The Science Reports of the Kanazawa University, Vol. II, No. 1, pp. 1—3, March, 1953.

A Note on Mr. Komm's Theorems

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(Received June, 27, 1952)

§ 1. Mr. Horace Komm [1] has proved that (1) *dim* $P_n'(E_n)=n$ *provided* $n\leqq\aleph_0$ by showing that (2) *dim* $P_n'(E_n)\leqq n\ (n\leqq\aleph_0)$ and (3) *if P is a denumerable partially ordered set of dimension n, there exits a subset* M_n *of* E_n *such that* $P\sim P_n'(M_n)$. Here E_n means the set of all sequences of real numbers $\{x_k\}$, $k=1, 2, \cdots\cdots, n$ where $n\leqq\aleph_0$ and $P_n'(M_n)$ for every $M_n\subseteqq E_n$ a partially ordered system in which $x=\{x_k\}>\{y_k\}=y$ if and only if $x_k\geqq y_k$ for all k and $x_i\neq y_i$ for some i. In the sequel we use the convenient abbreviation "poset" for partially ordered set and the notation $D[P]$ for dim P. Now let us define a cardinal product of posets. Let X_s be a poset associated with each element s of a set S, $x=\{x_s\,|\,s\in S\}$ a set of x_s selected one at a time from each set X_s and X the set of all such x. Then

$$\boldsymbol{X}=\{(x,\ y)\,|\,(x_s,\ y_s)\in\boldsymbol{X}_s \text{ for all } s\in S\}$$

is a partial order[1] on the set X where $\boldsymbol{X}_s$ is the partial order associated with each poset X_s. By the cardinal product $\Pi_S X_s$ we mean the set X together with the partial order $\boldsymbol{X}$. When X_s are all isomorphic with a poset Y, $\Pi_S X_s$ is the cardinal power Y^S which will also be denoted by Y^m where m is the number of elements of S. Thus if the chain of real numbers is denoted by R the three propositions of Mr. Komm will be formulated as follows. (1*) $D[R^m]=m$, $m\leqq\aleph_0$ (2*) $D[R^m]\leqq m$, $m\leqq\aleph_0$ (3*) *Every poset P with* $n[P]^{*)}\leqq\aleph_0$ *and* $D[P]=m$ *is isomorphic with a subposet of* R^m.

The purpose of this note is to prove that the propositions (1*), (2*) and (3*) hold in more general forms. That is

Theorem 1. *Let* $\Pi_S X_s=X$ *be a cardinal product of posets* X_s, $s\in S$ *and* $\mathfrak{R}_s=\{M_{t(s)}\,|\,t(s)\in T_s\}$ *a minimal realizer*[2] *of* X_s. *Then* $D[X]\leqq n[T]$ *where* $T=\bigcup_S T_s$. (A generalization of (2*).)

Theorem 2. *If* X_s *is a chain for every* $s\in S$, $D[\Pi_S X_s]=n[S]$. (A generalization of (1*).)

Theorem 3. *Every poset P with* $D[P]=m$ *is isomorphic with a subposet of some cardinal product of m chains whose dimension is m.* (A generalization of (3*).)

Theorem 3 is a special case of

Theorem 4. *If* X_s *is a poset which is d-reducible*[3] *to a poset which is isomorphic with* 2^{T_s} *for every* $s\in S$, *then* $D[\Pi_S X_s]=n[T]$ *where* $T=\bigcup_S T_s$.

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*) $n[\ \ ]$ means the number of elements of a set written in the brackets.

1), 2), 3), 4) As to the terminologies refer to the author's previous paper [2].

§ 2. 1. *Proof of the theorem* 1. It may be assumed that S and T_s are well-ordered sets without loss of generality. Let $t_0(s)$ be the first element of T_s. Then

$$L_s^{t(s)} = \{(x,x) \mid x \in X\} \cup \{(x, y) \mid x_s \neq y_s, (x_s, y_s) \in M_{t(s)}\}$$
$$\cup \{(x, y) \mid x_s = y_s, (x_\sigma, y_\sigma) \in M_{t_0(\sigma)} \text{ for the first } \sigma \text{ such that } x_\sigma \neq y_\sigma\}$$

is a linear order[4] on the set X. That the reflexivity and antisymmetry hold is evident. To show that the transitivity holds let $(x, y) \in L_s^{t(s)}$ and $(y, z) \in L_s^{t(s)}$. Then there are following four cases:

(1) $x_s \neq y_s$, $(x_s, y_s) \in M_{t(s)}$;
$y_s \neq z_s$, $(y_s, z_s) \in M_{t(s)}$.

(2) $x_s \neq y_s$, $(x_s, y_s) \in M_{t(s)}$;
$y_s = z_s$, $(x_\sigma, y_\sigma) \in M_{t_0(\sigma)}$ for the first σ such that $y_\sigma \neq z_\sigma$.

(3) $x_s = y_s$, $(x_\sigma, y_\sigma) \in M_{t_0(\sigma)}$ for the first σ such that $x_\sigma \neq y_\sigma$;
$y_s \neq z_s$, $(y_s, z_s) \in M_{t(s)}$

(4) $x_s = y_s$, $(x_\sigma, y_\sigma) \in M_{t_0(\sigma)}$ for the first σ such that $x_\sigma \neq y_\sigma$;
$y_s = z_s$, $(y_{\sigma'}, z_{\sigma'}) \in M_{t_0(\sigma')}$ for the first σ' such that $y_{\sigma'} \neq z_{\sigma'}$.

When one of the first three cases occurs it is evident that $(x_s, z_s) \in M_{t(s)}$. Hence $(x, z) \in L_s^{t(s)}$. When the last case occurs $x_s = z_s$. If $\sigma \leqq \sigma'$, then σ is the first suffix such that $x_\sigma \neq z_\sigma$ and for this $(x_\sigma, z_\sigma) \in M_{t_0(\sigma)}$. Similarly if $\sigma' < \sigma$, then σ' is the first suffix such that $x_{\sigma'} \neq z_{\sigma'}$, and for this $(x_{\sigma'}, z_{\sigma'}) \in M_{t_0(\sigma')}$. Hence in either case $(x,z) \in L_s^{t(s)}$. Thus the transitivity is verified. The chain $L_s^{t(s)}$ obtained by associating $L_s^{t(s)}$ with X is a linear extension of the poset X. In fact, if $(x, y) \in X$ and $x \neq y$, then either $x_s \neq y_s$, $(x_s, y_s) \in M_{t(s)}$ or $x_s = y_s$, $(x_\sigma, y_\sigma) \in M_{t_0(\sigma)}$ for the first σ such that $x_\sigma \neq y_\sigma$; hence $(x, y) \in L_s^{t(s)}$. Moreover the system

$$\mathfrak{R} = \{L_s^{t(s)} \mid s \in S,\ t(s) \in T_s\}$$

of all linear extensions $L_s^{t(s)}$ is a realizer of the poset X. To show this let $x \not\phi\, y$ in X. Then either (1) $x_s \not\phi\, y_s$ in X_s for some $s \in S$ or (2) $x_s \neq y_s$, $(x_s, y_s) \in X_s$ for some s and $y_{s'} \neq x_{s'}$, $(y_{s'}, x_{s'}) \in X_{s'}$ for some $s' \neq s$. If (1), then $(x_s, y_s) \in M_{t(s)}$ for some $t(s) \in T_s$ and $(y_s, x_s) \in M_{t'(s)}$ for some $t'(s) \in T_s$; hence $(x, y) \in L_s^{t(s)}$ and $(y,x) \in L_s^{t'(s)}$. If (2), then $(x_s, y_s) \in M_{t(s)}$ and $(y_s, x_s) \in M_{t(s')}$; hence $(x, y) \in L_s^{t(s)}$ and $(y, x) \in L_s^{t(s')}$. Thus $\mathfrak{R}$ is a realizer of X. Therefore $D[X] = n\,[T_s]$.

2. *Proof of the theorem* 2. By the theorem 1 we have the inequality $D\,[\Pi_S X_s] \leqq n[S]$. To have the inverse inequality it suffices to show that $\Pi_S X_s$ contains

a subposet of dimension $n[S]$. But evidently 2^S is a subposet of $\Pi_S X_s$. To prove that $D[2^S]=n[S]$ let 2 be {0, I}, a_s the element of 2^S whose s-component is I and other components are 0 and b_s the element of 2^S whose s-component is 0 and other components are I. Then the subposet of 2^S composed of all a_s and all b_s is isomorphic with the poset composed of all elements of S and their complements in S and ordered by the relation of set-inclusion. It has been known [3] that the dimension of the latter poset is $n[S]$. Hence we have $D[2^S]=n[S]$.

3. *Proof of the theorem* 3. Let a minimal realizer of P be $\mathfrak{R}=\{L_s \mid s\in S\}$, $n[s]=m$. Then P is isomorphic with a subposet of $\Pi_S L_s$. In fact P is isomorphic with the subposet P^* of $\Pi_S L_s$ composed of all elements such that all the components are equal to an element $x\in P$.

4. *Proof of the theorem* 4.

$D[\Pi_S X_s]\leqq n[T]$ is evident by the theorem 1. On the other hand

$$D[\Pi_S X_s]\geqq D[\Pi_S 2^{T_s}]=D[2^T]=n[T].$$

REFERENCES

1. H. Komm, "*On the dimension of partially ordered se.'s*", American Journal of Mathematics, vol. **71** (1948), pp. 507-520.
2. T. Hiraguchi, "*On the dimension of partially ordered sets*", Science Reports of the Kanazawa University, vol. 1, No. 2 (1951), pp. 77-94.
3. B. Dushnik and E. W. Miller, "*Partially ordered sets*" American Journal of Mathematics, vol. 63(1941), pp. 660-610.