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Dirac Particles with Anomalous Magnetic Moment in Arbitrary External Electromagnetic Fields

By

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1. Introduction

The suggestion of Lee and Yang [1] that the neutrino is to be described by a two-component theory indicates that the electrons emitted in β decay and the muons emitted in $\pi-\mu$ decay will be longitudinally polarized. As pointed out by Case [2], it is very interest to see to what extent it is possible to give a rigorous description of the behaviour of such particles in an external electromagnetic field. The results obtained up to the present day would certainly be expected [3,4] on a nonrelativistic model for spinning particle. Recently, Case [2] and Tolhoek-Groot [5] have shown independently that the same conclusions can be carried over to relativistic energies for Dirac particles.

In this paper we treat generally with Dirac particles taking into account an anomalous magnetic moment. Here, we introduce phenomenologically terms $\rho_3\Delta\mu\vec{\sigma}\cdot\vec{H}$ in the Hamiltonian where $\Delta\mu$ represents an anomalous magnetic moment.

2. Calculation

The treatment is simplest in the Heisenberg representation. We use the Heisenberg equation of motion for an arbitrary operator $\mathfrak{O}$.

$$d\mathfrak{O}/dt=(\partial\mathfrak{O}/\partial t)+(i/\hbar)[\mathfrak{H},\ \mathfrak{O}] \tag{1}$$

and we take the Hamiltonian in the form

$$\mathfrak{H}=c\rho_1\vec{\sigma}\cdot\vec{\pi}+\rho_3 mc^2+e\varphi+\rho_3\Delta\mu\vec{\sigma}\cdot\vec{H}. \tag{2}$$

Here, the last term $\rho_3\Delta\mu\vec{\sigma}\cdot\vec{H}$ is the interaction between an external magnetic field $\vec{H}$ and the anomalous magnetic moment $\Delta\mu$, σ_i's are (4×4) Pauli's spin matrices

$$\sigma_1=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&-i\\0&0&i&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}; \tag{3}$$

ρ_i's are following (4×4) matrices

$$\rho_1=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix},\quad \rho_2=\begin{pmatrix}0&0&-i&0\\0&0&0&-i\\i&0&0&0\\0&i&0&0\end{pmatrix},\quad \rho_3=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}; \tag{4}$$

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and $\vec{\pi}=\vec{p}-(e/c)\vec{A}$ and $\vec{A}$, φ are vector and scalar potentials for the external field.

In the following calculation, we make use of the commutation relations for σ_i's and ρ_i's:

$$\begin{aligned}
&\sigma_k\sigma_l+\sigma_l\sigma_k=2\delta_{kl} \qquad (k,\ l=1,\ 2,\ 3),\\
&\rho_k\rho_l+\rho_l\rho_k=2\delta_{kl} \qquad (k,\ l=1,\ 2,\ 3),\\
&\sigma_k\rho_l-\rho_l\sigma_k=0 \qquad (k,\ l=1,\ 2,\ 3);\\
&\sigma_k\sigma_l=i\sigma_m, \quad [\sigma_k,\sigma_l]=2i\sigma_m;\\
&\rho_k\rho_l=i\rho_m, \quad [\rho_k,\rho_l]=2i\rho_m\\
&[(k,\ l,\ m) \text{ is a cyclic permutation of } (1,\ 2,\ 3)].
\end{aligned} \tag{5}$$

(I) Longitudinal spin : $\Theta=\vec{\sigma}\cdot\vec{\pi}$.

If in Eq. (1) we put $\Theta=\vec{\sigma}\cdot\vec{\pi}$, we have

$$\begin{aligned}
\frac{d}{dt}(\vec{\sigma}\cdot\vec{\pi})&=\frac{\partial}{\partial t}(\vec{\sigma}\cdot\vec{\pi})+(i/\hbar)[\mathcal{H},\ \vec{\sigma}\cdot\vec{\pi}]=-\frac{e}{c}\vec{\sigma}\cdot\frac{\partial\vec{A}}{\partial t}-e\vec{\sigma}\cdot\nabla\varphi\\
&\quad+\rho_3\Delta\mu[-i\vec{\sigma}\cdot(\vec{\nabla}\times\vec{H})-\frac{ie}{\hbar c}i\vec{\sigma}\cdot(\vec{H}\times\vec{A}-\vec{A}\times\vec{H})]\\
&=e\vec{\sigma}\cdot\vec{E}-i\Delta\mu\rho_3\vec{\sigma}\cdot[(\vec{\nabla}\times\vec{H})-\frac{ie}{\hbar c}(\vec{H}\times\vec{A}-\vec{A}\times\vec{H})].
\end{aligned} \tag{6}$$

(II) Momentum : $\Theta=\vec{\pi}$.

Let $\Theta=\vec{\pi}$ in Eq. (1). We obtain

$$\begin{aligned}
\frac{d}{dt}\vec{\pi}&=\frac{\partial}{\partial t}\vec{\pi}+(i/\hbar)[\mathcal{H},\ \vec{\pi}]=e\vec{E}+c\rho_1\left\{\frac{e}{c}\vec{\sigma}\times\vec{H}-\frac{ie^2}{\hbar c^2}[\vec{\sigma}\cdot\vec{A},\ \vec{A}]\right\}\\
&\quad-\frac{ie^2}{\hbar c}[\varphi,\ \vec{A}]-\rho_3\Delta\mu\left\{\vec{\nabla}(\vec{\sigma}\cdot\vec{H})+\frac{ie}{\hbar c}[\vec{\sigma}\cdot\vec{H},\ \vec{A}]\right\}.
\end{aligned} \tag{7}$$

(III) Spin : $\Theta=\vec{\sigma}$.

When we take $\Theta=\vec{\sigma}$, it leads to

$$\frac{d}{dt}\vec{\sigma}=\frac{\partial}{\partial t}\vec{\sigma}+(i/\hbar)[\mathcal{H},\ \vec{\sigma}]=-\frac{2c}{\hbar}\rho_1\vec{\sigma}\times\vec{\pi}-\frac{2\Delta\mu}{\hbar}\rho_3\vec{\sigma}\times\vec{H}. \tag{8}$$

In deriving these equations we make use of the relations

$$\begin{aligned}
&\vec{E}=-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}-\vec{\nabla}\varphi,\\
&\vec{H}=\vec{\nabla}\times\vec{A},\\
&\vec{\nabla}\cdot\vec{H}=0;
\end{aligned} \tag{9}$$

and the useful formulas summarized in appendix.

3. Discussions

In Eq.'s (6), (7), and (8) the electromagnetic potential $A_\mu=(\vec{A},\ \varphi)$ is the external field i. e. c-number quantity. Therefore commutators between each other of these vanish and then Eq.'s (6), (7), and (8) are lead to the following :

$$\frac{d}{dt}\vec{\sigma}\cdot\vec{\pi}=e\vec{\sigma}\cdot\vec{E}-i\Delta\mu\rho_3\vec{\sigma}\cdot(\vec{\nabla}\times\vec{H}), \tag{10}$$

$$\frac{d}{dt}\vec{\pi}=e\vec{E}+e\rho_1\vec{\sigma}\times\vec{H}-\Delta\mu\rho_3\nabla(\vec{\sigma}\cdot\vec{H}), \tag{11}$$

and

$$\frac{d}{dt}\vec{\sigma}=-\frac{2c}{\hbar}\rho_1\vec{\sigma}\times\vec{\pi}-\frac{2\Delta\mu}{\hbar}\rho_3\vec{\sigma}\times\vec{H}. \tag{12}$$

In these equations last terms depending on $\Delta\mu$ represent the effect of the anomalous magnetic moment to be contrasted the results given by Case and others.

(A) First let us consider a pure magnetic field. Then,

$$\frac{d}{dt}\vec{\sigma}\cdot\vec{\pi}=-i\Delta\mu\rho_3\vec{\sigma}\cdot(\nabla\times\vec{H}),$$

$$\frac{d}{dt}\vec{\pi}=e\rho_1\vec{\sigma}\times\vec{H}-\Delta\mu\rho_3\nabla(\vec{\sigma}\cdot\vec{H}),$$

and

$$\frac{d}{dt}\vec{\sigma}=-\frac{2c}{\hbar}\rho_1\vec{\sigma}\times\vec{\pi}-\frac{2\Delta\mu}{\hbar}\rho_3\vec{\sigma}\times\vec{H}. \tag{13}$$

From Eq. (13) $\vec{\sigma}\cdot\vec{\pi}$, $\vec{\pi}$ and $\vec{\sigma}$ are no longer constants of motion. Then, for example, it is possible to change a state of longitudinal polarization to one of transverse polarization using purely magnetic fields and a longitudinally polarized beam will be depolarized on passing through purely magnetic fields.

(B) Second we consider a pure electric field. In this case we have

$$\frac{d}{dt}\vec{\sigma}\cdot\vec{\pi}=e\vec{\sigma}\cdot\vec{E},$$

$$\frac{d}{dt}\vec{\pi}=e\vec{E},$$

and

$$\frac{d}{dt}\vec{\sigma}=-\frac{2c}{\hbar}\rho_1\vec{\sigma}\times\vec{\pi}. \tag{14}$$

From Eq. (14) $\vec{\sigma}\cdot\vec{\pi}$, $\vec{\pi}$ and $\vec{\sigma}$ are not constants of motion. Then particles passing through a purely electric field are depolarized.

These conclusions can be applied, of course, so long as the Hamiltonian is given by Eq. (2). Finally we expect that for the baryons the effect of anomalous magnetic moment will be very important. From the degree of the depolarization of particles passing through a external field, we can estimate the order of the magnitude of their anomalous magnetic moment.

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Appendix

In this appendix, we summarize the useful formulas of which we make use :

$$(\vec{\sigma}\cdot\vec{B})(\vec{\sigma}\cdot\vec{C})=(\vec{B}\cdot\vec{C})+i\vec{\sigma}\cdot(\vec{B}\times\vec{C}), \tag{A1}$$

$$\vec{\sigma}(\vec{\sigma}\cdot\vec{B})\vec{C}=(\vec{B}\cdot\vec{C})-i\vec{\sigma}\cdot(\vec{B}\times\vec{C}), \tag{A2}$$

$$\vec{B}(\vec{\sigma}\cdot\vec{C})\vec{\sigma}=(\vec{B}\cdot\vec{C})-i\vec{\sigma}\cdot(\vec{B}\times\vec{C}), \tag{A3}$$

$$\vec{\sigma}\cdot(\vec{\sigma}\times\vec{B})=2i\vec{\sigma}\cdot\vec{B}, \tag{A4}$$

$$(\vec{\sigma}\times\vec{B})\cdot\vec{\sigma}=-2i\vec{\sigma}\cdot\vec{B}, \tag{A5}$$

$$\vec{\sigma}\ (\vec{\sigma}\cdot\vec{B})=\vec{B}-i\vec{\sigma}\times\vec{B}, \tag{A6}$$

$$(\vec{\sigma}\cdot\vec{B})\vec{\sigma}=\vec{B}+i\vec{\sigma}\times\vec{B}, \tag{A7}$$

where $\vec{B}$ and $\vec{C}$ are arbitrary operators which don't commute each other but commute with spin matrices σ_i's.

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