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On the Convergence of Some Gap Series

By

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1. Introduction

Let $f(x)$ be an L^2–integrable and periodic function with period 1, and satisfy the following conditions,

(1.1) $$\int_0^1 f(x)dx=0,$$

(1.2) $$\int_0^1 f^2(x)dx=1.$$

The Fourier series of $f(x)$ and the n-th partial sum of it are respectively,

(1.3) $$f(x)\sim\sum_{k=-\infty}^{\infty} c_k e^{2\pi ikx},$$

and

$$S_n(x)=\sum_{k=-n}^{n} c_k e^{2\pi ikx},$$

and $R(n)$ denotes

(1.4) $$R(n)=\left(\int_0^1 |f(x)-S_n(x)|^2dx\right)^{1/2}=\left(2\sum_{k>n}|c_k|^2\right)^{1/2}.$$

In this note we shall prove the convergence of a gap series

(1.5) $$\sum_{k=1}^{\infty}\frac{1}{L_p(k)}f(n_kx),$$

under some restrictions on $R(n)$, where $\{n_k\}$ is a sequence of positive integral numbers satisfying

(1.6) $$0<n_1<n_2<\cdots\cdots<n_k<\cdots\cdots,$$

and $L_p(k)$ is defined for arbitrary numbers $\alpha>0$, $\beta>1$ and a non-negative integral number p, by the following

(1.7) $$\begin{cases} L_0(x)=x^{1-\alpha}(log x)^{\beta}, \\ L_1(x)=x(log x)^{1-\alpha}(log_2 x)^{\beta}, \\ L_p(x)=x(log x)\cdots\cdots(log_{p-1}x)(log_p x)^{1-\alpha}(log_{p+1}x)^{\beta} \qquad (p\geqq 2). \end{cases}$$

In the sequel we suppose that $\log_0 x$ means x.

Theorem 1. If for any $\alpha>0$ and an integral number $p\geqq 1, f(x)$ satisfies

(1.8) $$R(n)=O\left(\frac{1}{(log_p n)^{\alpha}}\right),$$

then for almost all x, (1.5) converges, where $\{n_k\}$ is a sequence of (1.6).

Theorem 2. If for $\alpha \geqq \frac{1}{2}$, $f(x)$ satisfies

$$R(n)=O(1/n^{\alpha}) \tag{1.8'}$$

and $\{n_k\}$ satisfies (1.6), then the convergence of

$$\sum a_k^2\sqrt{k}\ \log k \log_2 k \cdots\cdots \log_{p-1}k(\log_p k)^{\beta}, \tag{1.9}$$

implies the almost everywhere convergence of

$$\sum_{k=1}^{\infty} a_k f(n_k x), \tag{1.10}$$

where $\beta>1$ and p is a positive integer.

Theorem 3. If for any $\alpha>0$ and an integral number $p\geqq 2$, or for any $0<\alpha<\frac{1}{2}$ and $p=1$, $f(x)$ satisfies

$$R(n)=O\left(\frac{1}{(\log_p n)^{2\alpha}}\right), \tag{1.11}$$

then for almost all x,

$$\sum_{k=1}^{\infty}\frac{1}{L_{p-1}(k)}\ f(n_k x) \tag{1.12}$$

converges, where $\{n_k\}$ is a sequence of positive integral numbers satisfying

$$\frac{n_{k+1}}{n_k}\geqq\theta>1 \qquad (k=1,2,\cdots\cdots). \tag{1.13}$$

The case $0<\alpha<\frac{1}{2}$ and $p=1$ of Theorem 3, was proved by M. Kac, R. Salem and A. Zygmund [1], and also by S. Izumi [2], whose method is the different way from them. We now give another proof of it. S. Izuim [2] proved weaker results of the cases $p=2$ and 3 of Theorem 3 than (1.12). Theorem 1 is proved by use of the methods of J. L. Koksma [3], which treated the law of large numbers of some sequence of $f(n_k x)$, and the proof of Theorem 3 is analogous to that of Theorem 1.

It is interesting that we discuss the almost everywhere convergence of (1.10) by use of $\{n_k\}$ which satisfies a stronger condition than (1.6). We have a theorem below in this sense that is a little generalized one of the theorem obtained by M. Kac [5] and generalized by S. Izumi [2].

Theorem 4. If for any $\alpha>0$, $f(x)$ satisfies (1.8') and $\{n_k\}$ satisfies

$$\sum_{k=1}^{\infty}\left(\frac{n_k}{n_{k+1}}\right)^{\lambda}<\infty, \tag{1.14}$$

where λ is an arbitrary positive number, then from the convergence of

$$\sum_{k=1}^{\infty} a_k^2,$$

the almost everywhere convergence of (1.10) follows.

2. Proof of Theorem 1

For the proof of Theorem 1 we need a lemma.

Lemma 1. If $f(x)$ and $\{n_k\}$ satisfy the hypotheses of Theorem 1, then for arbitrary integral numbers $z(1\leqq z)$ and $N(1\leqq N)$, it holds that

$$\int_0^1 \Big|\sum_{k=z+1}^{z+N} \frac{1}{L_p(k)} f(n_k x)\Big|^2 dx \leqq \frac{N}{L_p(z)^2} + O\Big(\frac{N^2 R(z/N)^2}{L_p(z)^2}\Big) \tag{2.1}$$

Proof. If the greatest common divisor of n_k and n_j is d, then there exist two numbers n_k' and n_j' such as

$$(n_k, n_j)=d, \quad (n_k', n_j')=1, \quad n_k=dn_k' \text{ and } n_j=dn_j'.$$

If we suppose that

$$z<n_j<n_k \leqq N+z,$$

then

$$d<(n_k'-n_j')d=n_k-n_j \leqq N,$$

and

$$\frac{z}{N} \leqq \frac{n_j}{d}=n_j'<n_k',$$

from which it follows that by the Parseval's relation,

$$\Big|\int_0^1 f(n_k x)f(n_j x)dx\Big| = \Big|\sum_{sn_k=tn_j} c_s\bar{c}_t\Big|$$

$$=2\Big|\sum_{v=1}^{\infty} c_{n_j' v}\bar{c}_{n_k' v}\Big| \leqq 2\Big(\sum_{v=1}^{\infty}|c_{n_j' v}|^2 \sum_{v=1}^{\infty}|c_{n_k' v}|^2\Big)^{1/2}$$

$$\leqq R(n_j')R(n_k') \leqq R\Big(\frac{z}{N}\Big)^2. \tag{2.2}$$

From the monotonity of $L_p(k)$ and (2.2), it holds that

$$\int_0^1 \Big|\sum_{k=z+1}^{z+N} \frac{1}{L_p(k)} f(n_k x)\Big|^2 dx$$

$$= \sum_{k=z+1}^{z+N} \frac{1}{L_p(k)^2}\int_0^1 f(n_k x)^2 dx + \sum_{k\neq j} \frac{1}{L_p(k)L_p(j)}\int_0^1 f(n_k x)f(n_j x)dx$$

$$\leqq \frac{N}{L_p(z)^2} + R\Big(\frac{z}{N}\Big)^2 \Big(\sum_{k=z+1}^{z+N} \frac{1}{L_p(k)}\Big)^2$$

$$\leqq \frac{N}{L_p(z)^2} + O\Big(\frac{N^2 R(z/N)^2}{L_p(z)^2}\Big).$$

Thus we obtain Lemma 1.

We now prove the theorem. In the inequality (2.1), if we put

$$z=\lambda^2 \text{ and } N=(\lambda+1)^2-\lambda^2=2\lambda+1(\equiv N(\lambda)),$$

then

$$\int_0^1 \Big|\sum_{k=z+1}^{z+N} \frac{1}{L_p(k)} f(n_k x)\Big|^2 dx \leqq \frac{N(\lambda)}{L_p(\lambda^2)^2} + O\Big(\frac{N(\lambda)^2}{L_p(\lambda^2)^2(log_p \lambda^2 N(\lambda)^{-1})^{2\alpha}}\Big)$$

$$=O\Big(\frac{1}{[\lambda\, log\,\lambda\, log_2\lambda \cdots\cdots log_p\lambda (log_{p+1}\lambda)^{\beta}]^2}\Big).$$

This contains the almost everywhere convergence of the series

$$\sum_{\lambda=1}^{\infty}\sum_{k=\lambda^2+1}^{(\lambda+1)^2} \frac{1}{L_p(k)} f(n_k x). \tag{2.3}$$

Whence it follows the almost everywhere convergence of (1.5), provided that for almost all x

(2.4) $$\lim_{\lambda\to\infty} \max_{\lambda^2<m\leq(\lambda+1)^2} \Big| \sum_{k=\lambda^2+1}^{m} \frac{1}{L_p(k)} f(n_k x)\Big| = 0.$$

For the proof of (2.4) we use the device of P. Erdös [4].

For every $\delta>0$ and

$$u=0,1,2,\cdots\cdots,\ 2^v,$$

and

$$v=1,2,\cdots\cdots,\ \frac{\log N(\lambda)}{\log 2}(\equiv V(\lambda)),$$

we have

$$meas\Big(\Big| \sum_{k=\lambda^2+2uN(\lambda)2^{-v}+1}^{\lambda^2+(2u+1)N(\lambda)2^{-v}} \frac{1}{L_p(k)} f(n_k x)\Big| \geq \frac{\delta}{v^2}\Big)$$

$$\leq \frac{v^4}{\delta^2} \int_0^1 \Big| \sum_{k=\lambda^2+2uN(\lambda)2^{-v}+1}^{\lambda^2+(2u+1)N(\lambda)2^{-v}} \frac{1}{L_p(k)} f(n_k x)\Big|^2 dx$$

$$= O\Big(\frac{\lambda 2^{-v} v^4}{[\lambda^2 \log\lambda \cdots (\log_p\lambda)^{1-\alpha} (\log_{p+1}\lambda)^{\beta}]^2}\Big)$$

$$+ O\Big(\frac{\lambda^2 2^{-2v} v^4}{[\lambda^2 \log\lambda \log_2\lambda \cdots (\log_p\lambda)^{1-\alpha} (\log_{p+1}\lambda)^{\beta}]^2 (\log_p\lambda)^{2\alpha}}\Big)$$

$$\leq O\Big(\frac{v^4 2^{-v}}{\lambda^3}\Big) + O\Big(\frac{v^4 2^{-2v}}{\lambda^2}\Big).$$

Whence it follows that

$$\sum_{\lambda=1}^{\infty} \sum_{v=1}^{V(\lambda)} \sum_{u=0}^{2^v} meas\Big(\Big| \sum_{k=\lambda^2+2uN(\lambda)2^{-v}+1}^{\lambda^2+(2u+1)N(\lambda)2^{-v}} \frac{1}{L_p(k)} f(n_k x)\Big| \geq \frac{\delta}{v^2}\Big)$$

$$\leq \sum_{\lambda=1}^{\infty} \frac{1}{\lambda^3} \sum_{v=1}^{V(\lambda)} v^4 + \sum_{\lambda=1}^{\infty} \frac{1}{\lambda^2} \sum_{v=1}^{V(\lambda)} v^4 2^{-v} < \infty,$$

and by the Borel-Cantelli lemma, we have for almost all x

$$\Big| \sum_{k=\lambda^2+1}^{m} \frac{1}{L_p(k)} f(n_k x)\Big| \leq \Big\{ \frac{\delta}{1^2} + \frac{\delta}{2^2} + \frac{\delta}{3^2} + \cdots\cdots \Big\} \leq 2\delta.$$

Since $\delta>0$ is an arbitrary number, we obtain (2.4).

3. Proof of Theorem 2

When $f(x)$ and $\{n_k\}$ satisfy the conditions of Theorem 2, and if

$$M+1 \leq n_k < n_j \leq M+N,$$

then it is seen that

$$\Big| \int_0^1 f(n_k x) f(n_j x) dx \Big| \leq R\Big(\frac{M}{N}\Big)^2 = O\Big(\frac{N^{2\alpha}}{M^{2\alpha}}\Big),$$

and then

(3.1) $$\int_0^1 \Big| \sum_{k=M+1}^{M+N} a_k f(n_k x)\Big|^2 dx \leq \sum_{k=M+1}^{M+N} |a_k|^2 + 2 \sum_{k=M+1}^{M+N-1} a_k \sum_{j=k+1}^{M+N} a_j R\Big(\frac{M}{N}\Big)^2$$

$$\leq \Big(\sum_{k=M+1}^{M+N} a_k^2\Big)\Big(1+O\Big(\frac{N^{2\alpha+1}}{M^{2\alpha}}\Big)\Big).$$

If we put in the above inequality

$$M = s^2 \text{ and } N \equiv N(s) = 2s+1,$$

then we have

$$\int_0^1 \Big| \sum_{k=s^2+1}^{(s+1)^2} a_k f(n_k x) \Big|^2 dx = O\Big(\sum_{k=s^2+1}^{(s+1)^2} a_k^2 \Big),$$

and the almost everywhere convergence of

$$\sum_{s=1}^{\infty} \sum_{k=s^2+1}^{(s+1)^2} a_k f(n_k x) \tag{3.2}$$

holds, whenever

$$\sum_{s=1}^{\infty} \Big(\sum_{k=s^2+1}^{(s+1)^2} a_k^2 \Big)^{1/2} < \infty. \tag{3.3}$$

However this is reduced from (1.9), i.e.,

$$\sum_{s=1}^{\infty} \Big(\sum_{k=s^2+1}^{(s+1)^2} a_k^2 \Big)^{1/2} = \sum_{s=1}^{\infty} \Big(\sum_{k=s^2+1}^{(s+1)^2} a_k^2 \sqrt{k}\, \log k \cdots\cdots (\log_p k)^{\beta} \frac{1}{\sqrt{k}\, \log k \cdots\cdots (\log_p k)^{\beta}} \Big)^{1/2}$$

$$\leqq \sum_{s=1}^{\infty} \frac{1}{\sqrt{s}\, \log s \cdots\cdots (\log_p s)^{\beta}} \sqrt{\sum_{k=s^2+1}^{(s+1)^2} a_k^2 \sqrt{k}\, \log k \cdots\cdots (\log_p k)^{\beta}}$$

$$\leqq \Big[\sum_{s=1}^{\infty} \frac{1}{s \log s \cdots\cdots (\log_p s)^{\beta}} \sum_{s=1}^{\infty} \sum_{k=s^2+1}^{(s+1)^2} a_k^2 \sqrt{k}\, \log k \cdots\cdots (\log_p k)^{\beta} \Big]^{1/2}$$

$$\leqq O(1) \Big[\sum_{k=1}^{\infty} a_k^2 \sqrt{k}\, \log k \cdots\cdots (\log_p k)^{\beta} \Big]^{1/2} < \infty.$$

In order to prove Theorem 2 it is only sufficient to prove that

$$\sum_{s=1}^{\infty} \int_0^1 \max_{s^2 < m < (s+1)^2} \Big| \sum_{k=s^2+1}^{m} a_k f(n_k x) \Big|^2 dx < \infty. \tag{3.4}$$

But the left hand member of (3.4) is less than by (3.1)

$$\sum_{s=1}^{\infty} \log N(s) \sum_{v=1}^{\log_N(s)} \sum_{u=0}^{N(s)2^{-v}} \int_0^1 \Big| \sum_{k=s^2+u2^v+1}^{s^2+(u+1)2^v} a_k f(n_k x) \Big|^2 dx$$

$$= \sum_{s=1}^{\infty} \log N(s) \sum_{v=1}^{\log_N(s)} N(s) 2^{-v} \Big(\sum_{k=s^2+1}^{(s+1)^2} a_k^2 \Big) \leqq \sum_{s=1}^{\infty} \Big(\sum_{k=s^2+1}^{(s+1)^2} a_k^2 \Big) N(s) \log N(s)$$

$$\leqq O\Big(\sum_{s=1}^{\infty} \sum_{k=s^2+1}^{(s+1)^2} a_k^2 \sqrt{k}\, \log k \Big) = O\Big(\sum_{k=1}^{\infty} a_k^2 \sqrt{k}\, \log k \Big) < \infty.$$

Thus if we make use of the Menchov's device [6], we verify (3.4), and from (3.2) and (3.4) we complete the proof of Theorem 2.

4. Proof of Theorem 3

For the proof of Theorem 3 we need a lemma corresponding to Lemma 1.

Lemma 2. If $f(x)$ and $\{n_k\}$ satisfy the hypotheses of Theorem 3, then it holds that

$$\int_0^1 \Big| \sum_{k=z+1}^{z+N} \frac{1}{L_{p-1}(k)} f(n_k x) \Big|^2 dx = \frac{N}{L_{p-1}(z)^2} + O\Big(\frac{N^2}{L_{p-1}(z)^2 (\log_{p-1} N)^{2\alpha}} \Big). \tag{4.1}$$

Proof. It is easy to prove the following formula

(4.2) $$\left|\int_0^1 f(n_k x) f(n_j x) dx\right| = O\left(\frac{1}{(log_{p-1}|k-j|)^{2\alpha}}\right).$$

In fact, if $k>j$, then by the Parseval's relation, we have

$$\left|\int_0^1 f(n_k x) f(n_j x) dx\right| = \left|\sum_{s n_k = t n_j} c_s \bar{c}_t\right|$$

$$= \left|\sum_{1 \leq s < \infty, n_j | s n_k} c_s \bar{c}_{s n_k / n_j}\right|$$

$$\leq \left(\sum_{s=1}^{\infty} |c_s|^2 \sum_{n_k \leq n_j s} |c_s|^2\right)^{1/2} = R\left(\frac{n_k}{n_j}\right) = O\left(\frac{1}{(log_{p-1}(k-j))^{2\alpha}}\right).$$

This is (4.2), and since $L_{p-1}(k)$ is non-decreasing as $k \to \infty$, we have

$$\int_0^1 \left|\sum_{k=z+1}^{z+N} \frac{1}{L_{p-1}(k)} f(n_k x)\right|^2 dx = \sum_{k=z+1}^{z+N} \frac{1}{L_{p-1}(k)^2} \int_0^1 f(n_k x)^2 dx$$

$$+ \sum_{k \neq j} \frac{1}{L_{p-1}(k)\, L_{p-1}(j)} \int_0^1 f(n_k x) f(n_j x) dx$$

$$\leq \frac{N}{L_{p-1}(z)^2} + O\left(\sum_{j=z+1}^{z+N-1} \frac{1}{L_{p-1}(j)} \sum_{k=j+1}^{z+N} \frac{1}{L_{p-1}(k)(log_{p-1}(k-j))^{2\alpha}}\right)$$

$$\leq \frac{N}{L_{p-1}(z)^2} + O\left(\sum_{r=1}^{N} \frac{1}{(log_{p-1} r)^{2\alpha}} \sum_{j=z+1}^{z+N-r} \frac{1}{L_{p-1}(j) L_{p-1}(j+r)}\right)$$

$$= \frac{N}{L_{p-1}(z)^2} + O\left(\frac{N^2}{L_{p-1}(z)^2 (log_{p-1} N)^{2\alpha}}\right).$$

Thus we complete the lemma.

If we proceed along the same line as the proof of Theorem 1, we obtain the proof of Theorem 3, but in this case we must put in (4.1)

$$z = N = 2^{\lambda}.$$

5. Proof of Theorem 4

We suppose that $f(x)$ and $\{n_k\}$ satisfy the hypotheses of Theorem 4, and if we put

$$\nu_k = \frac{n_k}{n_{k+1}}, \qquad \tau = \left[\frac{\lambda}{2a}\right]$$

(5.1) $$\mu_{s,k} = \left[\frac{1}{q} \min_{0 \leq i < \tau} \left(\frac{n_{s+\tau k+i+1}}{n_{s+\tau k+i}}\right)^{\tau}\right],$$

then for each s ($s = 1, 2, 3, \cdots\cdots, \tau$),

$$\sum_{k=0}^{\infty} |a_{s+\tau k}| \int_0^1 |f(n_{s+\tau k} x) - S_{\mu_{s,k}}(n_{s+\tau k} x)| dx$$

$$\leq \left(\sum_{k=0}^{\infty} a_{s+\tau k}^2\right)^{1/2} \left(\sum_{k=0}^{\infty} \int_0^1 |f(n_{s+\tau k} x) - S_{\mu_{s,k}}(n_{s+\tau k} x)|^2 dx\right)^{1/2}$$

$$\leq \left(\sum_{k=1}^{\infty} a_k^2\right)^{1/2} \left(\sum_{k=0}^{\infty} \int_0^1 |f(x) - S_{\mu_{s,k}}(x)|^2 dx\right)^{1/2}$$

$$\leqq\Big(\sum_{k=1}^{\infty}a_k^2\Big)^{1/2}\Big(\sum_{k=0}^{\infty}\frac{1}{\mu_{s,k}^{2\alpha}}\Big)^{1/2}$$

$$=O(1)\Bigg(\sum_{k=0}^{\infty}\frac{1}{\underset{0\leq i<\tau}{min}\Big(\frac{n_{s+\tau k+i+1}}{n_{s+\tau k+i}}\Big)^{\tau 2\alpha}}\Bigg)^{1/2}\leqq O(1)\Big(\sum_{k=0}^{\infty}\nu_k^{\lambda}\Big)^{1/2}<\infty.$$

It shows the almost everywhere convergence of each series

$$\sum_{k=0}^{\infty}a_{s+\tau k}\{f(n_{s+\tau k}x)-S_{\mu_{s,k}}(n_{s+\tau k}x)\}\qquad (s=1,2,\cdots\cdots,\tau)\tag{5.2}$$

On the other hand we can easily verify the convergence in the L^2-sense of two series $\sum a_k f(n_k x)$ and (5.2), and then each series

$$\sum_{k=0}^{\infty}a_{s+\tau k}S_{\mu_{s,k}}(n_{s+\tau k}x)\qquad (s=1,2,\cdots\cdots,\tau)$$

is so also.

Now by (5.1), we have

$$\frac{n_{s+\tau k+\tau}}{n_{s+\tau k}}=\frac{n_{s+\tau k+1}}{n_{s+\tau k}}\cdot\frac{n_{s+\tau k+2}}{n_{s+\tau k+1}}\cdots\cdots\frac{n_{s+\tau k+\tau}}{n_{s+\tau k+\tau-1}}$$

$$\geqq\Big(\underset{0\leq i<\tau}{min}\frac{n_{s+\tau k+i+1}}{n_{s+\tau k+i}}\Big)^{\tau}\geqq q\mu_{s,k},$$

and then we obtain by the Kolmogoroff's theorem [7], the almost everywhere convergence of each series

$$\sum_{k=0}^{\infty}a_{s+\tau k}S_{\mu_{s,k}}(n_{s+\tau k}x),\qquad (s=1,2,\cdots\cdots,\tau)\tag{5.3}$$

whence Theorem 4 follows from (5.2) and (5.3).

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