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On the Convergence of Some Gap Series

By

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1. Let $\varphi(x)$ be a periodic function with the period 2π satisfying

$$\varphi(x) \in \text{Lip } \alpha \ (0<\alpha\leqq 1) \text{ and } \int_0^{2\pi} \varphi(x)dx=0.$$

We consider the convergence of the series

$$(1) \qquad \sum_{n=1}^{\infty} c_n \varphi(\lambda_n x),$$

where $\{\lambda_n\}$ is the properly defined increasing sequence, but not necessary the sequence of integers. For this purpose we consider the integral

$$(2) \qquad I\equiv\int_{-\infty}^{\infty} S_{n(x)}(x)d\sigma(x), \qquad \sigma(x)=\frac{\sin^2 x/2}{2\pi x^2},$$

where $S_n(x)$ is the n–th partial sum of (1) and $n(x)$ is any bounded $(1\leqq n(x)\leqq N)$ and integral valued function.

Kawata[1] obtained the following theorem as the generalization of the results of Kac[2] and Hartman.[3]

Theorem. Let $\{\lambda_n\}$ satisfy

$$(3) \qquad \lambda_{n+1}/\lambda_n \geqq n^c \qquad (c>0),$$

then for any $p>1$ there exists A_p such as

$$\int_{-a}^{a} \max_{1\leqq n<\infty} |S_n(x)|^p d\sigma(x) \leqq A_p\left(\sum_{n=1}^{\infty} c_n^2\right)^{p/2}.$$

where $a>0$, and $A_p=A_{p,a}$ depends only on a and p.

We consider the case $p=1$ of this theorem.

Theorem 1. Let $\omega(n)\uparrow\infty$ and

$$n/\omega(n)=0(1).$$

If

$$(4) \qquad \lambda_{n+1}/\lambda_n \geqq \lambda > 1,$$

then there exists a constant A such as

$$\left|\int_{-\infty}^{\infty} S_{n(x)}(x)d\sigma(x)\right| \leqq A\left(\sum_{n=1}^{\infty} c_n^2 \omega(n)\right)^{1/2}.$$

If $\omega(n)$ satisfies merely $\omega(n)\uparrow$, we have the following theorems by use of the methods of Salem.[4]

Theorem 2. Suppose that $\{\lambda_n\}$ satisfies (3) and $\omega(n)\uparrow\infty$. If $\{n_k\}$ satisfies the following condition for some constant $B>1$,

$$\sum_{k=1}^{\infty}\frac{1}{\omega(n_k)}\left(\frac{1}{B}\right)^{\omega(n_k)}<\infty, \tag{5}$$

then there exists an absolute constant A such as

$$\left|\int_{-\infty}^{\infty} S_{n(x)}(x)d\sigma(x)\right|\leqq A(\sum_{n=1}^{\infty} c_n^2\omega(n))^{1/2},$$

where $n(x)$ is any bounded measurable function $(1\leqq n(x)\leqq N)$ having a sub-set of $\{n_k\}$ as its range.

On the other hand if $\{\lambda_n\}$ is a sequence of integers, we have

Theorem 3. Suppose that the sequence of integers $\{\lambda_n\}$ satisfies (4), and the sequence of integers $\{n_k\}$ satisfies (5) for some constant $B>1$, and

$$n_{k+1}-n_k\geqq\log k \qquad (k=1, 2, \cdots\cdots), \tag{6}$$

then

$$\left|\int_{0}^{2\pi} S_{n(x)}(x)dx\right|\leqq A(\sum_{n=1}^{\infty} c_n^2\omega(n))^{1/2},$$

where $n(x)$ is any bounded measurable function $(1\leqq n(x)\leqq N)$ having a sub-set of $\{n_k\}$ as its range.

2. **Lemma 1.** Let $\varphi(x)$ satisfy the above mentioned conditions, and $\{\lambda_n\}$ satisfy (4), then

$$\left|\int_{-\infty}^{\infty}\varphi(\lambda_p x)\varphi(\lambda_q x)d\sigma(x)\right|\leqq\frac{A}{\lambda^{\alpha|p-q|}}.$$

This was proved by Kawata and Udagawa.[5)]

Lemma 2. If $\varphi(x)$ and $\{\lambda_n\}$ satisfy the conditions of Theorem 2, then

$$\int_{-\infty}^{\infty}\left|\sum_{i=1}^{n} c_i\varphi(\lambda_i x)\right|^p d\sigma(x)\leqq A\left(\frac{p}{2}\right)^{p/2}(\sum_{1}^{\infty} c_n^2)^{p/2}.$$

This is Theorem 1 of Kawata.

Lemma 3. If $\varphi(x)$ and $\{\lambda_n\}$ satisfy the conditions of Lemma 1, then

$$\int_{-\infty}^{\infty}(\sum_{p=1}^{n} c_p\varphi(\lambda_p x))^2 d\sigma(x)\leqq A(\sum_{p=1}^{n} c_p^2).$$

Remark. If $\{\lambda_n\}$ is the sequence of integers, then we have

$$\int_{0}^{2\pi}(\sum_{p=1}^{n} c_p\varphi(\lambda_p x))^2 dx\leqq A(\sum_{p=1}^{n} c_p^2).$$

Proof.

$$J\equiv\int_{-\infty}^{\infty}(\sum_{p=1}^{n} c_p\varphi(\lambda_p x))^2 d\sigma(x)=\sum_{p=1}^{n} c_p^2\int_{-\infty}^{\infty}\varphi(\lambda_p x)d\sigma(x)$$

$$+2\sum_{p=1}^{n}\sum_{q=p+1}^{n} c_p c_q \int_{-\infty}^{\infty} \varphi(\lambda_p x)\varphi(\lambda_q x)d\sigma(x).$$

Since $\varphi(x)$ is the bounded function, $|\varphi(x)|\leqq M$ and

$$J\leqq \sum_{p=1}^{n} c_p^2 \int_{-\infty}^{\infty} M^2 d\sigma(x)+2\sum_{p=1}^{n}\sum_{q=p+1}^{n} c_p c_q \frac{A}{\lambda^{\alpha(q-p)}}$$

$$\leqq M^2 \sum_{p=1}^{n} c_p^2 + 2A(\sum_{p=1}^{n} c_p^2)^{1/2}\left\{\sum_{p=1}^{n}\left(\sum_{q=p+1}^{n} c_q \frac{1}{\lambda^{\alpha(q-p)}}\right)^2\right\}^{1/2}$$

$$\leqq M^2 \sum_{p=1}^{n} c_p^2 + 2A(\sum_{p=1}^{n} c_p^2)^{1/2}\left\{\sum_{r=1}^{n}\frac{1}{\lambda^{\alpha r}}\left(\sum_{p=1}^{n-r} c_{p+r}^2\right)^{1/2}\right\}$$

$$\leqq\left(\sum_{p=1}^{n} c_p^2\right)\left(M^2+2A\sum_{r=1}^{\infty}\frac{1}{\lambda^{\alpha r}}\right)\leqq A\left(\sum_{p=1}^{n} c_p^2\right).$$

3. Let $n(x)$ be a bounded $(1\leqq n(x)\leqq N)$ and integral valued function, and if we put

$$E_n \equiv (x;\ n(x)\geqq n) \qquad n=1, 2, \cdots\cdots, N,$$

then

$$I=\int_{-\infty}^{\infty} S_{n(x)}(x)d\sigma(x)=\int_{-\infty}^{\infty}\left(\sum_{n=1}^{N} c_n\varphi(\lambda_n x)\psi_n(x)\right)d\sigma(x)$$

$$=\int_{-\infty}^{\infty}\left(\sum_{n=1}^{N} c_n'\varphi(\lambda_n x)\frac{\psi_n(x)}{\sqrt{\omega(n)}}\right)d\sigma(x),$$

where $\psi_n(x)$ is the characteristic function of E_n and

$$c_n'=c_n\sqrt{\omega(n)}.$$

If we put

$$T_n(x)\equiv\sum_{i=1}^{n} c_i'\varphi(\lambda_i x),$$

then by the Abel's transformation

$$I=\int_{-\infty}^{\infty}\sum_{n=1}^{N} T_n(x)\Delta\left(\frac{\psi_n(x)}{\sqrt{\omega(n)}}\right)d\sigma(x)$$

$$=\int_{-\infty}^{\infty}\left(\sum_{n=1}^{N} T_n(x)\frac{1}{\sqrt{\omega(n)}}\Delta\psi_n(x)\right)d\sigma(x)+\int_{-\infty}^{\infty}\left(\sum_{n=1}^{N} T_n(x)\psi_{n+1}(x)\Delta\left(\frac{1}{\sqrt{\omega(n)}}\right)\right)d\sigma(x)$$

$$(7) \qquad \equiv I_1+I_2.$$

Now we can suppose $\sum c_n'^2<\infty$, otherwise our theorems are trivial. By the use of Schwarz' inequality and Lemma 3.

$$|I_2|\leqq\sum_{n=1}^{N}\Delta\left(\frac{1}{\sqrt{\omega(n)}}\right)\left(\int_{-\infty}^{\infty} T_n^2(x)d\sigma(x)\right)^{1/2}$$

$$(8) \qquad \leqq A\left(\sum_{n=1}^{\infty} c_n'^2\right)^{1/2}\sum_{n=1}^{N}\left(\frac{1}{\sqrt{\omega(n)}}\right)\leqq A\left(\sum_{n=1}^{\infty} c_n'^2\right)^{1/2}.$$

Let the Fourier series of $\varphi(x)$ be

$$\varphi(x)\sim\sum_{n=1}^{\infty}(a_n\cos nx+b_n \sin nx)$$

and the n-th partial sum of it be

$$s_n(x)=\sum_{i=1}^{n}(a_i \cos ix+b_i \sin ix),$$

then

$$T_n(x)=\sum_{i=1}^{n}c_i'\varphi(\lambda_i x)=\sum_{i=1}^{n}c_i's_{\mu_i}(\lambda_i x)+\sum_{i=1}^{n}c_i'(\varphi(\lambda_i x)-s_{\mu_i}(\lambda_i x))$$

$$\equiv\sum_{i=1}^{n}c_i's_{\mu_i}(\lambda_i x)+\sum_{i=1}^{n}c_i'R_{\mu_i}(\lambda_i x),$$

where

$$R_n(x)=\varphi(x)-s_n(x),$$

and $\{\mu_i\}$ is the properly defined sequence of integers. Thus we have

(9) $$I_1=\int_{-\infty}^{\infty}\left\{\sum_{n=1}^{N}\frac{1}{\sqrt{\omega(n)}}\left(\sum_{i=1}^{n}c_i's_{\mu_i}(\lambda_i x)\right)\Delta\psi_n(x)\right\}d\sigma(x)$$

$$+\int_{-\infty}^{\infty}\left\{\sum_{n=1}^{N}\frac{1}{\sqrt{\omega(n)}}\left(\sum_{i=1}^{n}c_i'R_{\mu_i}(\lambda_i x)\right)\Delta\psi_n(x)\right\}d\sigma(x)\equiv P+Q.$$

Since by the hypothesis

$$|\varphi(x)-s_n(x)|\leqq A\log n/n^{\alpha}$$

uniformly in x,

$$|R_{\mu_i}(\lambda_i x)|\leqq A\log\mu_i/\mu_i^{\alpha}$$

uniformly in x. Consequently

$$|Q|\leqq\sum_{n=1}^{N}\frac{1}{\sqrt{\omega(n)}}\int_{-\infty}^{\infty}\left|\sum_{i=1}^{n}c_i'R_{\mu_i}(\lambda_i x)\right|\Delta\psi_n(x)d\sigma(x)$$

$$\leqq A\sum_{n=1}^{N}\frac{1}{\sqrt{\omega(n)}}\int_{-\infty}^{\infty}\left(\sum_{i=1}^{N}c_i'^2\right)^{1/2}\left\{\sum_{i=1}^{N}\left(\frac{\log\mu_i}{\mu_i^{\alpha}}\right)^2\right\}^{1/2}\Delta\psi_n(x)d\sigma(x)$$

$$\leqq A\left\{\sum_{i=1}^{N}\left(\frac{\log\mu_i}{\mu_i^{\alpha}}\right)^2\right\}^{1/2}\left(\sum_{i=1}^{\infty}c_i'^2\right)^{1/2}\left(\sum_{n=1}^{N}\frac{1}{\sqrt{\omega(n)}}\Delta\sigma(E_n)\right).$$

Now if we put $\mu_i=[i^{(1+\epsilon)/2\alpha}]$ for any $\epsilon>0$, then

$$\sum_{i=1}^{\infty}\left(\frac{\log\mu_i}{\mu_i^{\alpha}}\right)^2<\infty,$$

and

(10) $$|Q|\leqq A\frac{1}{\sqrt{w(1)}}\left(\sum_{i=1}^{\infty}c_i'^2\right)^{1/2}\leqq A\left(\sum_{i=1}^{\infty}c_i'^2\right)^{1/2}.$$

Lastly we consider P. Since $\Delta\psi_n(x)\ \Delta\psi_m(x)=0$ for $n\neq m$,

$$P^2 \leqq \int_{-\infty}^{\infty}\Big\{\sum_{n=1}^{N}\frac{1}{\sqrt{\omega(n)}}\Big(\sum_{i=1}^{n} c_i' s_{\mu_i}(\lambda_i x)\Big)\Delta\psi_n(x)\Big\}^2 d\sigma(x)$$

$$=\int_{-\infty}^{\infty}\Big\{\sum_{n=1}^{N}\frac{1}{\omega(n)}\Big(\sum_{i=1}^{n} c_i' s_{\mu_i}(\lambda_i x)\Big)^2 \Delta\psi_n(x)\Big\} d\sigma(x)$$

$$\leqq \sum_{n=1}^{N}\frac{1}{\omega(n)}\Big(\sum_{i=1}^{n} c_i'\Big)^2\int_{-\infty}^{\infty}\Big(\sum_{i=1}^{n} s_{\mu_i}^2(\lambda_i x)\Big)\Delta\psi_n(x)d\sigma(x).$$

Since $|\varphi(x)|\leqq M$ uniformly in x and

$$|\varphi(x)-s_n(x)|\leqq A\frac{\log n}{n^\alpha}$$

we may suppose

$$|s_n(x)|\leqq 2M$$

uniformly in x. Thus by the hypothesis concerning $\omega(n)$

$$P^2\leqq\Big(\sum_{i=1}^{\infty} c_i'^2\Big)\sum_{n=1}^{N}\frac{n}{\omega(n)}(2M)^2\int_{-\infty}^{\infty}\Delta\psi_n(x)d\sigma(x) \tag{11}$$

$$\leqq A\Big(\sum_{n=1}^{N}\Delta\sigma(E_n)\Big)\Big(\sum_{i=1}^{\infty} c_i'^2\Big)\leqq A\Big(\sum_{i=1}^{\infty} c_i'^2\Big).$$

Consequently from (7)–(11) we obtain Theorem 1.

Since by (8) $|I_2|\leqq A\Big(\sum_{i=1}^{\infty} c_i'^2\Big)^{1/2}$, we must prove for Theorem 2 that

$$|I_1|\equiv\Big|\int_{-\infty}^{\infty}\Big(\sum_{n=1}^{N} T_n(x)\frac{1}{\sqrt{\omega(n)}}\Delta\psi_n(x)\Big)d\sigma(x)\Big|\leqq A\Big(\sum_{i=1}^{\infty} c_i'^2\Big)^{1/2}. \tag{12}$$

Let $\{n_k\}$ and $n(x)$ satisfy the hypothesis of Theorem 2, then there exists k' such as

$$n_{k'}\leqq N< n_{k'+1}, \tag{13}$$

and

$$I_1=\sum_{k=1}^{k'}\frac{1}{\sqrt{\omega(n_k)}}\int_{-\infty}^{\infty} T_{n_k}(x)\Delta\psi_{n_k}(x)d\sigma(x).$$

Now

$$I_1^2\leqq\int_{-\infty}^{\infty}\Big\{\sum_{k=1}^{k'}\frac{1}{\sqrt{\omega(n_k)}}T_{n_k}(x)\Delta\psi_{n_k}(x)\Big\}^2 d\sigma(x)\cdot\int_{-\infty}^{\infty} d\sigma(x)$$

$$=\int_{-\infty}^{\infty}\sum_{k=1}^{k'}\frac{1}{\omega(n_k)}T_{n_k}^2(x)\Delta\psi_{n_k}(x)d\sigma(x).$$

If we put

$$K\equiv\Big(\sum_{n=1}^{\infty} c_n'^2\Big)^{1/2}=\Big(\sum_{n=1}^{\infty} c_n^2\omega(n)\Big)^{1/2},$$

then

$$I_1^2\leqq K^2\sum_{k=1}^{k'}\frac{1}{\omega(n_k)}\int_{-\infty}^{\infty}\Big(\sum_{i=1}^{n_k}\frac{c_i'}{K}\varphi(\lambda_i x)\Big)^2\Delta\psi_{n_k}(x)d\sigma(x).$$

Let

$$l_k=\frac{\omega(n_k)}{e},\quad \text{and}\quad \frac{1}{l_k}+\frac{1}{l_k'}=1 \qquad (l_k\geqq 2)$$

then by the Young's inequality[6] and Lemma 2

$$I_1^2\leqq K^2\sum_{k=1}^{k'}\int_{-\infty}^{\infty}\frac{1}{l_k\omega(n_k)^{l_k}}\Big(\sum_{i=1}^{n_k}\frac{c_i'}{K}\varphi(\lambda_i x)\Big)^{2l_k}d\sigma(x)$$

$$+K^2\sum_{k=1}^{k'}\int_{-\infty}^{\infty}\frac{1}{l_k'}(\Delta\psi n_k)d\sigma(x)$$

$$\leqq AK^2\sum_{k=1}^{k'}\frac{l_k^{l_k}}{l_k\omega(n_k)^{l_k}}\Big(\sum_{i=1}^{n_k}\frac{c_i'^2}{K^2}\Big)^{l_k}+K^2\sum_{k=1}^{k'}\Delta\sigma(E_{n_k})$$

$$\leqq AK^2\sum_{k=1}^{k'}\frac{e}{\omega(n_k)}\Big(\frac{1}{e}\Big)^{\omega(n_k)/e}+K^2.$$

$$\leqq AK^2\Big\{1+\sum_{k=1}^{\infty}\frac{1}{\omega(n_k)}\Big(\frac{1}{e^{1/e}}\Big)^{\omega(n_k)}\Big\}.$$

Thus by (5)

$$I_1^2\leqq AK^2=A\Big(\sum_{n=1}^{\infty}c_k^2\omega(n)\Big),$$

and from (7) we obtain Theorem 2.

4. We now prove Theorem 3. By the same way as the proof of Theorem 2

$$I=\int_0^{2\pi}S_{n(x)}(x)dx$$

$$=\int_0^{2\pi}\sum_{n=1}^{N}T_n(x)\frac{1}{\sqrt{\omega(n)}}\Delta\psi_n(x)dx+\int_0^{2\pi}\sum_{n=1}^{N}T_n(x)\psi_{n+1}(x)\Delta\Big(\frac{1}{\sqrt{\omega(n)}}\Big)dx$$

$$\equiv I_1+I_2.$$

Then we have by the same process and Remark of Lemma 3.

$$|I_2|\leqq AK=A\Big(\sum_{n=1}^{\infty}c_n^2\omega(n)\Big)^{1/2},$$

and

$$I_1=\sum_{k=1}^{k'}\frac{1}{\sqrt{\omega(n_k)}}\int_0^{2\pi}Tn_k(x)\Delta\psi n_k(x)dx$$

$$=\sum_{k=1}^{k'}\frac{1}{\sqrt{\omega(n_k)}}\int_0^{2\pi}S_{\mu_{n_k}}(x)\Delta\psi n_k(x)dx+\sum_{k=1}^{k'}\frac{1}{\sqrt{\omega(n_k)}}\int_0^{2\pi}R_{\mu_{n_k}}(x)\Delta\psi n_k(x)dx,$$

$$\equiv P+Q.$$

Where k' has the same meaning with (13), $S_{\mu_{n_k}}(x)$ is the μ_{n_k}-th partial sum of the Fourier series of $Tn_k(x)$ and

$$R_{\mu_{n_k}}(x)=Tn_k(x)-S_{\mu_{n_k}}(x),\quad S_{\mu_{n_k}}=\sum_{i=1}^{n_k}c_i'\sum_{\lambda_i p\leqq\mu_{n_k}}(a_p\cos\lambda_i px-b_p\sin\lambda_i px)$$

Thus if we put $\mu_{n_k,i}=[\mu n_k/\lambda_i]$, then

$$R^2_{\mu_{n_k}}(x)=\sum_{i=1}^{n_k} c_i'\{\varphi(\lambda_i x)-S_{\mu_{n_k},i}(x,\varphi(\lambda_i x))\},$$

and

$$(14)\quad \left(\int_0^{2\pi} R^2_{\mu_{n_k}}(x)dx\right)^{1/2}=\left(\int_0^{2\pi}\left|\sum_{i=1}^{n_k} c_i'\{\varphi(\lambda_i x)-S_{\mu_{n_k},i}(x,\varphi(\lambda_i x))\}\right|^2 dx\right)^{1/2}$$

$$\leqq\sum_{i=1}^{n_k}|c_i'|\left(\int_0^{2\pi}|\varphi(\lambda_i x)-S_{\mu_{n_k},i}(x,\ \varphi(\lambda_i x))|^2 dx\right)^{1/2}$$

$$\leqq\left(\sum_{i=1}^{n_k} c_i'^2\right)^{1/2}\left(\sum_{i=1}^{n_k}\left|\int_0^{2\pi}\varphi(\lambda_i x)-S_{\mu_{n_k},i}(x,\ \varphi(\lambda_i x))\right|^2 dx\right)^{1/2}$$

$$\leqq AK\left(\sum_{i=1}^{n_k}\left(\frac{\lambda_i}{\mu_{n_k}}\right)^{2\alpha'}\right)^{1/2}\qquad (0<\alpha'<\alpha).$$

Hence by (14)

$$|Q|\leqq\sum_{k=1}^{k'}\frac{1}{\sqrt{\omega(n_k)}}\left(\int_0^{2\pi} R^2_{\mu_{n_k}}(x)\,dx\right)^{1/2}\left(\int_0^{2\pi}\Delta\psi_{n_k}(x)dx\right)^{1/2}$$

$$\leqq AK\sum_{k=1}^{k'}\frac{1}{\sqrt{\omega(n_k)}}\left(\sum_{i=1}^{n_k}\left(\frac{\lambda_i}{\mu_{n_k}}\right)^{2\alpha'}\right)^{1/2}$$

$$\leqq AK\sum_{k=1}^{k'}\frac{1}{\sqrt{\omega(n_k)}}\left(\frac{\lambda_{n_k}}{\mu_{n_k}}\right)^{\alpha'}\left\{\sum_{i=1}^{n_k}\left(\frac{\lambda_i}{\lambda_{n_k}}\right)^{2\alpha'}\right\}^{1/2}.$$

Now if we put for any $\epsilon>0$

$$(15)\qquad \mu_{n_k}=[k^{(1+\varepsilon)/\alpha'}\lambda_{n_k}]\qquad (k=1,\ 2,\cdots\cdots),$$

then

$$\sum_{k=1}^{k'}\frac{1}{\sqrt{\omega(n_k)}}\left(\frac{\lambda_{n_k}}{\mu_{n_k}}\right)^{\alpha'}\left\{\sum_{i=1}^{n_k}\left(\frac{\lambda_i}{\lambda_{n_k}}\right)^{2\alpha'}\right\}^{1/2}$$

$$\leqq A\sum_{k=1}^{k'}\frac{1}{\sqrt{\omega(n_k)}}\frac{1}{k^{1+\varepsilon}}\left(\sum_{i=1}^{n_k}\frac{1}{\lambda^{2\alpha'(n_k-i)}}\right)^{1/2}<\infty,$$

and

$$|Q|\leqq AK.$$

Let ν be a fixed integer such as

$$\nu>(1+\varepsilon)/\alpha\log\lambda,$$

then by (6) and (15)

$$\lambda_{n_{k+\nu}}\geqq\lambda^{n_{k+\nu}-n_k}\lambda_{n_k}\geqq\frac{\lambda^{n_{k+\nu}-n_k}}{k^{(1+\varepsilon)/\alpha'}}\mu_{n_k}$$

$$=\mu_{n_k}\exp\left((n_{k+\nu}-n_k)\log\lambda-\frac{1+\varepsilon}{\alpha'}\log k\right)=\mu_{n_k}\exp\left[\left(\frac{n_{k+\nu}-n_k}{\log k}-\frac{1+\varepsilon}{\alpha'\log\lambda}\right)\log\lambda\cdot\log k\right]$$

$$\geqq\mu_{n_k}\exp\left[\left(\frac{\log k(k+1)\cdots\cdots(k+\nu-1)}{\log k}-\frac{1+\varepsilon}{\alpha'\log\lambda}\right)\log\lambda\cdot\log k\right]$$

$$\geqq\mu_{n_k}\exp\left[\left(\nu-\frac{1+\varepsilon}{\alpha'\log\lambda}\right)\log\lambda\cdot\log k\right]\geqq\mu_{n_k}.$$

Hence there exists a positive integer ν_0 such as

$$\lambda_{n_{k+\nu}} > \mu_{n_k} \tag{16}$$

for $k=1, 2, \cdots\cdots$ and $\nu \geqq \nu_0 = \left[\frac{1+\varepsilon}{\alpha' \log \lambda}\right]+1.$

Now we consider P with μ_{n_k} defining by (15).

$$P = \sum_{k=1}^{k'} \frac{1}{\sqrt{\omega(n_k)}} \int_0^{2\pi} S_{\mu_{n_k}}(x, \ T n_k) \Delta \psi n_k(x) dx$$

$$= \sum_{k=1}^{k'} \frac{1}{\sqrt{\omega(n_k)}} \int_0^{2\pi} T_{n_k}(x) S_{\mu_{n_k}}(x, \ \Delta \psi n_k) dx$$

$$= \sum_{k=1}^{k'} \frac{1}{\sqrt{\omega(n_k)}} \int_0^{2\pi} \{T n_{k'}(x) + (T n_{k'-1} - T n_{k'}) + (T n_{k'-2} - T n_{k'-1}) + \cdots\cdots + (T n_k - T n_{k+1})\} \cdot S_{\mu_{n_k}}(x, \ \Delta \psi n_k) dx$$

$$= \sum_{k=1}^{k'} \frac{1}{\sqrt{\omega(n_k)}} \int_0^{2\pi} T n_{k'}(x) S_{\mu_{n_k}}(x, \ \Delta \psi n_k) dx$$

$$+ \sum_{k=1}^{k'} \frac{1}{\sqrt{\omega(n_k)}} \int_0^{2\pi} \sum_{i=\nu_0+1+k}^{k'} (T n_{i-1} - T n_i) S_{\mu_{n_k}}(x, \ \Delta \psi n_k) dx$$

$$+ \sum_{k=1}^{k'} \frac{1}{\sqrt{\omega(n_k)}} \int_0^{2\pi} \sum_{i=k+1}^{\nu_0+k} (T_{n_{i-1}} - T_{n_i}) S_{\mu_{n_k}}(x, \ \Delta \psi n_k) dx$$

$$\equiv P_1 + P_2 + P_3.$$

By (16) $P_2 = 0$, and

$$|P_3| \leqq \sum_{k=1}^{k'} \frac{1}{\sqrt{\omega(n_k)}} \sum_{i=1}^{\nu_0} \left(\int_0^{2\pi} |T n_{k+i-1}(x) - T n_{k+i}(x)|^2 dx \right)^{1/2} \left(\int_0^{2\pi} S_{\mu_{n_k}}(x, \ \Delta \psi n_k)^2 dx \right)^{1/2}$$

$$\leqq \sum_{k=1}^{k'} \frac{1}{\sqrt{\omega(n_k)}} \sum_{i=1}^{\nu_0} \left(\sum_{j=n_{k+i-1}}^{n_{k+i}} c_j'^2 \right)^{1/2} \left(A \int_0^{2\pi} \Delta \psi n_k dx \right)^{1/2}$$

$$= \sum_{i=1}^{\nu_0} \sum_{k=1}^{k'} \left(\sum_{j=n_{k+i-1}}^{n_{k+i}} c_j'^2 \right)^{1/2} \frac{A(\Delta |E n_k|)^{1/2}}{\sqrt{\omega(n_k)}}$$

$$\leqq \sum_{\nu=1}^{\nu_0} \left(\sum_{k=1}^{k'} \sum_{n_{k+i-1}}^{n_{k+i}} c'^2 \right)^{1/2} \left(\sum_{k=1}^{k'} \frac{A^2 |\Delta E n_k|}{\omega(n_k)} \right)^{1/2}$$

$$\leqq A \sum_{i=1}^{\nu_0} \left(\sum_{j=n_i}^{n_{k'}+i} c_j'^2 \right)^{1/2} \leqq A r_0 K$$

Now

$$P_1^2 = \left[\int_0^{2\pi} \left\{ T n_{k'}(x) \sum_{k=1}^{k'} \frac{1}{\sqrt{\omega(n_k)}} S_{\mu_{n_k}}(x, \ \Delta \psi n_k) \right\} dx \right]^2$$

$$\leqq \left(\int_0^{2\pi} T^2_{n_{k'}}(x) dx \right) \left(\int_0^{2\pi} \left\{ \sum_{k=1}^{k'} \frac{1}{\sqrt{\omega(n_k)}} S_{\mu_{n_k}}(x, \ \Delta \psi n_k) \right\}^2 dx \right)$$

$$\leqq A K^2 U,$$

where

$$U \equiv \int_0^{2\pi} \left\{ \sum_{k=1}^{k'} \frac{1}{\sqrt{\omega(n_k)}} S_{\mu_{n_k}}(x, \Delta\psi_{n_k}) \right\}^2 dx.$$

$$U = \int_0^{2\pi} \left\{ \sum_{k=1}^{k'} \frac{1}{\omega(n_k)} S^2_{\mu_{n_k}}(x, \Delta\psi_{n_k}) \right\} dx$$

$$+2 \sum_{p=1}^{k'} \sum_{q=p+1}^{k'} \int_0^{2\pi} \frac{S_{\mu_{n_k}}(x, \Delta\psi_{n_p}) S_{\mu_{n_q}}(x, \Delta\psi_{n_q})}{\sqrt{\omega(n_p)\ \omega(n_q)}} dx \equiv U_1 + U_2.$$

Since $\mu_{n_q} > \mu_{n_p}$ for $q > p$,

$$U_2 = 2 \sum_{p=1}^{k'} \sum_{q=p+1}^{k'} \int_0^{2\pi} \frac{\Delta\psi_{n_q} S_{\mu_{n_p}}(x, \Delta\psi_{n_p})}{\sqrt{\omega(n_p)\ \omega(n_q)}} dx$$

$$\equiv 2 \sum_{p=1}^{k'} \int_0^{2\pi} \frac{1}{\omega(n_p)} S_{\mu_{n_p}}(x, \Delta\psi_{n_p}) \chi_p(x) dx,$$

where

$$0 \leqq \chi_p(x) \equiv \sqrt{\omega(n_p)} \sum_{q=p+1}^{k'} \frac{\Delta\psi_{n_q}(x)}{\sqrt{\omega(n_q)}} \leqq 1.$$

Whence

$$|U_2| \leqq 2 \sum_{p=1}^{k'} \left| \int_0^{2\pi} \frac{S_{\mu_{n_p}}(x, \chi_p)}{\omega(n_p)} \Delta\psi_{n_p}\, dx \right|$$

$$\leqq 2 \sum_{p=1}^{k'} \frac{1}{l_p} \frac{1}{\omega(n_p)^{l_p}} \int_0^1 |S_{\mu_{n_p}}(x, \chi_p)|^{l_p} dx + 2 \sum_{p=1}^{k'} \frac{1}{l_p'} \int_0^1 |\Delta\psi_{n_p}|^{l_p'} dx,$$

where l_p and l_p' satisfy

$$\frac{1}{l_p} + \frac{1}{l_p'} = 1, \qquad 1 < l_p' \leqq 2 \leqq l_p.$$

Now by the Riesz' Theorem

$$|U_2| \leqq 2 \sum_{p=1}^{k'} \frac{1}{l_p} \frac{2\pi(4l_p)^{l_p}}{\omega(n_p)^{l_p}} + 2 \sum_{p=1}^{k'} \frac{1}{l_p'} \Delta |E_{n_p}|$$

$$\leqq 4\pi \sum_{p=1}^{k'} \frac{(4l_p)^{l_p}}{l_p(\omega(n_p))^{l_p}} + 4\pi.$$

If we use the specified sequence

$$l_p = \frac{\omega(n_p)}{4e}$$

as the sequence $\{l_p\}$, then by (5)

$$|U_2| \leqq 4\pi \left\{ 4e \sum_{p=1}^{k'} \frac{1}{\omega(n_p)} \left(\frac{1}{e} \right)^{\omega(n_p)/4e} + 1 \right\}$$

$$\leqq 16\pi e \sum_{p=1}^{k'} \frac{1}{\omega(n_p)} \left(\frac{1}{e^{1/4e}} \right)^{\omega(n_p)} + 4\pi < \infty.$$

On the other hand

$$U_1=\sum_{k=1}^{k'}\frac{1}{\omega(n_k)}\int_0^{2\pi}S_{\mu n_k}(x,\ \Delta\psi n_k)^2\,dx$$

$$\leqq A\sum_{k=1}^{k'}\frac{1}{\omega(n_k)}\int_0^{2\pi}\Delta\psi n_k dx\leqq A\sum_{k=1}^{k'}\frac{1}{\omega(n_k)}\Delta\,|\,E n_k\,|$$

$$\leqq A\frac{1}{\omega(n_1)}2\pi.$$

Thus U is bounded and

$$P_1\leqq AK.$$

Hence we obtain

$$\left|\int_0^{2\pi}S_{n(x)}(x)dx\right|\leqq A\left(\sum_{n=1}^{\infty}c_n^2\omega(n)\right)^{1/2},$$

and this is the desired result.

Remark. In Theorem 3 we can replace $\varphi\in$ Lip α, $0<\alpha\leqq1$ by

$$\left(\int_0^{2\pi}|\,\varphi(x+h)-\varphi(x)\,|^2\,dx\right)^{1/2}\leqq Ah^{\alpha}$$

$0<\alpha\leqq1$, and Lemma 1 was verified by Isumi.[7]

References

[1] T. Kawata, On gap series, Kodai Math. Sem. Reports, 1950, pp. 21—40.

[2] M. Kac, Convergence of certain gap series, Ann. of Math., **44** (1943), pp. 411—415.

[3] P. Hartman, The divergence of non-harmonic gap series, Duke Math. Jour., **9** (1942), pp. 404—405.

[4] R. Salem, Essais sur les series trigonometriques, Paris 1940, pp. 68—72.

[5] T. Kawata and M. Udagawa, Some gap theorems, Kodai Math. Sem. Reports, 1949, pp. 19—22.

[6] A. Zygmund, Trigonometrical series, Wilno 1935, p. 65.

[7] S. Izumi, On some series with the functional terms, The Monthly of Real Analysis, **2** (1948).