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On the λ-Dimension of the Product of Orders

By

Tosio HIRAGUCHI

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The purpose of this note is to demonstrate the following three theorems.

THEOREM 1. *Let A and S be enumerably infinite sets. If L is a linear order defined on the set A, then* $D_\lambda[L^S]=\aleph_0$.

THEOREM 2. *If S is an enumerably set, then* $D_\lambda[\lambda^S]=\aleph_0$.

THEOREM 3. *The cardinal product* $\Pi_S P_s$ $(|S|\leqq\aleph_0)$ *of a system of enumerable number of orders having λ-dimensions has the λ-dimension. And if* $R_s=\{L_{t(s)}\,|\,t(s)\in T_s\}$ *is a minimal λ-realizer of* P_s *and* Φ *is the set of all mappings* φ *of S into* $\cup_S R_s$ *such that* $\varphi(s)\in R_s$ *for all* $s\in S$, *then* $D_\lambda[\Pi_S P_s]\leqq\Sigma_\Phi D_\lambda[\Pi_S\varphi(s)]$.

(As to the terminology and the notations see §1.)

THEOREM 2 is nothing but a different formulation of the theorem, first demonstrated by Mr. Ginsburg in [1], that the λ-dimension of $P'(E_\infty)$ is $\aleph_0$. But the proof is less cumbersome, in which THEOREM 1 plays an improtant role. Previously the author demonstrated that the dimension of a cardinal product of a system of orders does not exceed the sum of the dimensions of the members [2], [3]. THEOREM 3 is an analogous theorem which estimates the λ-dimension of the cardinal product of a system of orders having λ-dimensions.

1. Preliminary.

It will be appropriate to give a brief account on the terminology and the notations used in this note. For further details refer to [3].

An *order* defined on a set A is a subset P of the Cartesian prduct $A\times A$ which satisfies the following conditions:

01: *$x\in A$ implies $(x,x)\in P$,*
02: *$(x,y)\in P$ and $(y,x)\in P$ imply $x=y$,*
03: *$(x,y)\in P$ and $(y,z)\in P$ imply $(x,z)\in P$.*

A *linear order* defined on a set A is an order L which satisfies the condition

04: *$(x,y)\in L$ or $(y,x)\in L$ for any $x,y\in A$.*

"$x\leqq y(P)$" means that $(x,y)\in P$. "$x<y(P)$" means that $(x,y)\in P$ and $x\neq y$. "*x and y are incomparable*(P)" means that $(x,y)\notin P$ and $(y,x)\notin P$. "*a* is *the least*(P) *element* of A" means that $a\in A$ and $(a,x)\in P$ for all $x\in A$.

Let P be an order defined on a set A and B a subset of A. The *suborder* of P restricted on the set B is the subset $P(B)$ of P specified by

$$P(B)=\{(x,y)\,|\,(x,y)\in P \text{ and } x,y\in B\}$$

An *extension* of an order P is an order Q defined on the same set as P such that $P\subseteqq Q$. An extension of an order is said a *linear extension* when it is an linear order. λ stands for the linear order defined on the real number system according to magnitude. A linear extension of an order is said a *λ-extension* when it is isomorphic to a suborder of λ.

A *realizer* of an order P is a set $R=\{L_s\,|\,s\in S\}$ of linear extensions L_s of P such that $P=\cap_S L_s$. In particular if L_s is a λ-extension for every $s\in S$, it is said a *λ-realizer* of P. A *minimal realizer* of an order is a realizer whose cardinality does not exceed the cardinality of any realizer of the order. A *minimal λ-realizer* of an order is defined correspondingly.

A *dimension* of an order is the cardinality of a minimal realizer of the order and a *λ-dimension* that of a minimal λ-realizer. The dimension and the λ-dimension of an order P are denoted by $D[P]$ and $D_\lambda[P]$ respectively. If $R=\{L_s\,|\,s\in S\}$ is a minimal realizer (λ-realizer resp.) of P, then $D[P]$ ($D_\lambda[P]$ resp.) is $|S|$ where $|\cdots\cdots|$ stands for the cardinality of the set $\cdots\cdots$.

Let $\{P_s\,|\,s\in S\}$ be a system of orders, each memebr P_s being defined on a set A_s, and F the set of all mappings f of S into $\cup_S A_s$ such that $f(s)\in A_s$ for every $s\in S$. The *cardinal product* of the system $\{P_s\,|\,s\in S\}$ is the order $\Pi_S P_s$ defined on F by

$$\Pi_S P_s=\{(f,f)\,|\,f\in F\}\cup\{(f,g)\,|\,f,g\in F \text{ and } (f(s),g(s))\in P_s \text{ for all } s\in S\}.$$

Let P be an order defined on a set A and F the set of all mappings of a set S into A. The *cardinal power* of P is the order P^S defined on F by

$$P^S=\{(f,f)\,|\,f\in F\}\cup\{(f,g)\,|\,f,g\in F \text{ and } (f(s),g(s))\in P \text{ for all } s\in S\}.$$

2. Proof of the theorems.

LEMMA 1. *Let L be a linear order defined on a set A. If $|S|=\aleph_0$, then $D[L^S]=\aleph_0$ provided $|A|\geqq 2$.*

This is a special case of 9.3 THEOREM on p. 18 of [3].

LEMMA 2. *Let P be an order defined on a set A and S a set such that $|S\times S|=|S|$, then $D[(P^S)^S]=D[P^S]$. Moreover if P has the λ-dimension, $(P^S)^S$ has also the λ-dimension and $D_\lambda[(P^S)^S]=D_\lambda[P^S]$.*

This follows immediately from the fact that $(P^S)^S$ is isomorphic to $P^{S\times S}$ and the latter in turn to P^S.

LEMMA 3. *Let A and S be enumerably infinite sets and L a linear order defined on the set A. Then there exists a suborder of $(L^S)^S$ which is isomorphic to λ^S.*

Proof. Let W be the linear order defined on the set N of all natural numbers according to magnitude and J the linear order defined on the set $\{x\,|\,0<x<1\}$ according to magnitude. Since λ is isomorphic to J and there exists a suborder of L isomorphic to W, there exists a suborder of L^S isomorphic to W^N. By the LEMMA 1.1 on p. 591 of [1] there exists a suborder of W^N isomorphic to J. Hence there exists a suborder of L^S isomorphic to λ. Let it be Q, then Q^S is isomorphic to λ^S.

Proof of THEOREM 1. Consider a well-order W defined on the set S and let L_s, for each element $s\in S$, be a subset of $F\times F$ specified by

$$L_s=\{(f,f)\,|\,f\in F\}\cup\{(f,g)\,|\,f,g\in F \text{ and } f(s)<g(s)(L)\}$$
$$\cup\{(f,g)\,|\,f,g\in F, f(s)=g(s) \text{ and } f(\sigma)<g(\sigma)(L)$$
$$\text{for the least}(W)\ \sigma\in S \text{ such that } f(\sigma)\neq g(\sigma)\}.$$

Then $R=\{L_s\,|\,s\in S\}$ is a λ-realizer of L^S, hence we have the inequality $D_\lambda[L^S]\leqq|S|=\aleph_0$. On the other hand we have, by LEMMA 1, the inverse inequality $D_\lambda[L^S]\geqq D[L^S]=\aleph_0$. Thus we obtain the equality to be demonstrated.

It is not hard to verify that L_s is a linear order defined on the set F and a extension of the order L^S and that R is a realizer of L^S. In order to verify that L_s is isomorphic to a suborder of λ put $W^*=s+W(S-s)$, $W(S-s)$ being the suborder of W restricted to the set $S-s$, then L_s will be written as follows:

$$L_s=\{(f,f)\,|\,f\in F\}\cup\{(f,g)\,|\,f,g\in F \text{ and } f(s)<g(s)(L)$$
$$\cup\{(f,g)\,|\,f,g\in F,\ f(s)=g(s) \text{ and } f(\sigma)<g(\sigma)(L)$$
$$\text{for the least}(W^*)\ \sigma\in S \text{ such that } f(\sigma)\neq g(\sigma)\}.$$

Thus we may take A as the set N of all natural numbers, L as the order defined on N according to magnitude and L_s as the lexicographical order Q defined on the set of all infinite sequences of natural numbers. To be demonstrated is that the order Q is isomorphic to a suborder of λ.

For a semi-closed interval $I=[a,b)$, let $D_n(I)$ mean the interval $[b-(b-a)/2^{n-1}, b-(b-a)/2^n)$ for each integer n and let I_n stand for the interval $[n, n+1)$ for every integer n. For a given sequence of natural numbers $n_1, n_2, \ldots, n_k, \ldots$, there is a decreasing sequence of intervals

$$I_{n_1}, I_{n_1 n_2}, \ldots, I_{n_1 n_2 \cdots n_k}, \ldots,$$

where $I_{n_1 n_2 \cdots n_k}$ stands for the interval $D_{n_k}(I_{n_1 n_2 \cdots n_{k-1}})$ for $k \geqq 2$. Since the length of the interval $I_{n_1 n_2 \cdots n_k}$ converges to 0 as $k \to \infty$, this sequence of intervals determines a real number $a_{n_1 n_2 \cdots n_k \cdots}$. Letting correspond this to the given sequence $n_1, n_2, \ldots, n_k, \ldots$, we obtain an isomorphic mapping of Q into λ^S.

Proof of THEREM 2. Since λ^S is, by LEMMA 3, isomorphic to a suborder of $(L^S)^S$ we have, by THEOREM 1 and LEMMA 1, the inequality $D_\lambda[\lambda^S] \leqq D_\lambda[(L^S)^S] = D_\lambda[L^S] = \aleph_0$. On the other hand we have, by LEMMA 1, the inverse inequality $D_\lambda[\lambda^S] \geqq D[\lambda^S] = \aleph_0$.

As an immediate result of THEORE 2 we have the

COROLLARY. *If L_s is, for each $s \in S$, a linear order isomorphic to a suborder of λ and $|S| \leqq \aleph_0$, then $\Pi_S L_s$ has the λ-dimension which does not exceed $\aleph_0$.*

Proof of THEOREM 3. Put $P = \Pi_S P_s$ and $Q\varphi = \Pi_S \varphi(s)$. $\varphi(s)$ being defined on A_s and a λ-extension of P_s, $Q\varphi$ is an order defined on F; moreover since $(f,g) \in P$ implies $(f(s), g(s)) \in P_s$ for all $s \in S$, it implies $(f(s), g(s)) \in \varphi(s)$, hence $P \subseteqq Q\varphi$ for all $\varphi \in \Phi$. By the COROLLARY to the THEOREM 2, $Q\varphi$ has the λ-dimension. Let $R\varphi = \{L_{t(\varphi)} | t(\varphi) \in T\varphi\}$ be a minimal λ-realizer of $Q\varphi$ for each $\varphi \in \Phi$, then $R = \cup_\Phi R\varphi$ is a λ-realizer of P. In fact: since $P \subseteqq Q$ for all $\varphi \in \Phi$ and every member of R is a λ-extension of $Q\varphi$ for some $\varphi \in \Phi$, each member of R is a λ-extension of P. In order to verify that R is a realizer of P, let f and g be two incomparable(P) elements of F. To be shown is that there exist two members L_1 and L_2 of R such that $(f,g) \in L_1$ and $(g,f) \in L_2$. Assume that $(f,g) \in L$ for all member L of R. Then $(f,g) \in L_{t(\varphi)}$ for all $\varphi \in \Phi$ and for all $t(\varphi) \in T\varphi$, hence $(f,g) \in Q\varphi$ for all $\varphi \in \Phi$, hence $(f(s), g(s)) \in \varphi(s)$ for all $\varphi \in \Phi$ and for all $s \in S$, hence $(f(s), g(s)) \in L_{t(s)}$ for all $s \in S$ and for all $t(s) \in T_s$, hence $(f(s), g(s)) \in P_s$ for all $s \in S$, hence $(f,g) \in P$. But this contradicts the hypothesis that f and g are incomparable(P). Consequently there exists a memeber L_2 such that $(g,f) \in L_2$, and similarly a member L_1 such that $(f,g) \in L_1$. Thus R is a λ-realizer of P and P has the λ-dimension. Clearly we have $D_\lambda[P] \leqq |R| \leqq \Sigma_\Phi |R\varphi| = \Sigma_\Phi D_\lambda[Q\varphi]$.

References

1 S. Ginsburg, *"On the λ-dimension and the A-dimension of partially ordered sets"*, American Journal of Mathematics, Vol. 76 (1954), pp. 590–598.

2 T. Hiraguchi, "*A note on Mr. Komm's Theorem*", Science Reports of Kanazawa University, Vol. **II**, No. **1** (1953), pp. 1–3.

3 T. Hiraguti, "*On the dimension of Orders*", ib., Vol. **IV** No. **1** (1955), pp. 1–20.

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T. Hiraguti, "*On the Diemension of Orders*", Vol. IV, No. 1 (1955), pp. 1–20.

page	line	read	instead of
2	17	A_s	A
2	23	into $A(=A_s)$	into P
10	8	$A'=A-(B-b_o)$	$A=A-(B-b_o)$
11	17	$L_{t,t(\sigma)}$	$L_{t,t(\sigma}$
12	1	$D[P_\sigma]<D[Q]$	$D[P_\sigma]>D[Q]$
13	24	$P_n(A_n-y_{n-1})$	$P_n(A_n-y_n\ 1)$
16	32	$\|A-a-b\|$	$\|A-a-b$
16	37	$\|A\|/2$	$\|A\|\ 2$
17	1	$\|A\|/2$	$A\|/2$
17	20	$D[P]\geqq 3$	$D[P]\leqq 3$
17	22	$n\geqq 3$	$n\leqq 3$
17	26	$n\geqq 2$	$n\leqq 2$
17	36	$\|A\|\geqq 4$	$\|A\|\leqq 4$
18	7	$M_{t(\sigma)}$	$M_{t\sigma)}$
18	9	$s<s'(W)$	$s\in s'(W)$
19	8	$\|S\|$	$\|S$
19	25	$\{L_s\|s\in S\}$	$\{L_s\ s\in S\}$