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On the Dimension of Partially Ordered Sets.

By

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The present paper is concerned with several problems on the dimension of posets*) (partially ordered sets) in the sense of Dushnik and Miller [1] and aims ultimately at the study of the least upper bound of the dimension of posets defined on a set or that of the number of elements which are necessary in order to define a poset having given dimension.

§1. Partial orders and posets.

We will distinguish a partial order from a partially ordered set.

A *partial order***) on a set P is a subset $\boldsymbol{P}$ of $P\times P$ which is not a subset of $P'\times P'$ for any proper subset P' of P and satisfies

$P1$: *For all* x, $(x, x) \in \boldsymbol{P}$.

$P2$: *If* $(x, y) \in \boldsymbol{P}$ *and* $(y, x) \in \boldsymbol{P}$, *then* $x=y$.

$P3$: *If* $(x. y) \in \boldsymbol{P}$ *and* $(x, z) \in \boldsymbol{P}$, *then* $(x, z) \in \boldsymbol{P}$.

A partial order $\boldsymbol{L}$ on a set P which satisfies

$P4$: *Given* x *and* y, *either* $(x, y) \in \boldsymbol{L}$ *or* $(y, x) \in \boldsymbol{L}$,

is said a *liner order* on P.

The *null-order* on P is the set $\{(x, x) \mid x \in P\}$.

A *poset* P is the set P associated with a partial order on it. When the associated partial order is a linear order it is said a *chain* on P and when the null-order is associated with it is said to be *unordered*.

When $(x, y) \in \boldsymbol{P}$ we say that x *includes* y in P and write $x \geqq y$ in P. When $(x, y) \notin \boldsymbol{P}$ and $(y, x) \notin \boldsymbol{P}$ we say x and y are *non-comparable* in P and write $x \phi y$ in P. Either notations will be used according to circumstances.

Let $\boldsymbol{P}'$ is a proper subset of a partial order $\boldsymbol{P}$ which satisfies the conditions $P1$, $P2$ and $P3$. Then there exists a subset P' of P on which the partial order $\boldsymbol{P}'$ is defined. There are two cases. When $P' \subset P$, $\boldsymbol{P}'$ is said a *partial suborder* of $\boldsymbol{P}$ and P' with $\boldsymbol{P}'$ a *subposet* of P. When $P'=P$, P is said a *extension* of P' and denoted by $P > P'$.

Let $\{P_q \mid q \in Q\}$ a system of non-overlapping posets associated with each element of

*) For the brevity's sake we adopt this convenient term suggested by G. Birkhoff.

**) The partial order thus defined is essentially no more than the oriented graph of the corresponding poset.

a poset Q. Then the set

$$\boldsymbol{P}=\bigcup_Q \boldsymbol{P}_q \cup \{(x, y) \mid x\in P_q, y\in P_{q'}, (q, q') \in \boldsymbol{Q}\}$$

is a partial order on the set $P=\bigcup_Q P_q$ which is said the sum of partial order $\boldsymbol{P}_q$ with the basic poset Q and denoted by $\sum_Q \boldsymbol{P}_q$. The poset P corresponding this partial order is said the *sum* of posets P_q with the basic poset Q and denoted by $\sum_Q P_q$.

Let $\{\boldsymbol{P}_t \mid t\in T\}$ be a system of partial orders on a set P and $\{P_t \mid t\in T\}$ the system of corresponding posets. Further let $\{P_t \mid t\in T\}$ be symply ordered by the relation of being an extension of or equal, that is, $P_t \geqq P_{t'}$ or $P_t \leqq P_{t'}$ for any $t, t'\in T$. Then

$$\boldsymbol{P}=\bigcup_T \boldsymbol{P}_t$$

is also a partial order on the set P. The corresponding poset is denoted by $\vee_T P_t$. It is obvious that $\vee_T P_t \geqq P_t$ for all $t \in T$.

§2. Linear extensions of a poset.

An extension L of a poset P is said a *linear extension* provided L is a chain on P. It is obvious that if $x \not\parallel y$ in P, then $x\neq y$ and $(x, y) \in \boldsymbol{L}$ or $(y, x) \in \boldsymbol{L}$ where $\boldsymbol{L}$ is the linear order on P associated with L, that is $x > y$ in L or $y > x$ in L. Conversely it has already been proved that

(2. 1) *If $p \not\parallel q$ in a partial order P, then there exists a linear extension L of P such that $(p, q) \in \boldsymbol{L}$ (Szpilrajn[2]).*

Proof. The following proof does not differ essentially from that of Szpilajn. For p and q such that $p \not\parallel q$ in P there exists an extension P^* such that $(p, q) \in \boldsymbol{P}^*$. In fact the poset corresponding to a partial order

$$\boldsymbol{P}^*=\boldsymbol{P} \cup \{(x, y) \mid (x, p) \in \boldsymbol{P}, (q, y) \in \boldsymbol{P}\}$$

on P is a required one. Now let $\mathfrak{S} = \{P_t \mid t\in T\}$ be the system of all extensions of P such that $(p, q) \in P$. $\mathfrak{S}$ is a poset by the relation of being an extension of or equal. Let $\{P_t \mid t \in T' \subseteqq T\}$ be any chain in $\mathfrak{S}$. Then $\vee_{T'} P_t \in \mathfrak{S}$ is an upper bound to the chain. Hence $\mathfrak{S}$ contains a maximal element P_m which is a required linear extension.

Let us prove further the existence of some particular linear extensions which are of use later.

A linear extension $L(a, P)$ of a poset P is said to be *left* with respect to an element a of P provided that $x \not\parallel a$ in P implies $(a, x) \in \boldsymbol{L}(a, P)$. Dually a *right* linear extension $\boldsymbol{M}(a, P)$ with respect to a is defined.

(2. 2) *For an element a of a poset P, $\mathfrak{L}(a, P) \neq 0$ ($\mathfrak{M}(a, P) \neq 0$) where $\mathfrak{L}(a, P)$ ($\mathfrak{M}(a, P)$) is the system of all left (right) linear extensions of P with respect to a.*

Proof. Let P, P^*, Q and Q^* be subposets of P on the sets $P_a=\{x \mid (x, a) \in \boldsymbol{P}\}$, $P_a^*=P-P_a$, $Q_a=\{x \mid (a, x) \in \boldsymbol{P}\}$ and $Q_a^*=P-Q_a$ respectively. Let $\mathfrak{L}_a$, $\mathfrak{L}_a^*$, $\mathfrak{M}_a$ and $\mathfrak{M}_a^*$ be the systems of all linear extensions of the subposets P, P^*, Q and Q^* respectively. Further let $L(a, P)=L_a^*+L_a$[1] be the ordinal sum of $L_a^* \in \mathfrak{L}_a^*$ and $L_a \in \mathfrak{L}_a$

and $M(a, P)=M_a+M_a^*$ that of $M_a \in \mathfrak{M}_a$ and $M_a^* \in \mathfrak{M}_a^*$. Then $L(a, P) \in \mathfrak{L}(a, P)$ and $M(a, P) \in \mathfrak{M}(a, P)$.

One sees easily that $\mathfrak{L}(a, P)=\{L_a^*+L_a \mid L_a^* \in \mathfrak{L}_a^*, L_a \in \mathfrak{L}_a\}$ and $\mathfrak{M}(a, P)=\{M_a+M_a^* \mid M_a \in \mathfrak{M}_a, M_a^* \in \mathfrak{M}_a^*\}$.

A linear extension L is said to be *left (right) with respect to a chain* C in P provided L is left (right) with respect to every element $c \in C$, that is, provided $L \in \bigcap_C \mathfrak{L}(c, P)$ $(L \in \bigcap_C \mathfrak{M}(c, P))$.

(2. 3) $\mathfrak{L}(C, P) \equiv \bigcap_C \mathfrak{L}(c, P) \neq 0$ $(\mathfrak{M}(C,P) \equiv \bigcap_C \mathfrak{M}(c, P) \neq 0)$ *if and only if* C *is a chin in* P.

Proof. Let $c, c' \in C$ and $L \in \mathfrak{L}(C, P)$ and assume that $c \not\phi c'$ in P. Then since $L \in \mathfrak{L}(c, P)$ and $L \in \mathfrak{L}(c', P)$, we have both $(c', c) \in \boldsymbol{L}$ and $(c, c') \in \boldsymbol{L}$. This is contradictory. Hence c and c' are comparable in P that is C is a chain in P.

Conversely let C be a chain in P. Decompose the set P to the sum of folloing three non-overlapping subsets

$P_1 = \{x \mid (x, c) \in \boldsymbol{P}$ for no $c \in C\}$,

$P_3 = \{x \mid (x, c) \in \boldsymbol{P}$ for all $c \in C\}-A$,

$P_2 = P-(P_1 \cup P_3)$.

Let L_i be any linear extension of the subposet P_i of P for $i=1$ and 3. A linear extension L_2 of the subposet P_2 of P is constructed in the following manner. For each element $x \in P_2$ put

$$C_x=\{c \in C \mid (x, c) \in \boldsymbol{P}_2\},$$

and let $x, y \in P_2$ be equivalent if $C_x=C_y$. Then the set P_2 is devided into classes $\{\boldsymbol{P}_\xi \mid \xi \in X\}$ where X is the set of representatives. Since C is a chain in P the system $\{C_\xi \mid \xi \in X\}$ is a chain by the relation of the set-inclusion. Hence X associated with the linear order

$$\boldsymbol{X}=\{(\xi, \xi') \mid C_\xi \supseteqq C_{\xi'}\}$$

is a chain. Let L_ξ be any linear extension of subposet P_ξ of P_2 and $L_2=\sum_X L_\xi$ the sum of L_ξ's with the base X. Then L_2 is a linear extension of P. Since L_2 is a chain on P_2 and $\sum_X \boldsymbol{P}_\xi \subseteqq \sum_X \boldsymbol{L}_\xi=\boldsymbol{L}_2$ in order to prove this it is sufficient to show that

$$\boldsymbol{P}_2 \subseteqq \sum_X \boldsymbol{P}_\xi .$$

Let $(x, y) \in \boldsymbol{P}_2$. When $x \in P_\xi$ and $y \in P_\xi$, $(x, y) \in \boldsymbol{P}_\xi \subset \sum_X \boldsymbol{P}_\xi$. When $x \in P_\xi$ and $y \in P_{\xi'}$ for $\xi \neq \xi'$, $(\xi, \xi') \in \boldsymbol{X}$. For otherwise there exists $c \in C$ such that $c \in C_{\xi'}$ but $c \notin C_\xi$. $c \in C_{\xi'}$ with $y \in P_{\xi'}$ implies $(y, c) \in \boldsymbol{P}_2$, hence $(x, c) \in \boldsymbol{P}$. On the other hand $c \notin C$ with $x \in P_\xi$ implies $(x, c) \notin \boldsymbol{P}_2$. This is contradictory. Hence $\boldsymbol{P}_2 \subseteqq \sum_X \boldsymbol{P}_\xi$.

Now let $L=\sum_3 L_i$ be the sum of above three linear extensions with the base $\mathbf{3}$. Then L is a linear extension of P. In order to show this it is sufficient to see that $\boldsymbol{P} \leqq \sum_3 \boldsymbol{P}_i$, since it is evident that L is a chain on P and $\sum_3 \boldsymbol{P}_i \leqq \boldsymbol{L}$. Let $(x, y) \in \boldsymbol{P}$. If $x, y \in P_i$, then $(x, y) \in \boldsymbol{P}_i \subset \sum_3 \boldsymbol{P}_i$. If $x \in P_i$, $y \in P_j$, then $i > j$. In

1) The sum $\sum_Q P_q$ where Q is the chain $1, 2, \ldots, n$ will be denoted by $\sum_n P_i$ or $P_1+P_2+\ldots+P_n$.

fact if $x \in P_1$ and $y \in P_2 \cup P_3$, then $(x, c) \notin \boldsymbol{P}$ for any c and $(y, c') \in \boldsymbol{P}$ for some c'. The latter implies $(x, c') \in \boldsymbol{P}_2$ which contradicts the former. If $x \in P_2$ and $y \in P_3$, then $(x, c') \notin \boldsymbol{P}$ for some c' and $(y, c) \in \boldsymbol{P}$ for all c. The latter implies $(x, c) \in \boldsymbol{P}$ for all c which contradicts the former. $i > j$ implies $(x, y) \in \sum_3 \boldsymbol{P}_i$. Hence $\boldsymbol{P} \subseteqq \sum_3 \boldsymbol{P}_i$ i. e., $P \leqq \sum_3 P_i$.

Moreover $L \in \mathfrak{L}(C, P)$. In order to show this let L_c and L_c^* be subchains of L on $P_c = \{x \mid (x, c) \in \boldsymbol{P}\}$ and on $P^* = P - P_c$ respectively. Evidently $L_c^* \in \mathfrak{L}_c^*$ and $L_c \in \mathfrak{L}_c$ and hence $L_c^* + L_c \in \mathfrak{L}(C, P)$ where $\mathfrak{L}_c^*$ and $\mathfrak{L}_c$ are systems of all linear extensions of subposet P_c^* and P_c respectively. Hence it remains only to show that $L = L_c^* + L_c$. For the purpose it is sfficient to show that $(c, x) \in \boldsymbol{L}$ whenever $(x, c) \notin \boldsymbol{P}$ and that $(x, c) \in \boldsymbol{L}$ whenever $(x, c) \in \boldsymbol{P}$. When $(x, c) \notin \boldsymbol{P}$, $x \in P_1 \cup P_2$. If $x \in P_1$, then $(c, x) \in \boldsymbol{L}$ since $c \in P_2$. If $x \in P_2$, then $x \in P_\xi$ and $c \in P_{\xi'}$ for $(\xi', \xi) \in \boldsymbol{X}$. Hence $(c, x) \in \boldsymbol{L}$. When $(x, c) \in \boldsymbol{P}$, $x \in P_2 \cup P_3$. If $x \in P_3$, then evidently $(x, c) \in \boldsymbol{L}$. If $x \in P_2$, then either $x, y \in P_\xi$ for some ξ or $x \in P_\xi$, $c \in P_\xi$ for $\xi' \neq \xi$. In the first case $(x, c) \in \boldsymbol{P}_\xi \subseteqq \boldsymbol{L}_2 \subset \boldsymbol{L}$. In the second case $(\xi, \xi') \in \boldsymbol{X}$, hence $(x, c) \in \boldsymbol{L}$.

$\mathfrak{M}(C, P) \neq 0$ may be proved dually. But let the gist be repeated. Decompose P to the sum of three non-overlapping subsets :

$P_1' = \{x \mid (c, x) \in \boldsymbol{P}$ for all $c \in C\} - C$,

$P_3' = \{x \mid (c, x) \in \boldsymbol{P}$ for no $c \in C\}$,

$P_2' = P - (P_1' \cup P_3')$.

Let M_i be any linear extension of the subposet $P_{i'}$ of P for $i = 1, 3$. For every element $x \in P_2'$ put

$$C_x' = \{c \in C \mid (c, x) \in \boldsymbol{P}_2'\}$$

By letting $x, y \in P_2$ be equivalent when $C_x' = C_y'$, P_2 is devided into classes $\{P_\xi' \mid \xi \in X'\}$ which is a chain by the relation of set-inclusion. Hence X' is a chain with the linear order

$$\boldsymbol{X}' = \{(\xi, \xi') \mid C_\xi' \supseteqq C_{\xi'}'\}$$

on it. Let M_ξ be any linear extension of the subposet P_ξ' of P_2'. Then $M_2 = \sum_{X'} M_\xi$ is a linear extension of P_2 and $M = \sum_3 M_{i'}$ that of P which is an element of $\mathfrak{M}(C, P)$.

(2. 4) *Let A and B two chains in a poset P such that $(b, a) \notin \boldsymbol{P}$ $((a, b) \notin \boldsymbol{P})$ for $a \in A$ and $b \in B$. Then*

$$\mathfrak{L}(A, P) \cap \mathfrak{M}(B, P) \neq 0 \ (\mathfrak{M}(A, P) \cap \mathfrak{L}(B, P) \neq 0).$$

Proof. Put

$P_1 = \{x \mid (x, a) \in P$ for no $a \in A\}$,

$P_3 = \{x \mid (x, a) \in P$ for all $a \in A\} - A$,

$P_2 = P - (P_1 \cup P_3)$;

$P_1' = \{x \mid (b, x) \in P$ for all $b \in B\} - B$,

$P_3' = \{x \mid (b, x) \in P$ for no $b \in B\}$,

$P_2' = P - (P_1' \cup P_3')$;

$P_{ij} = P_i \cap P_j'$.

As is easily seen $P_{21} = P_{22} = P_{31} = P_{32} = 0$. Hence

$P_{11} \cup P_{12} \cup P_{13} = P_1, \quad P_{23} = P_2, \quad P_{33} = P_3$;

$P_{11} = P_1', \quad P_{12} = P_2'. \quad P_{13} \cup P_{23} \cup P_{33} = P_3'$.

Let L_{ij} be a linear extension of the subposet P_{ij} of P where L_{11}, L_{13} and L_{33} may be chosen arbitrarily, but let L_{23} and L_{12} be that which are constructed in the same manner as in the proof of (2. 3) L_2 and M_2 were constructed respectively. Then it is easily seen that

$$L_{11}+L_{12}+L_{13} \in \mathfrak{L}(P_1),\ L_{23} \in \mathfrak{L}(P_2),\ L_{33} \in \mathfrak{L}(P_3)\ ;$$

$$L_{11} \in \mathfrak{L}(P_1'),\ L_{12} \in \mathfrak{L}(P_2'),\ L_{13}+L_{23}+L_{33} \in \mathfrak{L}(P_3'),$$

where $\mathfrak{L}(P)$ means the system of all linear extensions of the poset in the brackets. Hence

$$L_{11}+L_{12}+L_{13}+L_{22}+L_{33} \in \mathfrak{L}(A, P) \cap \mathfrak{M}(B, P).$$

Two chains A and B in a poset P are said to be non-comparable provided both $(b, a) \notin \boldsymbol{P}$ and $(a, b) \notin \boldsymbol{P}$ for $a \in A$ and $b \in B$. As a corollary of (2. 4) we have

(2. 5) *If two chains A and B are non-comparable, then* $\mathfrak{L}(A, P) \cap \mathfrak{M}(B, P) \neq 0$ *and* $L(B, P) \cap \mathfrak{M}(A, P) \neq 0$.

§3. Dimension of posets.

Let $\{P_s \mid s \in S\}$ be a sytem of posets which are defined on a set P and $\boldsymbol{P}_s$ the partial order associated with P_s. Then $\bigcap_s \boldsymbol{P}_s$ is a partial order on P. Hence P associated with $\bigcap_s \boldsymbol{P}_s$ is a poset which is denoted by $\wedge_s P_s$.

Let $\mathfrak{R} = \{L_s \mid s \in S\}$ be a system of linear extensions of a poset P. $\mathfrak{R}$ is said a *realizer* of P provided $P = \wedge_s L_s$. It is evident that $\mathfrak{R}$ *is a realizer of P if and only if $x \not\geq y$ in P implies the existence of s, $s' \in S$ such that* $(x, y) \in \boldsymbol{L}_s$ *and* $(y, x) \in \boldsymbol{L}_{s'}$. By this remrk and (2. 1) every poset has its realizer.

Among the realizers $\mathfrak{R}_t = \{L_s \mid s \in S_t\}$ $(t \in T)$ of a poset P a realizer $\mathfrak{R}_\tau = \{L_s \mid s \in S_\tau\}$ such that $n[S_\tau] \leqq n[S_t]$ for all $t \in T$ is said to be *minimal* where $n[S]$ denotes the number of elements of a set S. The number of elements of a minimal realizer $\mathfrak{R} = \{L_s \mid s \in S\}$ of a poset P is said the *dimension* of P and denoted by $D[P]$. Of course $D[P] = n[S]$.

One sees easily that $D[P]$ is finite provided $n[P]$ is finite and $D[P] \leqq n[P]$ provided $n[P]$ is a transfinite cardinal. But we assert that

(3. 1) *For every poset P, $D[P] \leqq n[P]$.*

Proof. The system $\mathfrak{R} = \{L(x, P) \mid x \in P\}$ of left linear extensions of P with respect to each element $x \in P$ is a realizer of P. Because $x \not\geq y$ in P implies $(x, y) \in \boldsymbol{L}(x, P)$ and $(y, x) \in \boldsymbol{L}(y, P)$. Hence $D[P] \leqq n[P]$.

Let $\mathfrak{S}$ be the system of all poset defined on a set P. Then (3. 1) says that $n[P]$ is an upper bound of $D(P)$ for $P \in \mathfrak{S}$. A question arise here as to what is the least upper bound of $D[P]$ over $\mathfrak{S}$ which will be answered in the last section. For a fixed $P \in \mathfrak{S}$ the estimation above obtained may be replaced by a more pricise one.

(3. 2) *Let* $\mathfrak{C}=\{C_t \mid t \in T\}$ *be a system of non-overlapping chains in a poset* P. *Then* $D[P] \leqq n[P-\bigcup_T C_t]+n[T]$.

Proof. Choose a $L(x, P) \in \mathfrak{L}(x, P)$ for each element $x \in P' = P-\bigcup_T C_t$ and a $L(C_t, P) \in \mathfrak{L}(C_t, P)$ for each $t \in T$. Then the system

$$\mathfrak{R}=\{L(x, P) \mid x \in P'\} \cup \{L(C_t, P) \mid t \in T\}$$

is a realizer of P. In fact for $x \not\sim y$ in P there are three cases.

(1) $x, y \in P'$. Then $(x, y) \in \boldsymbol{L}(x, P)$ and $(y, x) \in \boldsymbol{L}(y, P)$.

(2) $x \in P', y \in C_t$. Then $(x, y) \in \boldsymbol{L}(x, P)$ and $(y, x) \in \boldsymbol{L}(C_t, P)$.

(3) $x \in C_t, y \in C_{t'}(t \neq t')$. Then $(x, y) \in \boldsymbol{L}(C_t, P)$ and $(y, x) \in \boldsymbol{L}(C_{t'}, P)$.

Hence $D[P] \leqq n\,[P']+n[T]$.

As a corollary of (3. 2) it follows that

(3. 3) *Let* $\mathfrak{C}=\{C_t \mid t \in T\}$ *be a system of non-overlapping chains in* P *such that* $\bigcup_T C_t=P$. *Then* $D[P] \leqq n[T]$.

Example 1. Let P be the poset represented by the diagram of Fig. 1. Elements of P are exhausted by three non-overlapping chains. Hence by (3. 3) $D[P] \leqq 3$. Let $\mathfrak{R}=\{L_i\}$ be a realizer of P. Then since $a' \not\sim b$, $b' \not\sim a$, $c \not\sim a'$ and $c \not\sim b'$ in P there exist $L_1, L_2, L_3, L_4 \in \mathfrak{R}$ such that $(b, a') \in \boldsymbol{L}_1$, $(a, b) \in \boldsymbol{L}_2$, $(a', c) \in \boldsymbol{L}_3$ and $(b', c) \in \boldsymbol{L}_4$. One sees easily that $L_1 \neq L_2$, $L_2 \neq L_4$, $L_1 \neq L_3$ and that $L_1=L_4$, $L_2=L_3$ do not hold simaltaneouly. Therefore either (1) $L_1 \neq L_2$, $L_2 \neq L_3$, $L_3 \neq L_1$ or (2) $L_1 \neq L_2$, $L_2 \neq L_4$, $L_4 \neq L_1$ occurs. Hence $D[P] \geqq 3$. Therefore $D[P]=3$.

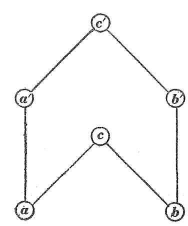

Fig. 1

Example 2. Let P be the poset of all integers with the inclusion relation : $2n > 2n-1$ and $2n > 2n+1$ $(n=0, \pm 1, \pm 2, \ldots\ldots)$. In order to exhaust the element of P enumerable number of non-overlapping chains are necessary, but $D[P]=2$. In fact P is realized by two linear extensions L_1 and L_2 defined by specifying that

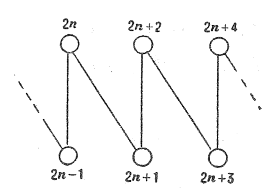

Fig. 2

$$2n < 2n+3,\ 2n-1 < 2n,\ 2n+1 < 2n \text{ in } L_1,$$
$$2(n+1) < 2n-1,\ 2n-1 < 2n,\ 2n+1 < 2n \text{ in } L_2.$$

§4. Dimension of the sum of posets.

(4. 1) *Let* $P=\sum_S P_s$ *be the sum of non-overlapping posets* P_s *associated with each element* s *of a poset* S, *that is the set* $\bigcup_S P_s$ *associated with the partial order*

$$\boldsymbol{P}=\bigcup_S \boldsymbol{P}_s \cup \{(x, y) \mid x \in P_s, y \in P_{s'}, (s, s')\ S\}$$

and σ *be an element of* S *such that* $D[P_\sigma] \geqq D[P_s]$ *for all* $s \in S$. *Then*

$$D[P]=Max\ (D[P_\sigma], D[S]).$$

Proof. Let $\mathfrak{R}=\{N_t \mid t \in T\}$ be a minimal realizer of S and $\mathfrak{R}_s=\{M_{t(s)} \mid t(s) \in T_s\}$

that of P_s. Since $n[T_\sigma] \geqq n[T_s]$ for all $s \in S$ there exists a single-valued mapping f_s of R_σ onto R_s. Let $\{f_s \mid s \in S\}$ be the system of such mappings where f_s may be chosen arbitrarily for $s \neq \sigma$, but for σ let it be the identical mapping. For each $t(\sigma) \in T_\sigma$ and for each $t \in T$ let

$$L^t_{t(\sigma)} = \sum_{N_t} f_s(M_{t(\sigma)})$$

be the sum of the system $\{f_s(M_{t(\sigma)}) \mid s \in S\}$ with the base N_t. Then $L^t_{t(\sigma)}$ is a cnain on $P = \bigcup_S P_s$ and moreover a linear extension of P. Because since $f_s(M_{t(\sigma)}) = M_{t(s)}$ for some $t(s) \in T_s$, $L^t_{t(\sigma)} = \sum_{N_t} M_{t(s)}$ that is

$$\boldsymbol{L}^t_{t(\sigma)} = \bigcup_S \boldsymbol{M}_{t(s)} \cup \{(x, y) \mid x \in M_{t(s)}, y \in M_{t(s')}, (s, s') \in \boldsymbol{S}\}.$$

Considering $\boldsymbol{P}_s \subset \boldsymbol{M}_{t(s)}$ and $M_{t(s)} = P_s$ as a set for all $s \in S$ we have $P < L^t_{t(\sigma)}$.

I. The case where $D[P_\sigma] \geqq D[S]$. Since $n[T_\sigma] \geqq n[T]$ there exists a single-valued mapping φ of T_σ onto T. Put $L^{\varphi(t(\sigma))}_{t(\sigma)} = L_{t(\sigma)}$. Then

$$\mathfrak{R}^* = \{L_{t(\sigma)} \mid t(\sigma) \in T\}$$

is a realizer of P. In order to show this let $x \not\phi\, y$ in P. There are two cases. (1) $x, y \in P_s$. Then since $x \not\phi\, y$ in P_s there exist $t(s)$, $t'(s) \in T_s$ such that $(x, y) \in \boldsymbol{M}_{t(s)}$ and $(y, x) \in \boldsymbol{M}_{t'(s)}$. Let $\boldsymbol{M}_{t(s)} = f_s(\boldsymbol{M}_{t(\sigma)})$ and $\boldsymbol{M}_{t'(s)} = f_s(\boldsymbol{M}_{t'(\sigma)})$, then $(x, y) \in f_s(\boldsymbol{M}_{t(\sigma)})$ and $(y, x) \in f_s(\boldsymbol{M}_{t'(\sigma)})$. Hence $(x. y) \in \boldsymbol{L}^t_{t(\sigma)}$ and $(y, x) \in \boldsymbol{L}^t_{t'(\sigma)}$ for all $t \in T$. In particular $(x, y) \in \boldsymbol{L}_{t(\sigma)}$ and $(y, x) \in \boldsymbol{L}_{t'(\sigma)}$. (2) $x \in P_s$, $y \in P_{s'}(s \neq s')$. Since $s \not\phi\, s'$ in S there exist$t$, $t' \in T$ such that $(s, s') \in \boldsymbol{N}_t$ and $(s', s) \in \boldsymbol{N}_{t'}$. Let $t = \varphi(t(\sigma))$ and $t' = \varphi(t'(\sigma))$. Considering $x \in P_s = f_s(M_{t(\sigma)}) = f_s(M_{t'(\sigma)})$ (as a set) and $y \in P_s' = f_{s'}(M_{t(\sigma)}) = f_{s'}(M_{t'(\sigma)})$ (as a set) $(x, y) \in \boldsymbol{L}^t_{t(\sigma)} = \boldsymbol{L}^{\varphi(t(\sigma))}_{t(\sigma)} = \boldsymbol{L}_{t(\sigma)}$ and similarly $(y, x) \in \boldsymbol{L}_{t'(\sigma)}$. Cosequently $\mathfrak{R}^*$ is a realizer of P. Hence $D[P] \leqq n[T_\sigma] = D[P_\sigma]$. Since $D[P] \geqq D[P_\sigma]$, $D[P] = D[P_\sigma]$.

II. The case where $D[P_\sigma] < D[S]$. Since $n[T_\sigma] < n[T]$ there exist a single-valued mapping ψ of T onto T_σ. Put $L^t_{\psi(t)} = L_t$, then

$$\mathfrak{R}^{**} = \{L_t \mid t \in T\}$$

is a realizer of P. The proof is similar as before. Hence $D[P] = D[S]$.

As the special case where the basic poset S is unordered we have the following corollary considering that the dimension of an unordered poset is 2.

(4. 2) *Let $P = \sum_S P_s$ be the sum of posets P_s where S is unordered and σ an element of S such that $D[P_\sigma] \geqq D[P_s]$ for all $s \in S$. Then $D[P] = D[P_\sigma]$ provided P_σ is not a chain and $D[P] = 2$ provided P_σ is a chain.*

§5. Dimension of subsets.

If some elements are removed from a poset preserving the order between remainig elements, then the dimension diminishes in general. The amount of this diminution will be estimated for several particular cases in this section.

(5. 1) *Let a be an element of a poset P. Then$D[P] \leqq D[P'] + 1$ where P' is the*

subposet of P on the set $P-a$.

Proof. Let $\mathfrak{N}'=\{N_s \mid s \in S\}$ be a minimal realizer of P' and put

$$P_1=\{x \mid (a, x) \in \boldsymbol{P}'\},\ P_3=\{x \mid (x, a) \in \boldsymbol{P}'\},\ P_2=P'-(P_1 \cup P_3).$$

Choose an element N_σ of $\mathfrak{N}'$ and let L_a^*, L_a, M_a and M_a^* be subchains of N_σ on the sets $P_1 \cup P_2$, $a \cup P_3$, $P_1 \cup a$ and $P_2 \cup P_3$ respectively. Then

$$L(a)=L_a^*+L_a \in \mathfrak{L}(a, P) \text{ and } M(a)=M_a+M_a^* \in \mathfrak{M}(a, P).$$

Put $N_{s1}=\{x \mid (x_1,x) \in \boldsymbol{N}_s$ for some $x_1 \in P_1\}$ and $N_{s2}=N_s-N_{s1}$ ($N_s=P'$ as a set). Then

$$\boldsymbol{L}_s=\boldsymbol{N}_s \cup (a, a) \cup \{(a, x) \mid x \subset N_{s1}\} \cup \{(x, a) \mid x \in N_{s2}\}$$

is a linear order on P and the corresponding chain L_s is a linear extension of P. And the sytem

$$\mathfrak{R}=\{L_s \mid s \in S-\sigma\} \cup L(a) \cup M(a)$$

is a realizer of P. In order to show this let $x \not\phi y$ in P. There are two essential cases. (1) $x=a$, $y \in P-a$. Then $(a, y) \in \boldsymbol{L}(a)$ and $(y, a) \in \boldsymbol{M}(a)$. (2) $x, y \in P-a$. Then since $x \not\phi y$ in P' there exist $s, s' \in S$ such that $(x, y) \in \boldsymbol{N}_s$, $(y, x) \in \boldsymbol{N}_{s'}$. Hence if $s \neq \sigma$, $s' \neq \sigma$, then $(y, x) \in \boldsymbol{L}_s$, $(x, y) \in \boldsymbol{L}_{s'}$. If either s or s', say s', is coincident with σ, then $(x, y) \in \boldsymbol{L}_s$ and $(y, x) \in \boldsymbol{L}(a)$ or $\boldsymbol{M}(a)$ according as $x, y \in P_1 \cup P_2$ or $x, y \in P_2 \cup P_3$. Hence $\mathfrak{R}$ is a realizer of P. Consequently $D[P] \leqq D[P']+1$.

(5. 2) *If C is a chain in a poset P, then* $D[P] \leqq D[P']+2$ *where* P' *is the subposet of P on the set* $P-C$.

Proof. Let $\mathfrak{N}'=\{N_s \mid s \in S\}$ be a minimal realizer of P'. For each $c \in C$ put

$$U_c=\{u \in P' \mid (c, u) \in \boldsymbol{P}\}$$

and for each $s \in S$

$$V_{cs}=\{x \in P' \mid) u, x) \in \boldsymbol{N}_s \text{ for some } u \in U_c\} \text{ or } V_{cs}=0$$

according as $U_c \neq 0$ or $U_c=0$ and $W_{cs}=N_s-V_{cs}$ ($N_s=P'$ as a set). Then $U_c \supseteqq U_{c'}$ and $V_{cs} = V_{c's}$ for $(c, c') \in \boldsymbol{C}$ and

$$\boldsymbol{L}_s=\boldsymbol{N}_s \cup \boldsymbol{C} \cup \{(c,x) \mid x \in V_{cs},\ c \in C\} \cup \{(x, c) \mid x \in W_{cs},\ c \in C\}$$

is a linear order on P. This is almost evident if we notice that the system $\{V_{cs} \mid c \in C\}$ is a chain with the relation of set-inclusion which is isomorphic to the chain C. To verify the conditions $P1$, $P2$ and $P4$ is easy. In order to verify $P3$ let $(z, y) \in \boldsymbol{L}$ and $(y, x) \in \boldsymbol{L}$. There are the following 8 cases.

(1) $x, y, z \in P-C$. Then since $(z, x) \in \boldsymbol{N}_s$, $(z, x) \in \boldsymbol{L}$.

(2) $x, y, z \in C$. Then since $(z, x) \in C$, $(z, x) \in \boldsymbol{L}$.

(3) $y, z \in P-C$, $x \in C$. Then $y \in W_{xs}$. When $U_x \neq 0$, $(y, u) \in \boldsymbol{N}_s$ for all $u \in U_x$; hence (z, u) for all $u \in U_x$, i. e., $z \in W_{xs}$. When $U_x=0$ evidently $z \in W_{xs}=P-C$. Hence in either case $(z, x) \in \boldsymbol{L}$.

(4) $x, z \in P-C$, $y \in C$. Then $x \in V_{ys}$. Hence $U_y \neq 0$ and $(u, x) \in \boldsymbol{N}_s$ for some $u \in U_y$. On the other hand $(z, u) \in \boldsymbol{N}_s$ for all $u \in U_y$. Consequently $(z, x) \in \boldsymbol{N}_s \subset \boldsymbol{L}$.

(5) $x, y \in P-C$, $z \in C$. Then $(y, x) \in \boldsymbol{N}_s$ and $(u, y) \in \boldsymbol{N}_s$ for some $u \in U_z$

($\neq 0$). Therefore $(u, x) \in \boldsymbol{N}_s$ for some $u \in U_z$, i. e., $x \in V_{zs}$ which implies $(z, x) \in \boldsymbol{L}_s$.

(6) $x, y \in C$, $z \in P-C$. Then $U_x \subsetneqq U_y$. If $U_x \neq 0$, then $U_y \neq 0$ and $(z, u) \in \boldsymbol{N}_s$ for all $u \in U_y$, a fortiori for all $u \in U_x$, i. e., $z \in W_{xs}$. If $U_x = 0$, then evidently $z \in W_{xs}$. Hence in either case $(z, x) \in \boldsymbol{L}_s$.

(7) $y, z \in C$, $x \in P-C$. Then $(u, x) \in \boldsymbol{N}_s$ for some $u \in U_y$ ($\neq 0$) and $U_y \subsetneqq U_z$. Hence $(u, x) \in \boldsymbol{N}_s$ for some $u \in U_z$ which implies $(z, x) \in \boldsymbol{L}_s$.

(8) $x, z \in C$, $y \in P-C$. Then $(u, y) \notin \boldsymbol{N}_s$ for some $u \in U_z$ and $(y, u) \in \boldsymbol{N}_s$ for all $u \in U_x$. Assuming that $(x, z) \in \boldsymbol{C}$ we have $(y, u) \in \boldsymbol{N}_s$ for all $u \in U_z \subsetneqq U_x$ which is contradictory. Hence $(z, x) \in \boldsymbol{C} \subset \boldsymbol{L}_s$.

The chain L_s on the set P with the linear order $\boldsymbol{L}_s$ is a linear extension of the poset P. Let $(y, x) \in \boldsymbol{P}$. Then there are three cases. (1) $x, y \in P-C$. Then $(y, x) \in \boldsymbol{P}' \subset \boldsymbol{N}_s \subset \boldsymbol{L}_s$. (2) $x \in P-C$, $y \in C$. Then $x \in V_{ys}$ which implies $(y, x) \in \boldsymbol{L}_s$. (3) $x \in C$, $y \in P-C$. Then $(y, u) \in \boldsymbol{P}$ for all $u \in U_x$ provided $U_x \neq 0$. Hence $y \in W_{xs}$. When $U_x = 0$ evidently $y \in W_{xs}$ Hence in either case $(y, x) \in \boldsymbol{L}$.

Now let $L(C) \in \mathfrak{L}(C, P)$ and $M(C) \in \mathfrak{M}(C, P)$. Then the system

$$\mathfrak{R} = \{L_s \mid s \in S\} \cup L(C) \cup M(C)$$

is a realizer of P. Hence $D[P] \leqq D[P'] + 2$.

(5. 3) *If A and B are non-comparable chains in a poset P, then $D[p] \leqq D[P']+2$ where P' is the subposet of P on the set $P-(A \cup B)$.*

Proof. Let $\mathfrak{R}' = \{N_s' \mid s \in S\}$ be a minial realizer of P'. Construct first a linear extension N_s of subposet P'' on the set $P-B$ from N_s' in the same manner as in the proof of (5. 2) L_s was constructed from N_s and then a linear extention L_s of P from N_s in like manner. Further let $L_1 \in \mathfrak{L}(A, P) \cap \mathfrak{M}(B, P)$ and $L_2 \in \mathfrak{M}(A, P) \cap \mathfrak{L}(B, P)$, then the system

$$\mathfrak{R} = \{L_s \mid s \in S\} \cup L_1 \cup L_2$$

is a realizer of P. Hence $D[P] \leqq D[P']+2$.

Let us confine ourselves to the finite posets. By the notation $b \gg a$ we mean that b covers a. The same notation will be used to denote the chain composed of such two elements which is said to be *elementary*. An elementary chain $b \gg a$ in a poset P is said to be of *rank* 0 when there exists no pair of non-comparable elements x and y such that $b \gg x$ and $y \gg a$. When such pair exists one and only one it is said to be of *rank* 1. When two and only two pairs of such elements exist it is said of *rank* 2 and so forth.

(5. 4) *If an elementary chain $b \gg a$ in a finite poset P is of rank 0 or 1, then $D[P] \leqq D[P']+1$ where P' is the subposet of P on the set $P-(a \cup b)$.*

Proof. Let $\mathfrak{R}' = \{N_1, N_2, N_n\}$ be a minimal realizer of P'. When the chain $b \gg a$ is of rank 0 choose arbitrarily an element, say N_n, of $\mathfrak{R}'$. When it is of rank 1 let x_0 and y_0 be the single pair of non-comparable elements such that $b \gg x_0$ and $y_0 \gg$ a in P. Then there exists an element of $\mathfrak{R}'$ in which $y_0 > x_0$. Let it be also N_n without loss of generality. Decompose $P-(a \cup b)$ to the sum of following non-overlapping subsets :

$$P_1 = \{x \mid a > x \text{ in } P\},$$
$$P_2 = \{x \mid x \not\phi\, a,\ b > x \text{ in } P\},$$
$$P_3 = \{x \mid x \not\phi\, a,\ x \not\phi\, b \text{ in } P\},$$
$$P_4 = \{x \mid x > a,\ x \not\phi\, b \text{ in } P\},$$
$$P_5 = \{x \mid x > b \text{ in } P\}.$$

Let M_i $(i=1, 2, 4, 5)$, M_3' and M_3'' be the subchains of N_n on the sets P_i $(i=1, 2, 4, 5)$, $P_1 \cup P_2 \cup P_3$ and $P_3 \cup P_4 \cup P_5$ respectively. Then

$$L = M_3' + a + M_4 + b + M_5$$

is a left linear extension of P with respect to the chain $b \gg a$ and

$$M = M_1 + a + M_2 + b + M_3''$$

is a right linear extension of P with respect to $b \gg a$. For each i $(\leqq n-1)$ construct a linear extension L_i of P from N_i in the same manner as in the proof of (5. 2) L_s was constructed from N_s. Then the system

$$\mathfrak{R} = \{L_1, L_2, \ldots, L_{n-1}, L, M\}$$

is a realizer of P. In order to verify this let $x \not\phi\, y$ in P. If $x \in a \cup b$ and $y \in P-(a \cup b)$, then $(x, y) \in \boldsymbol{L}$ and $(y, x) \in \boldsymbol{M}$. If $x, y \in P-(a \cup b)$, then there exist N_i and N_j such that $(x, y) \in \boldsymbol{N}_i$ and $(y, x) \in \boldsymbol{N}_j$. Hence when $i \neq n$ and $j \neq n$, $(x, y) \in \boldsymbol{L}_i$ and $(y, x) \in \boldsymbol{L}_j$. When $i=n$ or $j=n$ it sufficies to consider the following cases :

(1) $x, y \in P_1 \cup P_2 \cup P_3$. Then either $(x, y) \in \boldsymbol{L}$, $(y, x) \in \boldsymbol{L}_j$ or $(x, y) \in \boldsymbol{L}_i$, $(y, x) \in \boldsymbol{L}$ according as $i=n$ or $j=n$.

(2) $x, y \in P_3 \cup P_4 \cup P_5$. Then either $(x, y) \in \boldsymbol{L}_i$, $(y, x) \in \boldsymbol{M}$ or $(y, x) \in \boldsymbol{L}_j$ and $(x, y) \in \boldsymbol{M}$ according as $j=n$ or $i=n$.

(3) $x \in P_2$, $y \in P_4$. Then considering $b \gg a$ being of rank 1 (when $b \gg a$ is of rank 0 this case does not occur) $(x_0, x) \in \boldsymbol{P}$ and $(y, y_0) \in \boldsymbol{P}$. Since $(y_0, x_0) \in \boldsymbol{N}_n$, $(y, x) \in \boldsymbol{N}_n$. This means that $j=n$. Hence $(x, y) \in \boldsymbol{L}_i$ and $(y, x) \in \boldsymbol{L}$. Hence $\mathfrak{R}$ is a realizer of P and $D[P] \leqq D[P'] + 1$.

(5. 5) *Let a be a minimal element and b a maximal one of a finite poset P. If* $a \not\phi\, b$ *in P*

$$D[P] \leqq D[P'] + 1$$

where P' *is the subposet of P on the set* $P-(a \cup b)$.

Proof. Decompose $P-(a \cup b)$ to the sum of the following subsets :

$$P_1 = \{x \mid x \not\phi\, a,\ b > x \text{ in } P\},$$
$$P_2 = \{x \mid x \not\phi\, a,\ b \not\phi\, x \text{ in } P\},$$
$$P_3 = \{x \quad x > a,\ x > b \text{ in } P\}.$$

Let M_i $(i=1, 2, 3)$ be a linear extension of the subposet P_i of P, then

$$L = M_1 + b + M_2 + a + M_3 \in \mathfrak{L}(a, P) \cap \mathfrak{M}(b, P).$$

Let $R' = \{N_1, N_2, \dots, N_n\}$ be a minimal realizer of P' and put

$$L_i = a + N_i + b \qquad (i = 1, 2, \dots, n).$$

Obviouly $L_i \in \mathfrak{M}(a, P) \cap \mathfrak{L}(b, P)$. And the system

$$\mathfrak{R} = \{L_1, L_2, \dots, L_n, L\}$$

is a realizer of P. Hence $D[P] \leqq D[P'] + 1$.

Example 3. Let P_i $(i=3, 4, 5)$ and P_i^* $(i= 2, 3, 4)$ be the posets represented by the diagrams in Fig. 3. Then

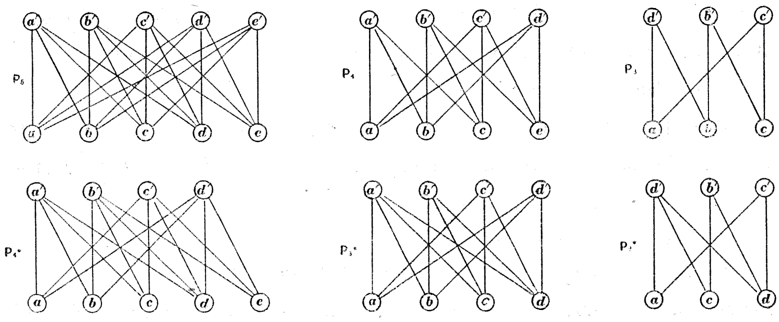

Fig. 3

(1) $D[P_5] = 5$

(2) $D[P_4^*] = 4$ $\quad (P_4^* = P_5 - e')$.

(3) $D[P_4] = 4$ $\quad (P_4 = P_5 - (d \cup e'))$.

(4) $D[P_3^*] = 3$ $\quad (P_3^* = P_5 - (e \cup e') = P_4^* - e)$.

(5) $D[P_3] = 3$ $\quad (P_3 = P_5 - (e' \cup e \cup a' \cup d) = P_3^* - (a' \cup d))$.

(6) $D[P_2^*] = 2$ $\quad (P_2^* = P_3^* - (b \cup c'))$.

As to (1), (3) and (5) refer to [1]. (6) will be easily verifyed by showing two linear extensions which realize P_2^*. Hence $D[P_3^*] \leqq D[P_2^*] + 1 = 3$. On the other hand $D[P_3^*] \geqq D[P_3] = 3$. Thus (4) is proved. Similarly $D[P_4^*] \leqq D[P_3^*] + 1 = 4$ and $D[P_4^*] \geqq D[P_4] = 4$. Hence (2) is proved.

§6. Reducible posets with respect to dimension.

Let P be a poset, A a subset of P and P' the subposet of P on the set $P - A$. If $D[P] = D[P']$, then the set A is said to be *removale* in P *with respect to dimension* or briefly *d-removable* in P. When P has at least a d-removable element P is said to be *reducible with respect to dimension* or briefly *d-reducible*. The following proposition is an immediate result of (4. 1).

(6. 1) *Let a poset P be decomposable to a sum $\sum_S P_s$ and σ an element of S such that $D[P_\sigma] \geqq D[P_s]$ for all $s \in S$. If $D[P_\sigma] \geqq D[S]$, then the set $P - P_\sigma$ is d-removable. If $D[P_\sigma] < D[S]$, then $P - P^*$ is d-removable where $P^* = \{x_s \mid s \in S\}$ is a set whose element x_s is an element selected arbitrarily from the set P_s.*

The following two propositions are the special cases of (6. 1).

(6. 2)*) *If a poset has the greatest (least) element, then it is d-removable.*

(6. 3)**) *If there exists no element other than maximal or minimal elements in*

a poset and if every maximal element is comparable with every minimal one, then all element except two maximal (minimal) elements are d-removable.

Further we shall give several examples of d-reducible poset.

(6. 4) *Let a chain C in a poset P and two elements $a, b \in P-C$ satisfy*

1° : *For every $c \in C$, $(b, c) \in \boldsymbol{P}$ and $(c, a) \in \boldsymbol{P}$.*

2° : *For $c \in C$ and for $x \in P-C$, $(x, c) \in \boldsymbol{P}$ implies $(x, b) \in \boldsymbol{P}$ and $(c, x) \in \boldsymbol{P}$ implies $(a, x) \in \boldsymbol{P}$.*

Then the set C is d-removable in P except at most one of its element.

Proof. Let c_0 be an element of C and $\mathfrak{R}' = \{M_s \mid s \in S\}$ a minimal realizer of the subposet P' of P on the set $P-(C-c_0)$. Then

$$\boldsymbol{L}_s = \boldsymbol{M}_s \cup \boldsymbol{C} \cup \{(x, c) \mid (x, c_0) \in \boldsymbol{M}_s,\ c \in C\} \cup \{(c, x) \mid (c_0, x) \in \boldsymbol{M}_s,\ c \in C\}$$

is a linar order on the set P and the corresponding chain L_s on P is a linear extension of the poset P. Moreover the sytem

$$\mathfrak{R} = \{L_s \mid s \in S\}$$

is a realizer. These to verify is not so hard. Hence $D[P] = D[P']$.

(6. 5) *Besides the conditions 1° and 2° in the last proposition let C, a and b satisfy*

3° : *For $x \in P-C$ either (i) $(x, a) \in \boldsymbol{P}$ implies $(x, b) \in \boldsymbol{P}$ or (ii) $(b, x) \in \boldsymbol{P}$ implies $(a, x) \in \boldsymbol{P}$.*

Then the set C is d-reducible as a whole.

Proof. We shall prove the proposition under the condition 3°, (i). On account of (6. 4) it may be supposed without loss of generality that C is the set of single element c. Let $\mathfrak{R}' = \{M_s \mid s \in S\}$ be a minimal realizer of the subposet P' on $P-c$. Then

$$\boldsymbol{L}_s = \boldsymbol{M}_s \cup \{(x, c) \mid (x, a) \in \boldsymbol{M}_s\}$$

is a liner order on the set P and the corresponding chain L_s is a linear extension of the poset P. Moreover the system $\mathfrak{R} = \{L_s \mid s \in S\}$ is a realizer of the poset P. In order to verify this let $x \not\phi\, y$ in P. When $x, y \in P-c$ since $x \not\phi\, y$ in P', $(x, y) \in \boldsymbol{M}_s \subset \boldsymbol{L}_s$, $(y, x) \in \boldsymbol{M}_{s'} \subset \boldsymbol{L}_{s'}$ for some s and s'. When $x = c$, $y \in P-c$ by the condition 2° and 3°, (i) $a \not\phi\, y$ in P' ; hence for some s, s' $(y, a) \in \boldsymbol{M}_s$ and $(a, y) \in \boldsymbol{M}_{s'}$. For these s and s'

$$(y, c) \in \boldsymbol{L}_s \text{ and } (c, y) \in \boldsymbol{L}_{s'}.$$

Consequently $\mathfrak{R}$ is a realizer of P. Therefore $D[P] = D[P']$.

(6. 6) *Under the same conditions as in the proposition* (6. 5) *either $a \cup C$ or $C \cup b$ is d-removable.*

By (6. 5) this may be reduced to the proposition that

(6. 7) *Let $b > a$ be an elementary chain in a poset P and satisfy either* (i) *for $x \in P-b$, $(x, a) \in \boldsymbol{P}$ implies $(x, b) \in \boldsymbol{P}$ or* (ii) *for $x \in P-a$, $(b, x) \in \boldsymbol{P}$ implies $(a, x) \in \boldsymbol{P}$. Then either b or a is d-removable according as* (i) *or* (ii) *occurs.*

Proof. Let $\mathfrak{R}' = \{M_s \mid s \in S\}$ be a minimal realizer of the subposet P' on the set

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*) (6. 2) is also a special case of (6. 11).

**) (6. 3) follows also from (6. 10).

$P'=P-b$ and L_s the chain on P associated the linear order

$$\boldsymbol{L}_s=\boldsymbol{M}_s \cup (b, b) \cup \{(x, b) \mid (x, a) \in \boldsymbol{M}_s\}.$$

Then the system $\mathfrak{R}=\{L_s \mid s \in S\}$ is a realizer of P. Hence $D[P]=D[P']$, i. e., b is d-removable in P.

In like manner it may be proved that

(6. 8) *Let a chain C in a poset and an element $a \in P-C$ satisfy*

1° : *For every $c \in C$, $(c, a) \in \boldsymbol{P}$ $((a, c) \in \boldsymbol{P})$.*

2° : *For $x \in P-C$, $(a, x) \notin \boldsymbol{P}$ $((x, a) \notin \boldsymbol{P})$ implies $x \not\!\phi\, c$ in P for all $c \in C$.*

Then C is d-removable except at most one of its elements. Further let C and a satisfy

3° : *For $x \in P-a$, $(x, a) \in \boldsymbol{P}$ $((a, x) \in \boldsymbol{P})$ implies $x \in c$. Then C is d-removable as a whole.*

(6. 9) *Let P be a poset and P_1, P_2, P_3 subsets of the set P satisfying*

1° : *For x, y $(x \neq y) \in P_i$, $x \phi y$ in P.*

2° : *For $x_i \in P_i$ $(i=1, 2, 3)$, $(x_i, x_j) \in \boldsymbol{P}$ if $i > j$.*

3° : *For $x \in P-P_2$, $(x, x_2) \in \boldsymbol{P}$ for some $x_2 \in P_2$ implies $(x, x_3) \in \boldsymbol{P}$ for some $x_3 \in P$ and $(x_3, x) \in \boldsymbol{P}_3$ for some $x_2 \in P_2$ implies $(x_1, x) \in \boldsymbol{P}$ for some $x_1 \in P_1$. Then P_2 is d-removable except at most an element p provided the subposet P' on the set $P-(P_2-p)$ is not a chain.*

Proof. Let $\mathfrak{R}'=\{M_s \mid s \in S\}$ be a realizer of P', L_2 a chain on P_2 and L_2^* the dual of L_2. Then

$$\boldsymbol{L}_s^1=\boldsymbol{M}_s \cup \boldsymbol{L}_2 \cup \{(x, y) \mid (x, p) \in \boldsymbol{M}_s, y \in P_2\} \cup \{(x, y) \mid x \in P_2, (p, y) \in \boldsymbol{M}_s\}$$

and

$$\boldsymbol{L}_s^2=\boldsymbol{M}_s \cup \boldsymbol{L}_2^* \cup \{(x, y) \mid (x, p) \in \boldsymbol{M}_s, y \in P_2\} \cup \{(x, y) \mid x \in P_2, (p, y) \in \boldsymbol{M}_s\}$$

are linear orders on P and corresponding chains L_s^1 and L_s^2 are linear extensions of the poset P. Since $n[S] \geqq 2$ there exist a system $\mathfrak{R}=\{L_s \mid S \in S\}$ satisfying

1° : $L_s=L_s^1$ or else $L_s=L_s^2$

2° : There exists $s, s' \in S$ such that $L_s=L_s^1$ and $L_s=L_s^2$.

$\mathfrak{R}$ is a realizer of P. Hence $D[P]=D[P']$.

In like manner it may be proved that

(6. 10) *Let P_2 (P_1) be a set of maximal (minimal) elements in a poset P and P_1 (P_2) a subset of P satisfying*

1° : *For $x_1 \in P_1$ and $x_2 \in P_2$, $(x_2, x_1) \in \boldsymbol{P}$.*

2° : *For $x \in P-P_2$ $(P-P_1)$, $(x_2, x) \in \boldsymbol{P}$ for some $x_2 \in P_2$ $((x, x_1) \in \boldsymbol{P}$ for some $x_1 \in P_1)$ implies $(x_1, x) \in \boldsymbol{P}$ for some $x_1 \in P_1$ $((x, x_2) \in \boldsymbol{P}$ for some $x_2 \in P_2)$.*

Then the set P_2 (P_1) is d-removable except at most an element p provided the subposet P' on $P-(P_2-p)$ $(P-(P_1-p))$ is not a chain.

(6. 11) *If an element a is comparable with every element of a poset, then a is d-removable.*

Proof. Let $\mathfrak{R}'=\{M_s \mid s \in S\}$ be a minimal realizer of the subposet P' on $P-a$ and put

$$P_a'=\{x \mid (a, x) \in \boldsymbol{P}\},\ P_a''=\{x \mid (x, a) \in \boldsymbol{P}\}.$$

Then

$$\boldsymbol{L}_s=\boldsymbol{M}_s \cup (a, a) \cup \{(a, x) \mid x \in M_s'\} \cup \{(x, a) \mid x \in M_s''\}$$

is a linear order on P where

$$M_s'=\{x \mid (x', x) \in \boldsymbol{M}_s \text{ for some } x' \in P_a'\},\ M_s''=M_s-M_s'.$$

The corresponding chain L_s on P is a linear extension of the poset P and the system $\mathfrak{R}=\{L_s \mid s \in S\}$ is a realizer of P.

(6. 12) *Let a be an element of a poset P and P_a', P_a'' mean the same as above. If for every element x such that $x \phi a$ in P there exist $p \in P_a'$ such that $x \phi p$ in P and $q \in P_a''$ such that $x \phi q$ in P. Then a is d-removable.*

Proof. Let $\mathfrak{R}'=\{M_s \mid s \in S\}$ be minimal realizer of the poset P' on the set $P-a$, then the system $\mathfrak{R}=\{L_s \mid s \in S\}$ constructed in the same way as in the proof of (6. 11) is a realizer of P. In order to verify this let $x \phi y$ in P. When $x, y \in P-a$ the existence of $s, s' \in S$ such that $(x, y) \in \boldsymbol{L}_s$ and $(y, x) \in \boldsymbol{L}_{s'}$ is obvious. When $x=a$, $y \in P-a$ there exist, by hypothesis, $p \in P_a'$ and $q \in P_a''$ such that $p \phi y$ and $q \phi$ y in P'. Hence there exist $s, s' \in S$ such that $(p, y) \in \boldsymbol{M}_s$ and $(y, q) \in \boldsymbol{M}_{s'}$. For these s and s' $(a, y) \in \boldsymbol{L}_s$ and $(y, a) \in \boldsymbol{L}_{s'}$.

Example 4. The poset P in Fig. 4 will be reduced to the poset P_5 by removing the removable elements applying the designated theorems one after another. Hence $D[P]=D[P_5]=3$. The poset P_5 is no longer d-reducible.

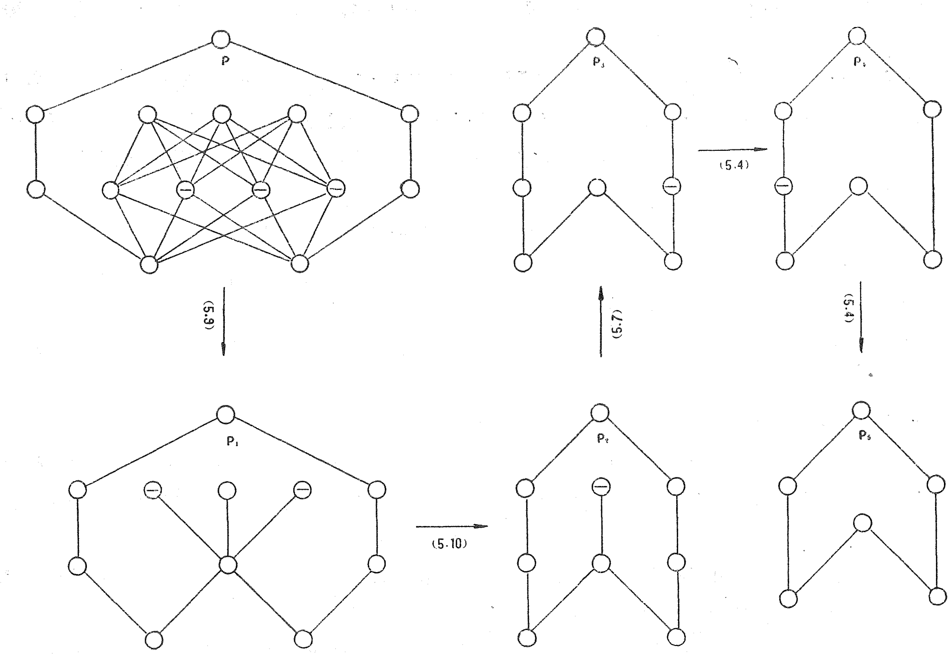

Fig. 4

§7. The greatest $D[P]$ over the system of posets on a set P.

It has already been known that for every cardinal number n there exists a poset, defined on a set of power $2n$, whose dimension is n (Dushnik and Miller[1]). In other words in order to define a poset of dimension n a set of power $2n$ is sufficient. The question that in order to define a poset of dimension n set of what power is necessary is equivalent to the question what is the greatest $D[P]$ over the system of posets defined on a set P. To answer to the question is the subject of this section. We shall begin with two lemmas on finite posets.

(7. 1) *Let P be a poset such that $n[P] \leqq 7$. Then there exists in P an elementary chain of rank 0 or of rank 1.*

Proof. Let us prove the contraposition : if every elementary chain in P is at least of rank 2, then $n[P] \geqq 8$. Put

$A_{1,2,3,\dots,k}(b)=\{a \mid b \gg a \text{ in } P,\ a \neq a_i\ (i=1, 2, 3,\dots, k)\}$,

$B_{1,2,3,\dots,k}(a)=\{b \mid b \gg a \text{ in } P,\ b \neq b_i\ (i=1, 2, 3,\dots, k)\}$.

Let $b_1 \gg a_1$ be an elementary chain in P. Since it is at least of rank 2 there exist three elements either (1) $a_2 \in A_1(a_1)$, $b_3 \in B_1(a_1)$, $b_4 \in B_{13}(a_1)$ such that $a_2 \not\!\phi\, b_i$ $(i=3, 4)$ or (2) $b_2 \in B_1(a_1)$, $a_3 \in A_1(b_1)$, $a_4 \in A_{13}(b_1)$ such that $b_2 \,\phi\, a_i$ $(i=3, 4)$. Assume, without loss of generality, that (1) occurs. Since the chains $b_1 \gg a_2$ and $b_i \gg a_1 (i=3, 4)$ are not of rank 0 there also exist three elements $b_2 \in B_1(a_2)$ and $a_i \in A_1(b_i)$ $(i=3, 4)$. Evidently $b_2 \neq b_i$ $(i=1, 3, 4)$ and $a_i \neq a_2$ $(i=3, 4)$. Hence $n[P] \geqq 8$ provided $a_3 \neq a_4$. Let $a_3 = a_4$ for every $a_3 \in A_1(b_3)$ and for every $a_4 \in A_1(b_4)$, i. e., $A_1(b_3)=A_1(b_4)$ be a set of single element. If there exists an element c such that $b_2 > c > a_3$, then evidently $n[P] \geqq 8$. If $b_2 \gg a_3$ or if $b_2 \,\phi\, a_3$, then since the chain $b_4 \gg a_3$ is of rank 2 thete exists at least an element $b_5 \in B_{3,4}(a_3)$. Since $b_5 \neq b_i$ $(i=1, 2)$, $n[P] \geqq 8$.

(7. 2) *Let P be a finite poset of length $d[P]=2$ in which every maximal element is comparable with every minimal element and no elementary chain is of rank 0. Then there exists a pair of two non-comparable elementary chains.*

Proof. Let B be the set of all maximal elements in P and M that of minimal elements. Since $d[P]=2$ and every elementary chain is not of rank 0, there must exist five elements $a_1 \notin M$, $b_1 \in B$, $a_2 \in A_1(b_1)$, $b_2 \in B_1(a_2)$, $b_3 \in B_1(a_1)$ such that $b_1 \gg a_1$ and $b_3 \,\phi\, a_2$ in P. $b_3 \in B$, $b_3 \,\phi\, a_2$ and $d[P]=2$ imply $a_2 \notin M$. Now if $b_2 \,\phi\, a_1$, then the chains $b_3 \gg a_1$ and $b_2 \gg a_2$ are non-comparable in P. Let b_2 and a_1 be comparable. Then $b_2 \gg a_1$ on account of $a_1 \notin M$ and $d[P]=2$. Since the chain $b_3 \gg a_1$ is not of rank 0, $A_1(b_3) \neq 0$. If for some $a_3 \in A_1(b_3)$ and for $i=1$ or $i=2$ $a_3 \,\phi\, b_i$ in P, then two chains $b_3 \gg a_3$ and $b_i \gg a_2$ are non-comparable. Let every element of $A_1(b_3)$ is comparable with every b_i $(i=1, 2)$. Then since $b_3 \,\phi\, a_2$ in P every $a_3 \in A_1(b_3)$ can not be coincident with a_2. The chain $b_3 \gg a_1$ not being of rank 0 there exists an element $b_4 \in B_{1,2,3}(a_1)$ such that $b_4 \,\phi\, a_3$ for some $a_3 \in A_1(b_3)$. $b_4 \,\phi\, a_3$ implies $a_3 \notin M$, and hence $b_i \gg a_3$ $(i=1, 2)$ on account of $d[P]=2$. Since the chain $b_4 \gg a_1$ is not of rank 0, $A_1(b_4) \neq 0$. If for some $a_4 \in A_1(b_4)$ and for a value of i (=1, 2, 3) $a_4 \not\!\phi\, b_i$, then two chains $b_4 \gg a_4$ and $b_i \gg a_3$ are non-comparable. Let every element of $A_1(b_4)$ be comparable with every b_i (i=1, 2, 3) for every pair of $b_4 \in B_{1,2,3}(a_1)$ and $a_3 \in A_1(b_3)$ such that $b_4 \phi a_3$. Apply the same reasoning as above and continue the same procedure. Since $b_k \in B_{1,2,\dots(k-1)}(a_1)$ and $n[P]$ is finite this procedure must cease after a finite number of times and we must come upon a pair of non-comparable chains.

On inspecting the process of the proof one sees easily that in a poset P of length $d[P]=1$ no chains of which is of rank 0 there exists a pair of non-comparable chains.

Now let us prove a theorem which is the main purpose of this paper.

(7. 3) *If for a poset P $n[P] \geqq 4$, then $D[P] \leqq [n[P]/2]$ where $[n[P]/2]$ means the integral part of $n[P]/2$ when $n[P]$ is finite and $n[P]$ itself when it is transfinite.*

Proof. When $n[P]$ is transfinite it is evident by (3. 1). When $n[P]$ is an integer we shall prove it by the mathematical induction. In the first place we shall prove that if $n[P] \leqq 5$, then $D[P] \leqq 2$. When $n[P]=2$ it is trivial. When $n[P] \geqq 3$ we may confine ourselves, without loss of generality, to the posets which are not decomposable to a sum of subposets with an unordered set as the base since they are d-reducible by (6. 1). Classifying all posets under consideration by the combination of number of maximal elements and that of minimal ones, we have the following tables.

$n[P]=3$	I_3	II_3
No. of max. el.	1	1
No. of min. el.	1	2

$n[P]=4$	I_4	II_4	III_4	IV_4
No. of max. el.	1	1	1	2
No. of min. el.	1	2	3	2

$n[P]=5$	I_5	II_5	III_5	IV_5	V_5	VI_5
No. of max. el.	1	1	1	1	2	2
No. of min. el.	1	2	3	4	2	3

On interchanging the number of maximal elements and that of minimal ones we obtain other classes than those which are listed in the tables, but on account of duality those may be left out of consideration so far as the dimension is concerned. Every poset which belongs to the other classes than IV_4, V_5 and VI_5 has the greatest element which is d-removable by (6. 2). Hence it may be left out of consideration. Every poset P which belongs to the class IV_4 possesses two non-overlapping chains and all elements of P are exhausted by the elements of these chains. Hence by (3. 3) $D[P] \leqq 2$. Every poset P which belongs to the class V_5 must have at least a chain of three elements. The remaining two elements are either comparable or non-comparable. In the former case since P is composed of two non-overlapping chains $D[P] \leqq 2$ by (3. 3). In the latter case since one of the remaining two elements must be a maximal element and another a minimal one $D[P] \leqq 2$ by (5. 5). Every partial order which belongs to the class VI_5 is isomorph to one of four posets represented by the diagrams in Fig. 5.

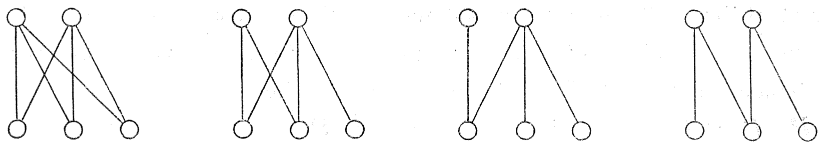

Fig. 5

The first three are reducible to the case where $n[P]=4$ by (6. 10). The last one is of dimension 2 since it is a subposet of the poset of the example 2. Thus our proposition is established for $n[P]=4$, 5. When $n[P]=6$, 7 there exists, by(7.1), an elementary chain of rank 0 or 1. Let it $b \gg a$ and P' the subposet on the set $P-(a \cup b)$. Since $n[P']=4$ or 5, $D[P']=2$. Hence by (5. 4)

$$D[P] \leqq D[P']+1 \leqq 3=[n[P]/2].$$

Thus the proposition is established for $n[P] \leqq 7$. Now let $n[P] \geqq 8$ and assume that the proposition is true for $n[P]-k$ where $k \geqq 1$. If P is d-reducible, there exists an element a such that $D[P]=D[P']$ where P' is the subposet on $P-a$. Hence by the assumption of induction

$$D[P]=D[P'] \leqq [n[P']/2] \leqq [n[P]/2].$$

Let P be d-irreducible. If there exists a pair a, b of non-comparable minimal and maximal elements, then by (5. 5) and the assumption of induction

$$D[P] \leqq D[P']+1 \leqq [n[P']/2]+1=[n[P]/2]$$

where P' is the subposet on $P-(a \cup b)$. Let every maximal element is comparable with every minimal element. If $d[P] \geqq 3$ there exists a chain C composed of four elements. Then by (5. 2) and the assumption of induction

$$D[P] \leqq D[P']+2=[n[P']/2]+2=[n[P]/2]$$

where P' is the subposet on $P-C$. Let $d[P] \leqq 2$. Since under the condition that every maximal element is comparable with every minimal one $d[P]=1$ implies the d-reducibility of P by (6. 3), it suffices to verify the inequality for $d[P]=2$. If there exist an elementary chain $b \gg a$ of rank 0, then by (5. 4) and the assumption

$$D[P] \leqq D[P']+1 \leqq [n[P']/2]+1=[n[P]/2]$$

where P' is the subposet on the set $P-(a \cup b)$. If every elementary chain is not of rank 0, then by (7. 2) there exist a pair of non-comparable elementary chains. Let it be A and B, and P' the subposet on $P-(A \cup B)$. By (5. 3) and the assumption of induction

$$D[P] \leqq D[P']+2 \leqq [n[P']/2]+2=[n[P]/2].$$

Thus our proposition is established completely.

The theorem of Dushnik and Miller at the beginging of this section will be generalized as follows.

(7. 4) *For every cardinal number* $n \geqq 2$ *there exist a poset on a set of power* n, *whose dimension is* $[n/2]$.

Proof. When n is transfinite or an even integer it is evident by the theorem of Dushnik and Miller. When n is an odd integer there exists a poset P', defined on a set of power $(n-1)$, whose dimension is $(n-1)/2$ since $n-1$ is even. Let a be a maximal element of P' and a^* an element $\notin P'$. Then

$$\boldsymbol{P} = \boldsymbol{P}' \cup (a^*, a^*) \cup \{(a^*, x) \mid (a, x) \in \boldsymbol{P}'\}$$

is a partial order on $P = P' \cup a^*$ and the corresponding poset P is a required poset since a^* is d-removable in P by (6. 7).

It is a immediate result of (7. 3) and (7. 4) that

(7. 5) *Let* $\mathfrak{S}$ *be the system of all posets defined on a set* P. *Then the greatest of* $D[P]$ *for* $P \in \mathfrak{S}$ *is* $[n[P]/2]$ *provided* $n[P] \geqq 4$.

One sees easily that without the restriction $n[P] \geqq 4$ it is $[n[P]/2]+1$.

As the converse of Dushnik-Miller's theorem, with slight restriction, we have the following proposition which is equivalent to (7. 3).

(7. 6) *If* $D[P] \geqq 3$, *then* $2D[P] \leqq n[P]$. *In other words, in order to define a poset of dimension* n *a set of power* $2n$ *is necessary.*

Proof of the equivalence. $D[P] \geqq 3$ implies $n[P] \geqq 6$. Hence $D[P] \leqq [n[P]/2]$, i. e., $2D[P] \leqq n[P]$. Conversely let $n[P] \geqq 4$ and assume that $D[P] > [n[P]/2]$. Then since $D[P] \geqq 3$ we have $2D[P] \leqq n[P]$ by (7. 6) which contradicts the assumption.

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