The Science Reports of the Kanazawa University, Vol. III, No. 1, pp. 35—43, March, 1955.

On Mean Values and Geometrical Probabilities in E_3

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(Received October 1, 1954)

Introduction.

It is said that the original problem of integral geometry was G. L. L. Comte De Buffon's needle problem [1]*, that is;

"*A floor is ruled with equidistant parallel lines; a rod, shorter than the distance between each pair, being thrown at random on the floor, to find the chance of its falling on one of the lines.*"

As the direct generalization of this problem we can point out the following modifications, that is, we may use two rows of parallel lines which construct a lattice-work by congruent rectangles, or take an oval instead of a needle.

Developing such a problem concering with the estimation of geometrical probabilities, Blaschke and his circle organized integral geometry. Then L. A. Santaló, using integral geometry, treated a problem in the Euclidean plane which contains Buffon's needle problem as a special case ([3] and [4]).

The aim of this paper is to generalize his results on this subject in the plane to the case in the Euclidian three-space E_3. For this purpose we shall first define the uniform-figure-systems in E_3 and then consider the subject under three classified cases.

§ 1. Definition.

1.1 Covering of Euclidean space E_3. —— In the first place, let us take a closed surface σ_0 in E_3 which admits the construction defined by a discontinuous rigid motion T_1 operating upon σ_0 which satisfies the conditions (α) and (β_1). Let the surface σ_0 be marked at its initial position in E_3. If T_1 be operated upon σ_0, it will move to the next position. Then we put the following condition (α) upon these two congruent surfaces.

(α) The two adjacent surfaces have a part of them in common but neither of them has any inner point of the domain enclosed by the other surface in common.

Suppose that the operation T_1 is repeated μ_1^+- times in the same direction and the inverse operation T_1^{-1} μ_1^--times in the opposite direction starting from the initial position of σ_0, then we obtain a tubiform surface which joins $\mu_1(=\mu_1^+ + \mu_1^-)$-pieces of σ_0 piece by piece. Let us denote this tubiform surface by σ_1, and put another condition (β_1) upon σ_1.

* Numbers in brackets refer to the references at the end of the paper.

(β_1) For any number μ_1 of the operation T_1, the surface σ_1 does not intercross itself, namely it is Jordan's curvic.

In the next place, let us consider another rigid motion T_2 operating upon σ_1 whose direction is different from that of T_1. Again upon T_2 and σ_1 we put the same condition as that of (α) for T_1 and σ_0. If we operate T_2 upon σ_1 μ_2^+-times in the same direction and the inverse operation T_2^{-1} μ_2^--times in the opposite direction, we obtain a leaf-like surface joining $\mu_2(=\mu_2^+ + \mu_2^-)$-pieces of σ_1 piece by piece. Let us denote this leaf-like surface by σ_2 and put the following condition (β_2) upon it.

(β_2) For any number μ_2 of the operation T_2, the surface σ_2 does not intercross itself.

In the last place we define a third rigid motion T_3 operating upon σ_2 whose direction is independent of those of T_1 and T_2, and which satisfies the condition (α) too. Then repreating the rigid motion T_3 upon σ_2 μ_3-times in the same manner as before, we obtain μ_3-pieces of σ_2 arranged in layers and the boundary surface formed by all the parts of surfaces which are not common to them is a closed surface in E_3. Let us denote this closed surface by σ_3 and put the following condition (β_3) upon it.

(β_3) For any number μ_3 of the operation T_3, the surface σ_3 does not intercross itself.

Thus we get a Jordan's domain enclosed by σ_3, or, in other words, the whole domain which is enclosed by σ_3 is divided into congruent $\mu_1\mu_2\mu_3$-pieces of the domain enclosed by σ_0 without overlapping or without leaving gaps.

Thereupon if we shall increase μ_1, μ_2 and μ_3 to infinity at the same time, the whole space E_3 will be covered by ∞^3-pieces of domains, each being congruent to the domain enclosed by σ_0, without overlapping or without leaving gaps.

Now let us call such a construction "a covering of E_3 by σ_0", the domain $\mathfrak{D}$ enclosed by σ_0 "a fundamental cell" in the covering and each domain which is arranged by the covering "a unit cell" in the covering.

1.2 Uniform figure system.— We now proceed to give the following constructions for three uniform-arrangements of points, curves and surfaces in E_3. For the construction, let us prepare three sets of figures as follows ;

(0) a set of points of a finite number,
(1) a set of curves of finite length,
(2) a set of surfaces of finite area.

Thereupon, let us cover E_3 by a fundamental cell σ_0 attached with a set of figures, and every unit cell in the so covered space by σ_0 will be allotted uniformly with a set of figures. Then we cross out the covering surfaces. Let us call each of the sets of figures "the fundamental figure" and choose it so that these figures allotted to each unit cell may not intersect with the others in its adjoining cells.

We classify such an arrangement of figures corresponding to the kind of the fundamental figure (0), (1) or (2) by calling

(0) a uniform-point-system in E_3,

(1) a uniform-curve-system in E_3,

(2) a uniform-surface-system in E_3.

Hereupon we only give an example for such uniform-figure-system, that is, a regular system of points in E_3 [5].

§ 2. A uniform-point-system in E_3.

2.1 Set of points of a finite number.— Let $\mathfrak{S}$ be a set of a finite number of points $(P_1, P_2, \cdots\cdots, P_\rho)$ which are fixed in E_3 and $\mathfrak{K}$ a closed surface of volume V, which moves in E_3. It is well known that the total measure of the positions of $\mathfrak{K}$ which contains some points of $\mathfrak{S}$ is given by the formula

$$\int_{\mathfrak{S} \frown \mathfrak{K} \neq 0} m\dot{\mathfrak{K}} = 8\pi^2 \rho V, \tag{1}$$

where $\dot{\mathfrak{K}}$ denotes the kinematic density for $\mathfrak{K}$ [2], m the number of the points of $\mathfrak{S}$ contained in $\mathfrak{K}$ at a position and the integration is extended all over the intersection points of $\mathfrak{S}$ and $\mathfrak{K}$.

2.2 A uniform-point-system and moving surface.— Let us take a uniform-point-system in E_3 which is defined by the fundamental cell carrying ρ–points and a closed surface $\mathfrak{K}$ of volume V which moves in the space E_3. Then the number m of points of the system, which are contained by $\mathfrak{K}$, depends upon the position of $\mathfrak{K}$ in the space E_3. Let us find the mean value of the numbers of such points.

1) Let us take a domain in E_3 which is composed of $\mu_1\mu_2\mu_3$–pieces of unit-cell. In this domain, μ_1, μ_2 and μ_3–pieces are lying in three rows respectively whose directions are mutually independent. Let us denote this domain by $\mathfrak{A}$ and the set of points of the system in $\mathfrak{A}$ by $\mathfrak{P}_{\mathfrak{A}}$. Now we move $\mathfrak{K}$ freely in E_3 with the sole condition that the origin $P(x, y, z)$ of the moving frame attached to it remains always in $\mathfrak{A}$. The set of the unit cells which be able to have common points with such a moving $\mathfrak{K}$ forms a domain including $\mathfrak{A}$ in it. We denote it by $\overline{\mathfrak{A}}$. In $\overline{\mathfrak{A}}$, if the numbers of the unit cells arranged in the three directions are $\mu_1+2\nu_1$, $\mu_2+2\nu_2$, $\mu_3+2\nu_3$ respectively, the total number of the unit cells is $(\mu_1+2\nu_1)(\mu_2+2\nu_2)(\mu_3+2\nu_3)$. Accordingly the number of the points of the system which are contained in $\overline{\mathfrak{A}}$ is $\rho(\mu_1+2\nu_1)(\mu_2+2\nu_2)(\mu_3+2\nu_3)$. Then let us denote the set of points contained in $\overline{\mathfrak{A}}$ by $\mathfrak{P}_{\overline{\mathfrak{A}}}$.

Next, if we take off ν_1, ν_2 and ν_3 layers of the cell of $\mathfrak{A}$ from both sides of each direction respectively, we get a domain formed by $(\mu_1-2\nu_1)(\mu_2-2\nu_2)(\mu_3-2\nu_3)$–cells. This domain is denoted by $\overline{\overline{\mathfrak{A}}}$ and the hollows domain, which is formed by taking off $\overline{\overline{\mathfrak{A}}}$ from $\overline{\mathfrak{A}}$, by $\overline{\mathfrak{A}}-\overline{\overline{\mathfrak{A}}}$. Further the set of the points of the system in $\overline{\mathfrak{A}}-\overline{\overline{\mathfrak{A}}}$ is denoted by $\mathfrak{P}_{\overline{\mathfrak{A}}-\overline{\overline{\mathfrak{A}}}}$. Obviously $\overline{\mathfrak{A}}-\overline{\overline{\mathfrak{A}}}$ contains

$\rho\{(\mu_1+2\nu_1)(\mu_2+2\nu_2)(\mu_3+2\nu_3)-(\mu_1-2\nu_1)(\mu_2-2\nu_2)(\mu_3-2\nu_3)\}$ –points of the uniform-point-system.

2) To obtain the mean value of the number of points of the system which are

contained by $\mathfrak{K}$, we apply the formula (1) in § 2.1 to the present case and we have

$$\int_{\mathfrak{P}_{\bar{\mathfrak{A}}} \frown \mathfrak{K} \neq 0} m\dot{\mathfrak{K}} = 8\pi^2 \rho V \prod_i (\mu_i + 2\nu_i). \tag{2}$$

We divide the integral at the left side into two parts; the one is a total measure of $\mathfrak{K}$ as the origin $P(x, y, z)$ attached to $\mathfrak{K}$ is contained in $\mathfrak{A}$, and the other is the rest of the integral. If we denote the former by $I_{P\in\mathfrak{A}}$ and the latter by $I_{P\overline{\in}\mathfrak{A}}$, we have

$$I_{P\in\mathfrak{A}} + I_{P\overline{\in}\mathfrak{A}} = 8\pi^2 \rho V \prod_i (\mu_i + 2\nu_i).$$

Then, denoting by $\int_{P\in\mathfrak{D}} m\dot{\mathfrak{K}}$ the total measure of positions of $\mathfrak{K}$ so that P attached to $\mathfrak{K}$ may be contained in a unit cell $\mathfrak{D}$ and $\mathfrak{K}$ may contain $m(\geqq 1)$-points of the system, we can express $I_{P\in\mathfrak{A}}$ as follows,

$$I_{P\in\mathfrak{A}} = \mu_1\mu_2\mu_3 \int_{P\in\mathfrak{D}} m\dot{\mathfrak{K}}.$$

On the other hand, we denote by $I_{P\overline{\in}\mathfrak{A}}$ the total measure of $\mathfrak{K}$ in positions that it has some points in common with $\mathfrak{P}_{\bar{\mathfrak{A}}-\underline{\mathfrak{A}}}$ on condition that P attached to $\mathfrak{K}$ be out of $\mathfrak{A}$. Therefore we can put

$$I_{P\overline{\in}\mathfrak{A}} \leqq 8\pi^2 \rho V \{\prod_i (\mu_i + 2\nu_i) - \prod_i (\mu_i - 2\nu_i)\}.$$

Dividing the both members by $\mu_1\mu_2\mu_3$ and letting $\mu_1, \mu_2, \mu_3 \to \infty$, we have,

$$\lim_{\mu_i\to\infty} \frac{I_{P\overline{\in}\mathfrak{A}}}{\mu_1\mu_2\mu_3} \leqq \lim_{\mu_i\to\infty} 8\pi^2\rho V \left\{\prod_i \left(1 + 2\frac{\nu_i}{\mu_i}\right) - \prod_i \left(1 - 2\frac{\nu_i}{\mu_i}\right)\right\} = 0.$$

Then, if we rewrite (2) as follows,

$$\mu_1\mu_2\mu_3 \int_{P\in\mathfrak{D}} m\dot{\mathfrak{K}} + I_{P\overline{\in}\mathfrak{A}} = 8\pi^2 \rho V \prod_i (\mu_i + 2\nu_i),$$

and divide both members by $\mu_1\mu_2\mu_3$, we have

$$\int_{P\in\mathfrak{D}} m\dot{\mathfrak{K}} + \frac{I_{P\overline{\in}\mathfrak{A}}}{\mu_1\mu_2\mu_3} = 8\pi^2 \rho V \prod_i \left(1 + 2\frac{\nu_i}{\mu_i}\right).$$

Consequently, the limit of the equation as $\mu_i \to \infty (i = 1, 2, 3)$ is

$$\int_{P\in\mathfrak{D}} m\dot{\mathfrak{K}} = 8\pi^2 \rho V. \tag{3}^*$$

On the other hand, the total measure of $\mathfrak{K}$ at such a position as $P(x, y, z)$ attached to $\mathfrak{K}$ is contained in a unit cell of the system is given by

$$\int_{P\in\mathfrak{D}} \dot{\mathfrak{K}} = 8\pi^2 C, \tag{4}$$

where C denotes the volume of a unit cell.

Dividing (3) with (4), we have the following theorem.

Theorem 1. When a closed surface $\mathfrak{K}$ of volume V moves in E_3 where a uniform-

* In order to save trouble of reexplanation of the same procedure, we shall speak of the equation (3) as "a limit equation by covering procedure of the whole space E_3 with the equation (1)".

point-system is defined, the mean value $\overline{m}$ of the number of points of the system contained in $\mathfrak{K}$, is given by

$$\overline{m}=\frac{\rho V}{C}, \tag{5}$$

where ρ denotes the number of points belonging to a unit cell and C the volume of a unit cell.

3) Geometrical probability (1) .

Let us suppose a special case in which the number of points contained by $\mathfrak{K}$ is exclusively limited to m_1 or m_2. Let $p_i(i=1,2)$ be the probabilities that the number in question is m_i $(i=1,2)$, $(m_1>m_2\geqq 0)$.

Then

$$m_1p_1+m_2p_2=\frac{\rho V}{C} \quad \text{and} \quad p_1+p_2=1.$$

Hence we have

$$p_1=\frac{\rho V-Cm_2}{C(m_1-m_2)},\quad p_2=\frac{Cm_1-\rho V}{C(m_1-m_2)}. \tag{6}$$

§ 3. A uniform-curve-system.

3.1 A uniform-curve-system and moving surface.— Let us take a uniform-curve-system in E_3 which is defined by a fundamental cell attached with rectifiable curves of tatal length U, and $\mathfrak{K}$ is a surface of area F which moves in the space. We denote by m the number of common points of $\mathfrak{K}$ at any position with curves of the system.

Let us find the mean value of m.

First, we take two domains $\mathfrak{A}$ and $\overline{\mathfrak{A}}$ in the space under the same manner as in §2.2, (1), namely $\mathfrak{A}$ and $\overline{\mathfrak{A}}$ contain $\prod_i \mu_i$ and $\prod_i(\mu_i+2\nu_i)$-pieces of unit cell respectively; and then denote two set of curves in $\mathfrak{A}$ and $\overline{\mathfrak{A}}$ by $\mathfrak{C}_{\mathfrak{A}}$ and $\mathfrak{C}_{\overline{\mathfrak{A}}}$ respectively. Denoting by U the total length of the curves allotted to a unit cell and using the kinematic density $\dot{\mathfrak{K}}=|\cos\theta|\dot{s}\dot{\sigma}\dot{\tau}\dot{\Omega}$ [6], we obtain

$$\int_{\mathfrak{C}_{\overline{\mathfrak{A}}}\frown\mathfrak{K}\neq 0} m\dot{\mathfrak{K}}=4\pi^2 UF\prod_i(\mu_i+2\nu_i).$$

In the same manner as in §2.2, (2), let us denote by $\int_{P\in\mathfrak{D}} m\dot{\mathfrak{K}}$ such a tatal measure of $\mathfrak{K}$ as the origin P attached to $\mathfrak{K}$ is contained in a unit cell $\mathfrak{D}$ and $\mathfrak{K}$ crosses with some of the curves in $\mathfrak{D}$, and by $J_{P\in\mathfrak{D}}$ the total measure of $\mathfrak{K}$ in positions where P is out of $\mathfrak{A}$. Then the above equation can be represented as follows,

$$\mu_1\mu_2\mu_3\int_{P\in\mathfrak{D}} m\dot{\mathfrak{K}}+J_{P\overline{\in}\mathfrak{A}}=4\pi^2 UF\prod_i(\mu_i+2\nu_i).$$

Divinding the both members by $\mu_1\mu_2\mu_3$ and letting $\mu_i\to\infty(i=1,2,3)$, we have

$$\int_{P\in\mathfrak{D}} m\dot{\mathfrak{K}}=4\pi^2 UF. \tag{7}$$

Again dividing the above equation with (4), we have a following theorem with regard to the mean value in question.

Theorem 2. When a closed surface $\mathfrak{K}$ *of area* F *moves in* E_3, *where a uniform-curve-system is defined, the mean value* $\bar{m}$ *of the number of common points to* $\mathfrak{K}$ *and curves of the system is given by*

$$\bar{m}=\frac{UF}{2C}, \tag{8}$$

where C *denotes the volume of a unit cell* $\mathfrak{D}$ *and* U *the total length of curves attached to a unit cell.*

3.2 Kinematic formula and geometrical probability (2).— Let us take a convex body $\mathfrak{G}_1$ of volum V, which is enclosed by a closed surface $\mathfrak{K}$ of area F. And apply to $\mathfrak{G}_1$, moving in the space E_3, the kinematic principal formula ([2] and [6]) by W. Blaschke, that is,

$$\int C(\mathfrak{G}_0,\mathfrak{G}_1)\dot{\mathfrak{G}}_1=8\pi^2\{C_0V_1+M_0S_1+S_0M_1+V_0C_1\}. \tag{9}$$

In this case, taking account of $C(\mathfrak{C}_{\mathfrak{A}},\mathfrak{G}_1)=4\pi$, $C_0=4\pi$, $V_1=V$, $M_0=\pi U$, $S_1=F$, $S_0=V_0=0$, we have

$$\int_{\mathfrak{C}_{\mathfrak{A}}\frown\mathfrak{G}_1\neq 0}\dot{\mathfrak{G}}_1=2\pi^2(4V+UF)\prod_i(\mu_i+2\nu_i). \tag{10}$$

Taking now the same procedure as in §2.2 (2), we can obtain the limit equation* by the covering procedure of the whole space E_3 with the kinematic formula (10) for the present case, as follows,

$$\int_{\mathfrak{C}\frown\mathfrak{G}_1\neq 0}\dot{\mathfrak{G}}_1=2\pi^2(4V+UF) \tag{11}$$

Now, we can classify $\mathfrak{G}_1$'s position with reference to the uniform-curve-system into three cases as follows;

(i) $\mathfrak{K}$ includes completely one or any pieces of curves belonging to the uniform-curve-system,

(ii) $\mathfrak{K}$ has common points with the uniform-curve-system,

(iii) $\mathfrak{G}_1$ has no common point with the uniform-curve-system.

Corresponding with the above classification, we divide the integral standing at the left side of (11) into two parts, that is,

$$\int_{\mathfrak{C}'\subset\mathfrak{G}_1}\dot{\mathfrak{G}}_1+\int_{\mathfrak{C}\frown\mathfrak{K}\neq 0}\dot{\mathfrak{G}}_1=2\pi^2(4V+UF),$$

where $\mathfrak{C}'$ represents any pieces of curves belonging to a unit cell. On the other hand, the integral $\int_{\mathfrak{C}\frown\mathfrak{K}\neq 0}\dot{\mathfrak{G}}_1$ can be written by (7) as follows,

$$\int_{\mathfrak{C}\frown\mathfrak{K}\neq 0}\dot{\mathfrak{G}}_1=\int_{P\in\mathfrak{D}}m\dot{\mathfrak{K}}=4\pi^2UF.$$

* see §2.2 (2)

By the above two equations, we have

$$\int_{\mathfrak{C}'\subset\mathfrak{G}_1} \dot{\mathfrak{G}}_1 = 2\pi^2(4V-UF). \tag{12}$$

Let us now denote by p_i $(i=1, 2, 3)$ the respective probability that one of the above classified cases (i), (ii) and (iii) may occur. Using the above equation (12), we have,

$$p_1=\frac{4V-UF}{4C},\quad p_2=\frac{UF}{2C},\quad p_3=\frac{4(C-V)-UF}{4C}. \tag{13}$$

3.3 Geometrical probability (3). — In regard to the preceeding problem, let us consider such a special case in which the number m of the intersection points is exclusively limited to m_1 and m_2. Then the equation (7) can be written as follows,

$$\int_{P\in\mathfrak{D}} m\dot{\mathfrak{K}} = m_1\int_{P\in\mathfrak{D}_1}\dot{\mathfrak{K}} + m_2\int_{P\in\mathfrak{D}_2}\dot{\mathfrak{K}} = 4\pi^2 FV, \tag{14}$$

where $\mathfrak{D}_i$ $(i=1, 2)$ is a sub-domain in a unit cell $\mathfrak{D}$, that is, so long as P is contained in $\mathfrak{D}_i$, the number of the intersection points of $\mathfrak{C}$ and $\mathfrak{K}$ is always m_i $(i=1, 2)$, $(m_1 > m_2 \geqq 0)$.

Let us now denote by p_i $(i=1, 2)$ the respective probabilities that the number of the intersection points in question is m_i $(i=1,2)$. Then

$$p_i=\frac{\int_{P\in\mathfrak{D}_i}\dot{\mathfrak{K}}}{\int_{P\in\mathfrak{D}}\dot{\mathfrak{K}}},\quad (i=1,\ 2)$$

where

$$\int_{P\in\mathfrak{D}}\dot{\mathfrak{K}} = 4\pi^2 C.$$

So we have

$$p_1=\frac{UF-2m_2C}{2C(m_1-m_2)},\quad p_2=\frac{2m_1C-UF}{2C(m_1-m_2)}. \tag{15}$$

For example, when a convex closed surface $\mathfrak{K}$ moves in a space E_3, where we take a uniform-curve-system defined by a set of parallel lines whose distances are greater than the diameter of $\mathfrak{K}$, $\mathfrak{K}$ will either intersect one of the curves or not. Let k be the number of curves and U the length of a curve in a fundamental domain. Now $m_1=2$ and $m_2=0$, hence the probability that $\mathfrak{K}$ will intersect one of the parallel curves is

$$p=\frac{kUF}{4C}. \tag{16}$$

§ 4. A uniform-surface-system.

4.1 A uniform-surface-system and moving curves. — Let us take a uniform-surface-system in E_3 which is defined by a fundamental cell attached with a set of surfaces of total area F, and move a set of curves of total length U, in the space. If we denote by $\overline{m}$ the mean value of the number of common points of the set of curves with surfaces of the system, by the same procedure as in § 3.1 we have the following theorem.

Theorem 3. When a set of curves of total length U moves in E_3 where a uniform-surface-system is defined, the mean value $\overline{m}$ of the number of common points to the set of curves and surfaces of the system is given by

$$\overline{m}=\frac{UF}{2C}, \tag{17}^*$$

where C denotes the volume of a unit cell and F the total area of a set of surfaces attaching to a unit cell.

4.2 Uniform-surface-system and moving surfaces. — Let us now consider that a set $\mathfrak{K}_1$ of surfaces of total area F_1 moves in the space E_3 where a uniform-surface-system is defined. In this case $\mathfrak{K}_1$ will cross with surfaces of the system along curves.

Let us represent by s the total length of intersecting curves. Let us now find the mean value of s. Using the kinematic density by L. A. Santaló [6]

$$\dot{s}\dot{\mathfrak{K}}_1=\sin^2\alpha\,\dot{\sigma}_0\dot{\sigma}_1\dot{\tau}_0\dot{\tau}_1\dot{\alpha},$$

we have a integral formula

$$\int_{\mathfrak{K}_0\frown\mathfrak{K}_1\neq 0}\dot{s}\dot{\mathfrak{K}}_1=4\pi^3F_0\cdot F_1, \tag{18}$$

where $\mathfrak{K}_0$ represents a set of surfaces contained in a unit cell and F_0 the sum of the surface area of $\mathfrak{K}_0$. Taking now the same procedure as in § 2.2 (2), we can obtain the limit-equation by covering procedure of the whole space E_3 with the equation (18), as follows,

$$\int_{P\in\mathfrak{D}}s\dot{\mathfrak{K}}_1=4\pi^3F_0\cdot F_1. \tag{19}$$

Now the above integration is extended to all over the positions in which $\mathfrak{K}_1$ always intersects some of the surfaces of the system and the origin $P(x, y, z)$ attached to $\mathfrak{K}_1$ is contained in $\mathfrak{D}$.

Then, dividing (19) by the equation (4), we have a following theorem in regard to the mean value of s.

Theorem 4. When a set of surfaces of total area F_1 moves in E_3 where a uniform-surface-system is defined, the mean value of the total length s of the intersection curves along which the moving surfaces cut some of the surfaces of the system is given by

$$\overline{s}=\frac{\pi F_0F_1}{2C}, \tag{20}$$

where C denotes the volume of a unit cell and F_0 the total area of a set of surfaces attaching to a unit cell.

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* This is same with (8).

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