R(T) - R(T_0)}{T^2 - T_0^2}\, dT.$$ (17b)

Case (ii)

$$R(T_0) = \frac{2T_0}{\pi}\int_0^\infty \frac{X(T) - X(T_0)}{T^2 - T_0^2}\, dT,$$ (18a)

$$X(T_0) = -\frac{2T_0}{\pi}\int_0^\infty \frac{R(T) - R(T_0)}{T^2 - T_0^2}\, dT.$$ (18b)

Case (iii)

$$R(T_0) = \text{the same as Eq. (18a)},$$ (19a)

$$X(T_0) = -\frac{2}{\pi}\int_0^\infty \frac{TR(T) - T_0 R(T_0)}{T^2 - T_0^2}\, dT.$$ (19b)

Case (iv)

$$R(T_0) = \text{the same as Eq. (17a)},$$ (20a)

$$X(T_0) = \text{the same as Eq. (19b)}.$$ (20b)

4. Temperature dependence of Reactance

The terperature dependence of resistance for 53 metals is listed in the available table[2] for us at this moment. However, they are measured only at 6 temperature points in the range $T = 78$ to $1473°K$(otherwise specified explicitly), they are not enough to test the theoretical expressions obtained in the previous section. One should be satisfied by

getting a qualitative feature found by our fit and by finding how to apply theory to the experimental study. There are two possibilities for the reactance Eqs. (17b) and (19b). In order to apply them one has to replace the integral by a sum. In practice we shall replace the lower and upper limit of the integral by the observed minimum and maximum of the data. If we make these approximations we find the modified expressions for reactance as

$$X'(T_0) = -\frac{2T_0}{\pi} \sum_i \frac{R(T_i)-R(T_0)}{T_i^2-T_0^2} (T_{i+1}-T_i) \qquad (21)$$

and

$$X'(T_0) = -\frac{2}{\pi} \sum_i \frac{T_i R(T_i)-T_0 R(T_0)}{T_i^2-T_0^2} (T_{i+1}-T_i), \qquad (22)$$

for Eqs. (17b) and (19b), respectively. We shall compare these expressions numerically at 6 terperature points. In order to perform our program a literal use of Eqs. (21) and (22) by computer study leads a trouble, because the computer does not care the cancellation of null in the numerator and the denominator in the sum. What T_i and T_{i+1} should we choose when we want to estimate $X'(T_{min})$ and $X'(T_{max})$, respectively? How about $X'(T)$ at an intermediate T_i? Some of these questions may be avoided if we have enough data points.

In order to avoid first questions we added fictitious data with negligible contributions just below and just above T_{min} and T_{max}, respectively. We shall be satisfied by the straight line approximation as an interpolating value in our crude estimate, so only five terperature intervals appear in the sum. If we shift T_0 by a small amount, say, 0.1°K at the interest of study for $X'(T_0)$, the second trouble can also be avoided by the same account, in a good approximation, for the present purpose.

In this study we have chosen three metals Al, Cu and Li as our examples. The experimental data for them are tabulated in Table 1.

Table 1. Temperature dependence of resistance for Al, Cu and Li.

$R \times 10^8$ (Ω)

T(°K)	78	273	373	573	973	1473
Al	0.21	2.50	3.55	5.9	24.7	32.1
Cu	0.2	1.55	2.23	3.6	6.7	21.3(1356)
Li	1.04	8.55	12.4	30	40.5	53

As seen from the table the resistance increases monotonically.
The same tendency applies to all other metals except for Zr.
Our estimated results are tabulated in Table 2.

Hisanobu ISHIBASHI and Syurei IWAO

Table 2. Estimated reactance $X'(T_0) \times 10^8$ (Ω)

T_0	78	273	373	573	973	1473
Al (21)	-1.41	-4.84	-5.84	-8.20	-15.5	-14.8
(22)	-12.8	-14.0	-14.2	-15.3	-21.0	-19.1
						(1356)
Cu (21)	$-.591$	-1.99	-2.15	-2.27	-3.13	-14.3
(22)	-4.62	-5.07	-4.92	-4.58	-4.87	-15.7
Li (21)	-4.23	-14.9	-18.3	-19.4	-15.9	-19.6
(22)	-34.1	-38.0	-39.2	-3.69	-29.1	-29.9

In the first column of the table the applied formula's number is indicated in addition to the specification of metal. In general Eq. (22) gives the bigger value of $X'(T_0)$ than that for Eq. (21). Our estimate is very approximate in nature, however, a more detailed study from the refined data may possibly discriminate which symmetry from (i) − (iv) should be preferred by nature.

5. Summary and Conclusion

We have developed the temperature dispersion relation for the impedance of metals in analogous manner to the known frequency one. A crude numerical estimate of the reactances of the three metals in given from the measured resistances at limited number of temperatures for them. A general tendency obtained from our study indicates a possibility to find out which symmetry from four possibilities suggested under the temperature reversal should be realized in nature.

We hope that this little work will induce the interest of the reader and put forward it to complete check of the symmetry problem under discussion.

References

[1] P.M. Morse and H. Feshbach, Methods of Theoretical Physics, Part I (McGRAW-Hill, 1953).
[2] Ed. National Observatory of Astronomy, Tables of Annual Scince, in Japanese (Nov., 1990).