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A Note on Yang-Mills Field

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Abstract A result of an attempt to obtain a solution of Yang-Mills field equation is presented under both a gauge condition $pA^a_z = qA^a_0$, where p and q are numerical constants, and a condition $(p\partial_z + q\partial_{ct})X^a = 0$, where X^a is arbitrary physical quantity.

§ 1 Introduction

Many people are now searching after exact classical solutions of the equation for nonabelian gauge field[1),2),3)]. In this note we present preliminary results for this attempt, namely limited results under severe constraints. The equations we shall investigate are

$$D_\nu G^a_{\mu\nu} = J^a_\mu \tag{1}$$

$$\varepsilon_{\mu\nu\lambda\rho} D_\nu G^a_{\lambda\rho} = 0, \tag{2}$$

where D_ν denotes covariant derivative, $G^a_{\mu\nu}$ field strength of field A^a_μ, and J^a_μ fermion current.

Throughout this note we shall fix the gauge with

$$pA^a_z = qA^a_0 \qquad (A^a_4 = iA^a_0), \tag{3}$$

where p and q are numerical constants. Also throughout this note we shall assume a condition

$$(p\frac{\partial}{\partial z} + q\frac{1}{c}\frac{\partial}{\partial t})X^a = 0, \tag{4}$$

where X^a is any arbitrary quantity. Eq.(4) is consitstent with the gauge condition eq.(3). Eq.(4) shows any quantity is function of $(qz\text{-}pct)$. We intend to have a solution of eqs.(1) and (2) under two conditions that a) J^a_x, J^a_y, $\tilde{J}^a_x$ and $\tilde{J}^a_y$ ($\tilde{J}^a_\mu$ will be defined in eq.(8)) always vanishes, respectively, for $Q = (p/q)^2 - 1 = 0$ case, and J^a_r and $\tilde{J}^a_r$ always vanishes, respectively, for $Q \neq 0$ case, and b) field A^a_μ and field strength $G^a_{\mu\nu}$ vanishes when x and y (or r) becomes infinitely remote, respectively. Here x, y, and z are cartesian coordinates and r, θ, z are cylindrical coordinates ($x = r\cos\theta$, $y = r\sin\theta$). We shall obtain a set of eqs.(14), (15) and (16) of which only a special situation will be considered. To get a solution of eqs.(14), (15) and (16) under lighter constraint is a future problem.

§ 2 General Equations

Under two conditions eqs.(3) and (4) we obtain following eqs.(5)~(13).

$$pH^a_x = -qE^a_y, \qquad E^a_z = 0$$
$$pH^a_y = qE^a_x, \tag{5}$$

where $G^a_{23}=H^a_x$ (cyclic) and $G^a_{i4}=-iE^a_i$.

$$J^a_x = D_yH^a_z + QD_zH^a_y$$
$$J^a_y = -D_xH^a_z - QD_zH^a_x$$
$$J^a_z = D_xH^a_y - D_yH^a_x \tag{6}$$
$$c\rho^a = (p/q)J^a_z,$$

where $Q=(p/q)^2-1$ and $J^a_\mu = (J^a_x, J^a_y, J^a_z, ic\rho^a)$.

$$\tilde{\tilde{J}}^a_x = -gf^{abc}(A^b_yH^c_z + QA^b_zH^c_y)$$
$$\tilde{\tilde{J}}^a_y = gf^{abc}(A^b_xH^c_z + QA^b_zH^c_x)$$
$$\tilde{\tilde{J}}^a_z = gf^{abc}(A^b_yH^c_x - A^b_xH^c_y) \tag{7}$$
$$c\tilde{\tilde{\rho}}^a = (p/q)\,\tilde{\tilde{J}}^a_z,$$

where $\tilde{\tilde{J}}^a{}_\rho=-gf^{abc}A^b_\nu G^c_{\rho\mu}$.

$$\tilde{J}^a_x = \partial_yH^a_z + Q\partial_zH^a_y$$
$$\tilde{J}^a_y = -\partial_xH^a_z - Q\partial_zH^a_x$$
$$\tilde{J}^a_z = \partial_xH^a_y - \partial_yH^a_x \tag{8}$$
$$c\tilde{\rho}^a = (p/q)\tilde{J}^a_z\,,$$

where $\tilde{J}^a_\mu = J^a_\mu+\tilde{\tilde{J}}^a_\mu$.

$$
\begin{aligned}
J_x^{aA} &= 0 \\
J_y^{aA} &= 0 \\
J_z^{aA} &= (p/q)\, c\rho^{aA} \\
c\rho^{aA} &= -gf^{abc}(A_x^b H_x^c + A_y^b H_y^c + A_z^b H_z^c),
\end{aligned}
\tag{9}
$$

where $J_\mu^{aA} = -(1/2)\, \epsilon_{\mu\nu\lambda\rho} g f^{abc} A_\nu^b G_{\lambda\rho}^c$ (with $\epsilon_{1234} = 1$).

$$
\begin{aligned}
K_x &= QH_y^a(D_x H_y^a - D_y H_x^a) - QH_z^a D_z H_x^a - H_z^a D_x H_z^a \\
K_y &= QH_x^a(D_y H_x^a - D_x H_y^a) - QH_z^a D_z H_y^a - H_z^a D_y H_z^a \\
K_z &= QH_y^a D_z H_y^a + QH_x^a D_z H_x^a + H_y^a D_y H_z^a + H_x^a D_x H_z^a \\
K_0 &= (p/q) K_z,
\end{aligned}
\tag{10}
$$

where $K_\mu = G_{\mu\nu}^a J_\nu^a = (K_x, K_y, K_z, iK_0)$.

$$
\begin{aligned}
K_x^A &= -\tilde{\tilde{\tilde{K}}}_x = -Q(q/p) H_x^a J_z^{aA} \\
K_y^A &= -\tilde{\tilde{\tilde{K}}}_y = -Q(q/p) H_y^a J_z^{aA} \\
K_z^A &= -\tilde{\tilde{\tilde{K}}}_z = (q/p) H_z^a J_z^{aA} \\
K_0^A &= -\tilde{\tilde{\tilde{K}}}_0 = H_z^a J_z^{aA},
\end{aligned}
\tag{11}
$$

where $\tilde{\tilde{\tilde{K}}}_\mu = G_{\mu\nu}^a \tilde{\tilde{\tilde{J}}}_\nu^a$ and $K_\mu^A = -(1/2) \varepsilon_{\mu\nu\lambda\rho} G_{\lambda\rho}^a J_\nu^{aA}$.

$$
\begin{aligned}
\theta_{4x}^G &= -i(p/q) H_x^a H_z^a \\
\theta_{4y}^G &= -i(p/q) H_y^a H_z^a \\
\theta_{4z}^G &= i(p/q)\, \{(H_x^a)^2 + (H_y^a)^2\}
\end{aligned}
\tag{12}
$$

and

$$
-\theta_{ij}^G = T_{ij} = \begin{pmatrix} (-Q/2)\, \{(H_x^a)^2 - (H_y^a)^2\} - (1/2)(H_z^a)^2, & -QH_x^a H_y^a, & H_x^a H_z^a \\ -QH_y^a H_x^a, & (Q/2)\, \{(H_x^a)^2 - (H_y^a)^2\} - (1/2)(H_z^a)^2, & H_y^a H_z^a \\ H_z^a H_x^a, & H_z^a H_y^a, & (-1/2)\, \{(p/q)^2 + 1\}\, \{(H_x^a)^2 + (H_y^a)^2\} + (1/2)(H_z^a)^2 \end{pmatrix},
\tag{13}
$$

where $\theta_{\mu\nu}^G$ denotes symmetrized energy-momentum tensor of field A_μ^a.

We intend to have a solution when J_x^a, J_y^a, $\tilde{J}_x^a$ and $\tilde{J}_y^a$ vanishes, respectively, for $Q=0$ case, and J_r^a and $\tilde{J}_r^a$ vanishes, respectively, for $Q \neq 0$ case. Axial currents J_x^{aA} and J_y^{aA} already vanishes, respectively, as shown in eq.(9). Then we have following three sets of equation which the field strength H^a_x, H^a_y, and H^a_z (or H^a_r, H^a_θ, and H^a_z) must satisfy.

$$\partial_y H^a_z = 0 \quad \text{and} \quad \partial_x H^a_z = 0 \quad \text{for } Q = 0$$
$$\partial_\theta H^a_z + Q\partial_z H^a_\theta = 0 \quad \text{for } Q \neq 0 \tag{14}$$

$$f^{abc} A^b_y H^c_z = 0 \quad \text{and} \quad f^{abc} A^b_x H^c_z = 0 \quad \text{for } Q = 0$$
$$f^{abc}(A^b_\theta H^c_z + Q A^b_z H^c_\theta) = 0 \quad \text{for } Q \neq 0 \tag{15}$$

$$\partial_x H^a_x + \partial_y H^a_y + \partial_z H^a_z = -g f^{abc}(A^b_x H^c_x + A^b_y H^c_y + A^b_z H^c_z)$$
$$\partial_x H^a_y - \partial_y H^a_x = J^a_z - g f^{abc}(A^b_x H^c_y - A^b_y H^c_x) \tag{16}$$

In the first place we treat $Q=0$ case and in the next $Q \neq 0$ case.

§ 3 Q=0 Case

From eqs.(14) field strength H^a_z is independent of coordinates x and y. We impose the boundary condition that field and field strength must vanish when coordinate x and y becomes infinitely remote, respectively. Therefore H^a_z must vanish everywhere. Eqs.(15) are satisfied when $Q=0$ and $H^a_z=0$. Eqs.(16) become

$$\partial_x H^a_x + \partial_y H^a_y = -g f^{abc}(A^b_x H^c_x + A^b_y H^c_y)$$
$$\partial_x H^a_y - \partial_y H^a_x = J^a_z - g f^{abc}(A^b_x H^c_y - A^b_y H^c_x). \tag{17}$$

When the internal symmetry group is $SU(2)$ and after we transform eqs.(17), we obtain a equation identical in form with Euler's equation of motion for rigid body, where $H^a_{x(y)}$, $A^a_{x(y)}$, and J^a_z is identified, respectively, with angular momentum, angular velocity, and moment of external force in internal isospace. From eqs.(17) we obtain

$$(1/2)\partial_x\{(H^a_x)^2-(H^a_y)^2\} + \partial_y(H^a_x H^a_y) = -H^a_y J^a_z$$
$$(-1/2)\partial_y\{(H^a_x)^2-(H^a_y)^2\} + \partial_x(H^a_x H^a_y) = H^a_x J^a_z, \tag{18}$$

where summations about indices a are understood. Eqs.(18) correspond with the space part of expression for energy-momentum conservation. If we restrict our gauge group to be $SU(2)$, we obtain from $H^a_z = 0$[3]

$$A^a_x = (-2/g)\varepsilon^{abc}\phi^b\partial_x\phi^c$$
$$A^a_y = (-2/g)\varepsilon^{abc}\phi^b\partial_y\phi^c. \tag{19}$$

It is difficult to obtain a general solution of eqs.(18) for given and assumed form of J^a_z function. So we consider a special case with axial symmetry (all derivatives with respect to θ vanish.). Eqs.(18) become

$$H^a_\theta(1/r)\partial_r(rH^a_\theta) = H^a_\theta J^a_z. \tag{20}$$

But we have $(1/r)\partial_r(rH^a_\theta) \neq J^a_z$. Also we obtain

$$A^a_\theta = 0 \text{ and } H^a_r = 0. \tag{21}$$

All components of Lorentz force K_μ vanish when $Q=0$ and $H^a_z = 0$. Fields A^a_r and A^a_z satisfy a relation

$$H^a_\theta = \partial_z A^a_r - \partial_r A^a_z + gf^{abc}A^b_z A^c_r. \tag{22}$$

The $Q=0$ case corresponds to the situation where all quantities move toward z-axis with light velocity.

§ 4 Q≠0 Case

In this section we consider the case $Q=(p/q)^2-1\neq 0$. Eqs.(14) and (15) become more complicated ones when $Q\neq 0$. So we solely treat special situation where all derivatives with respect to coordinate z vanish. This is the situation where all derivatives with respect to time t vanish, too. Then we obtain from eqs.(14)

$$\partial_\theta H^a_z = 0 \tag{23}$$

We have only one eq.(23) in general as compared with two eqs.(14) for $Q=0$ case, because of conservation of total vector current. So $\partial_r H^a_z \neq 0$ in general. We can not have vanishing H^a_z in general in contrast with $Q=0$ case. But we are now treating a situation where all $\partial_z=0$. In this situation only we can assume $\partial_r H^a_z=0$ without conflict with conservation of total vector current. This is an *ad hoc* assumption. Then, as in the preceding section, H^a_z must vanish everywhere. So eq.(15) gives

$$f^{abc}A^b_z H^c_\theta = 0. \tag{24}$$

In the present section we have also eqs.(17), (18) and (19). Now we confront eqs.(18) with additional constraint eqs.(24), as compared with $Q=0$ case. Here also we treat a special case with axial symmetry. Then we have eqs.(20), (21), (22) and further eq.(24). Component K_θ, K_z, and K_0 of Lorentz force acting on fermion vanishes, respectively, and sole non-vanishing component K_r becomes

$$K_r = QH^a_\theta(1/r)\partial_r(rH^a_\theta) = QH^a_\theta J^a_z \tag{25}$$

Lorentz force K_r is negative when $Q<0$, $H^a_\theta>0$, and $J^a_z>0$.

In summary we have the following nonvanishing quantities when $Q\neq 0$, $\partial_z=0$, $\partial_t=0$, $\partial_\theta=0$ and $\partial_r H^a_z=0$:

$$A^a_r,\ A^a_z,\ A^a_0 = (p/q)A^a_z,\ H^a_\theta, \text{ and } E^a_r = (p/q)H^a_\theta$$

$$J^a_z = (q/p)c\rho^a = (1/r)D_r(rH^a_\theta)$$

$$\begin{aligned}
&\tilde{\tilde{J}}^a_z = (q/p)c\tilde{\rho}^a = -gf^{abc}A^b_r H^c_\theta \\
&\tilde{J}^a_z = (q/p)c\tilde{\rho}^a = (1/r)\partial_r(rH^a_\theta) \\
&K_r = QH^a_\theta J^a_z \\
&\theta^G_{4z} = i(p/q)(H^a_\theta)^2 \\
&T_{rr} = -T_{\theta\theta} = (Q/2)(H^a_\theta)^2 \qquad \text{and} \\
&T^{zz} = (-1/2)(Q+2)(H^a_\theta)^2.
\end{aligned} \tag{26}$$

In addition we have a relation $f^{abc}A^b_z H^c_\theta = 0$. Axial current and Lorentz forces acting on boson vanish.

$$J^{aA}_\mu = 0 \text{ and } K^A_\mu = -\tilde{\tilde{K}}_\mu = 0$$

§ 5 Remarks

We suppose our $Q \neq 0$ case represent a model of hadron which is infinitely long in z-direction. In order to have a closed string instead of an infinitely long one, we must have a solution of eqs.(14), (15) and (16) without the constraint $\partial_\theta = 0$ and $\partial_z = 0$ (or $\partial_t = 0$). In our $Q \neq 0$ case Lorentz force K_r ($K_r < 0$) balances the pressure P due to many pairs of quark and antiquark which exist inside of and outside of the cylindrical surface of our string. We have following relations:

$$\boldsymbol{K} = \nabla P,\ \boldsymbol{H}^a \cdot \nabla P = 0, \text{ and } \boldsymbol{J}^a \cdot \nabla P = 0. \tag{27}$$

We are now trying to have a solution of eqs.(14), (15) and (16) under less severe constraints than those done in this note, —a solution which may also have, we hope, the property shown in eqs.(27)—.

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