Some Remarks on Bochner Curvature Tensors

by

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Abstract

This paper deals with the study of Kähler manifolds via Bochner curvature tensors. Similar study has been done by Kulkarni 4), 5), 6), where he has shown that many curvature preserving mappings between Rieamannian manifolds (or Kähler manifolds) reduce to conformal or isometric mappings.

In the first section, we shall study K-curvature-like tensors on almost Hermitian manifolds which include Riemannian curvature tensors, Bochner curvature tensors, and so on.

In the second section, we shall consider Bochner curvature tensors on Kähler manifolds. The characterizations of the Kähler manifold with vanishing Bochner curvature tensor have been obtained by many authors. We shall give some other conditions for vanishing Bochner curvature tensor in Theorem 2. In Theorem 3, we show an analogous result to Kulkarni for a H_B-preserving mapping.

1. K-curvature-like tensors in almost Hermitian manifolds

In this paper all manifolds are assumed to be connected. We usually denote the vector fields on the manifold by the capital letters $X, Y, \cdots$, and the tangent vectors at a point by the small letters $x, y, \cdots$.

Let (M,g,J) be an almost Hermitian manifold with almost complex structure J and almost Hermitian metric tensor g, and let $T_p(M)$ be a tangent space to M at a point p. On an almost Hermitian manifold (M,g,J) we consider a (1,3)-tensor field T such that

(a) $T(X,Y)=-T(Y,X)$,
(b) $g(T(X,Y)Z,W)=g(T(Z,W)X,Y)$,
(c) $T(X,Y)Z+T(Y,Z)X+T(Z,X)Y=0$,
(d) $T(X,Y)\circ J=J\circ T(X,Y)$

for any vector fields X, Y, Z, W on M. We shall call the above tensor field T *a K-curvature-like tensor* after Ogitsu and Iwasaki 7). (The tensor T which satisfy the above (a), (b), (c) is called *a curvature structure* 5), or *a semi-curvature-like tensor* 2).)

For a K-curvature-like tensor T, we define a (0,2)-tensor field T_1 and a (1,1)-tensor field T^1 by setting

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$$T_1(X,Y)=\text{trace of the map } [Z \longmapsto T(X,Z)Y] \ ,$$

and

$$g(T^1X,Y)=T_1(X,Y) \ .$$

Then it is easy to verify that for any K-curvature-like tensor T,

$$\begin{aligned} T(JX,JY) &= T(X,Y), \\ T_1(JX,JY) &= T_1(X,Y), \\ T^1\circ J &= J\circ T^1. \end{aligned}$$

For a given T, we also define *a T-sectional curvature* K_T and *a holomorphic T-sectional curvature* H_T by

$$K_T(x,y)=\frac{g(T(x,y)x,y)}{\| x\wedge y \|^2}$$

for linearly independent vectors x, y, and

$$H_T(x)=K_T(x,Jx)=\frac{g(T(x,Jx)x,Jx)}{\| x \|^4} \quad ,$$

where $\| x \|^2=g(x,x)$ and $\| x\wedge y \|^2=\| x \|^2 \| y \|^2-g(x,y)^2$. It is easy to check that the K_T is determined by the plane section spanned by x and y, independent of the choice of x, y.

Given a K-curvature-like tensor T on an almost Hermitian manifold M, the notion of the constancy of K_T and of H_T can be considered in the same way as the ordinary Riemannian curvature. For instance, the following two propositions are easily seen by the same methods as the Kählerian (or Riemannian) case:

Proposition 1. *Let T be a K-curvature-like tensor on an almost Hermitian manifold M. Then H_T is constant at a point p of M, if and only if T is of the form*

$$T(x,y)z=\frac{\alpha}{4}\{g(x,z)y-g(y,z)x+g(Jx,z)Jy-g(Jy,z)Jx+2g(Jx,y)Jz\}$$

for every x, y, $z \in T_p(M)$, where α is constant.

Proposition 2. (Tanno 9)). *Assume that* $\dim M \geqq 4$. *An almost Hermitian manifold M with K-curvature-like tensor T is of constant holomorphic T-sectional curvature at a point p of M, if and only if*

$$T(x,Jx)x \ \textit{ is proportional to } \ Jx$$

for any tangent vector $x \in T_p(M)$.

Remark. The key of the proof of Proposition 1 is the following formula (Bishop-Goldberg 1), see also Sawaki-Watanabe-Sato 8)):

$$g(T(x,y)x,y)=\frac{1}{32}\{3Q(x+Jy)+3Q(x-Jy)-Q(x+y)-Q(x-y)-4Q(x)-4Q(y)\} \ ,$$

where $Q(x)=g(T(x,Jx)x,Jx)=H_T(x)\| x \|^4$.

Now let $(\widetilde{M},\widetilde{g},\widetilde{J})$ be another almost Hermitian manifold with K-curvature-like tensor $\widetilde{T}$, and $f: M \longrightarrow \widetilde{M}$ be an almost complex diffeomorphism, that is, f is a diffeomorphism satisfying $f_*\circ J=\widetilde{J}\circ f_*$. Then, f is called *H_T-preserving* if the following equality holds at each $p \in M$:

$$H_T(x)=H_{\widetilde{T}}(f_* x) \qquad \text{for all } x \in T_p(M) .$$

About the H_T-preserving diffeomorphism, we can prove the following

Theorem 1. *Let (M,g,J) and $(\widetilde{M},\widetilde{g},\widetilde{J})$ be almost Hermitian manifolds with K-curvature-like tensors T and $\widetilde{T}$, respectively. Suppose that* $\dim M=2n \geqq 4$, *and the set*

$$\{p \in M \mid H_T \neq \text{constant at } p\}$$

is dense in M. Then H_T-preserving diffeomorphism $f: M \longrightarrow \widetilde{M}$ is conformal.

We note only the following lemma, since the rest of proof is obtained by the same arguments as Kulkarni 4), 6).

Lemma 1. *If* $\dim M=4$, *and H_T is not constant at $p \in M$, then there exists an orthonormal basis $\{e_1, e_2, Je_1, Je_2\}$ for $T_p(M)$ such that*

$$g(T(e_1,Je_1)e_1,Je_2)^2+(H_T(e_1)-H_T(e_2))^2\neq 0 ,$$

and

$$g(T(e_2,Je_2)e_2,Je_1)^2+(H_T(e_1)-H_T(e_2))^2\neq 0 .$$

Proof. Since $H_T \neq$ constant at p, taking account of Proposition 2, we can find a pair of unit vectors x, y in $T_p(M)$ such that $g(x,y)=g(x,Jy)=0$ and $g(T(x,Jx)x,Jy) \neq 0$.

If $H_T(x)\neq H_T(y)$ or $g(T(y,Jy)y,Jx)\neq 0$, the basis $\{x, y, Jx, Jy\}$ is the desired one. So we suppose $H_T(x)=H_T(y)$ and $g(T(y,Jy)y,Jx)=0$, and set

$e_1=\cos\theta\cdot x+\sin\theta\cdot y,\quad e_2=-\sin\theta\cdot x+\cos\theta\cdot y \qquad$ for some θ, $0<\theta<\frac{\pi}{2}$.

Then we have

$$\begin{aligned}H_T(e_1)=&\cos^4\theta\, H_T(x)+\sin^4\theta\, H_T(y)+4\cos^3\theta\sin\theta\, g(T(x,Jx)x,Jy)\\&+\cos^2\theta\sin^2\theta\,\{2g(T(x,Jx)y,Jy)+4g(T(x,Jy)x,Jy)\} ,\\H_T(e_2)=&\sin^4\theta\, H_T(x)+\cos^4\theta\, H_T(y)-4\cos\theta\sin^3\theta\, g(T(x,Jx)x,Jy)\\&+\cos^2\theta\sin^2\theta\,\{2g(T(x,Jx)y,Jy)+4g(T(x,Jy)x,Jy)\} .\end{aligned}$$

Thus

$$\begin{aligned}H_T(e_1)-H_T(e_2)&=(\cos^4\theta-\sin^4\theta)(H_T(x)-H_T(y))+4\cos\theta\sin\theta\, g(T(x,Jx)x,Jy)\\&=2\sin 2\theta\, g(T(x,Jx)x,Jy)\\&\neq 0 \qquad \text{if } 0<\theta<\tfrac{\pi}{2} .\end{aligned}$$

Hence the basis $\{e_1, e_2, Je_1, Je_2\}$ satisfies the conditions in Lemma 1. Q. E. D.

2. Bochner curvature tensors in Kähler manifolds

In this section, we consider the Kähler manifold (M,g,J) of dimension $m=2n \geqq 4$, and the

Bochner curvature tensor on M which is one of the K-curvature-like tensors. By R we denote the Riemannian curvature tensor defined by $R(X,Y)Z=\nabla_{[X,Y]}Z-[\nabla_X,\nabla_Y]Z$, where ∇ is the covariant differentiation with respect to the Kähler metric g. The Ricci curvature tensors R_1, R^1 and the scalar curvature r are defined by usual way, that is,

$$R_1(X,Y)=\text{trace of the map } [Z\longmapsto R(X,Z)Y],$$
$$g(R^1X,Y)=R_1(X,Y),$$
$$r=\text{trace } R^1.$$

It is well-known that R is a K-curvature-like tensor. It is also easy to see that the following tensors are K-curvature-like tensors:

$$I(X,Y)Z=g(X,Z)Y-g(Y,Z)X+g(JX,Z)JY-g(JY,Z)JX+2g(JX,Y)JZ,$$
$$\begin{aligned}S(X,Y)Z=&R_1(X,Z)Y-R_1(Y,Z)X+g(X,Z)R^1Y-g(Y,Z)R^1X\\&+R_1(JX,Z)JY-R_1(JY,Z)JX+2R_1(JX,Y)JZ\\&+g(JX,Z)R^1JY-g(JY,Z)R^1JX+2g(JX,Y)R^1JZ.\end{aligned}$$

Making use of the above tensors, *the Bochner curvature tensor* B is defined by

$$B=R-\frac{1}{2(n+2)}S+\frac{r}{4(n+1)(n+2)}I.$$

It is clear that the Bochner curvature tensor is a K-curvature-like tensor, since R, S and I are all so. Thus, we have following

Proposition 3. *For a Bochner curvature tensor B on a Kähler manifold, we have*

$$\begin{aligned}K_B(x,y)=K_R(x,y)-&\frac{1}{2(n+2)\|x\wedge y\|^2}\{\|y\|^2R_1(x,x)-2g(x,y)R_1(x,y)+\|x\|^2R_1(y,y)\\&+6g(Jx,y)R_1(Jx,y)\}+\frac{r}{4(n+1)(n+2)\|x\wedge y\|^2}\{\|x\wedge y\|^2+3g(Jx,y)^2\},\end{aligned}$$

and

$$H_B(x)=H_R(x)-\frac{4}{(n+2)\|x\|^2}R_1(x,x)+\frac{r}{(n+1)(n+2)}.$$

Proof. From the definition of the T-sectional curvature, we have

$$K_S(x,y)=\frac{1}{\|x\wedge y\|^2}\{\|y\|^2R_1(x,x)-2g(x,y)R_1(x,y)+\|x\|^2R_1(y,y)+6g(Jx,y)R_1(Jx,y)\},$$

$$K_I(x,y)=\frac{1}{\|x\wedge y\|^2}\{\|x\wedge y\|^2+3g(Jx,y)^2\}.$$

The first formula is obtained at once from these formulas. The second follows immediately from the first. Q. E. D.

Lemma 2. $B_1=0$.

Proof. Recall that $B_1(x,y)=$ trace of $[z\longmapsto B(x,z)y]$. Let $\{e_1,\cdots,e_m\}$ be an orthonormal basis for $T_p(M)$. Taking account of the fact $R^1\circ J=J\circ R^1$, we have for any tangent vectors x, y in $T_p(M)$

$$\begin{aligned}S_1(x,y)&=\text{trace of } [z\longmapsto S(x,z)y]\\&=\sum_{i=1}^{m}g(S(x,e_i)y,e_i)\\&=\sum_{i=1}^{m}\{R_1(x,y)-g(x,e_i)R_1(e_i,y)+g(x,y)R_1(e_i,e_i)-g(e_i,y)R_1(x,e_i)\\&\quad+g(Je_i,e_i)R_1(Jx,y)-g(Jx,e_i)R_1(Je_i,y)+2g(Jy,e_i)R_1(Jx,e_i)\\&\quad+g(Jx,y)R_1(Je_i,e_i)-g(Je_i,y)R_1(Jx,e_i)+2g(Jx,e_i)R_1(Jy,e_i)\}\\&=2(n+2)R_1(x,y)+rg(x,y).\end{aligned}$$

Similarly

$$I_1(x,y)=2(n+1)g(x,y).$$

Therefore, for any $x,\ y\in T_p(M)$

$$\begin{aligned}B_1(x,y)&=R_1(x,y)-\frac{1}{2(n+2)}S_1(x,y)+\frac{r}{4(n+1)(n+2)}I_1(x,y)\\&=R_1(x,y)-\frac{1}{2(n+2)}\{2(n+2)R_1(x,y)+rg(x,y)\}+\frac{r}{2(n+2)}g(x,y)\\&=0.\end{aligned}$$

Q. E. D.

Now, we show the following theorem which gives some characterizations of vanishing Bochner curvature tensor.

Theorem 2. *On a Kähler manifold* (M,g,J) *of dimension* $2n\geqq 4$, *the following statements are equivalent:*

(1) $B=0$ *at a point* p *of* M,

(2) H_B *is constant at* p,

(3) *For every vector* x *of* $T_p(M)$,

$$R(x,Jx)x-\frac{2}{n+2}\|x\|^2R^1Jx \quad \textit{is proportional to} \quad Jx.$$

Proof. It is evident that (1) $\Longrightarrow$ (2).

(2) $\Longrightarrow$ (1) : From the Remark in the previous section, we get the formula

$$\begin{aligned}g(B(x,y)x,y)=\frac{1}{32}\{&3H_B(x+Jy)\|x+Jy\|^4+3H_B(x-Jy)\|x-Jy\|^4-H_B(x+y)\|x+y\|^4\\&-H_B(x-y)\|x-y\|^4-4H_B(x)\|x\|^4-4H_B(y)\|y\|^4\}.\end{aligned}$$

If $H_B=\alpha$ (constant) at $p\in M$, then we have

$$\begin{aligned}g(B(x,y)x,y)&=\frac{\alpha}{32}\{3\|x+Jy\|^4+3\|x-Jy\|^4-\|x+y\|^4-\|x-y\|^4-4\|x\|^4-4\|y\|^4\}\\&=\frac{\alpha}{4}\{\|x\|^2\|y\|^2-g(x,y)^2+3g(x,Jy)^2\}.\end{aligned}$$

Using an orthonormal basis $\{e_1,\cdots,e_n,Je_1,\cdots Je_n\}$ of $T_p(M)$, we have

$$\begin{aligned}B_1(x,x)&=\sum_{i=1}^{n}\{g(B(x,e_i)x,e_i)+g(B(x,Je_i)x,Je_i)\}\\&=\frac{\alpha}{4}\sum_{i=1}^{n}\{\|x\|^2+g(x,e_i)^2+g(x,Je_i)^2\}\end{aligned}$$

$$=\frac{n\alpha}{2}\|x\|^2+\frac{\alpha}{2}\|x\|^2$$
$$=\frac{(n+1)\alpha}{2}\|x\|^2 .$$

Thus, by Lemma 2, we see

$$\alpha=0,$$

and

$$g(B(x,y)x,y)=0 \qquad \text{for all } x,\ y \in T_p(M) .$$

Now, $g(B(x,y)z,w)=0$ is an immediate consequence of the following relation which is checked by straightforward computation:

$$\begin{aligned} g(B(x,y)z,w)=\frac{1}{6}\{&B^*(x+z,y+w)-B^*(x+w,y+z)+B^*(x,y+z)-B^*(x,y+w)\\ &+B^*(y,x+w)-B^*(y,x+z)+B^*(z,x+w)-B^*(z,y+w)+B^*(w,y+z)\\ &-B^*(w,x+z)+B^*(x,w)-B^*(x,z)+B^*(y,z)-B^*(y,w)\} ,\end{aligned}$$

where $B^*(x,y)=g(B(x,y)x,y)$.

(2) $\Longleftrightarrow$ (3): Applying Proposition 2 to the Bochner curvature tensor B, we see that the condition (2) holds if and only if

(*) $\qquad B(x,Jx)x$ is proportional to Jx.

On the other hand, we have

$$B(x,Jx)x=R(x,Jx)x-\frac{2}{n+2}R_1(x,x)Jx-\frac{2}{n+2}\|x\|^2R^1Jx+\frac{r}{(n+1)(n+2)}\|x\|^2Jx$$

by definition. Seeing that the second and the last term of the right hand side of the above equation are proportional to Jx, we can see that (*) is equivalent to the condition (3). Q. E. D.

As an application of Theorem 2, we obtain the following

Corollary. *A Kähler manifold is of constant holomorphic sectional curvature if and only if the manifold is an Einstein Kählerian with vanishing Bochner curvature tensor.*

Proof. It is well-known that the Kähler manifold M of constant holomorphic sectional curvature is an Einstein, that is, $R^1=\frac{r}{m}$ id (id stands for the identity transformation). In this case, on account of Proposition 2, both $R(X,JX)X$ and R^1JX are proportional to JX. Hence $B=0$ from the equivalence (1) $\Longleftrightarrow$ (3) in Theorem 2.

Conversely, if $B=0$ and $R^1=\frac{r}{m}$ id, from the equivalence (1) $\Longleftrightarrow$ (3) once more, we have

$R(x,Jx)x$ is proportional to Jx at each point of M,

or equivalently

H_R is constant at each point of M.

Owing to the Schur's theorem for Kählerian case (Kobayashi-Nomizu 3) II, p.162), we see that M is a space of constant holomorphic sectional curvature. Q. E. D.

As for a H_B-preserving diffeomorphism of a Kähler manifold onto another, the following is an analogous fact to Kulkarni.

Theorem 3. *Let* (M,g,J), $(\widetilde{M},\widetilde{g},\widetilde{J})$ *be two complete Kähler manifolds with Bochner* *curvature tensors B, $\widetilde{B}$ respectively. Suppose that the Bochner curvature B does not vanish identically. Then H_B-preserving diffeomorphism $f: M \longrightarrow \widetilde{M}$ is an isometry.*

Proof. First we observe that the set of points where B is not zero is dense in M by the analyticity of the manifold. And by equivalence in Theorem 2, the assumption $B \not\equiv 0$ implies that the set

$$\{p \in M \mid H_B \neq \text{constant at } p\}$$

is dense in M.

As the assumptions in Theorem 1 are fulfilled, f is conformal, that is, $f^*\widetilde{g} = \varphi \cdot g$ for some positive real-valued function φ on M.

Next, let ω, $\widetilde{\omega}$ be the fundamental 2-forms on M, $\widetilde{M}$, respectively (ω is defined by $\omega(X, Y) = g(X, JY)$). Then $f^*\widetilde{g} = \varphi \cdot g$ implies $f^*\widetilde{\omega} = \varphi \cdot \omega$. Since ω and $\widetilde{\omega}$ are closed, we have $d\varphi \wedge \omega = 0$. This means $d\varphi = 0$ and φ is constant, because of dim $M \geqq 4$ and connectedness of the manifold. Therefore, we see that f is a homothety.

Furthermore, considering that the Kähler manifold M is not locally Euclidean when B does not vanish everywhere, we can see that the homothety f is an isometry (Kobayashi-Nomizu 3) I, p242, Lemma 2).

Q. E. D.

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