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On the Dimension of Orders

By

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1. Introduction.

The present paper is a summary of the author's two previous ones [1], [2], revised and supplemented. Throughout the sequel the term "order" is used in place of the term "partial order". It seems to the author that, so far as the linear extension and the dimension of orders are concerned, it is convenient to consider an order as a subset of a Cartesian product. Thus we have the following definitions.

1.1. By an *order* defined on a set A we mean a subset P of the product $A\times A$ which satisfies the following conditions:

01: $\{(x,x)\mid x\in A\}\subseteqq P$,

02: $(x,y)\in P$ *and* $(y,x)\in P$ *imply* $x=y$,

03: $(x,y)\in P$ *and* $(y,z)\in P$ *imply* $(x,z)\in P$.

By the above definition $\{(x,x)\mid x\in A\}$ is itself an order on A, which is said a *null-order.*

By a *linear order* defined on a set A we mean an order L which satisfies the condition

04: *For every* $x,y\in A$, *either* $(x,y)\in L$ *or* $(y,x)\in L$.

By the *domain* of an order we mean the set on which the order is defined. By an *ordered set* $A(P)$ we mean a set A considered together with an order P difined on it.

We shall use the usual terminologies in the theory of the ordered sets, accompanied by "(P)" which may be interpreted in the obvious fashion. For example: "x and y are comparable(P)" means that either $(x,y)\in P$ or $(y,x)\in P$; x and y are incomprable (P)" means that neither $(x,y)\in P$ nor $(y,x)\in P$, which will be abbreviated by "$x\phi y(P)$"; "x procedes(P) y" means that $(x,y)\in P$, but $x\neq y$, which will be abbreviated by "$x<y$ (P)"; "a is a maximal(P) element of A" means that $a<x(P)$ for no element $x\in A$; "B is a linear(P) subset of A" means that B is a subset of A which is linearly ordered by the order P, etc.

1.2. Let P be an order on a set A and B a subset of A. Then $P(B)=\{(x,y\mid(x,y)\in P$ *and* $x,y\in B\}$ is an order defined on the set B which is said a *suborder* of P on B. The suborder of P on a set of single element b will be denoted by $P(b)$ or simply by b. When $P(B)$ is a linear order it is said a *linear suborder* and B a *linear*(P) *subset* of A.

1.3. By an *extension* of an order P we mean an order Q defined on the domain of P such that $P \subseteqq Q$. An extension L of an order P is said a *linear extesion* of P provided that L is a linear order.

Let $\{P_s \mid s \in S\}$ be a system of orders defined on a *fixed* set A. The intersection $\cap_{s \in S} P_s$ is also an order on A and each P_s is an extention of it. The union $\cup_{s \in S} P_s$ is not always an order, but if either $P_s \subseteqq P_{s'}$ or $P_{s'} \subseteqq P_s$ for every $s,s' \in S$, then it is an order on A and an extension of each P_s.

1.4. Let $\{A_s \mid s \in S\}$ be a system of pairwise disjoint sets, P_s an order difined on A_s for each $s \in S$ and Q an order defined on the set S. Then

$$\cup_{s \in S} P_s \cup \{(x_s, x_{s'}) \mid x_s \in A_s, x_{s'} \in A_{s'} \textit{ and } s < s'(Q)\}$$

is an order defined on the set $\cup_{s \in S} A_s$, which is said the *ordinal sum* of the system of orders $\{P_s \mid s \in S\}$ according to the order Q and denoted by $\Sigma_{Q(S)} P_s$. In particular when Q is the null-order on S, the sum coincides with $\cup_{s \in S} P_s$ and is said the *cardinal sum* and denoted by $\Sigma_{s \in S} P_s$. The ordinal sum of finite number of summands $A_1, A_2, \ldots, A_n$ according to the order of the index numbers will be denoted by $A_1 + A_2 + \ldots + A_n$.

1.5. Let $\{P_s \mid s \in S\}$ be a system of orders P_s defined on a set A for each $s \in S$, and F the set of all mappings f of S into $\cup_{s \in S} A_s$ such that $f(s) \in A_s$ for every $s \in S$. Then

$$\{(f,f) \mid f \in F\} \cup \{(f,g) \mid f,g \in F \textit{ and } (f(s),g(s)) \in P_s \textit{ for very } s \in S\}$$

is an order defined on the set F which is said the *cardinal product* of the system and denoted by $\Pi_{s \in S} P_s$. In particular, if $P_s = P$ for every $s \in S$, F becomes the set of all mappings of S into P. In this case the product is denoted by P^S. If all P_s are isomorphic to a fixed order P, then the product is isomorphic to P^S.

1.6. Let $\{P_s \mid s \in S\}$, A_s and F mean the same as in 1.5 and let W be a well-order defined on the set S. Then

$$\{(f,f) \mid f \in F\} \cup \{(f,g) \mid f,g \in F \textit{ and } f(\sigma) < g(\sigma)\ (P_\sigma) \textit{ for the least}(W) \textit{ element } \sigma \textit{ such that } f(\sigma) \neq g(\sigma)\}$$

is an order defined on F which is said the *ordinal product* of the system $\{P_s \mid s \in S\}$ according to the well-order W and denoted by $\Pi_{W(S)} P_s$.

2. Linear Extensions of an Order.

The following theorem has been already known [3].

2.1. THEOREM. *Let P be an order defined on a set A and a,b any two incomparable(P) elements of A, then there esists a linear extension L_1 such that $(a,b) \in L_1$ and a linear extension L_2 such that $(b,a) \in L_2$.*

This theorem will be generalized as follows:

2.2. THEOREM. *Let P be an order defined on a set A and B a null-ordered(P)*

subset of A, then there exists, for any linear order $L(B)$ defined on B, a linear extension L of P such that $L(B)\subseteqq L$.

Proof. For every $(a,b)\in L(B)$ put

$$P_{ab}=\{(x,y)\mid x,y\in A \textit{ and } (x,a),(b,y)\in P\},$$

then $Q=P\cup(\bigcup_{(a,b)\in L(B)} P_{ab})$ is an extension of P such that $L(B)\subseteqq Q$. Let $\mathfrak{A}$ be the set of all extensions Q of P such that $L(B)\subseteqq Q$. By the fact just mentioned $\mathfrak{A}\neq 0$ and the subset $\mathfrak{Q}$ of $\mathfrak{A}\times\mathfrak{A}$ specified by

$$\mathfrak{Q}=\{(Q,Q')\mid Q,Q'\in\mathfrak{A} \textit{ and } Q\subseteqq Q'\}$$

is an order defined on $\mathfrak{A}$. Let $\mathfrak{L}$ be any linear suborder of $\mathfrak{Q}$ and $\mathfrak{B}$ its domain. By the remark stated at the end of 1.3, $\bigcup_{Q\in\mathfrak{L}} Q$ is an order on A and is an extension of each $Q\in\mathfrak{L}$, i.e. it is an upper bound($\mathfrak{Q}$) of $\mathfrak{B}$. Therefore, by the Zorn's lemma, there exists an element L in $\mathfrak{Q}$ which has no proper extension. L is necessarily a linear order on A. Thus L is a linear extension of P which contains $L(B)$ as a suborder.

By a *right* linear extension of an order P with respect to an element a we mean a linear extension L satisfiying the condition

(α) if $a\phi x(P)$, then $(a,x)\in L$.

Dually a *left* linear extension L' with respect to a is defined by the condition

(β) if $a\phi x(P)$, then $(x,a)\in L'$.

We have the following theorem.

Theorem. *For every order P defined on a set A and for every element $a\in A$, there exist a right linear extention and a left linear extension with respect to a.*

Proof. Put

$$A_1=\{x\mid x\in A \textit{ and } (x,a)\in P\},\quad A_2=A-A_1,$$
$$A_3=\{x\mid x\in A \textit{ and } (a,x)\in P\},\quad A_4=A-A_3,$$

and let L_i be a linear extension of $P(A_i)$ for each $i=1,2,3,4$. Then the ordinal sum

(*) $$L=L_1+L_2 \textit{ and } L'=L_4+L_3$$

are right and left linear extensions of P with respect to a respectively.

One sees easily that every right linear extension L and every left linear extension L' with respect to an element a have the forms (*) respectively.

A linear extension of a given order which is right (left) with respect to *every* element of a subset of the domain of the order is said *right (left) with respect to the subset.*

2.4. Theorem. *Let P be an order defined on a set A and B a subset of A. There exists a right (left) linear extension of P with respect to B, if and only if B is a linear(P) subset of A.*

Proof. Let L be a right(left) linear extension of P with respect to B, and assume that $b\phi b'(P)$ for some $b,b'\in B$. Then we have both $(b,b')\in L$ and $(b',b)\in L$, hence $b=b'$ which contradicts $b\phi b'(P)$. Therefore either $(b,b')\in P$ or $(b',b)\in P$, i.e. B is a linear(P) subset of A.

Conversely let B is a linear(P) subset of A, and split the set A to the following 3 pairwise disjoint subsets:

$$A_1 = \{x \mid (x,b) \in P \text{ for all } b \in B\} - B,$$
$$A_3 = \{x \mid (x,b) \in P \text{ for no } b \in B\},$$
$$A_2 = A - (A_1 \cup A_3).$$

Let L_1 and L_3 be any linear extention of the suborders $P(A_1)$ and $P(A_3)$ respectively and let L_2 that of $P(A_2)$ constructed in the following manner.

For each element $x \in A_2$ put $B_x = \{b \mid b \in B \text{ and } (x,b) \in P(A_2)\}$, and define a binary relation "$\sim$" on A_2 by writing $x \sim y$ if and only if $B_x = B_y$. The relation thus defined is an equivalent relation by which the set A_2 is devided into classes $\{A_\xi \mid \xi \in X\}$, the set of representatives X being a subset of A_2. As is easily verified the subset $Q = \{(\xi,\eta) \mid \xi,\eta \in X \text{ and } B_\xi \subseteqq B_\eta\}$ of $X \times X$ is a linear order defined on X. Now let L_ξ be any linear extension of $P(A_\xi)$ for each $\xi \in X$ and $L_2 = \sum_{Q(X)} L_\xi$ the ordinal sum of L_ξ's according to the linear order Q on X. Then L_2 is a linear extension of $P(A_2)$. Since it is evident that L_2 is a linear order on A_2, it remains only to show that $P(A_2) \subseteqq L_2$. But since $\sum_{Q(X)} P(A_\xi) \subseteqq L_2$, it suffices to show that $P(A_2) \subseteqq \sum_{Q(X)} P(A_\xi)$. Let $(x,y) \in P(A_2)$. If $x,y \in A_\xi$ for a $\xi \in X$, then $(x,y) \in P(A_\xi) \subseteqq \sum_Q P(A_\xi)$. If $x \in A_\xi$, $y \in A_\eta$ for distinct $\xi,\eta \in X$, then we have necessarily $(\xi,\eta) \in Q$. For otherwise there exists an element $b \in B$ such that $b \in B_\eta$ but $b \notin B_\xi$. $y \in A_\eta$ implies $B_y = B_\eta \ni b$, hence $(y,b) \in P(A_2)$ by the definition of B_y. This, together with $(x,y) \in P(A_2)$, implies $(x,y) \in P(A_2)$, and hence $b \in B_x$. But on the other hand $x \in A_\xi$ implies $B_\xi = B_x \ni b$ which contradicts $b \notin B_\xi$. By the definition of the ordinal sum (1.4), $(\xi,\eta) \in Q$ implies $(x,y) \in \sum_{Q(X)} P(A_\xi)$. Thus we have $P(A_2) \subseteqq \sum_{Q(X)} P(A_\xi)$.

Now let $L = L_1 + L_2 + L_3$ be the ordinal sum of L_1, L_2 and L_3, then L is a right linear extension of P with respect to B. Since it is evident that L is a linear order defined on A, it remains only to show that $P \subseteqq L$. But since we have

$$\sum P(A_i) = P(A_1) + P(A_2) + P(A_3) \subseteqq L_1 + L_2 + L_3,$$

it suffices to show that $P \subseteqq \sum P(A_i)$. Let $(x,y) \in P$. If $x,y \in A_i$ for some i, $(x,y) \in P(A_i) \subseteqq \sum P(A_i)$. If $x \in A_i$, $y \in A_j$ for $i \neq j$, we have necessarily $i < j$, thence $(x,y) \in \sum P(A_i)$. In fact: If $x \in A_3$ and $y \in A_1 \cup A_2$, then $(x,b) \in P$ for no $b \in B$ and $(y,b) \in P$ for some $b \in B$. But latter, together with $(x,y) \in P$, implies $(x,b) \in P$ which contradicts the former. If $x \in A_2$ and $y \in A_1$, then $(x,b) \in P$ for some $b \in B$ and $(y,b) \in P$ for all $b \in B$. But the latter, together with $(x,y) \in P$, implies $(x,b) \in P$ for all $b \in B$ which contradicts the former. Hence we have $i < j$.

In order to show that L is right with respect to B, it is sufficient to show that $(b,x) \in L$ whenever $(x,b) \notin P$ for any fixed $b \in B$. But $(x,b) \in P$ implies $x \in A_2 \cup A_3$. If $x \in A_3$, evidently $(b,x) \in L$ since $b \in B \subseteqq A_2$. If $x \in A_2$, we have necessarily $x \in A_\xi$, $b \in A_\eta$ for ξ,η such that $(\eta,\xi) \in Q$ which implies $(b,x) \in L_2 \subseteqq L$. In fact: $(\xi,\eta) \in Q$ implies $B_x = B_\xi \subseteqq B_\eta = B_b$, hence $(x,b) \in P(A_2) \subseteqq P$ which contradicts $(x,b) \notin P$.

Thus the existence of a right linear extension with respect to B is estsblished. Dually the existence of a left one may be established. The gist is as follows. Split A to the subsets

$$A'_1 = \{x \mid (b,x) \in P \text{ for no } b \in B\},$$

$$A'_3=\{x \mid (b,x)\in P \text{ for all } b\in B\}-B,$$
$$A'_2=A-(A'_1\cup A'_3),$$

and let L'_1, L'_3 be any linear extensions of $P(A'_1)$ and $P(A'_3)$ respectively. Put $B'_x=\{b \mid b\in B \text{ and } (b,x)\in P(A'_2)\}$ for each $x\in A'_2$. The set A'_2 will be divided into classes $\{A'_\xi \mid \xi\in X'\}$ by the equivalent relation "$\sim$" defined by putting $x\sim y$ if and only if $B'_x=B'_y$. Then $Q'=\{(\xi,\eta) \mid \xi,\eta\in X' \text{ and } B'_\xi\subseteqq B'_\eta\}$ is a linear order on X'. Let L'_ξ be any linear extension of $P(A'_\xi)$ and put $L'_2=\sum_{Q'(X')} L'_\xi$, then L'_2 is a linear extension of $P(A'_2)$ and $L'=L'_1+L'_2+L'_3$ is a left linear extension of P with respect to B.

Let B and B' be two subsets of the domain of an order P. B is said *order-disjoint* (P) *upwards* (*downwards*) to B' provided that $(B'\times B)\cap P=0$ $((B\times B')\cap P=0)$. When B and B' are order-disjoint upwards to each other, they are simply said *order-disjoint* (P). We have the following theorem and the corollary.

2.5. THEOREM. *Let P be an order defined on a set A and B and B' two linear(P) subsets of A such that B is order-disjoint (P) upwards (downwards) to B'. Then there exists a linear extension of P which is both right(left) with respect to B and left (right) with respect to B'.*

Proof. Let $A_i(i=1,2,3)$ and $A'_j(j=1,2,3)$ be the same partitions of A as in the proof of the last theorem, and put $A_{ij}=A_i\cap A'_j$. Then considering the condition that $(B'\times B)\cap P=0$, one sees easily that

$$A_{12}=A_{13}=A_{22}=A_{23}=0$$

Hence we have

$$A_1=A_{11},\qquad A_2=A_{21},\qquad A_3=A_{31}\cup A_{32}\cup A_{33}.$$
$$A'_1=A_{11}\cup A_{21}\cup A_{31},\qquad A'_2=A_{32},\qquad A'_3=A_{33}.$$

Let $L_1=L_{11}$, L_{31} and $L'_3=L_{33}$ be any linear extensions of $P(A_1)=P(A_{11})$, $P(A_{31})$ and $P(A'_3)=P(A_{33})$ respectively and $L_2=L_{21}$, $L'_2=L_{32}$ the right and left linear extensions of $P(A_2)=P(A_{21})$ and $P(A'_2)=P(A_{32})$ respectively constructed in the same manners as in the proof of the last theorem. Then as is easily seen

$$L_3=L_{31}+L_{32}+L_{33} \qquad \text{and} \qquad L'_1=L_{11}+L_{21}+L_{31}$$

are linear exresions of $P(A_3)$ and $P(A'_1)$ respectively. Now put

$$L=L_{11}+L_{21}+L_{31}+L_{32}+L_{33},$$

then we have

$$L=L_1+L_2+L_3=L'_1+L'_2+L'_3$$

which shows that L is right with respect to B and left with respect to B'. The remaining part of the theorem may be proved dually.

2.6. COROLLARY. *Let P be an order difined on a set A and B and B' linear(P) subsets of A which are order-disjoint(P). Then there exists a linear extension which is right with respect to B and left with respect to B', and a linear extension which is left with respect to B and right with respect to B'.*

3. Dimension of Orders.

3.1. By a *realizer* of an order P we mean a system $\{L_s \mid s \in S\}$ of linear extensions of P such that $\cap_S L_s = P$, and by a *minimal realizer* a realizer $\{L_t \mid t \in T\}$ such that the cardinality $|T|$ of its index-set is less than that of every realizer of P. By the *dimension* of an order P we mean the cardinality of the index-set of its minimal realizer. The dimension of an order P is denoted by $D[P]$. It is evident that a system $\{L_s \mid s \in S\}$ of linear extensions of an order P is a realizer of P, if and only if there exists, for any incomparable(P) elements x and y, $s, s' \in S$ such that $(x,y) \in L_s$ and $(y,x) \in L_{s'}$. By this remark and the theorem 2.1, every order has a realzer and hence the dimension.

Evidently the dimension of a linear order is 1 and that of a null-order is 2.

The following theorem gives an estimation of the dimension of orders difined on a fixed set A.

3.2. Theorem. *The dimension of an order P does not exceed the cardinality of its domain A, i.e. $D[P] \leq |A|$.*

Proof. Let L_x, for each element $x \in A$, a right linear extension of P with respect to x. Then the system $\{L_y \mid x \in A\}$ is a realizer of P, since $(x,y) \in L_y$ and $(y,x) \in L_y$ for any incomparable(P) elements x and y.

The estimation given above is not the sharpest. The sharpest one will be given in section 8. The last theorem will be generalized as follows.

3.3. Theorem. *Let P be an order defined on a set A and $\{A_s \mid s \in S\}$ be a system of pairwise disjoint linear(P) subsets of A. Then $D[P] \leq |A - \cup_S A_s| + |S|$.*

Proof. Let L_x, for each element $x \in A - \cup_S A_s$, a right linear extension of P with respect to x and L_s, for each element $s \in S$, a left linear extension of P with respect to A_s. Then the system $\{L_x \mid x \in A - \cap_S A_s\} \cup \{L_s \mid s \in S\}$ is an realizer of P.

3.4. Corollary. *Let P be an order defined on a set A and $\{A_s \mid s \in S\}$ a system of pairwise disjoint linear(P) subset of A satisfying $A = \cup_S A_s$. Then $D[P] \leq |S|$.*

4. Dimension of Suborders.

4.1. Let P be an order defined on a set A and B a subset of A. Then $D[P(B)] \leq D[P]$ where $P(B)$ is the suborder of P on B. Thus if some elements are deleted from the domain of an order, the dimension diminishes in general. In this section the amount of the diminution caused by the deletion of elements will be estimated. We shall begin with showing that the diminution caused by deleting an element is at most 1.

4.2. Theorem. *Let P be an order defined on a set A and a an element of A. Then $D[P] \leq D[P(A-a)] + 1$.*

Proof. Let $\{L'_s \mid s \in S\}$ be a minimal realizer of $P(A-a)$, and split the set $A-a$ to the following subsets:

$$A_1=\{x \mid x\in A-a,\ (x,a)\in P\},$$
$$A_3=\{x \mid x\in A-a,\ (a,x)\in P\},$$
$$A_2=(A-a)-(A_1\cup A_3).$$

Take an element $\sigma\in S$ and fix it. Put

$$L_1=L'_\sigma(A_1)+a, \qquad L_2=L'_\sigma(A_2\cup A_3),$$
$$L_3=L'_\sigma(A_1\cup A_2), \qquad L_4=a+L'_\sigma(A_3).$$

Then $L=L_1+L_2$ and $L^*=L_3+L_4$ are right and left linear extensions of P with respect to a respectively.

For each $s\in S-\sigma$ put

$$A_{s1}=\{x \mid x\in A \text{ and } (x,x_1)\in L'_s \text{ for some } x_1\in A_1\},$$
$$A_{s2}=(A-a)-A_{s1}.$$

Then

$$L_s=\cup_S L'_s\cup\{(a,a)\}\cup\{(x,a) \mid x\in A_{s1}\}\cup\{(a,x) \mid x\in A_{s2}\}$$

is a linear extension of P and the system $\{L_s \mid s\in S-\sigma\}\cup\{L,L^*\}$ is a realizer of P. Hence we have

$$D[P]\leqq|S-\sigma|+2=|S|+1=D[P(A-a)]+1.$$

The last theorem shows that if n elements are deleted from the domain of an order the dimension diminishes, in general, by n. But when the deleted elements satisfy a particular condition the diminution will be lessend.

4.3. Theorem. *Let P be an order defined on a set A and a a minimal(P) element and b a maximal(P) element of A. If $a\phi b(P)$, then $D[P]\leqq D[P']+1$, P' being the suborder of P on the set $A'=A-\{a,b\}$.*

Proof. Split the set A' to the following subsets:

$$A_1=\{x \mid x\in A',\ x\phi a(P) \text{ and } (x,b)\in P\},$$
$$A_2=\{x \mid x\in A',\ x\phi a(P) \text{ and } x\phi b(P)\},$$
$$A_3=\{x \mid x\in A',\ (a,x)\in P \text{ and } x\phi b(P)\},$$

and let L_i be a linear extension of $P(A_i)$ for each $i=1,2,3$. Then

$$L=L_1+b+L_2+a+L_3$$

is a linear extension of P which is left with respect to a and right with respect to b.

Now let $\{L'_s \mid s\in S\}$ be a minimal realizer of P'. Then, for each $s\in S$, $L_s=a+L'_s+b$ is a linear extension of P which is right with respect to a and left with respect to b, and the system $\{L_s \mid s\in S\}\cup L$ is a realizer of P. Therefore we have

$$D[P]\leqq D[P']+1.$$

It is noteworthy that the diminution of the dimension caused by deleting a linear subset, whatever the cardinality may be, is at most 2, as the following theorem shows.

4.4. Theorem. *Let P be an order defined on a set A and C a linear(P) subset of A. Then $D[P]\leqq D[P(A-C)]+2$.*

Proof. Let $\{L'_s \mid s\in S\}$ be a minimal realizer of $P(A-C)$. For each element $c\in C$, put $U_c=\{u \mid u\in A-C \text{ and } (u,c)\in P\}$. Obviously $U_c\subseteqq U_{c'}$ if $(c,c')\in P(C)$. For given $s\in S$

and $c \in C$, put either $A_{sc} = \{x \mid x \in A - C \text{ and } (x,u) \in L'_s \text{ for some } u \in U_c\}$ or $A_{sc} = 0$ according as $U_c \neq 0$ or $U_c = 0$, and put $A^*_{sc} = (A-C) - A_{sc}$. Then

$$L_s = L'_s \cup P(C) \cup \{(x,c) \mid c \in C \text{ and } x \in A_{sc}\} \cup \{(c,x) \mid c \in C \text{ and } x \in A^*_{sc}\}$$

is a linear extension of P. Now let L_1 be a right linear extension of P with respect to C and L_2 a left one with respect to C. Then the system $\{L_s \mid s \in S\} \cup \{L_1, L_2\}$ is a realizer of P. Hence we have $D[P] \leq |S| + 2 = D[P(A-C)] + 2$.

To verify that L_s satisfies the conditions 01, 02, 04 is not hard. To verify that it satisfies 03 let (x,y), $(y,z) \in L_s$. There are the following 8 cases:

(1) $x,y,z \in A-C$. Then (x,y), $(y,z) \in L'_s$, hence $(x,z) \in L'_s \subseteq L_s$.

(2) $x,y,z \in C$. Then (x,y), $(y,x) \in P(C)$, hence $(x,z) \in P(C) \subseteq L_s$.

(3) $x,y \in A-C$ and $z \in C$. Then $(x,y) \in L'_s$ and $y \in A_{sz}$ that is $(y,u) \in L'_s$ for some $u \in U_z$. Therefore $(x,u) \in L'_s$ for some $u \in U_z$ that is $x \in A_{sz}$. Hence we have $(x,z) \in L_s$.

(4) $x,z \in A-C$ and $y \in C$. Then $(x,z) \in L'_s$ and $x \in A_{sy}$ that is $(x,u) \in L'_s$ for some $u \in U_y$ and $z \in A^*_{sy}$ that is $(u,z) \in L'_s$ for all $u \in U_y$. Hence we have $(x,z) \in L_s$.

(5) $y,z \in A-C$ and $x \in C$. Then $(y,z) \in L'_s$ and $y \in A^*_{sx}$. Hence when $U_x \neq 0$, $(u,y) \in L_s$ for all $u \in U_x$ which implies $(u,z) \in L'_s$ for all $u \in U_x$ that is $z \in A^*_{sx}$. When $U_x = 0$, $z \in A - C = A^*_{sx}$ since $A_{sx} = 0$. Thus in either case we have $(x,z) \in L_s$.

(6) $x \in A-C$ and $y,z \in C$. Then $x \in A_{sy}$ that is $(x,u) \in L'_s$ for some $u \in U_x (\neq 0)$, hence $(x,u) \in L'_s$ for some $u \in U_z$ which implies $(x,z) \in L_s$.

(7) $y \in A-C$ and $x,z \in C$. Then $y \in A^*_{sx}$ and $y \in A_{sz}$ that is $(u,y) \in L'_s$ for all $u \in U_x$ and $(y,u) \in L'_s$ for some $u \in U_z$. Assuming that $(z,x) \in P(C)$ and $x \neq z$ we have $U_z \subseteq U_x$ which is impossible. Hence $(x,z) \in P(C) \subseteq L_s$.

(8) $z \in A-C$ and $x,y \in C$. Then $U_x \subseteq U_y$ and $z \in A^*_{sy}$. When $U_x \neq 0$, $U_y \neq 0$ and we have $(u,z) \in L'_s$ for all $u \in U_y$, a fortiori for all $u \in U_x$. Therefore $z \in A^*_{sx}$. When $U_x = 0$, evidently $z \in A^*_{sx}$ since $A_{sx} = 0$. Hence in either case we have $(x,z) \in L_s$.

In order to verify that L_s is an extension of P, let $(x,y) \in P$. There are the following three cases:

(1) $x,y \in A-C$. Then $(x,y) \in P(A-C) \subseteq L'_s \subseteq L_s$.

(2) $x \in A-C$ and $y \in C$. Then $x \in U_y$, hence $x \in A_{sy}$ which implies $(x,y) \in L_s$.

(3) $y \in A-C$ and $x \in C$. If $U_x \neq 0$, $(u,y) \in P$ for all $u \in U_x$ which implies $y \in A^*_{sx}$. If $U_x = 0$, evidently $y \in A^*_{sx}$. Hence in either case $(x,y) \in L_s$.

4.5. THEOREM. *Let P be an order defined on a set A and C_1 and C_2 be two linear (P) subsets of A which are order-disjoint(P). Then $[P] \leq D[P(A-C_1-C_2)] + 2$.*

Proof. Let $\{L''_s \mid s \in S\}$ be a minimal realizer of $P(A-C_1-C_2)$. Let L'_s be a linear extension of $P(A-C_1)$ constructed from L''_s in the same manner as L_s was constructed from L'_s in the proof of the theorem 4.4. Then let L_s be a linear extension of P constructed from L'_s in like manner. On the other hand since C_1 and C_2 are order-disjoint(C) there exists a linear extension both left with respect to C_1 and right with respect to C_2, and a linear extension both right with respect to C_1 and left with respect to C_2. Let it be L_1 and L_2 respectively. Then the system $\{L_s \mid s \in S\} \cup \{L_1, L_2\}$

is a realizer of P. Hence we have $D[P] \leqq D[P(A-C_1-C_2)]+2$.

4.6. Let P be an order defined on a set A and a,b two distinct elements of A. When $(a,b) \in P$, $(a,x) \in P$ and $(x,b) \in P$ for no $x \in A$, it is said that b *covers*(P) a or a and b are *consecutive*(P) and denoted by $(a:b) \in P$. When $(a:b) \in P$, the pair of consecutive elements a and b is denoted by $(a:b)$. A pair $(a:b)$ is said of *rank* n if there exist n pairs of elements $x,y \in A$ such that $(x:b) \in P$, $(a:y) \in P$ and $x \phi y(P)$.

We have the following theorem.

4.7. THEOREM. *Let P be an order defined on a set A and $(a:b)$ a pair of consecutive elements of rank 0 or 1. Then we have $D[P(A-a-b)]+1$.*

Proof. Let $\mathfrak{R}'=\{L'_s \mid s \in S\}$ be a minimal realizer of $P(A-a-b)$. When $(a:b)$ is of rank 0 choose arbitrarily an element of $\mathfrak{R}'$ and let it be L'_σ. When $(a:b)$ is of rank 1 there exists a single pair x_o and y_o such that $(x_o:b)$, $(a:y_o) \in P$ and $x_o \phi y_o(P)$. Hence there exists an element of $\mathfrak{R}'$ which contains (x_o,y_o) as an element. Let it be also L'_σ. Split the set $A'=A-a-b$ to the following five disjoint subsets:

$$A_1=\{x \mid x \in A' \text{ and } (x,a) \in P\},$$
$$A_2=\{x \mid x \in A',\ x \phi a(P) \text{ and } (x,b) \in P\},$$
$$A_3=\{x \mid x \in A',\ x \phi a(P) \text{ and } x \phi b(P)\},$$
$$A_4=\{x \mid x \in A',\ (a,x) \in P \text{ and } x \phi b(P)\},$$
$$A_5=\{x \mid x \in A' \text{ and } (b,x) \in P\}.$$

Put

$$A_{123}=A_1 \cup A_2 \cup A_3, \qquad A_{345}=A_3 \cup A_4 \cup A_5,$$
$$L_{123}=L'_\sigma\,(A_{123}), \qquad L_{345}=L'_\sigma\,(A_{345}),$$

and

$$L_i=L'_\sigma\,(A_i) \qquad \text{for } i=1,2,3,4,5.$$

Then $L=L_1+a+L_2+b+L_{345}$ is a right linear extension of P with respect to $\{a,b\}$ and $L^*=L_{123}+a+L_4+b+L_5$ is a left linear extension of P with respect to $\{a,b\}$.

Now let L_s, for $s \in S-\sigma$, be the linear extension of P constructed from L'_s in the same manner as in the proof of the theorem 4.4 for the case where $C=(a:b)$, then the system $\{L_s \mid s \in S-\sigma\} \cup \{L,L^*\}$ is a realizer of P. Hence we have

$$D[P] \leqq |S-\sigma|+2=|S|+1=D[P(A-a-b)]+1.$$

5. D-reducible Orders.

5.1. Let P be an order defined on a set A. A subset B of A is said to be *d-removable* provided $P[D(A-B)]=D[P]$. An order P is said to be *d-reducible* if there exists at least a d-removable element in its domain. On estimating the dimension of a given order, it will be often convenient to delete beforhand the d-remvable set if it exists.

As a criterion for the d-removabilty of a subset we have the following very comprehensive theorem.

5.2. THEOREM*. *Let P be an order defined on a set A and B a subset of A satisfying the following conditions:*

1° *If there exists an element $x \in A-B$ such that $(x,b) \in P$ for an element $b \in B$, then $(x,b) \in P$ for all elements $b \in B$.*

2° *If there exists an element $x \in A-B$ such that $(b,x) \in P$ for an element $b \in B$, then $(b,x) \in P$ for all elements $b \in B$.*

Then B is d-remvable except at most an element b_o (chosen arbitrarily), *provided $D[P(A')] \geqq D[P(B-b_o)]$ where $A=A-(B-b_o)$.*

Proof. Let $\mathfrak{R}_1=\{L'_s \mid s \in S\}$ be a mimimal realizer of $P(A')$ and $\mathfrak{R}_2=\{L'_t \mid t \in T\}$ that of $P(B-b_o)$. Since $D[P(A')] \geqq D[P(B-b_o)]$ means that $|S| \geqq |T|$, there exists a mapping of $\mathfrak{R}_1$ onto $\mathfrak{R}_2$. Let it be f. Then

$$L_s=L'_s \cup f(L'_s) \cup \{(x,y) \mid x \in A',\ y \in B-b_o \text{ and } (x,b_o) \in L'_s\}$$
$$\cup \{(x,y) \mid x \in B-b_o,\ y \in A' \text{ and } (b_o,y) \in L'_s\}$$

is a linear extension of P for each $s \in S$ and the system $\mathfrak{R}=\{L_s \mid s \in S\}$ is a realizer of P. Hence we have $D[P]=D[P(A')]$.

In order to verify that L_s satisfies 03, let (x,y), $(y,z) \in L_s$.
There are the following 8 cases:

1) $x,y,z \in A'$, 2) $x,y,z \in B-b_o$,
3) $x,y \in A'$; $z \in B-b_o$, 4) $y,z \in A'$; $x \in B-b_o$,
5) $x,z \in A'$; $y \in B-b_o$, 6) $x \in A'$; $y,z \in B-b_o$,
7) $y \in A'$; $x,z \in B-b_o$, 8) $z \in A'$; $x,y \in B-b_o$.

If 1) or 2), evidently $(x,z) \in L_s$. If 3), we have $(x,y) \in L'_s$ and $(y,b_o) \in L'_s$, hence $(x,b_o) \in L'_s$ which implies $(x,z) \in L_s$. If 4), we have (y,z), $(b_o,y) \in L'_s$ hence $(b_o,z) \in L'_s$ which implies $(x,z) \in L_s$. If 5), we have (x,b_o) $(b_o,z) \in L'_s$, hence $(x,z) \in L'_s \subseteqq L_s$. If 6), $(x,y) \in L_s$ implies $(x,b_o) \in L'_s$ which implies $(x,z) \in L'_s$. If 7), we have $y=b_o$ since (b_o,y), $(y,b_o) \in L'_s$. Hence $(x,b_o) \in L'_s$ and $(b_o,z) \in L'_s$ which impliy $(x,z) \in L'_s \subseteqq L_s$. If 8), $(y,z) \in L_s$ implies $(b_o,z) \in L'_s$ which implies $(x,z) \in L_s$.

That L_s satisfies the conditions 01, 02 and 04 may be easily verified. Therefore L_s is a linear order on A.

In order to verify that L_s is an extension, let $(x,y) \in P$. There are the following 4 cases:

1) $x,y \in A'$, 2) $x \in A'$, $y \in B-b_o$,
3) $y \in A'$, $x \in B-b_o$, 4) $x,y \in B-B_o$.

If 1), evidently $(x,y) \in L'_s \subseteqq L_s$. If 2), by the condition 1°, necessarily $(x,b_o) \in P$. Hence we have $(x,b_o) \in L'_s$ which implies $(x,y) \in L_s$. If 3), we have $(b_o,y) \in P \subseteqq L'_s$ which implies $(x,y) \in L_s$. If 4), $(x,y) \in f(L'_s) \subseteqq L_s$.

In order to verify that $\mathfrak{R}$ is a realizer of P, let $x \phi y(P)$. If $x,y \in A'$, since $x \phi y(P(A))$, $(x,y) \in L'_s \subseteqq L_s$ and $(y,x) \in L'_{s'} \subseteqq L_{s'}$ for some $s,s' \in S$. If $x \in A'$ and $y \in B-b_o$, we have necessarily $x \phi b_o(P)$ by the conditions 1° and 2°. Therefore there exist $s,s' \in S$ such that

* (6.4), (6.5), (6,6), (6.8), (6.9) and (6.10) in [1] are all special cases of this theorem.

$(x,b_o)\in L'_s$ and $(b_o,y)\in L'_{s'}$, hence $(x,y)\in L_s$ and $(y,x)\in L'_s$. If $x,y\in B-b_o$, there exist $t,t'\in T$ such that $(x,y)\in L'_t$ and $(y,x)\in L'_{t'}$. But since there exist $s,s'\in S$ such that $L'_t=f(L'_s)$ and $L'_{t'}=f(L'_{s'})$, there exist $s,s'\in S$ auch that $(x,y)\in L_s$ and $(y,x)\in L_{s'}$.

It must be noticed that the set B satisfying the conditions 1° and 2° is not always removable as a whole.

6. Dimension of the Sum of Orders.

6.1. THEOREM. *Let Q be an order defined on a set S, $\{A_s \mid s\in S\}$ a system of pairwise disjoint sets, A_s an order defined on P_s and σ an element of S such that $D[P_\sigma]\geqq D[P_s]$ for all $s\in S$. Then*

$$D[Q]\leqq D[\Sigma_{Q(S)}P_s]=Max\ (D[P_\sigma],\ D[Q]).$$

Proof. Let B be a set of representstives slected from each set A_s. Then suborder of $\Sigma_{Q(S)}P_s$ on the set B is isomorphic to Q. Therfore $D[Q]\leqq D[\Sigma_{Q(S)}P_s]$.

Let $\mathfrak{R}=\{L_t \mid t\in T\}$ be a minimal realizer of Q and $\mathfrak{R}_s=\{L_{t(s)} \mid t(s)\in T_s\}$ that of P_s for each $s\in S$. Since $D[P_\sigma]\geqq D[P_s]$ means $|T_\sigma|\geqq|T_s|$, there exists a mapping f_s of $\mathfrak{R}$ onto $\mathfrak{R}_s$. Let $\{f_s \mid s\in S\}$ be a system of such mappings where f_s may be chosen arbitrarily for $s\neq\sigma$, but for σ let it be the identical mapping. For each $t(\sigma)\in T_\sigma$ and for each $t\in T$ let $L_{t\cdot t(\sigma}=\Sigma_{Lt}f_s(L_{t(\sigma)})$, be the sum of $f_s(L_{t(\sigma)})$'s according to the linear order L_t on S. Then $L_{t\cdot t(\sigma)}$ is a linear order defined on $\cup_S A_s$ and a linear extension of $\Sigma_{Q(S)}P_s$. In fact, considering $f_s(L_{t(\sigma)})=L_{t(s)}$ for some $t(s)\in T_s$, $L_{t(s)}\supseteqq P_s$ and $L_t\supseteqq Q$, we have

$$\begin{aligned}L_{t\cdot t(\sigma)}&=\cup_S f_s(L_{t(\sigma)})\cup\{(x,y) \mid x\in A_s,\ y\in A_s,\ and\ (s,s')\in L_t\}\\&\supseteqq\cup_S P_s\cup\{(x,y) \mid x\in A_s,\ y\in A_s,\ and\ (s,s')\in Q\}\\&=\Sigma_{Q(S)}P_s.\end{aligned}$$

If $D[P_\sigma]\geqq D[Q]$, there exists a mapping of T_σ onto T. Let it be φ, then $\mathfrak{R}_I=\{L_{\varphi(t(\sigma)),\ t(\sigma)} \mid t(\sigma)\in T_\sigma\}$ is a realizer of $\Sigma_{Q(S)}P_s$, and we have $D[\Sigma_{Q(S)}P_s]=D[P]$.

If $D[P_\sigma]<D[Q]$, there exists a mapping of T onto T_σ. Let it be ψ, then the system $\mathfrak{R}_{II}=\{L_{t,\psi(t)} \mid t\in T\}$ is a realizer of $\Sigma_{Q(S)}P_s$, and we have $D[\Sigma_{Q(S)}P_s]=D[Q]$.

In order to verify that $\mathfrak{R}_I$ is a realizer of $\Sigma_{Q(S)}P_s$, let $x\phi y(P)$. If $x,y\in A_s$ for some $s\in S$, then $x\phi y(P_s)$. Hence there exist $t(s)$, $t'(s)\in T_s$ such that $(x,y)\in L_{t(s)}=f_s(L_{t(\sigma)})$ and $(y,x)\in L_{t'(s)}=f_s(L_{t'(\sigma)})$. Therefore $(x,y)\in L_{t\cdot t(\sigma)}$ and $(y,x)\in L_{t,t'(\sigma)}$ for all $t\in T$. In particular $(x,y)\in L_{\varphi(t(\sigma)),t(\sigma)}$ and $(y,x)\in L_{\varphi(t'(\sigma)),t'(\sigma)}$. If $x\in A_s$ and $y\in A_s$, for $s\neq s'$, then $s\phi s'(Q)$. Hence there exist $t,t'\in T$ such that $(s,s')\in L_t$ and $(s',s)\in L_{t'}$. Let $t=\varphi(t(\sigma))$ and $t'=\varphi(t'(\sigma))$. Considering that the domains of $f_s(L_{t(\sigma)})$ and $f_{s'}(L_{t(\sigma)})$ are A_s and $A_{s'}$ respectively, we have $(x,y)\in L_{\varphi(t(\sigma)),t(\sigma)}$ and $(y,x)\in L_{\varphi(t'(\sigma)),t'(\sigma)}$. That $\mathfrak{R}_{II}$ is a realizer of $\Sigma_{Q(S)}P_s$ is verified similarly.

The following theorem is an immediate result of the theorem 6.1.

6.2. THEOREM. *Let P be an order defined on $A=\cup_S A_s$ which is decomposable to a sum $\Sigma_{Q(S)}P_s$, Q and P_s being orders defined on S and A_s respectively, and σ be an element of S such that $D[P_\sigma]\geqq D[P_s]$ for all $s\in S$. Then $A-A_\sigma$ is d-removable* *provided* $D[P_\sigma] \geqq D[Q]$, *and* $A-\{f(s) \mid s \in S\}$ *is d-removable provided* $D[P_\sigma] > D[Q]$, *f being a function which selects one element from each set* A_s.

As a particular case of 6.2, we have the following corollary.

6.3. COROLLARY. *Let P be an order difined on a set A. If a is an element of A comparable(P) to each element of A, then a is d-removable. In particular, the greatest(P) element and the least(P) element are d-removable.*

7. Some Examples.

In this section some particular orders will be studied

7.1. *Let* $X=\{x_i \mid i \in I\}$, $Y=\{y_i \mid i \in I\}$ *be two disjoint sets where I is the set of all integers, and let W be the order on* $X \cup Y$ *specified by*

$$W=\{(x_i,x_i) \mid i \in I\} \cup \{(y_i,y_i) \mid i \in I\} \cup \{x_i,y_i) \mid i \in I\} \cup \{(x_{i+1},y_i) \mid i \in I\}.$$

Then $D[W]=2$.

Proof. Let L_i and L'_i be linear orders on the sets $\{x_{i+1},y_i\}$ and $\{x_i,y_i\}$ respectively specified by

$$L_i=\{(x_{i+1}, x_{i+1}), (y_i,y_i), (y_{i+1},y_i)\}$$

and

$$L'_i=\{(x_i,x_i), (y_i,y_i), (x_i,y_i)\}$$

and let $Q(I)$ be the linear order defined on I in the natural fashion and $Q'(I)$ the inverse order of $Q(I)$. Then $L=\Sigma_{Q(I)}L_i$ and $L'=\Sigma_{Q'(I)}L_i$ are linear extensions of W and $\{L,L'\}$ is a realizer of W. Since $D[W] \geqq 2$ is evident, we have $D[W]=2$.

7.2. Let $X=\{x_s \mid s \in S\}$ and $Y=\{y_s \mid s \in S\}$ be two disjoint set and P_s the order on $X \cup Y$ specified by

$$P_s=\{(x_s,x_s) \mid s \in S\} \cup \{(y_s,y_s) \mid s \in S\} \cup \{(x_s,y_{s'}) \mid s,s' \in S \text{ and } s \neq s'\}.$$

P_S is isomorphic to the order on the set composed of all elements of S and their compliments in S defined by the relation of set inclusion. As is already known [4] the dimension of the latter is $|S|$. Hence $D[P_S]=|S|$. Therefore if $|S|$ is a transfinite cardinal number, P_S is d-reducible. But if $|S|$ is an integer it is d-irreducible in general, i.e. we have the following theorem.

7.3. THEOREM. *The order* P_n *defined on the set*

$$A_n=\{x_1,x_2,\dots,x_n;y_1,y_2,\dots,y_n\}$$

by specifying that

$$P_n=\{(x_i,x_i) \mid i=1,2,\dots,n\} \cup \{(y_i,y_i) \mid i=1,2,\dots,n\} \cup \{(x_i,y_j) \mid i,j=1,2,\dots,n;\ i \neq j\}$$

is d-irreducible provided that $n \geqq 3$.

Proof. We shall begin with proving the following lemma.

LEMMA. *If any pair of consecutive(P_n) elements is deleted from A_n, the dimension diminishes by 2 provied that $n \geqq 4$.*

One sees easily that whatever pair $(x:y)$ may be deleted the suborder $P_n(A_n - x - y)$ is isomorphic to the suborder $P_n(A_n - x_n - y_{n-1})$. Therefore to be proved is that

$$D[P_n(A_n - x_n - y_{n-1})] = n-2,$$

since $D[P_n]=n$ by 7.2. The mathmatical induction is used for the proof. In the first place we have $D[P_4]=4$ and $D[P_4(A_4 - x_4 - y_3)]=2$, since $P_4(A_4 - x_4 - y_3) = W(x_1, x_2, x_3, y_1, y_2, y_4)$ is a suborder of the order W defined in 7.1. Hence the lemma is ture for $n=4$. In the next place assume that the lemma is true for $n=k-1$, i.e. $D[P_{k-1}(A_{k-1} - x_{k-1} - y_{k-2})]=k-3$, and show that $D[P_k(A_k - x_k - y_{k-1})]=k-2$ holds. For the brevity put

$$P' = P_{k-1}(A_{k-1} - x_{k-1} - y_{k-2}) = P_k(A_k - x_k - y_k - x_{k-1} - y_{k-2}),$$
$$P'' = P_k(A_k - x_k - y_{k-1}),$$
$$P''' = P_k(A_k - x_k - y_{k-1} - x_{k-2} - y_{k-2}).$$

P' and P''' being isomorphic, we have

$$k-3 = D[P'] = D[P'''], \qquad (1)$$

and by the theorem 4.4 we have $k = D[P_k] \leqq D[P'''] + 2$, hence

$$k-2 \leqq D[P'']. \qquad (2)$$

Since P''' is obtained by delting x_{k-2} and y_{k-2} from P'' which are incomparable minimal (P'') and maximal(P'') elements, we have by the theorem 4.3

$$D[P''] \leqq D[P'''] + 1 \qquad (3)$$

From (1),(2) and (3) we obtain $D[P'']=k-2$.

In order to prove the theorem it suffices to show that the equality $D[P_n(A_n - y_{n-1})] = n-1$ holds, since $P_n(A_n - x)$ is isomorphic or inversely isomorphic to $P_n(A_n - y_{n\ 1})$ for any $x \in A_n$. When $n=3$, $D[P_3]=3$ and $D[P_3(A_3 - y_2)]=2$ since $P_3(A_3 - y_2) = W(x_1, x_2, x_3, y_1, y_3)$, a suborder of W in 7.1. Hence the theorem is true for $n=3$. When $n \geqq 4$, we have by the theorem 4.2 and the lemma,

$$n-1 \leqq D[P_n(A_n - y_{n-1})] \leqq D[P_n(A_n - x_n - y_{n-1})] + 1 = n-1,$$

hence $$D[P_n(A_n - y_{n-1})] = n-1.$$

We shall close this section with proving the following theorem.

7.4. THEOREM. *Let A be a set whose cardinality does not exceed 5. Then for every order P defined on A, $D[P] \leqq 2$. In other words in order to define an order whose dimension is greater than 2, a set whose cardinality is greater than 5 is necessary.*

Proof. When $|A|=2$ it is trivial. Let $3 \leqq |A| \leqq 5$. Every order which is decomposable to a cardinal sum may be left out of consideration, since it is d-reducible by the theorem 6.2. Classifying all orders under consideration by the combination of the number of maximal elements and that of minimal ones, we obtain the following table:

	$\|A\|=3$		$\|A\|=4$				$\|A\|=5$					
	I_3	II_3	I_4	II_4	III_4	IV_4	I_5	II_5	III_5	IV_5	V_5	VI_5
No. of max. el.	1	1	1	1	1	2	1	1	1	1	2	2
No. of min. el.	1	2	1	2	3	2	1	2	3	4	2	3

By interchanging the number of maximal elements and that of minimal ones we obtain other classes than those listed in the table which may be left out of consideration on account of the duality. Every order belonging to the classes other than IV_4, V_5 and VI_5 may be also left out of consideration, since it has the greatest element which is d-removable by the corollary 6.3. The domain of each order belonging to the class IV_4 is decomposable to a union of two disjoint linear subset. Therefore the dimension is at most 2 by the corollary 3.4. The domain of each order belonging to the class V_5 contains a linear set of three elements. The remaining two elements are either comparable or incomparable. If they are comparable, then the domain of the order is a union of two disjoint linear subsets, hence the dimension is at most 2 by 3.4. If they are incomparable, then one of them is maximal and the other minimal. Therefore the dimension is, by the theorem 4.3, at most 2 since the dimension of the order obtained by delting them is 1. Every order belonging to the class VI_5 is isomorphic to one of four orders represented by the following diagrams:

The first three of them are d-reducible by the theorm 5.2. The last is isomorphic to the suborder $W(x_1,x_2,y_1,y_2)$ of the order W in 7.1, hence the dimension is 2. Thus the theorem is proved completely.

8. The Least Upper Bound of the Dimensions of the Orders difined on a fixed Set.

We know that for the dimensions of the orders defined on a fixed set A, $|A|$ is an upper bound but not the least one (3.2) and that for every cardinal number n there exists an order of dimension n defined on a set of cardinality $2n$ (7.2), which shows that in order to define an order of dimension n a set of cardinality $2n$ is sufficient. Here arise the following two questions closely connected with each other: "What is the least upper bound for the dimensions of the orders defined on a fixed set A?" and "Is a set of cardinality $2n$ necessary in order to define an order of dimension n?". To answer to these questions is the subject of this section.

We shall begin with proving two lemmas on the finite orders.

8.1. LEMMA 1. *For every order P defined on a set A whose cardinality is at most 7, there exists at leas a pair of two consective(P) elements whose rank is at most 1.*

Proof. Let us prove the contraposition: If the rank of every pair of consecutive elements is at least 2, then $|A|\geqq 8$. Denote the set $\{a|(a:b)\in P\}$ by $A(b)$ for $b\in A$ and the set $\{b|(a:b)\in P\}$ by $B(a)$ for $a\in A$. Now let $(a_1:b_1)$ be a pair of consecutive elements. Since its rank is, by hypothesis, at least 2, there exists three elements either

1) $a_2\in A(b_1)-a_1$, $b_3\in B(a_1)-b_1$ and $b_4\in B(a_1)-b_1-b_3$ such that $a_2\phi b_3(P)$ and $a_2\phi b_4$(P)

or

2) $b_2\in B(a_1)-b_1$, $a_3\in A(b_1)-a_1$ and $a_4\in A(b_1)-a_1-a_3$ such that $a_3\phi b_2(P)$ and $a_4\phi b_2(P)$.

We may assume, without loss of generality, that the case 1) occurs.

Since $(a_2:b_1)$, $(a_1:b_1)$ and $(a_1:b_2)$ are at least of rank 2, there must exist three elements

$$b_2\in B(a_2)-b_1 \text{ such that } a_1\phi b_2(P),$$
$$a_3\in A(b_3)-a_1 \text{ such that } a_3\phi b_1(P),$$
$$a_4\in A(b_4)-a_1 \text{ such that } a_4\phi b_1(P).$$

Evidently $b_i(i=1,2,3,4)$ are pairwise distinct and $a_1\neq a_i$ for $i=2,3,4$ and $a_2\neq a_i$ for $i=3,4$. Hence when $a_3\neq a_4$, a_i and b_j $(i,j=1,2,3,4)$ are pairwise distinct. When $a_3=a_4$ for every $a_3\in A(b_3)-a_1$ and for every $a_4\in A(b_4)-a_1$, necessarily $A(b_3)-a_1=A(b_4)-a_1$. Hence if $(a_3,b_2)\in P$ and there exists an element c such that $a_3<c<b_2(P)$, then $a_i(i=1,2,3)$, $b_j(j=1,2,3,4)$ and c are pairwise distinct. And if either $(a_3:b_2)\in P$ or $a_3\phi b_2(P)$, then, since the rank of $(a_3:b_4)$ is at least 2, ther exists at least an element $b_5\in B(a_3)-b_3-b_4$ such that $b_5\phi a_1(P)$. Since $b_5\neq b_i$ for $i=1,2,3,4$, $a_i(i=1,2,3)$ and $b_j(j=1,2,3,4,5)$ are pairwise distinct. Thus A must contain at least 8 distinct elements.

8.2. LEMMA 2. *Let P be an order defied on a set A satisfying the following conditions:*

1° *Every linear subset of A is composed of at most 3 elements.*

2° *There exists at least a linear subset of A composed of 3 elements.*

3° *No pair of consecutive elements is of rank 0.*

4° *Every minimal element is comparable with every maximal element.*

Then there exists at least a pair of two order disjoint linear subset of A.

Proof. Denote the set of all maximal elements by B and the set of all minimal elements by M. By the condition 2° let $\{a,a_1,b_1\}$ be a linear subset composed of three elements where $a\in M$, $b_1\in B$ and $(a:a_1)$, $(a_1:b_1)\in P$. Since $(a_1:b_1)$ is not of rank 0, there exist two elements $a_2\in A(b_1)-a_1$ and $b_3\in B(a_1)-b_1\subset B$ such that $a_2\phi b_3(P)$. Then since $(a_2:b_1)$ is not of rank 0, there exists an element $b_2\in B(a_2)-b_1$ such that $b_2\phi a_2(P)$. Evidently b_1,b_2 and b_3 are pairwise distinct, and $A(b_3)-a_1\neq 0$ since $(a_1:b_3)$ is not of rank 0. Hence if there exists an $a_3\in A(b_3)-a_1$ such that either $a_3\phi b_1(P)$ or $a_3\phi b_2(P)$, then $(a_3:b_3)$ is order-disjoint with either $(a_2:b_1)$ or $(a_2:b_2)$. In case

$(a_3:b_1)$, $(a_3:b_2)\in P$ for all $a_3\in A(b_3)-a_1$, take any a_3 and let it be fixed. The rank of $(a_1:b_3)$ not being 0, there must be a $b_4\in B(a_1)-b_1-b_2-b_3$ such that $b_4\phi a_3(P)$. Evidently $b_4\neq b_i$ for $i=1,2,3$ and $a_3\notin M$ by the condition 4°. Therefore $(a_1:b_i)\in P$ for all $i=1,2,3$ by the condition 1°. The rank of $(a_1:b_4)$ not being 0, $A(b_4)-a_1\neq 0$. Hence if there exists an $a_4\in A(b_4)-a_1$ such that $a_4\phi b_i(P)$ for a value of $i=1,2,3$, then $(a_4:b_4)$ is order disjoint with $(a_4:b_i)$ for a value of $i=1,2,3$. In case $(a_4:b_i)\in P$ for every $a_4\in A(b_4)-a_1$ and for every value of $i=1,2,3$, take an a_4 and let it be fixed. The rank of $(a_1:b_4)$ not being 0, there must be a $b_5\in B(a_1)-b_1-b_2-b_3-b_4$ such that $b_5\phi a_4(P)$. Evidently $b_5\neq b_i$ for every $i\leqq 4$ and $a_4\notin M$ by the condition 4°. Therefore $(a_4:b_i)\in P$ for $i\leqq 3$ by the condition 1°. The rank of $(a_1:b_5)$ not being 0, $A(b_5)-a\neq 0$. Hence if there exists an $a_5\in A(b_5)-a_1$ such that $a_5\not\phi b_i(P)$ for a value of $i\leqq 4$, $(a_5:b_5)$ is order-disjoint with $(a_4:b_i)$ for a value of $i=1,2,3,4$. In case $(a_5:b_i)$ for every $i\leqq 4$, apply the same reasning and continue the same procedure as above. We will obtain two sequences

$$b_1,b_2,\dots,b_i,\dots(b_i\in B(a_1)-b_1-b_2\dots-b_{i-1}),$$

and $a_1,a_2,\dots,a_i,\dots$ such that $a_i\not\phi b_i(P)$ for every i and $(a_{i-1}:b_k)\in P$ for a fixed i and for every $k\leqq i-1$. But since A is a finite set and b's are all distinct the sequence$\{b_i\}$ must be finite. Let the last term be b_n. Then there must be an $a_n\in A(b_n)-a_1$ such that $a_n\phi b_i(P)$ for a value of $i\leqq n-1$, and hence $(a_n:b_n)$ is order-disjoint with $(a_{n-1}:b_i)$ for a value of $i\leqq n-1$. In fact, assume that $(a_n,b_i)\in P$ for all $i\leqq n-1$ and for all $a_n\in A(b_n)-a_1$. Then since $(a_1:b_n)$ is not of rank 0, there must be a $b_{n+1}\in B(a_1)-\cup_{i\ n}b_i$. This contradicts the definition of b_n.

8.3. THEOREM. *Let A be a set whose cardinality is greater than 3 and P any order defined on A. Then $D[P]\leqq[|A|/2]$ where $[|A|/2]$ means the integral part of $|A|/2$ in case $|A|$ is finite and $|A|$ itself in case it is transfinite.*

Proof. When $|A|$ is transfinite it is evident by the theorem 3.2. When $|A|$ is finite we shall prove it by the mathematical induction according to the cardinality of A. By the theorem 6.5 the proposition is true for $|A|=4,5$. Let $|A|=6$ or 7, then there exists, by the lemma 1, a pair of two consecutive elements $(a:b)$ the rank of which is 0 or 1. Hence by the theorem 4.7 we have

$$D[P]\leqq D[P(A-a-b)]+1\leqq 3=[|A|/2]$$

considering that $|A-a-b|=4$ or 5. Thus the preposition is true for $|A|\leqq 7$. Now let $|A|\geqq 8$ and assume that the proposition is ture for the sets whose cardinality are less than that of A. If P is d-reducible, then there exists an element a such that $D[P]=D[P(A-a)]$. Considering that the proposition is true for $P(A-a)$ by the assumption we have

$$D[P]=D[P(A-a)]\leqq[|A-a|/2]\leqq[|A|\ 2].$$

Let P be d-irreducible. If there exists a pair of incomparable minimal and maximal elements a and b, then we have, by the theorem 4.3 and the assumption of the induction,

$$D[P]\leqq D[P(A-a-b)]+1\leqq[|A-a-b|/2]+1=[|A|/2].$$

Let every maximal element is comparable with every minimal element. If there exists linear subset C composed of 4 elements, then by the theorem 4.4 and the assumption of the induction we have

$$D[P]\leqq D[P(A-C)]+2\leqq[|A-C|/2]+2=[|A|/2].$$

Let every linear subset of A be composed of at most three elements. If therer exists no linear subset of three elements, then by the theorem 5.2, P is d-reducible since in this case there is no element other than maximal or minimal elements and every maximal element is comparable with every minimal element. Hence it suffices to consider the case where at least a linear subset of three elements exists. Now if there exists a pair of consecutive elements $(a:b)$ of rank 0, then by the theorem 4.7 we have

$$D[P]\leqq D[P(A-a-b)]+1\leqq[|A-a-b|/2]+1=[|A|/2].$$

Let every pair of consecutive elements is not of rank 0. Then there exists, by the lemma 2, two order-disjoint linear subsets B and C. Hence we have, by the theorem 4.5,

$$D[P]\leqq D[P(A-B-C)]+2\leqq[|A-B-C|/2]+\leqq 2[(|A|-4)/2]+2=[|A|/2].$$

Thus the proposition is established completely.

The following proposition is equivalent to the last theorem.

8.4. THEOREM. *Let P be an order defined on a set A. If $D[P]\leqq 3$, then $2D[P]\leqq|A|$. In other words, in order to define an order of dimension n a set of cardinality $2n$ is necessary, provided $n\leqq 3$.*

The example 7.2 shows that for every cardinal number n (finite or transfinite), there exists an order of dimension n defined on a set of cardinality $2n$. But this will be generalized as follows.

8.5. THEOREM. *For every cardinal number $n\leqq 2$, there exists an order of dimension $[n/2]$ difined on a set of cardinality n.*

Proof. It suffices to consider the case where n is an odd integer. Let P' be an order of dimension $(n-1)$ 2 defined on a set A' of cardinality $n-1$. Then the order, for example, specified by

$$P=P'\cup\{(b,b)\}\cup\{(x,b)\,|\,x\in A\};b\notin A$$

is one of required orders.

As an immediate result of the theorems 8.3 and 8.5 we have the following theorem which is the answer to the first question mentioned at the begining of this section.

8.6. THEOREM. *Among the dimenosions of the orders defined on a fixed set A, $[|A|/2]$ is the greatest, provided $|A|\leqq 4$.*

9. The dimension of the Product of Orders.

9.1. Theorem. *Let $\Pi_{W(S)} P_s$ be the product of the orders P_s according to a well-order W difined on a set S, where P_s is difined on a set A_s for each $s \in S$. If $D[P_\sigma] \geqq D[P_s]$ for all $s \in S$, then $D[\Pi_{W(S)} P_s] = D[P_\sigma]$.*

Proof. Let $\mathfrak{R}_s = \{L_{t(s)} \mid t(s) \in T_s\}$ be a minimal realizer of P_s. Sinse $D[P_\sigma] \geqq D[P_s]$ there exists a mapping φ_s of $\mathfrak{R}_\sigma$ onto $\mathfrak{R}_s$. Let $\{\varphi_s \mid s \in S\}$ be a system of such mappings, where φ_σ is the identical mapping, and for $s \neq \sigma$, φ_s may be taken arbitrarily. Then $M_{t(\sigma)} = \Pi_{W(S)}\, \varphi_s(L_{t(\sigma)})$ is a linear extension of $\Pi_{W(S)} P_s$ and $\mathfrak{R} = \{M_{t(\sigma)} \mid t(\sigma) \in T_\sigma\}$ is a realizer of $\Pi_{W(S)} P_s$. In fact, let $f < g(\Pi_{W(S)} P_s)$, then there exists an element $s' \in S$ such that $f(s) = g(s)$ for all $s \in s'(W)$ and $f(s') < g(s')\ (P_{s'})$. Since $\varphi_{s'}(L_{t(\sigma)})$ is a linear extension of $P_{s'}$, we have $(f(s'), g(s')) \in \varphi_{s'}(L_{t(\sigma)})$ for every $t(\sigma) \in T_\sigma$. Hence we have $(f,g) \in L'_{t(\sigma)}$ for every $t(\sigma) \in T_\sigma$. Now let $f \phi g(\Pi_{W(S)} P_s)$. Then there exists $s' \in S$ such that $f(s') \phi g(s')(P_{s'})$ and

$$(a) \qquad f(s) = g(s) \text{ for every } s < s'(W).$$

Hence there exist $t(s'),\ t'(s') \in T_{s'}$ such that $(f(s'), g(s')) \in L_{t(s')}$ and $(g(s'), f(s')) \in L_{t'(s')}$. Let $L_{t(s')} = \varphi_{s'}(L_{t(\sigma)})$ and $L_{t'(s')} = \varphi_{s'}(L_{t'(\sigma)})$. Then $(f(s'), g(s')) \in \varphi_{s'}(L_{t(\sigma)})$ and $(g(s'), f(s')) \in \varphi_{s'}(L_{t'(\sigma)})$, which imply, together with (a), $(f,g) \in M_{t(\sigma)}$ and $(g,f) \in M_{t'(\sigma)}$. Thus $\mathfrak{R}$ realizer $\Pi_{W(S)s} D$. Hence $D[\Pi_{W(S)} P_s] \leqq D[P_\sigma]$ and since it is evident that the inverse inequality holds we have $D[\Pi_{W(S)} P_s] = D[P_\sigma]$.

9.2. Theorem. *Let $\{A_s \mid s \in S\}$ be a system of sets, P_s an order defined on A_s, $\mathfrak{R}_s = \{L_{t(s)} \mid t(s) \in T_s\}$ a minimal realizer of P_s and $P = \Pi_S P_s$ the cardinal product of the system $\{P_s \mid s \in S\}$. Then $D[P] \leqq |\cup_S T_s|$ under the condition that T_s's are pairwise disjoint, i.e. $D[P] \leqq \Sigma_S D[P_s]$.*

Proof. Consider any well-order W and W_s defined on S and on each T_s respectively and let $t_0(s)$ the least(W_s) element of T_s. Then

$$\begin{aligned} L_s^{t(s)} = \{(f,f) \mid f \in F\} \cup \{(f,g) \mid f,g \in F \textit{ and } f(s) < g(s)(L_{t(s)})\} \\ \cup \{(f,g) \mid f,g \in F,\ f(s) = g(s) \textit{ and } f(\sigma) < g(\sigma)\ (L_{t_0(\sigma)}) \\ \textit{for the least}(W) \textit{ element } \sigma \textit{ such that } f(\sigma) \neq g(\sigma)\}, \end{aligned}$$

F being the set of all mappings f of S into $\cup_S A_s$ such that $f(s) \in A_s$ (see 1.5), is a linear order on F and a linear extension of P and the system $\mathfrak{R} = \{L_s^{t(s)} \mid s \in S \textit{ and } t(s) \in T_s\}$ is a realizer of P. Hence we have $D[P] \leqq |\cup_S T_s|$.

In the conclusion of the last theorem the equality does not always hold.

9.3. Theorem. *If P_s is a linear order for each $s \in S$, then $D[\Pi_S P_s] = |S|$, provided that the domain of each P_s contains at least two elements.*

In order to prove this we shall prove the following lemma.

9.4. Lemma. *If Q be a linear order defined on a set of two elements, then $D[Q^S] = |S|$.*

Proof. Let Q be the order defined on the set $\{a,b\}$ so that $(a,b)\in Q$ and let f_s and g_s be the functions specified by

$$f_s(s)=b,\ f_s(s')=a \text{ for every } s'\neq s;$$
$$g_s(s)=a,\ g_s(s')=b \text{ for every } s'\neq s$$

respectively. Put $F=\{f_s \mid s\in S\}$ and $G=\{g_s \mid s\in S\}$. Then the suborder $Q^S(F\cup G)$ of Q^S on $F\cup G$ is equal to the order

$$\{(f_s,f_s) \mid s\in S\}\cup\{(g_s,g_s) \mid s\in S\}\cup\{(f_s,g_s) \mid s,s'\in S \text{ and } s\neq s'\}$$

whose dimension is, by 7.2, $|S|$. Hence we have $D[Q^S]=|S|$. On the other hand we have by the theorem 9.2, the inverse inequality.

Now since the linear order P_s contains a suborder which is isomorphic to the order Q, the order P contains a suborder which is isomorphic to the order Q^S. Hence we have the inequality $D[\Pi_S P_s]\geqq|S|$, and the inverse inequality by the theorem 9.2.

Mr. H. komm has proved that $D[P'_n(E_n)]=n$ for $n\leqq\aleph_0$ [5]. Putting $P_s=R$ where R is the linear order defined on the set of all real numbers in the natural fashion, we have, by 9.3, $D[R^S]=|S|$. And when $|S|=n\leqq\aleph_0$, $R^S=P'_n(E_n)$. Thus the Mr. Komm's theorem is a special case of the theorem 8.3.

The theorem 9.3 will be generalized as follows:

9.5. THEOREM. *If P_s is an order which contains a suborder isomorphic to the ordes Q^{T_s}, where T_s is a set whose cardinality is equal to the dimension of P_s and Q is the same order as in the last lemma, then $D[\Pi_S P_s]=|\cup_S T_s|$ under the condition that T_s's are pairwise disjoint.*

9.6. THEOREM. *For every order P of dimension m (finite or transfinite), there exists a cardinal product of m linear orders which contains a suborder isomorphic to P.*

Proof. Let a minimal realizer of P be $\{L_s\ s\in S\}$ where $|S|=m$, and f_x the constant mapping such that $f_x(s)=x$ for each $x\in A$, A being the domain of P. Then the suborder of the product $\Pi_S L_s$ on $\{f_x \mid x\in A\}$ is isomorphic to P.

Problems still open.

The author conjectures the following two propositions, but can not prove nor disprove.

1. *Let P be an order defined on a set A and a a maximal(P) element of A. If there exists one and only one element b such that $(b:a)\in P$, then a is d-removable.*

It may be proved easily that if, moreover, either no element other than b exceeds (P) a or the suborder $P(A-a)$ is d-irreducible, then a is d-removable.

2. *It is not possible to define a d-irreducible order on a set whose cardinality is an odd integer.*

References

1. T. Hiraguti, "*On the dimension of partially ordered sets*" Science Reports of the Kanazawa University, vol. I, No. 2 (1951), pp. 77–94.
2. T. Hiraguti, "*A note on Mr. Komm's Theorems*", ib., vol. II, No. 1 (1953), pp. 1-3.
3. E. Szpilrajn, "*Sur l'extension de l'ordre partial*", Fundamenta Mathematica, vol. 16 (1930), pp. 386–389.
4. B. Dushnik and E. W. Miller, "*Partially ordered sets*", American Journal of Mathematics, vol. 63 (1941), pp. 600-610.
5. H. Komm, "*On the dimension of partially ordered sets*", American Journal of Mathematics, vol, 71 (1948), pp. 507–520.