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"Theoretical Consideration for the Measurements of Attenuation of Millimetre and Centimetre Waves in the Rain Fall"

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and

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I. Introduction

During the World War II, study of microwave region in U. S. A. made a rapid and remarkable progress through various and extensive researches. After the War, the results of measurements of attenuation of electro-magnetic waves in the rain-fall region have been reported in succession : the results for 3.2*cm.* and 1.09*cm.* wave length by Robertson and King[1] in April 1946, those for 1.25*cm.* wave length by Lloyd and Anderson[2] in April 1947, and those for 0.62*cm.* wave length by Mueller[3] in April 1946. In either case attenuation which took place between transmitter and receiver about a hundred feet apart, was measured in *db per mile*, and rain precipitation at that time was also measured. These measurements are represented in Figure 1 to 4 by small circles.

Meanwhile theoretical researches related to this subject have also been made and propounded : computation for the colour of colloid by G. von Mie[4] in 1908, theoretical contribution for dielectric constant of water by P. Debye[5] in 1927 and research for attenuation of electro-magnetic wave in clouds and fogs by K. Fränz[6] in 1940. G. von Mie, solving Maxwell's equation exactly for the case where there is a dielectric sphere of arbitrary dielectric constant in plane wave field, discussed the phenomena of scattering and absorption of light by dilute colloidal dispersive medium. In this case it was assumed that effect of a number of particles is equal to that of one particle multiplied by the number of particles.

The ratio of dimension of rain drop as dispersed particle to centi-metre wave, is comparable to that of colloidal particle to visual ray ; hence Mie's theory is applicable to our present study.

Debye's paper has discussed dielectric constant and other material constants of liquid composed of dipole molecules and how it changes as varying frequency, and deduced Debye's Formulae.

Fränz has computed the attenuation of short wave in clouds and fogs, on the basis of the computation of G. von Mie, and with the dielectric constant of water gained from Debye theory of molecular dispersion. The fact that Fränz has worked on the clouds or fogs instead of rain drops, means that diameter of water drop is far smaller than the wave length of electro-magnetic wave and that he could take up only the first term of power series of *diameter/wave length*. Our case is of rain drop. And in the millimetre and centimetre region drop size is in the same order with the wave length, so that if we assume Rayleigh scattering after Fänz, the theory is contradictory to the observation.

To discuss the comparison of theory and experiments, we must compute theoretical values going back to Mie's paper. Further, since only attenuation and precipitation are measured in the experiments, we must obtain concentration of rain drop from precipitation by assuming drop size or falling speed, because it is only drop concentration in the wave path that is essential in the theoretical treatment. We have assumed drop size on the ground of several data, since there is a relation between drop size and rain-fall velocity, we have been able to estimate concentration of rain drop suspending in air from rain precipitation.

It is the substance of this article to compare the theory of absorption and scattering of electromagnetic wave by rain-fall, thus gained, with the experiments.

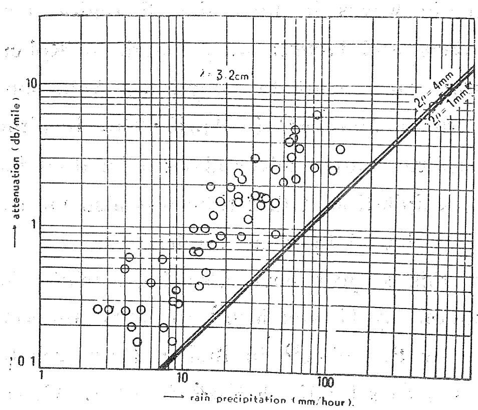

Fig. 1

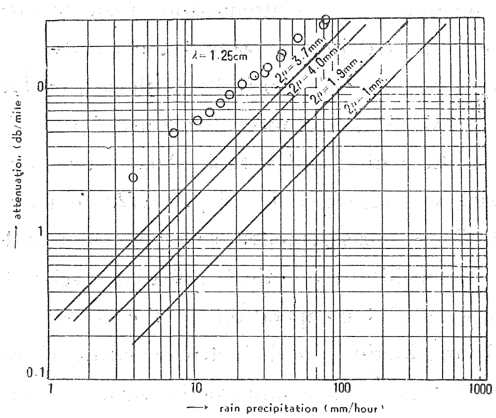

Fig. 2

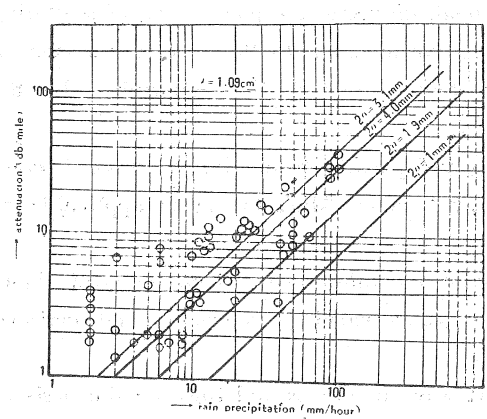

Fig. 3

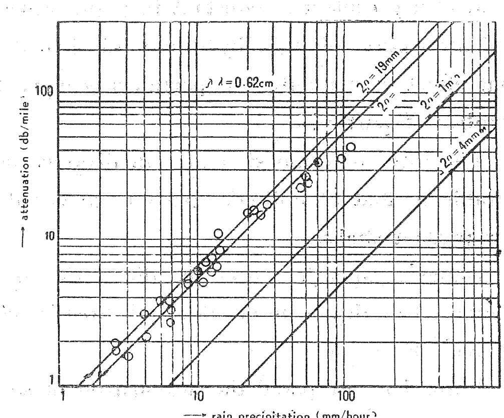

Fig. 4

II. Calculation of attenuation coefficient.

To compute the attenuation theoretically, we must begin with calculation of attenuation coefficient. After the general and exact calculation, Mie gave following formula as the absorption coefficient of colloidal solution;

$$k=N\frac{\lambda_2}{2\pi}I_m\left\{\sum_{\nu=1}^{\infty}(-1)^\nu(a_\nu-p_\nu)\right\}. \tag{1}$$

Where N is number of particle *per* cm^3, λ is wave length of electromagnetic wave in cm, $I_m\{\quad\}$ represents the imaginary part of $\{\quad\}$. This Equ. (1) represents total attenuation involving absorption and scattering. Herein attenuation due to scattering is given by

$$k=N\frac{\lambda^2}{2\pi}\sum_{\nu=1}^{\infty}\frac{|a_\nu|^2+|p_\nu|^2}{2\nu+1}. \tag{2}$$

In Equs. (1) and (2) k, k' are attenuation coefficients in unit of *per cm*, a_ν, p_ν are relative amplitude of electromagnetic field in the particle to incident electromagnetic wave, which correspond to the coefficients of expansion of electromagnetic field after surface spherical harmonics, having two sorts of terms a and p respectively, as a results of expressing the field as a sum of that have only either of electric or magnetic radial component. We call a_1, a_2,...p_1, p_2,...as electric dipole, quadrapole,...and magnetic dipole, quadrapole,...respectively, according to Mie.

This a_ν and p_ν are given by

$$a_\nu=(2\nu+1)\cdot i^\nu\cdot\frac{I_\nu'(\alpha)\cdot I_\nu(\beta)\cdot\beta-I_\nu'(\beta)\cdot I_\nu(\alpha)\cdot\alpha}{K_\nu''(-\alpha)\cdot I_\nu(\beta)\cdot\beta-I_\nu'(\beta)\cdot K_\nu(-\alpha)\cdot\alpha} \tag{3}$$

$$p_\nu=-(2\nu+1)\cdot i^\nu\cdot\frac{I_\nu(\alpha)\cdot I'_\nu(\beta)\cdot\beta-I_\nu(\beta)\cdot I_\nu'(\alpha)\cdot\alpha}{K_\nu(-\alpha)\cdot I_\nu'(\beta)\cdot\beta-I_\nu(\beta)\cdot K_\nu'(-\alpha)\cdot\alpha} \tag{4}$$

here,

$$\alpha=\frac{2\pi\rho}{\lambda} \tag{5}$$

$$\beta=\frac{2\pi\rho}{\lambda}\cdot n, \tag{6}$$

and ρ is radius of rain drop, λ is wave length in cm, n is refractive index of water, and therefore a_ν, p_ν are represented as functions of ρ/λ and n.

I_ν and K_ν are functions deduced from Bessel function of half an odd integer order, and I_ν', K_ν' are first derivatives of these :

$$I_\nu(x)=x^{\nu+1}\sqrt{\frac{\pi}{2x}}\cdot J_{\nu+\frac{1}{2}}(x), \tag{7}$$

$$K_\nu(x)=i^{\nu+1}x^{\nu+1}\sqrt{\frac{\pi}{2x}}\cdot H^{(1)}_{\nu+\frac{1}{2}}(x). \tag{8}$$

Concrete forms and various expanded forms of these functions are given in Mie's paper, but it is excessively laborious to give precision numerical calculation of a_ν, p_ν.

But a_ν, p_ν can also be represented, using power series of α, β ; u_ν, v_ν, w_ν, which have unity as initial term; as follows,

$$a_\nu=(-1)^{\nu-1}\frac{\nu+1}{\nu}\cdot\frac{\alpha^{2\nu+1}}{1^2\cdot3^2\cdot(2\nu+1)^2}\cdot u_\nu\cdot\frac{n^2-v_\nu}{n^2+\frac{\nu+1}{\nu}w_\nu}\cdot e^{i\alpha} \tag{9}$$

$$p_\nu = (-1)^\nu \frac{\nu+1}{\nu} \cdot \frac{\alpha^{\nu+1}}{1^2 \cdot 3^2 \cdot (2\nu+1)} \cdot u_\nu \cdot \frac{1-v_\nu}{1+\dfrac{\nu+1}{\nu} w_\nu} \cdot e^{i\alpha}, \quad (10)$$

so since these are of order of $\alpha^{2\nu+1}$, $\alpha^{2\nu+3}$ respectively for the case of $\alpha \leqslant 1$, $\beta \leqslant 1$, it comes into consideration, term by term from lower mode, as α and β become lager. If we put u_1, v_1, w_1, equal unity for a_1 (electric dipole), it becomes,

$$a_1 = 2\alpha^3 \cdot e^{i\alpha} \cdot \frac{n^2-1}{n^2+2} \quad (11)$$

So it reduces to so-called Rayleigh scattering formula.

However, for the case which is now under consideration, since we can not regard as $\alpha \leqslant 1$, $\beta \leqslant 1$, higher order terms of expansion of a_1 as well as higher mode terms of Equs. (1) and (2) are essential.

Therefore, we intended, returning to Equs. (3) and (4), to calculate electric dipole, quadrapole, magnetic dipole etc., and for the first place we have made exact calculation for a_1. As the result of this, calculated values of k and k' due to a_1 only are shown in Fig. 6. According to this, attenuation by no means increase proportional to *(drop radius)*^{3} like Rayleigh scattering formula, but it shows maximum attenuation for some dropsize $\left(\text{about } 2\rho \doteqdot \frac{1}{3}\lambda.\right)$ with all wave lengths.

At this calculation we employed following original function forms,

$$\left.\begin{aligned} I_1(x) &= -\cos x + \frac{\sin x}{x} \\ I_1'(x) &= +\sin x + \frac{\cos x}{x} - \frac{\sin x}{x^2} \\ K_1(-x) &= -\frac{i}{x} \cdot e^{-ix} \cdot (1+ix) \\ K_1'(-x) &= +\frac{i}{x^2} \cdot e^{-ix} \cdot \{(1-x^2)+ix\}\,. \end{aligned}\right\} \quad (12)$$

And as for value of refractive index n included in β, since it has dispersing region around about $2cm.$, as it is well known, it becomes naturally an imaginary number which varies with wave length. Therefore with our calculation, we employed dielectric constant and $\tan\delta$ resulting from Debye's formula, which has been described in Fränz's paper as showed in Fig. 5, calculating refractive index from

$$n = \sqrt{\varepsilon} = \sqrt{\varepsilon' - i\varepsilon''}$$

with every wave lengths. For the values of attenuation showed in Fig. 6, this n values are employed.

III. Considerations of Rain Drop Size.

Exact estimation of rain drop size is a very hard problem, because it differs with rain character, and also because even in one rain fall it has complicated distribution ; therefore it becomes difficult to get proper conclusion in comparison of theory and measurements. Fortunately, among papers of measurements, that of Lloyed and Anderson gave results of efforts to get correlation between rain drop size and rain precipitation. This is

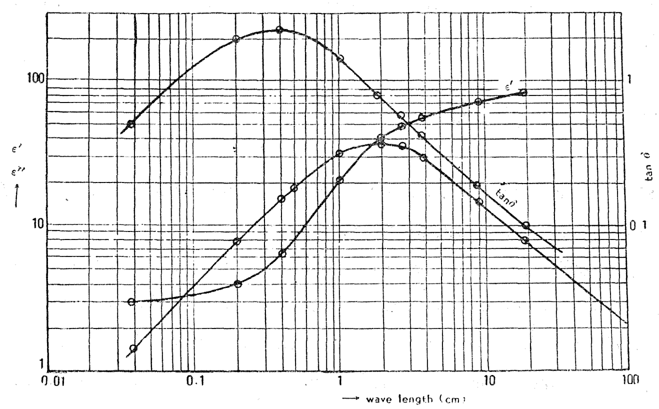

Fig. 5

shown in Fig. 7. According to this, mean drop diameters are mostly between 1 to 2 *mm*. Here we have to pay attention to statistical treatment in evaluating the mean drop diameter in one rain fall. Mean diameter is gained from frequency distribution function, but frequency distribution function differs with independent variable employed. Frequency curve is defined by

$$f(x)=\frac{dn}{dx}, \tag{13}$$

when we take the number of measured value which fall in x to $x+dx$ as dn. Therefore, frequency curve for the function of x ; $y=\phi(x)$ must be obtained from the relation

$$f(x)dx=g(y)dy=g\{\phi(x)\}\cdot\frac{dy}{dx}\cdot dy.$$

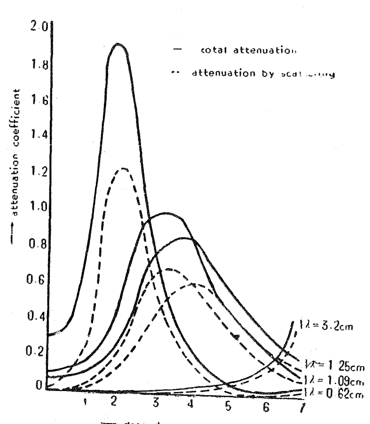

Fig. 6

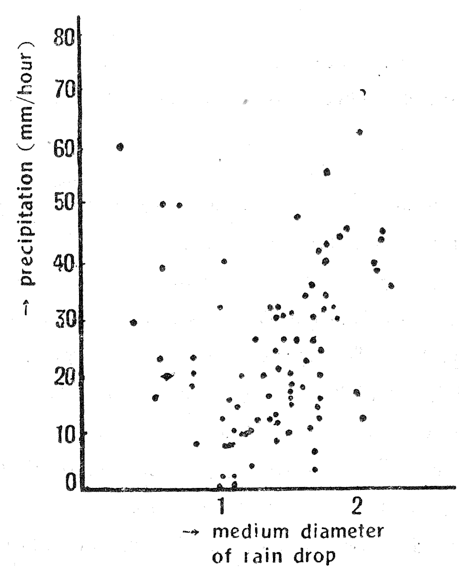

Fig. 7

Therefore, to obtain the approximate mean diameter, there is a method to define medium diameter as x_0 gained from

$$\int_0^{\infty} f(x)dx = 2\int_0^{x_0} f(x)dx \tag{14}$$

According to this, for the other variables medium value amount to $y_0 = \phi(x_0)$, and for the case in which homogeneous random value is unknown, we can avoid contradiction that mean values are different for variable used. By the paper of Lloyed and Anderson medium diameter is employed according to this method. However, it is necessary to take a mean diameter which is effective to the phenomena of scattering and absorption. Now putting frequency curve for drop radius as $f(\rho)$, attenuation constant k, function of ρ as $k(\rho)$,

$$k(\rho) = CK(\rho)$$

$$C = N\frac{4}{3}\pi\rho^3 \qquad \text{concentration } gr/cm^3$$

and effective mean radius is ρ_0 gained from

$$\int_0^{\infty} K(\rho)\cdot\rho^3 f(\rho)\, d\rho = K(\rho_0)\rho_0^3\int_0^{\infty} f(\rho)\cdot d\rho. \tag{15}$$

Strictly, it can be gained only after determination of $K(\rho)$ curve, since, however, variation of $K(\rho)$ is small, ρ_0 is obtained as mean value for the frequency curve $\rho^3 f(\rho)d\rho$.

$$\int_0^{\infty} \rho^3\cdot f(\rho)\cdot d\rho = 2\int_0^{\rho_0'} \rho^3 f(\rho)\, d\rho \tag{16}$$

And we take medium value in place of this. We call this ρ_0' as mass medium radius for convenience sake. So the ratio of this to the conventional medium radius ;

$$\frac{\rho_0'}{\rho_0} = \eta \tag{17}$$

is constant for the definite frequency distributlon curve.

It is difficult to determine the form of frequency distribution curve, but authors obtaining ρ_0', ρ_0 from two results of observation at hand (we express to Mr. Takahashi, director of Nagoya Local Meteorological Observatory, for offering the data generously) and obtained 1.5 and 2.1 for η of Equ. (17). An example is shown in Fig. 8. Comparing this values of η with Fig. 7, it is appropriate to regard that medium diameter falls in 1 to 4 mm.

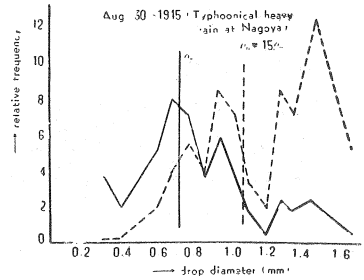

Fig. 8

In the next place, as for the correlationship between rain drop size and rain fall velocity, theoretically it can be obtained from Stokes's and

Newton's resistance law, but actually it approaches to Newton's law when drop size is large and to Stokes's law when it is small. And usually measurements of Mr. Schmidt of Austraria, or the experimental formula which is combination of two theories is in use. This relation between drop diameter and rain fall velocity is shown in Fig. 9.

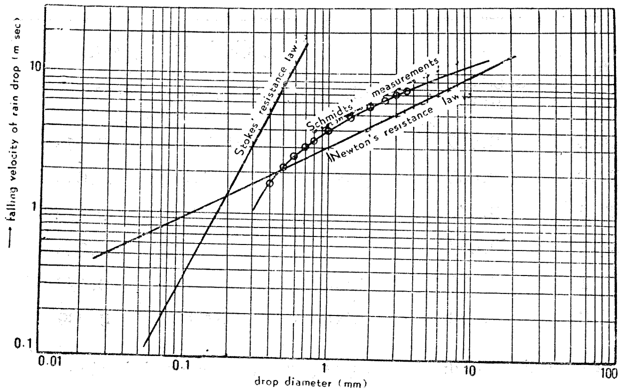

Fig. 9

IV. Theoretical Value of Attenuation.

As we mentioned in preceding section, medium diameter effective for attenuation is 1—4 *mm*, and from Fig. 9. rain fall velocity v *cm*/sec is fixed, so concentration of rain drops C *gr/cm*^{3} will be found from precipitation h *mm/hour* as follows ;

$$C=\frac{0.1h}{60\times60\times v\times100}\ gr/cm^3. \qquad (18)$$

While C is also

$$C=N\frac{4}{3}\pi\rho^3.$$

Therefore

$$k=N\frac{\lambda^2}{2\pi}I_m\{-a_1\}=\frac{C\lambda^2}{\frac{4}{3}\pi\rho^3\cdot2\pi}I_m\{-a_1\}=\frac{3C\lambda^2}{8\pi^2\rho^3}I_m\{-a_1\}$$

$$k'=N\frac{\lambda^2}{2\pi}\frac{|a_1|^2}{3}=\frac{C\lambda^2}{8\pi^2\rho^3}|a_1|^2$$

So putting (18) in thses,

$$k=\frac{3\lambda^2}{8\pi^2\rho^3}\frac{h}{36\cdot10^5v}I_m\{-a_1\} \qquad (19)$$

$$k'=\frac{\lambda^2}{8\pi^2\rho^3}\frac{h}{36\cdot10^5v}|a_1|^2 \qquad (20)$$

Where k, k' is in *naper/cm*, reducing this into *db/mile* which is adopted in mesurements,

$$k_0=10log_{10}e\times1.61\times\frac{\lambda^2h}{12\cdot8\pi^2\rho^3v}I_m\{-a_1\}=\beta_0h \qquad (21)$$

$$k_0'=10log_{10}e\times1.61\times\frac{\lambda^2h}{36\cdot8\pi^2\rho^3v}|a_1^2|=\beta_0'h \qquad (22)$$

There attenuation is proportional to precipitation.

From this formula and from Fig. 5, β_0 is obtained as in Tab. 1. For the comparison with mesurements of Fig. 1 to 4, the lines of Fqu. (21) $k_0=\beta_0 h$ obtaind from the table are drawn in corresponding figures.

Table 1

2ρ *mm*	$3.6\times v$	0.1×10^6 / $3600\times v\times100$	β_0 (in *db per mile/mm* prec) $\lambda=0.62$	$\lambda=1.09$	$\lambda=1.25$	$\lambda=3.2$
1.0	14.4	0.059	0.179	0.069	0.048	0.014
1.9	20.0	0.050	0.650	0.158	0.093	0.008
3.1	25.2	0.040		0.285	0.218	
3.7	27.0	0.037			0.228	
4.0	28.1	0.036	0.058	0.205	0.177	0.014
7.0	35.0	0.028	0.005	0.024	0.039	0.072

attenuation coefficient β_0 (*db per mile/mm* precipitation)

V. Comparison of Theory with Experiments and General Consideration.

Comparing theory and experiments on Figs. 1 to 4, these are in good agreement in the order of magnitude. And in experiments the attenuation is propotional to the precipitation, in harmony with the theory. But looking over these precisely, in Fig. 1, for the wave of 3.2 *cm* wave length attenuation is greater by the theory than by measurement, and to adapt one to the other we must take the drop size as 7 *mm* for rain drop size. In Fig. 2 for 1.25 *cm* wave, observed values are also greater and observed values are still a little greater than the dotted line of $2\rho=3.7$ *mm* in the figure which gives the maximum theoretical attenuation. In Fig. 3 for 1.09 *cm* wave, we see better agreement than for 3.2 and 1.25 *cm* wave, but line of $2\rho=3.1mm$ in the figure giving the maximum theoretical attenuation, still appears to be a little less than the observation. In Fig. 4 for the 0,62 *cm* wave, line of $2\rho=1.9mm$ in the figure which gives the maximum theoretical attenuation shows good agreement with the observation.

Now we have calculated only electric dipole a_1 exactly, which is no more complete one, so we will discuss the points which is to be considered for further exactness.

For the first place as for the calculation of attenuation constant k, according to the assumption employed by Mie ;

1. Assumption of sphere shaped dielectrics appears to be proper for the case of rain. But the assumption that k is N times that of one particle, is not always right for our case, but come into question at two points as follows :
2. Rain drops recieve secondary waves reflected from other drops in addition to the incident plane wave, so that there exist so-called multiple diffraction.
3. Putting the scattered electric and magnetic field from *i–th* individual rain drop to E_i and H_i, the total energy lossed by scattering is

$$L=\int_0^1 dt\int\int(\sum_i E_i)\times(\sum H_i)\,d\omega \qquad (22)$$

and this is not equal to the sum of individual scattered energy ;

$$L'=\int_0^1 dt\int\int\sum_i(E_i\times H_i)\,d\omega \qquad (23)$$

For these points there is the researches by Ros Gans and the other, and further discussions are necessary for our case, but assumption of N times to be still fit for the first approximtion.

Next for the refractive index of the water ;

4. Although Debye's theory have been verified by mesurements, theory and measurements are both about the water as a vapour or a dilution to the other solvent, but in the actual water there is some association, so it seems not proper to employ the vapour value of refractive index. In this place, however, it is difficult to find the value which include the existence of association.
5. For our cace, since $\alpha \ll 1$, $\beta \ll 1$ is not hold, higher mode terms ; electric quadrapole, magnetic dipole etc. becomes essential. We are now calculating these terms.

Last place, relating to the treatment of drop size and presipitation ;

6. The rain drop size which employd here is not that of the case when the measurement was carried out, but estimated from the other data, and therefore it may be possible that the estimation is not appropriate.
7. As it is mentioned in the papers of measurements, precipitation is not uniform throughout the path in which the measurements are carried out, measured precipitation does not represent that contributed to the attenuation, without sufficient number of measuring point.
8. Since it is difficult to measure the precipitation instantaneously, on the rain fall which varies rapidly with time, the precipitation at the instant of mesurement does'nt coincide with the mean precipitation before and after the time.

In adition to these aspects ;

9. In the centimetre and millimetre wave region, the very measurement of attenuation is considerably difficult, and it seems unable to avoid the inclusion of more or less systematic error.

VI. Conclusion.

Because phenomena of absorption and scattering of centimeter and millimeter waves by rain drop are very interesting in the present state that the application of microwaves to the meteorology is having more activity, and because the rain drop size amounts to the same order with the wave length, we have calculated the electric dipole term of scattering and absorption, under the plan of calculating the electric dipole, quadrapole and magnetic dipole successively going back to the Mie's paper ; and deduced the concentration of rain drop, assuming the drop size as 1 to 4 *mm*, from various data ; and compared the theoretical attenuation with the experiments in U.S.A.. As the result, the experiments agree with the theory in the order of magnitude. But these doesn't agree precisely, so we have discussed various conceivable origins of errors. We are now calculating electric quadrapole and magnetic dipole term, and after the performance of this, we may be able to verify the better agreement of theory and expreiments.

References :

1) S. D. Robertson and A. D. King : I. R. E. 34. (1946) 278.
2) L. Lloyed and J. Anderson : I. R. E. 35. (1947) April.
3) G. E. Mueller : I. R. E. 34. (1946) April.
4) G. von Mie : Ann. D. Physik. 25. (1908) 377.
5) P. Debye : Polare Molekln. Leipzig. (1929) S. 104-108.
6) K. Fränz : H. F. T. E. A. 55. (1940) 141.