Shape Optimization Approach to a Free Boundary Problem

Maharani Ahsani Ummi[a,b], Masato Kimura[b],
Hideyuki Azegami[c], Kohji Ohtsuka[d], Armanda Ikhsan[a,b]

[a]Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jl. Ganesha 10, Bandung 40132 Indonesia, E-mail: maharaniahsani@gmail.com, armandaikhsan@gmail.com
[b]Institute of Science and Engineering, Kanazawa University, Kakuma, Kanazawa 920-1192 Japan, E-mail: mkimura@se.kanazawa-u.ac.jp
[c]Graduate School of Information Science, Nagoya University, Furo-cho, Chigusa-ku, Nagoya, 464-8601 Japan, Email: azegami@is.nagoya-u.ac.jp
[d]Department of Computer Science, Hiroshima Gakuin University, Aki-ku, Nakano, Hiroshima,739-0321 Japan, Email: ohtsuka@hgk.ac.jp

Abstract. *We take a shape optimization approach to solve a free boundary problem of the Poisson equation numerically. A numerical method called traction method invented by one of the authors are applied. We begin by changing the free boundary problem to a shape optimization problem and define a least square functional as a cost function. Then shape derivative of the cost function is derived by using Lagrange multiplier method. Detail structures and profiles of exact solutions to a concrete free boundary problem due to A. Henrot are also illustrated with proofs. They are used to check the efficiency of the traction method.*

Keywords: *shape optimization, free boundary problem, traction method*

1 Introduction

Free Boundary Problem (FBP) deals with solving partial differential equations in a domain whose boundary is partially unknown; that the portion of boundary is called a free boundary. The study about free boundary problem is an important branch of partial differential equations (PDEs). In most cases, it is difficult to obtain analytical exact solution of free boundary problem. Therefore numerical analysis is needed to compute the approximation of the solutions.

Shape optimization approach can be used as one of the methods to solve free boundary problem numerically. A numerical method called traction method was developed for solving many shape optimization problems. However, the exact solution (optimal shape) is usually unknown even for a simple problem since this method is often applied only in engineering field. Our aim in this paper is to apply the traction method to obtain a numerical solution of free boundary problems. Then to check the efficiency of the traction method, we consider the following free boundary problem, since its exact solutions are analytically derived by using conformal mapping due to the idea of A. Henrot [2].

Problem 1.1 *Let μ be a given function in $\mathbb{R}^2$ with compact support. Find (u, Ω) such that $supp(\mu) \subset \Omega$ and*

$$\begin{cases} -\Delta u = \mu & \text{in } \Omega \\ u = 0 & \text{on } \Gamma := \partial\Omega \\ \dfrac{\partial u}{\partial n} = -1 & \text{on } \Gamma. \end{cases}$$

where μ is a combination of Dirac functions

$$\mu := \sum_{j=1}^{N} \alpha_j \delta_{\xi_j},$$

with $\alpha_j > 0$ and $\xi_j \in \mathbb{C} \cong \mathbb{R}^2$.

The organization of this paper is as follows. In Section 2, the detail structures and profiles of exact solutions to a concrete free boundary problem due to A. Henrot [2] are illustrated with proofs. Then we change this free boundary problem to a shape optimization problem by defining a cost function. Cost function is a function that we want to minimize it. Afterwards, we derive variation formula of the cost function using Lagrange multiplier method and an adjoint problem. Finally we can apply the traction method and compare its result with the exact solutions from the previous section.

2 Exact Solutions

We solve Problem 1.1 analytically by using conformal mapping. In this section, we identify $\mathbb{R}^2 \cong \mathbb{C}$. Especially, we denote a $\mathbb{R}^2$-coordinate in Ω by $x = (x_1, x_2)$ and its complex representation by $\xi = x_1 + ix_2 \in \mathbb{C}$. But we often mix these notation if no confusion occurs. For a complex variable $\xi = x_1 + ix_2 \in \mathbb{C}$, we denote the two dimensional Lebesgue measure by $d\mathcal{L}^2_\xi$. Let

$$G_0 := \{\Omega \,|\, \Omega \text{ is a bounded open set in } \mathbb{R}^2,\ \mathrm{supp}(\mu) \subset \Omega, \partial\Omega \text{ is Lipschitz}\}.$$

We define a cut-off function $\eta \in \mathbb{C}^\infty(\Omega)$ such that $\eta(x) = 1$ in a neighborhood of $\partial\Omega$ and $\eta(x) = 0$ in neighborhood of $\mathrm{supp}(\mu)$. We call (u, Ω) a weak solution of Problem 1.1 if $\Omega \in G_0$ and they satisfy,

$$\begin{cases} \displaystyle\int_\Omega \nabla u \cdot \nabla \varphi \, dx = -\int_{\partial\Omega} \varphi ds \quad (\forall \varphi \in H^1(\Omega), \mathrm{supp}(\mu) \cap \mathrm{supp}(\varphi) = \emptyset) \\ \displaystyle u(x) - \sum_{j=1}^{N} \alpha_j E(x - \xi_j) \text{ is harmonic function in } \Omega \\ \eta u \in H^1_0(\Omega), \end{cases}$$

where $E(x) = -\frac{1}{2\pi} \log |x|$ is the fundamental solution for $-\Delta$.

Lemma 2.1 *Let Ω_1 and Ω_2 are bounded domains. We suppose that $u \in H^1_0(\Omega_1)$ and set $\Phi(z)$ as a conformal mapping that maps Ω_0 to Ω_1, and $w(z) := u(\Phi(z))$ for $z \in \Omega_0$. Then $w \in H^1_0(\Omega_0)$.*

Proof. We first remark the following equality:

$$\|\nabla(f \circ \Phi)\|_{L^2(\Omega_0)} = \|\nabla f\|_{L^2(\Omega_1)}.$$

For $z \in \Omega_0$, we set $\xi = \Phi(z) \in \Omega_1$. Then $d\mathcal{L}^2_\xi = |\Phi'(z)|^2 d\mathcal{L}^2_z$ holds. Since $|\nabla(f \circ \Phi)(z)| = |\nabla f(\xi)||\Phi'(z)|$, we have

$$\begin{aligned}
\|\nabla(f \circ \Phi)\|_{L^2(\Omega_0)} &= \int_{\Omega_0} |\nabla(f \circ \Phi)(z)|^2 d\mathcal{L}^2_z \\
&= \int_{\Omega_0} |\nabla f(\xi)|^2 |\Phi'(z)|^2 d\mathcal{L}^2_z \\
&= \int_{\Omega_1} |\nabla f(\xi)|^2 d\mathcal{L}^2_\xi \\
&= \|\nabla f\|^2_{L^2(\Omega_1)}.
\end{aligned}$$

We choose a sequence $\{u_n\} \subset C_0^\infty(\Omega_1)$ which satisfies

$$\lim_{n\to\infty} \|u - u_n\|_{H^1(\Omega_1)} = 0$$

and define $w_n := u_n \circ \Phi \in C_0^\infty(\Omega_0)$. Since $\{u_n\}$ is a Cauchy sequence in $H_0^1(\Omega_1)$, from the Poincaré inequality we have

$$\begin{aligned}
\|w_m - w_n\|_{H^1(\Omega_0)} &\leq C(\Omega_0)\|\nabla(w_m - w_n)\|_{L^2(\Omega_0)} \\
&= C(\Omega_0)\|\nabla(u_m - u_n)\|_{L^2(\Omega_1)} \\
&\leq C(\Omega_0)\|(u_m - u_n)\|_{H^1(\Omega_1)},
\end{aligned}$$

and it follows that $\{w_n\}$ is a Cauchy sequence in $H^1(\Omega_0)$. Hence, there exists $w_* \in H_0^1(\Omega_0)$ such that

$$\lim_{n\to\infty} \|w_* - w_n\|_{H^1(\Omega_0)} = 0.$$

For an arbitrary subdomain D with $\bar{D} \subset \Omega_0$, we have

$$\begin{aligned}
\|w - w_*\|_{L^2(D)} &\leq \|w - w_n\|_{L^2(D)} + \|w_n - w_*\|_{L^2(D)} \\
&\leq \|w - w_n\|_{L^2(D)} + \|w_n - w_*\|_{H^1(\Omega_0)},
\end{aligned}$$

where the second term tends to 0 as $n \to \infty$. On the other hand, the first term also converges to 0 as $n \to \infty$ as follows:

$$\begin{aligned}
\|w - w_n\|^2_{L^2(D)} &= \int_D |(w - w_n)(z)|^2 \, d\mathcal{L}^2_z \\
&= \int_{\Phi(D)} |(u - u_n)(\xi)|^2 \frac{1}{|\Phi'(z)|} \, d\mathcal{L}^2_\xi \\
&\leq C_D^2 \int_{\Phi_D} |u - u_n|^2 \, d\mathcal{L}^2_\xi \\
&= C_D^2 \|(u - u_n)(\xi)\|^2_{L^2(\Phi(D))} \\
&\leq C_D^2 \|u - u_n\|^2_{L^2(\Omega_1)},
\end{aligned}$$

where $C_D = (\min_{z\in\bar{D}} |\Phi'(z)|)^{-1}$. Hence we have $w = w_*$ in $L^2(D)$ for an arbitrary domain D with $\bar{D} \subset \Omega_0$. This implies

$$w(z) = w_*(z) \qquad \mathcal{L}^2_z\text{-a.e. } z \in \Omega_0,$$

and we conclude that $w \in H_0^1(\Omega_0)$. ■

The following theorems are given in [2] without detail of their proofs. We give a proof here for the readers convenience.

Theorem 2.1 (A. Henrot [2]) *Suppose $N = 1$. Then (u, Ω) is a weak solution of Problem 1.1, if and only if $\alpha_1 > 0$ and*

$$\begin{cases} \Omega = \left\{ x \in \mathbb{R}^2 \big| \, |x - \xi_1| < \dfrac{\alpha_1}{2\pi} \right\} \\ u = \dfrac{\alpha_1}{2\pi} \log \dfrac{\alpha_1}{2\pi |x - \xi_1|}. \end{cases} \tag{1}$$

Proof. It is easy to show that (1) is a solution of Problem 1.1. We suppose that (u, Ω) is a weak solution of Problem 1.1. We show that Ω is connected. Let Ω_0 is an open component of Ω with $\xi_1 \notin \Omega_0$, then $\Delta u = 0$ on Ω_0 and

$$\int_{\Omega_0} \nabla u \cdot \nabla \varphi \, dx = -\int_{\partial\Omega_0} \varphi \, ds \qquad \forall \varphi \in H^1(\Omega_0).$$

We choose $\varphi = 1$ in Ω_0, then we have

$$-|\partial\Omega_0| = -\int_{\partial\Omega_0} \varphi \, ds = \int_{\Omega_0} \nabla u \cdot \nabla \varphi \, dx = 0.$$

This contradicts to $|\partial\Omega_0| > 0$. Hence, all of the component of Ω has to include ξ_1. Therefore, Ω is connected.

Let $\Phi(z)$ be a conformal mapping from the unit disc $D_0 := \{z \in \mathbb{C} \big| |z| < 1\}$ to Ω with $\Phi(0) = \xi_1$ and $\Phi'(0) > 0$. We define

$$\Psi(z) := \begin{cases} \dfrac{\Phi(z) - \Phi(0)}{z} & (z \neq 0, z \in D_0) \\ \Phi'(0) & (z = 0), \end{cases} \tag{2}$$

then $\Psi(z)$ is holomorphic in D_0 and $\Psi(z) \neq 0$ for $z \in D_0$.
We define $w(z) = u(\Phi(z))$ $(z \in D_0)$. From the conditions $\Delta u_0 = 0$ in Ω,

$$u(\xi) = \alpha_1 E(\xi - \xi_1) + u_0(\xi) \qquad (\xi \in \Omega \setminus \{\xi_1\}),$$

we have

$$\begin{aligned} w(z) &= \alpha_1 E(\Phi(z) - \xi_1) + u_0(\Phi(z)) \\ &= \alpha_1 E(\Phi(z) - \Phi(0)) + u_0(\Phi(z)) \\ &= \alpha_1 E(z\Psi(z)) + u_0(\Phi(z)) \\ &= \alpha_1 E(z) - \frac{\alpha_1}{2\pi} \log |\Psi(z)| + u_0(\Phi(z)). \end{aligned}$$

Since the second and third terms of the equation above are harmonic in D_0, we obtain $-\Delta w = \alpha_1 \delta_0$ in D_0. We define $\tilde{w}(z) := \eta(\Phi(z))w(z) = (\eta u) \circ \Phi(z)$. From Lemma 2.1, $\tilde{w} \in H_0^1(D_0)$. Since $\tilde{w} = w$

in a neighborhood of ∂D_0 and is harmonic, from the theory of elliptic regularity [4], w is smooth up to ∂D_0 and $w = 0$ on ∂D_0. Hence, the following equations holds

$$\begin{cases} -\Delta w = \alpha_1 \delta_0 & \text{in } D_0 \\ w = 0 & \text{on } \partial D_0 \\ \dfrac{\partial w}{\partial n} = -|\nabla w| = -|\nabla u||\Phi'| = -|\Phi'| & \text{on } \partial D_0. \end{cases} \tag{3}$$

From the first two equations of (3), we have $w(z) = \alpha_1 E(z)$ and

$$\frac{\partial w}{\partial n} = -\frac{\alpha_1}{2\pi}\Big(\frac{\partial}{\partial r}\log r\Big)\Big|_{r=1} = -\frac{\alpha_1}{2\pi}.$$

By the third condition of (3), we obtain $|\Phi'(z)| = \frac{\alpha_1}{2\pi}$ on ∂D_0. We set $v(x,y) := \mathrm{Re}[\log|\Phi'(z)|] = \log|\Phi'(z)|$ and $v(z)$ is harmonic since Φ is holomorphic and $\Phi'(z) \neq 0$ in D_0 . Then $\Delta v = 0$ in D_0 and $v = \log\frac{\alpha_1}{2\pi}$ on ∂D_0 hold and these imply that $v = \log\frac{\alpha_1}{2\pi}$ in D_0. Since $\mathrm{Re}[\log \Phi'(z)] = v = \log\frac{\alpha_1}{2\pi}$ in D_0, we obtain that $\log\Phi'(z) = \log\frac{\alpha_1}{2\pi} + i\beta$ for $\beta \in \mathbb{R}$. From the condition $\Phi'(0) > 0$, $\beta = 0$ follows. Hence we have $\Phi'(z) = \frac{\alpha_1}{2\pi}$ and conclude that

$$\Phi(z) = \frac{\alpha_1}{2\pi}z + \xi_1,$$

where we used $\Phi(0) = \xi_1$. Therefore by the conformal mapping $\Phi(z)$ we obtain

$$\Omega = \{x \in \mathbb{R}^2 \big| |x - \xi_1| < \frac{\alpha_1}{2\pi}\}$$

as a solution of Problem 1.1.

We know that $u = 0$ on $\partial\Omega$, then we have $u_0 = \frac{\alpha_1}{2\pi}\log\frac{\alpha_1}{2\pi}$. Hence we can conclude that

$$u(x) = -\frac{\alpha_1}{2\pi}\log|x - \xi_1| + \frac{\alpha_1}{2\pi}\log\frac{\alpha_1}{2\pi} = \frac{\alpha_1}{2\pi}\log\frac{\alpha_1}{2\pi|x - \xi_1|}.$$

■

Let us consider the case $N = 2$. We suppose that $c > 0$ and $\xi_1, \xi_2 \in \mathbb{C} \cong \mathbb{R}^2$ are given as $\xi_1 = c$ and $\xi_2 = -c$. We denote the Dirac function at ξ_1 and ξ_2 by δ_c and δ_{-c}, respectively, we consider

$$\mu = \alpha\delta_c + \alpha\delta_{-c} \tag{4}$$

for same $\alpha > 0$. Then we define a conformal mapping on D_0

$$\Phi_a(z) := \frac{\alpha(1 - a^4)}{4\pi a^2}\left[\frac{-2z}{z^2 - 1/a^2} + a\log\frac{1/a + z}{1/a - z}\right] \tag{5}$$

for $0 < a < 1$.

Theorem 2.2 (A.Henrot [2]) *We suppose a function μ as in (4),*

1. *(u, Ω) is a weak solution to Problem 1.1 and Ω is connected, if and only if there exists $a \in (0, 1)$ such that $c = \Phi_a(a)$, $\Omega = \Phi_a(D_0)$, and $u(\xi) = w(\Phi_a^{-1}(\xi))$, $\xi \in \Omega$, where*

$$w(z) := \frac{\alpha}{2\pi}\log\left|\frac{1 - a^2z^2}{z^2 - a^2}\right| \qquad (z \in D_0). \tag{6}$$

2. *(u,Ω) is a weak solution to Problem 1.1 and Ω is disconnected if and only if $\frac{\alpha}{2\pi} < c$ and the solution is given by*

$$\begin{cases} \Omega = B\Big(\xi_1, \dfrac{\alpha}{2\pi}\Big) \cup B\Big(\xi_2, \dfrac{\alpha}{2\pi}\Big), \\ u = \begin{cases} \dfrac{\alpha}{2\pi} \log \dfrac{1}{|x-\xi_1|} & \Big(x \in B\Big(\xi_1, \dfrac{\alpha}{2\pi}\Big)\Big) \\ \dfrac{\alpha}{2\pi} \log \dfrac{1}{|x-\xi_2|} & \Big(x \in B\Big(\xi_2, \dfrac{\alpha}{2\pi}\Big)\Big). \end{cases} \end{cases} \tag{7}$$

Proof. Let $\tilde{g}$ be a conformal mapping from $D_0 := \{z \in \mathbb{C} : |z| < 1\}$ to Ω with $\tilde{g}(0) = \xi_1$, and set $\tilde{g}(\xi_2) = be^{i\theta}$ $(0 < b < 1)$. We define $g(z) := \tilde{g}(e^{i\theta_0}z)$ and $f : D_0 \to D_0$ be a Möbius transform. Since $f(a) = 0$ and $f(-a) = b$

$$\begin{aligned} a &:= \frac{1-\sqrt{1-b^2}}{b} \in (0,1), \\ f(z) &:= \frac{a-z}{1-az}. \end{aligned}$$

We can define a conformal mapping $\Phi(z) := g(f(z))$ which maps D_0 to the domain Ω with $\Phi(a) = \xi_1$ and $\Phi(-a) = \xi_2$. Set $w(z) = u \circ \Phi(z) = u(\Phi(z))$, then by using the similar argument in the proof of Theorem 2.1, we have

$$\begin{cases} -\Delta w = \alpha\delta_a + \alpha\delta_{-a} & \text{in } D_0 \\ w = 0 & \text{on } \partial D_0 \\ \dfrac{\partial w}{\partial n} = -|\nabla w| = -|\nabla u||\Phi'| = -|\Phi'| & \text{on } \partial D_0. \end{cases} \tag{8}$$

We define

$$w_0(z) := \frac{1}{\alpha} w(z) - E(z-a) - E(z+a) \qquad (z \in D_0).$$

Then $w_0(z)$ becomes a harmonic function in D_0. Since $w(z) = 0$ on the boundary, for $z \in \partial D_0$, we obtain

$$\begin{aligned} w_0(z) &= -E(z-a) - E(z+a) \\ &= \frac{1}{2\pi}(\log|z-a| + \log|z+a|) \\ &= \frac{1}{2\pi}\log|z^2 - a^2| \\ &= \frac{1}{2\pi}\log|1 - a^2z^2|. \end{aligned}$$

Hence we have $w_0(z) = \frac{1}{2\pi}\log|1-a^2z^2|$ in D_0. Then

$$w(z) = \frac{\alpha}{2\pi}\log\frac{1}{|z-a|} + \frac{\alpha}{2\pi}\log\frac{1}{|z+a|} + \frac{\alpha}{2\pi}\log|1-a^2z^2| = \frac{\alpha}{2\pi}\log\frac{|1-a^2z^2|}{|z^2-a^2|}.$$

holds. From the third condition of (8), for $z = e^{i\theta} \in \partial D_0$, we have

$$
\begin{aligned}
|\Phi'(z)| &= -\frac{\partial w}{\partial n}(z) \\
&= -\frac{\partial}{\partial r} w(re^{i\theta})\Big|_{r=1} \\
&= -\frac{\partial}{\partial r}\left(\frac{\alpha}{2\pi}\log\left|\frac{1-a^2r^2e^{2i\theta}}{r^2e^{2i\theta}-a^2}\right|\right)\Big|_{r=1} \\
&= -\frac{\alpha}{2\pi}\frac{\partial}{\partial r}\Bigg(\log|1-a^2r^2e^{2i\theta}| - \log|r^2e^{2i\theta}-a^2|\Bigg)\Bigg|_{r=1} \\
&= -\frac{\alpha}{4\pi}\left(\frac{\partial}{\partial r}(\log|1-2a^2r^2\cos 2\theta + a^4r^4| - \log|r^4 - 2a^2r^2\cos 2\theta + a^4|)\right)\Bigg|_{r=1} \\
&= -\frac{\alpha}{4\pi}\left(\frac{4a^2(a^2-\cos 2\theta) - 4(1-a^2\cos 2\theta)}{|e^{2i\theta}-a^2|^2}\right) \\
&= \frac{\alpha}{\pi}\frac{1-a^4}{|1-a^2z^2|^2}.
\end{aligned}
$$

Similarly to the proof of Theorem 2.1, the harmonic function $v(z) := \mathrm{Re}[\log \Phi'(z)]$ in D_0 satisfies

$$
v(z) = \mathrm{Re}\Big[\log\Big(\frac{\alpha}{\pi}\frac{1-a^4}{|1-a^2z^2|^2}\Big)\Big] \qquad (z \in \partial D_0).
$$

Hence it follows that

$$
\log \Phi'(z) = \log(\frac{\alpha}{\pi}\frac{1-a^4}{(1-a^2z^2)^2}) + i\beta \ (z \in D_0),
$$

for some $\beta \in \mathbb{R}$. Then we have

$$
\Phi'(z) = \frac{\alpha}{\pi}e^{i\beta}\frac{1-a^4}{(1-a^2z^2)^2}. \tag{9}
$$

We define

$$
\Phi_0(z) = \frac{(1-a^4)}{4\pi a^2}\left[\frac{-2z}{z^2-1/a^2} + a\log\frac{1/a+z}{1/a-z}\right].
$$

Then, integrating (9), we have $\Phi(z) = e^{i\beta}\Phi_0(z) + \gamma$, where $\gamma \in \mathbb{C}$. Since $\Phi(\pm a) = \pm c$, using $\Phi_0(a) + \Phi_0(-a) = 0$, we obtain

$$
0 = \Phi(a) + \Phi(-a) = e^{i\beta}\alpha(\Phi(a)_0 + \Phi_0(-a)) + 2\gamma = 2\gamma,
$$

and $\gamma = 0$. Also from $\Phi_0(a) > 0$ (see Figure 1), $\beta = 0$ follows. Therefore we have

$$
\Phi(z) = \Phi_a(z),
$$

where $\Phi_a(z)$ is defined in (5).

It is easy to show that (u, Ω) defined in (7) is a solution of Problem 1.1 with μ as in (4) for $\frac{\alpha}{2\pi} < c$. Let us suppose (u, Ω) is a weak solution and Ω is disconnected. Then, from the same argument of the proof of Theorem 2.1, each open component of Ω should contain ξ_1 or ξ_2 and Ω should have exactly two components Ω_1 and Ω_2 ($\xi_1 \in \Omega_1$, $\xi_2 \in \Omega_2$). Then from Theorem 2.1, we obtain (7). ■

Remark In the Henrot's paper [2], equation (6.5) has a typo. The correct expression of (6.5) is

$$\begin{cases} \Delta v = 0 & \text{in } \Omega_0 \\ v = \log\left(\dfrac{\alpha}{\pi}\dfrac{1-a^4}{|1-a^2z^2|^2}\right) & \text{on } \partial\Omega_0. \end{cases}$$

We define $l = 2c$, based on the conformal mapping Φ_a in (5), we know that $\frac{\alpha}{l} = \frac{1}{2\Phi_0(a)}$. Then we can plot a graph α/l versus a as in Figure 1.

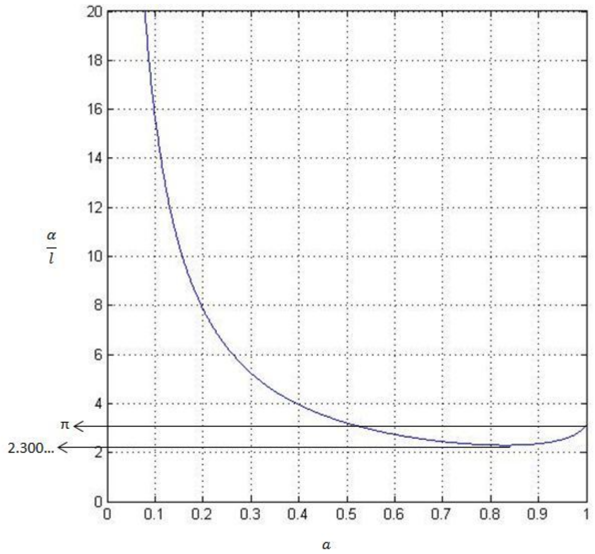

Figure 1: α/l vs a graph

From the graph in Figure 1, we can see that for 2.300.. $< \alpha/l < \pi$ there exist two connected solution and that for $\alpha/l > \pi$ there exists a unique solution (which is connected). Table 1 shows the number of the exact solutions of Problem 1.1 where μ as in (3). Although this table was shown in [2], we present it in more detailed form, particularly for the values $\alpha/l = 2.300...$ and $\alpha/l = 2.827....$ According to [2], Ω is convex if and only if $a \leq 1/\sqrt{3}$. We have $\frac{\alpha}{l} = 2.827...$ for $a = 1/\sqrt{3}$.

Table 1: Table of number of the solutions

α/l	0	...	2.300...	...	2.827...	...	π	...
# connected convex solution	0	0	0	0	1	1	1	1
# connected non-convex solution	0	0	1	2	1	1	0	0
# connected solution	0	0	1	2	2	2	1	1
# disconnected solution	0	1	1	1	1	1	0	0

Example of connected solutions are given in Figure 2 - 7 for $\alpha = 3$, where we use MATLAB to draw them. We change $\alpha/l = 2.3007, 2.400, 2.827, 3.000, \pi, 3.300$. Then the number of solutions becomes $1, 2, 2, 2, 1, 1$ for each figure.

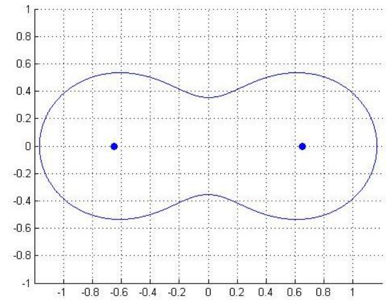

Figure 2: $\alpha/l = 2.300$

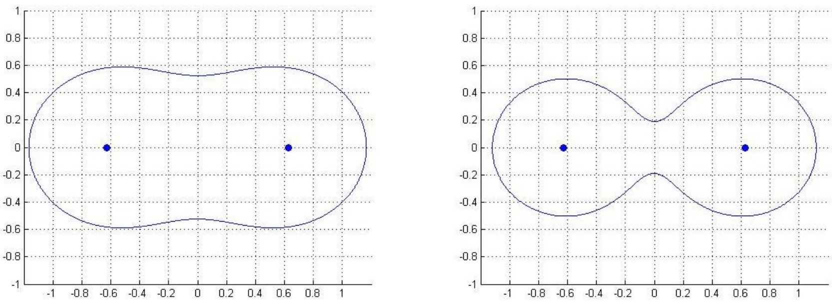

Figure 3: $\alpha/l = 2.400$

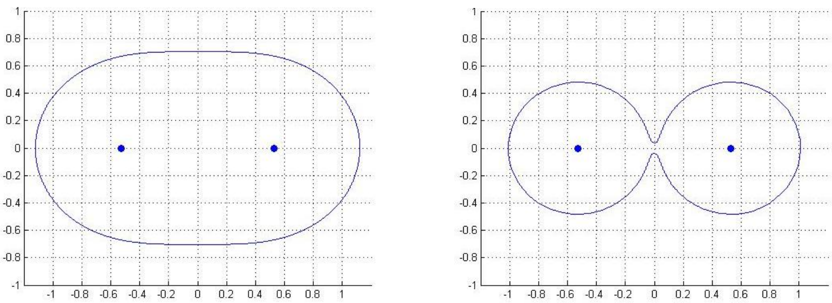

Figure 4: $\alpha/l = 2.827$

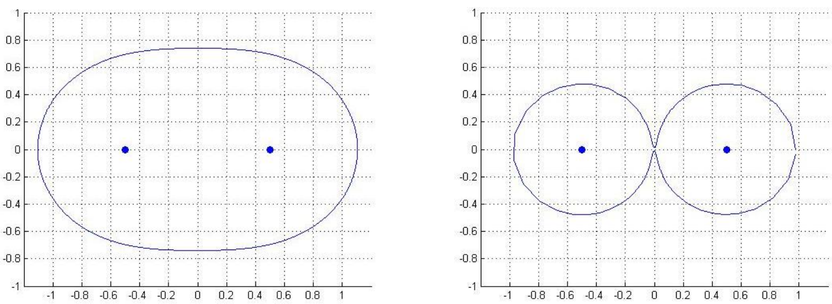

Figure 5: $\alpha/l = 3.000$

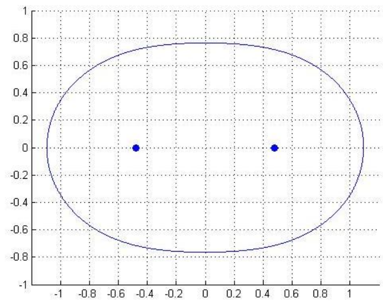

Figure 6: $\alpha/l = \pi$

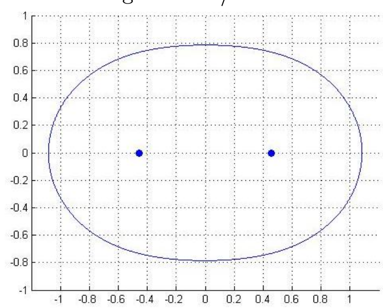

Figure 7: $\alpha/l = 3.300$

3 Shape Optimization Approach

We consider Problem 1.1 with μ as in (4). Then we replace μ by

$$\mu(x) = \alpha\delta^{\varepsilon}(x - \xi_1) + \alpha\delta^{\varepsilon}(x - \xi_2) \tag{10}$$

for sufficiently small $\varepsilon < 0$, where

$$\delta^{\varepsilon}(x) := \begin{cases} \dfrac{1}{\pi\varepsilon^2} & |x| < \varepsilon \\ 0 & |x| \geq \varepsilon. \end{cases}$$

We remark that the problems for $\mu = \alpha(\delta_c + \delta_{-c})$ and for (10) are equivalent except for $u(x)$ in $D := B(\xi_1, \varepsilon) \cup B(\xi_2, \varepsilon)$.

We fix $\beta > 0$, and rewrite Problem 1.1 in the following equivalent form with a Robin boundary condition

$$\begin{cases} -\Delta u = \mu & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \\ \beta u + \dfrac{\partial u}{\partial n} = -1 & \text{on } \Gamma. \end{cases}$$

We define

$$G := \{\Omega \,|\, \Omega \text{ is a bounded domain in } \mathbb{R}^2,\ \bar{D} \subset \Omega,\ \partial\Omega: \text{ Lipschitz } \}.$$

Then for given $\Omega \in G$ with $\Gamma = \partial\Omega$, we can find a unique solution $u_\Omega \in H^1(\Omega)$ to the following problem

$$u_\Omega : \begin{cases} -\Delta u = \mu & \text{in } \Omega \\ \beta u + \dfrac{\partial u}{\partial n} = -1 & \text{on } \Gamma. \end{cases}$$

If $u_\Omega = 0$ on Γ then (u_Ω, Ω) is a solution for Problem 1.1. Then we define a cost function as follows

$$J(\Omega) = \frac{1}{2}\int_\Gamma |u_\Omega|^2 ds.$$

We want to minimize $J(\Omega)$ among $\Omega \in G$. We remark here that (Ω, u) is a solution if and only if $J(\Omega) = 0$ and $u = u_\Omega$.

4 Variation Formula of Cost Function

We use Lagrange multiplier method [5] to derive the variation formula of cost function $J(\Omega)$ with respect to the domain $\Omega \in G$. For a vector field $\mathbf{V} \in W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)$, we define

$$\Omega(t) := \{x + t\mathbf{V}(x) \big| x \in \Omega\} \qquad (0 \le t < t_0).$$

Then we introduce an adjoint problem as follows

$$v_\Omega : \begin{cases} \Delta v = 0 & \text{in } \Omega \\ \beta v + \dfrac{\partial v}{\partial n} = -u_\Omega & \text{on } \Gamma. \end{cases}$$

By using the Lagrange multiplier method and the adjoint problem, under some regularity conditions, we obtain a variation formula of the cost function $J(\Omega)$:

$$\frac{d}{dt}J(\Omega(t))\Big|_{t=0} = \int_\Gamma \Big((\mathbf{V}\cdot\mathbf{n})f + \mathbf{V}\cdot\nabla g + (\mathbf{V}_s\cdot\tau)g\Big)ds, \tag{11}$$

where τ is a counter clockwise tangential unit vector on Γ, $\mathbf{V}_s = \frac{\partial \mathbf{V}}{\partial \tau}$, and

$$\begin{aligned} f &= \nabla u_\Omega \cdot \nabla v_\Omega \\ g &= \frac{1}{2}u_\Omega^2 + \alpha u_\Omega v_\Omega + v_\Omega. \end{aligned}$$

A proof of (11) will be given in our forthcoming paper.

5 Traction Method

The main idea of traction method is to treat the domain Ω as an elastic body and iterate small deformation by a boundary traction given by the variational formula of $J(\Omega)$. In order to solve Problem 1.1 using the traction method, we have to solve the following artificial elasticity problem

$$\begin{cases} -\text{div } \sigma[w] = 0 & \text{in } \Omega \setminus \overline{D} \\ \sigma[w]n = -B & \text{on } \Gamma \\ w = 0 & \text{on } \partial D, \end{cases} \tag{12}$$

where $w(x) \in \mathbb{R}^2$ is a displacement field on $\bar{\Omega}$.

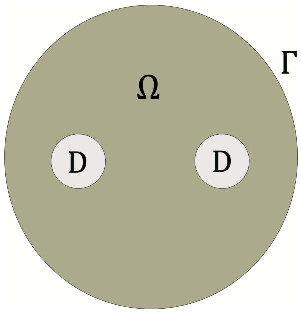

Figure 8: The initial domain

We put B as boundary force, which is implicitly defined by

$$\int_\Gamma B \cdot \mathbf{V} = \frac{d}{dt} J(\Omega(t)) = \int_\Gamma \Big((\mathbf{V} \cdot \mathbf{n}) f + \mathbf{V} \cdot \nabla g + (\mathbf{V}_s \cdot \tau) g \Big) ds \qquad ({}^\forall\, \mathbf{V} \in \mathrm{W}^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)).$$

The complete procedure of solving Problem 1.1 by using the traction method can be summarized as follows:

1. Define an initial domain Ω as in figure (8) and generate a finite element mesh on Ω.

2. Solve u_Ω and v_Ω by finite element method.

3. Solve the artificial elasticity problem (12) by finite element method.

4. Modify the domain $\Omega_{new} := \{x + \eta w(x) | x \in \Omega\}$ for sufficiently small $\eta > 0$, together with the nodal points of the mesh.

5. Repeat step 2-4 until the domain Ω converges.

6 Numerical Examples

To study the efficiency of the traction method, we apply it into a free boundary problem as in Problem 1.1 with μ as in (4). Figure 9 shows the numerical result of Problem 1.1 with $\alpha = 3$ and $c = 0.47727$ ($\alpha/l = \pi$) where we use FreeFem++ [3] for the simulation. We also summarize the value of the cost function for some iterations in Table 2.

Table 2: Table of the cost function

Iteration	500	1000	1500	2000	2500	2964
Cost function	0.00355...	0.000273...	0.0000541...	0.0000279...	0.0000218...	0.00000513...

From Table 2 we can see that the cost function becomes smaller with more iterations and it is almost equal to zero ($J(\Omega) = 0.00000513...$) after 2964 iteration.

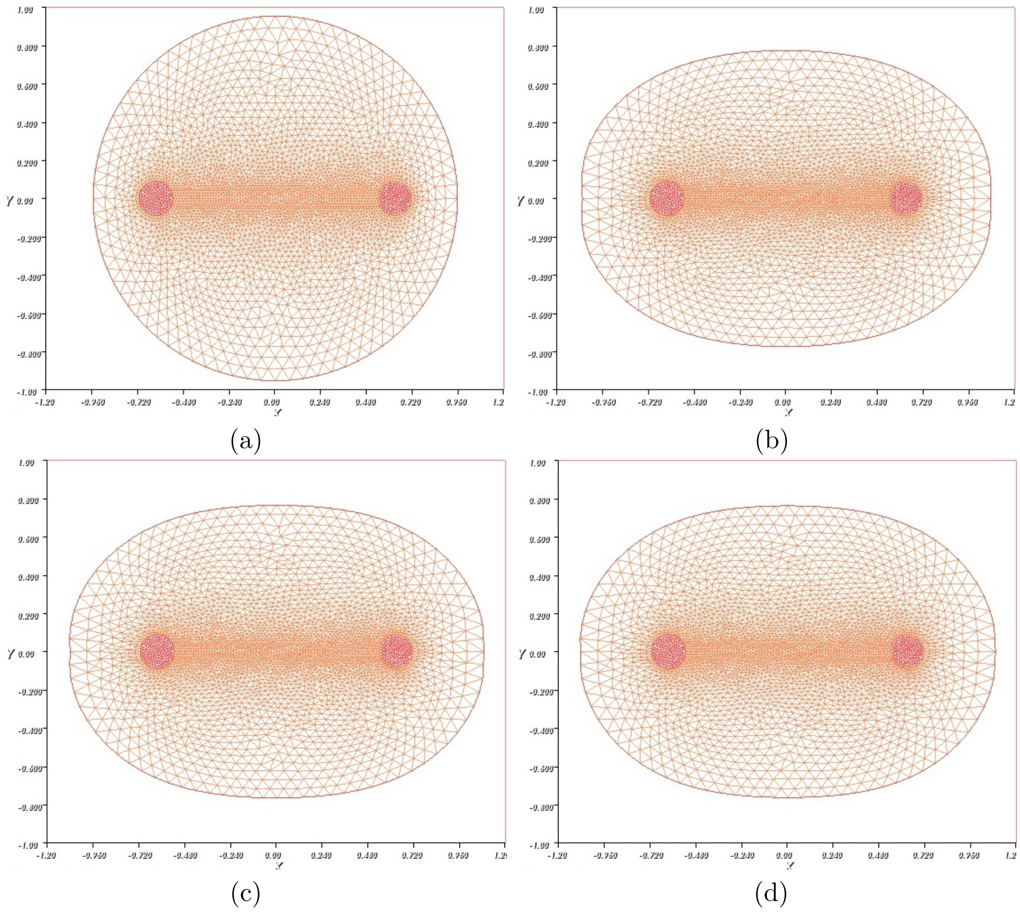

Figure 9: Numerical result of Problem 1.1 with $\alpha = 3$ and $c = 0.47727$ (a) initial domain (b) iteration 1000 (c) iteration 2000 (d) iteration 2964

Comparing with the exact solution in Figure 6, we can observe that the numerical result after 2964 iteration in Figure 9 gives an accurate solution.

7 Conclusion

This paper has presented a complete construction of exact solutions of a free boundary problem by means of the conformal mapping based on the paper of A. Henrot [2]. We could classify all the exact solution into connected/disconnected and convex/non-convex ones and specified the number of each solutions for the case that μ is the combination of two Dirac function as shown in Theorem 2.2 and Table 1. The figures of some exact solutions are also presented in this paper using MATLAB. Hence we can use it as an comparison to the numerical result.

We also solved Problem 1.1 numerically using a shape optimization approach, specifically using the traction method. First we changed the free boundary problem in Problem 1.1 to a shape optimization problem as described in section 3. Then we derived the variation formula of the cost function $J(\Omega)$. Under some regularity, the variation formula in (11) could be obtained using the Lagrange multiplier method and the adjoint problem. The numerical result of Problem 1.1 (where

μ is a combination of two Dirac function with same coefficient α) for $\alpha/l = \pi$ obtained by the traction method was shown in Figure 9. It was observed that by comparing with the exact solution in Figure 6, the traction method could give the numerical result with good accuracy.

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