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A FORMULA FOR THE RADIAL PART OF THE LAPLACE -BELTRAMI OPERATOR ON THE RIEMANNIAN FOLIATION

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1. Introduction

Let (M, g) be a $(p+q)$-dimensional riemannian manifold and H a compact subgroup of the Lie group of all isometries of M. We suppose that all orbits of H have the same dimension p (codimension q). Then H defienes a riemannian foliation $\mathcal{F}$ of codimension q. The leaf space $M/\mathcal{F}=:N$ is a riemannian Satake manifold. Let $\Box$ be the Laplace -Beltrami operator on M and $\Delta(\Box)$ its radial part in the sense of S. Helgason ([H]). We shall prove.

Theorem. *Let $\Box$ be the Laplace-Beltrami operator on M and Δ_N the Laplace-Beltrami operator on N associated with the riemannian metric defined by transversal component of the metric on M. If the mean curvature vector field H of $\mathcal{F}$ is an infinitesimal automorphism, then the radial part $\Delta(\Box-H)$ is given by*

$$\Delta(\Box-H)=\delta^{-1/2}\Delta_N\cdot\delta^{1/2}-\delta^{-1/2}\Delta_N(\delta^{1/2}),$$

where δ is the function given by (3.5).

H. Kitahara and S. Yorozu ([KY]) proved the similar theorem by means of the second connection defined by I. Vaisman ([V]). Moreover, in this paper we shall study some formulae of Laplace-Beltrami operators, which give relations between Ph. Tonduer's formulae ([T]) and I. Vaisman-H. Kitahara's ones ([V]), [K]).

Finally, we shall give a simple proof about eigenvalues of Y. Muto ([M]).

2. The basic Laplacian on $\Omega_B{}^*$

Let $(M, \mathcal{F}, g)$ be a $(p+q)$-dimensional manifold with a riemannian foliation $\mathcal{F}$ of codimension $q=n-p$. The foliation $\mathcal{F}$ defines the integrable subbundle L of TM. The normal bundle Q of fiber-dimension q is the quotient bundle $Q:=TM/L$. Equivalently, Q Appears in the exact sequence of vector bundles;

(2.1) $\quad 0\rightarrow L\rightarrow TM\underset{\sigma}{\overset{\pi}{\rightleftarrows}}Q\rightarrow 0.$

By means of the riemannian metric g, TM splits orthogonally as

$$TM=L\oplus L^{\perp}\quad,$$

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with $\sigma: Q \simeq L^{\perp}$ splitting (2.1). The metric g on TM is a direct sum;

$$g = g_L \oplus g_{L^{\perp}} .$$

With $g_Q = \sigma^* g_{L^{\perp}}$, the splitting map $\sigma : (Q, g_Q) \to (L^{\perp}, g_{L^{\perp}})$ is isometric. Let$(U, (x^i, x^\alpha))$ be a distinguished coordinate chart, i. e., x^α=constants defines $\mathcal{F}$ locally. Hereafter we use indices as the following ranges; $1 \leqq i, j, k, \cdots \leqq p$; $1 \leqq \alpha, \beta, \gamma, \cdots \leqq q$. Let (X_i, X_α) be the basic adapted framing to $\mathcal{F}$ and $(\omega^i, \theta^\alpha)$ its dual framing. Recall that X_i is tangent to the leaves of $\mathcal{F}$ and $g(X_i, X_\alpha)=0$. We set $g_{ij} := g(X_i, X_j)$ and $g_{\alpha\beta} := g(X_\alpha, X_\beta)$. Then the metric g is locally written as

$$g \mid_U = \Sigma_{ij} g_{ij}(x^i, x^\alpha) \omega^i \otimes \omega^j + \Sigma_{\alpha\beta} g_{\alpha\beta}(x^\gamma) \theta^\alpha \otimes \theta^\beta.$$

Let Ω_M^t be the space of t-forms on M. $\varphi \in \Omega_M^t$ is a (s, r)-from if φ is written as locally

$$\varphi \mid_U = \frac{1}{s!r!} \Sigma \varphi_{i_1 \cdots i_s \alpha_1 \cdots \alpha_r}(x^k, x^\gamma) \omega^{i_1} \wedge \cdots \wedge \omega^{i_s} \wedge \theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_r}.$$

Then we have the decomposition of forms;

$$\Omega_M^t = \Sigma_{s+r=t} \Omega_M^{s,r},$$

where $\Omega_M^{s,r}$ is the space of (s, r)-forms on M. Let $\pi_{s,r}: \Omega_M^t \to \Omega_M^{s,r}$ be a projection operator for each s and r. The above decomposition induces the decomposition of the exterior differential operator d and its formal adjoint δ;

(2.2) $\quad d = d' + d'' + d'''$ and $\delta = \delta' + \delta'' + \delta'''$.

Note that $d'': \Omega_M^{s,r} \to \Omega_M^{s,r+1}$ and $\delta'' := \pm * d''$ $*$ is the Hodge star operator with respect to g. An operator $\square := \delta d + d\delta$ acting on Ω_M^* is the Laplace-Beltrami operator. Moreover we can define two operators $\square' := \delta' d' + d' \delta'$ and $\square'' := \delta'' d'' + d'' \delta''$. A form $\omega \in \Omega_M^t$ is a basic form on M if $d'\omega=0$, equivalently, $i(X)\omega \doteq 0$ and $L_X \omega = 0$ for all vector fields X tangent to the leaves, where $i(\ )$ and $L_{(\ )}$ are the interior product and the Lie derivative respectively.

Let $\Omega_B{}^*$ be the space of the basic forms on M. Let a form $\nu \in \Omega_B^q$ be the transversal (closed) volume form associated to the transversal holonomy invariant riemannian metric $ds_Q^2 := \Sigma\ g_{\alpha\beta} \theta^\alpha \otimes \theta^\beta$. Then the characteristic form $\chi_{\mathcal{F}}$ of $\mathcal{F}$ is the p-form defined by $\chi_{\mathcal{F}} := * \nu$. Thus the riemannian volume form μ is given by $\nu \wedge \chi_{\mathcal{F}}$. Then we have the Rummler's Formula ;

(2.3) $\quad d\chi_{\mathcal{F}} + \kappa \wedge \chi_{\mathcal{F}} = d''' \chi_{\mathcal{F}},$

where κ is the dual form of the mean curvature vector field H of $\mathcal{F}$. Suppose that $\kappa \in \Omega_B^1$. Then we have $d\kappa = 0$.

Let $\bar{*} : \Omega_B^r \to \Omega_B^{q-r}$ be the star operator associated to ds_Q^2. Then we have

(2.4) $\quad \bar{*} \alpha = (-1)^{p(q-r)} * (\alpha \wedge \chi_{\mathcal{F}}),\ \alpha \in \Omega_B^r$

(2.5) $\quad * \alpha = \bar{*}\alpha \wedge \chi_{\mathcal{G}},\ \alpha \in \Omega_B^r.$

We set $d_B := d \mid \Omega_B^* = d''$. For $\alpha \in \Omega_B^r$,

$$\delta'' \alpha = (-1)^{n(r+1)+1} * d'' * \alpha$$

$$=(-1)^{n(r+1)+1} * (d''(\bar{*}\alpha)\wedge\chi_{\mathcal{G}}) + (-1)^{n(r+1)+1} * ((-1)^{q-r}\bar{*}\alpha\wedge d''\chi_{\mathcal{G}})$$

The first term in the $RHS=(-1)^{n(r+1)+1}(-1)^{p(r-1)}\bar{*}d''\bar{*}\alpha$ $=(-1)^{q(r+1)+1}\bar{*}d''\bar{*}\alpha$ and the second term in the $RHS=(-1)^{n(r+1)+1}(-1)^{q-r} * (\bar{*}\alpha\wedge d''\chi_{\mathcal{G}})= i(H)\alpha$.

Then we have, for $\alpha \in \Omega_B^r$,

(2.6) $\quad \delta''\alpha = (-1)^{q(r+1)+1}\bar{*}d''\bar{*}\alpha + i(H)\alpha.$

Moreover, note that for $\alpha \in \Omega_B^r$, we have

$$d'(\bar{*}\alpha)\wedge\chi_{\mathcal{G}} = (\pi_{(1,0)}d(\bar{*}\alpha))\wedge\chi_{\mathcal{G}} \in \Omega_M^{p+1,q-r} = \{0\},$$

$$d''(\bar{*}\alpha)\wedge\chi_{\mathcal{G}} = (\pi_{(0,1)}d(\bar{*}\alpha))\wedge\chi_{\mathcal{G}} \in \Omega_M^{p,q-r+1},$$

$$d'''(\bar{*}\alpha)\wedge\chi_{\mathcal{G}} = (\pi_{(-1,2)}d(\bar{*}\alpha))\wedge\chi_{\mathcal{G}} = \{0\},$$

and

$$(-1)^{n(r+1)+1}(-1)^{q-r} * (\bar{*}\alpha\wedge d'''\chi_{\mathcal{G}})$$
$$= \quad * (-1)^{p(r+1)+qr+1-r}(\bar{*}\alpha\wedge d'''\chi_{\mathcal{G}}) = *\gamma(\alpha),$$

where $\gamma(\alpha) := (-1)^{(p+1)(r+1)+qr}\bar{*}\alpha\wedge d'''\alpha$. Then we have, for $\alpha \in \Omega_B^r$,

(2.7)
$$\begin{aligned} \delta\alpha &= (-1)^{n(r+1)+1} * d * \alpha \\ &= (-1)^{n(r+1)+1} * d(\bar{*}\alpha\wedge\chi_{\mathcal{G}}) \\ &= (-1)^{n(r+1)+1} * (d(\bar{*}\alpha)\wedge\chi_{\mathcal{G}} + (-1)^{q-r}\bar{*}\alpha\wedge d\chi_{\mathcal{G}}) \\ &= \delta''\alpha + *\gamma(\alpha), \end{aligned}$$

On the other hand, we have, for $\alpha \in \Omega_M^{0,r}$,

(2.8) $\quad \delta\alpha = \delta''\alpha + \delta'''\alpha \quad$ (note that $\delta'\alpha = 0$).

Therefore, we have

(2.9) $\quad \delta''\alpha = *\gamma(\alpha) \quad$ for $\alpha \in \Omega_B^r$.

Suppose that H is an infinitesimal automorphism of $\mathcal{G}$, equivalently, $\varkappa$ is a basic one form. Then we have

(2.10) $\quad \delta''\alpha = \delta_B\alpha \quad$ for $\alpha \in \Omega_B^*$

where $\delta_B\alpha := (-1)^{q(r+1)+1}\bar{*}(d_B - \varkappa\wedge)\bar{*}\alpha$. Thus we have for $\alpha \in \Omega_B^r$, (2.11) $\square\alpha = \square_B\alpha + \eta(\alpha)$,

where

(2.12) $\square_B := \delta_B d_B + d_B\delta_B$,

(2.13) $\eta(\alpha) := *\gamma(d''\alpha) + d*\gamma(\alpha)$.

On the other hand, we have (cf. [TY])

(2.14) $\square\alpha = \square''_0\alpha + \pi_{0,r}\cdot L_H\alpha + \eta(\alpha)$,

where

(2.15) $\square''_0 := \delta_\chi d + d\delta_\chi$,

(2.16) $\delta_\chi := (-1)^{q(r+1)+1}\bar{*}\, d''\,\bar{*}$.

Then we have, for $\alpha \in \Omega_B^r$,

(2.17) $\square_B\alpha = \square''_0\alpha + \pi_{0,r}\cdot L_H\alpha$.

Fact 2.1. $\square''_0 = \pi_U{}^*\Delta_N$ *where* $\pi_U : U \to N$ *is a local riemannian submersion defining* $\mathcal{F}$ *and* Δ_N *is the Laplace-Beltrami operator on* N.

Fact 2.2([TY]). Let $(M, \mathcal{F}, g)$ be a $(p+q)$-dimensional riemannian manifold with a riemannian foliation $\mathcal{F}$ of codimension q. Then the Laplace-Beltrami operator $\square$ acting on Ω_M^0 has a decomposition ;

$$\square = \square' + \square''_0 + H.$$

3. Proof of Theorem

Let M be a $(p+q)$-dimensional riemannian manifold and H compact subgroup of the Lie group of all isometries of M. We suppose that all orbits of H are of the same dimenaion p(codimension q). Then H defines a riemannian foliation $\mathcal{F}$ of codimension q whose each compact leaf is H/H_m, where H_m is the isotropy group at $m \in M$.

Key Lemma ([Mo]). *Let* $(M, \mathcal{F}, g)$ *be a* $(p+q)$*-dimensional riemannian manifold with a riemannian foliation* $\mathcal{F}$ *of codimension q with compact leaves. Then* $M/\mathcal{F} =: N$ *admits a natural structure of a q-dimensional Satake manifold such that the natural projection* π : $M \to N$ *is a morphism of Satake manifolds.*

Let Ω_N^0(resp. $\Omega_{N,o}^0$) be the space of smooth functions (resp. smooth functions with compact support) on N. We may define a map $\Phi : \Omega_B^0 \to \Omega_N^0$ by $\Phi(f)(\pi(m)) := f(\mathrm{m})$ which is injective. Let $\Omega_{M,B}^0 := \Phi^{-1}(\Omega_N^0)$. It is clear that $f \in \Omega_{M,B}^0$ if and only if $f \in \Omega_B^0$.

Lemma. *Suppose that the mean curvature vector field H is an infinitesimal automorphism.*

If $f \in \Omega^0_{M,B}$, *then* $(\Box - H)(f) \in \Omega^0_{M,B}$.

Proof. For $f \in \Omega^0_{M,B}$, we have $(\Box - H)(f) = (\Box' + \Box''_0)(f) = \Box''_0 f \in \Omega^0_{M,B}$.

By means of the riemannian metric $g := g_{ij}(x,y)\omega^i \otimes \omega^j + g_{\alpha\beta}(y)\theta^\alpha \otimes \theta^\beta$, the volume element dvol_M of M is given by

(3.2) $\mathrm{dvol}_M = [\det\begin{pmatrix} g_{ij} & 0 \\ 0 & g_{\alpha\beta}\end{pmatrix}]^{1/2}\omega^1 \wedge \cdots \wedge \omega^p \wedge \theta^1 \wedge \cdots \wedge \theta^q.$

For a distinguished chart $(U,(x,y))$ and the natural projection $\pi : U \to N$

(3.3) $d\sigma = |\det(g_{\alpha\beta})|^{1/2}\theta^1 \wedge \cdots \wedge \theta^q$

may be regarded as the volume elememt dvol_N of N, since $\{U, \Gamma, \pi(U)\}$ is a local uniformizing system for $\pi(U)$ in N. And $|\det(g_{ij}(x,y))|^{1/2}\omega^1 \wedge \cdots \wedge \omega^p$ is the volume element $d\Sigma_m$ on the leaf $H \cdot m$ through a point $m := (x,y)$. For $f \in \Omega^0_{B,o}$ ($:=$ the space of basic functions on M with compact support), the Fubini Theorem implies that

(3.4) $\int_M f\,\mathrm{dvol}_M = \int_N \left[\underline{\int_{H \cdot m} f\, d\Sigma_m}\right] \mathrm{dvol}_N(\pi(m)),$

where "______" denotes the image under Φ. Since $d\Sigma_m$ is invariant under H, it must be a scalar multiple of $d\mathring{h}$;

$$d\Sigma_m = \bar{\delta}(m)d\mathring{h},$$

where $d\mathring{h}$ is an H-invariant measure on each orbit $H \cdot m$. Then the function $\bar{\delta}$ is in $\Omega^0_{M,B}$. We set

(3.5) $\delta := \Phi(\bar{\delta}).$

Therefore we have

(3.6) $\int_M f\,\mathrm{dvol}_M = \int_N \left[\int_{H \cdot m} f(h \cdot m)d\mathring{h}\right]\delta(\pi(m))\mathrm{dvol}_N(\pi(m)).$

Δ_N is defined by the Levi-Civita connection associated with the normal component ds^2_Q. Then we see that

(3.7) $\Delta(\Box - H) = \Delta_N +$ lower order terms.

It follows from **Fact 2.2** that the operator $\Box - H$ restricted to $\Omega^0_{B,o}$ is symmetric with respect to dvol_M, i.e.,

(3.8) $\int_M [(\Box - H)f_1] f_2\,\mathrm{dvol}_M = \int_M f_1 (\Box - H) f_2\,\mathrm{dvol}_M$

for $f_1, f_2 \in \Omega^0_{B,o}$.

For $f \in \Omega^0_{B,o}$ and $m \in M$, we have

(3.9) $\int_{H \cdot m} f\, d\mathring{h} = \underline{f}(\pi(m))c,$

where $c := \int_{H\cdot m} f d\mathring{h} \neq 0$. Setting $\underline{f_1} := \Phi(f_1)$, $\underline{f_2} := \Phi(f_2)$ for $f_1, f_2 \in \Omega^0_{B,o}$, we have

$$\int_M (\square - H)(f_1) f_2 \, \mathrm{dvol}_M$$
$$= \int_N \left[\int_{H\cdot m} (\square - H)(f_1) f_2 d\mathring{h} \right] \delta \, \mathrm{dvol}_N$$
$$= \int_N \left[\underline{\int_{H\cdot m} (\square - H)(f_1) d\mathring{h}} \right] c\delta \, \underline{f_2} \, \mathrm{dvol}_N$$
$$= c^2 \int_N (\underline{(\square - H)(f_1) f_2}) \delta \, \mathrm{dvol}_N.$$

Then we have

$$\int_N (\underline{(\square - H)(f_1)}) \underline{f_2} \delta \, \mathrm{dvol}_N \tag{3.10}$$
$$= \int_N \underline{f_1} ((\square - H)(\underline{f_2})) \delta \, \mathrm{dvol}_N$$

for $f_1, f_2 \in \Omega^0_{B,o}$. We have, by definition, $\underline{(\square - H)(f)} = \Delta((\square - H))(\underline{f})$ for $f \in \Omega^0_B$, so that

$$\int_N \left[\Delta((\square - H))(\underline{f_1}) \right] \underline{f_2} \delta \, \mathrm{dvol}_N = \int_N \underline{f_1} \left[\Delta((\square - H)(\underline{f_2})) \right] \delta \, \mathrm{dvol}_N.$$

This equality implies that $\Delta((\square - H))$ is symmetric with respect to $\delta \mathrm{dvol}_N$ and it clearly agrees with Δ_N up to lower order terms. The symmetric operators $\Delta((\square - H))$ and $\delta^{-1/2} \Delta_N \cdot \delta^{1/2}$ agree up to an operator of order ≤ 1, so that this operator, being symmetric, must be a function. Applying the operators to the constant function 1, we have

$$\Delta((\square - H))\,(1) - \delta^{-1/2} \Delta_N \cdot \delta^{1/2} = -\delta^{-1/2} \Delta_N (\delta^{1/2}).$$

Therefore, we have

$$\Delta((\square - H)) = \delta^{-1/2} \Delta_N \cdot \delta^{1/2} - \delta^{-1/2} \Delta_N (\delta^{1/2}). \tag{3.11}$$

□

4. Examples

(1) **Let** $M := B \times_f F$ be a warped product of dimension $p+q$. Then we have

$$\square h = \Delta_B h + (q/f)(\mathrm{grad}\, f) h,$$

so that $\Delta(\square - (q/f)(\mathrm{grad}\, f)) = \Delta_B$.

(2) Let $0(n)$ be the orthogonal group acting on (R^n, can). In $R^n \setminus$ {the origin} , we set $f := r :=$ the geodesic distance from x to the origin. Then we have

$$\mathrm{can} = dr^2 + r^2 d\theta^2.$$

Setting $f := r$ in (1), we have

$$\Delta(\square - H) = -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r}.$$

5. The minimal foliations

Let $(M, \mathcal{F}, g)$ be a $(p+q)$-dimensional riemannian manifold with a riemannian foliation $\mathcal{F}$ with compact minimal leaves of codimension q. It follows from (2.14) and **Fact 2.1** that, for a basic 1-form α,

$$(\square - \eta)(\alpha) = \square''_0 \alpha = (\pi^* \Delta_N)\alpha. \tag{5.1}$$

Note that $* \gamma(\alpha)$ is orthogonal to Ω_B^*, then we have

Fact 5.1. *Let $\alpha := \pi^* \alpha_N$ be a basic 1-form pulled back from a 1-form α_N on N. Then $(\square\alpha - \pi^*(\Delta_N \alpha_N))(X) = 0$ if and only if $(L_H \alpha)(X) = 0$ for any basic vector field X.*

Corollary 1 ([M]). *Let $(M, \mathcal{F}, g)$ be as above. Let $\alpha := \pi^* \alpha_N$ be a basic 1-form pulled back from a 1-form α_N. such that $\Delta_N \alpha_N = \lambda \cdot \alpha_N$. Then $(\square\alpha - (\lambda \cdot \pi)\alpha(X) = 0$ for any basic vector field X.*

Corollary 2 ([P)]. *Let $\pi : M \to N$ be a riemannian submersion with minimal fibres. Then the pull-back $\alpha : \pi^* \alpha_N$ of a harmonic 1-form α_N on N is a harmonic 1-form on M if M is compact.*

In fact, it is sufficient to note that $\alpha \in \Omega_B^1$.

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