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Weak and Strong Pseudoconvexities

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Abstract. In this paper, we shall study a relation between weak and strong pseudoconvexities. In §1, we consider the weakly 1-complete manifold whose canonical line bundle is negative outside some compact set on X. In §1 and §2, we shall obtain a relation between weak and strong pseudoconvexities by assuming some topological (or algebraic) conditions. Moreover, its topological condition is closely related to Oka's principle and the globalization of solutions for some differential equation.

§0. Preliminaries

Let X be a n-dimensional paracompact complex manifold and let $\pi; E \to X$ be a holomorphic vector bundle of rank$=m$ on X. Let E be defined by 1-cocycle

$$e_{ij} : U_i \cap U_j \to GL(m, C)$$

in such a way that $(z_i, \zeta_i) \in U_i \times C^m$ and $(z_j, \zeta_j) \in U_j \times C^m$ are identified if and only if $z_i = z_j$, $\zeta_i = e_{ij}(z_j) \cdot \zeta_j$. There exists a hermitian metric $\{h_i = (h_{i\lambda\bar{\mu}})\}_{i \in I}$ for an open covering $\mathfrak{U} = \{U_i\}_{i \in I}$ of X. The ∂-connection θ with respect to the metric h is defined locally by $\theta_i = h_i^{-1} \partial h_i$ on each U_i. The (λ, μ)-component of (m, m)-matrix of θ_i is as follows:

$$\theta_{i\mu}^{\lambda} = \sum_{\alpha=1}^{n} \sum_{\tau=1}^{m} h_i^{\bar{\tau}\lambda} \frac{\partial h_{i\mu\bar{\tau}}}{\partial z_i^{\alpha}} dz_i^{\alpha}$$

where (z_i^{α}) is the local coordinates on U_i. The curvature form is given by $\Theta_i = \sqrt{-1}\, \bar{\partial} \theta_i$, whose (λ, μ)-component is as follows: $\Theta_{i\mu}^{\lambda} = \sqrt{-1} \sum_{\alpha,\beta=1}^{n} \Theta_{i\mu\alpha\bar{\beta}}^{\lambda}\, dz_i^{\alpha} \wedge d\bar{z}_i^{\beta}$

$$\Theta_{i\mu\alpha\bar{\beta}}^{\lambda} = \sum_{\rho,\tau,\sigma} h_i^{\bar{\sigma}\lambda} \left(h_i^{\bar{\tau}\rho} \frac{\partial h_{i\mu\bar{\tau}}}{\partial z_i^{\alpha}} \frac{\partial h_{i\rho\bar{\sigma}}}{\partial \bar{z}_i^{\beta}} - \frac{\partial^2 h_{i\mu\bar{\sigma}}}{\partial z_i^{\alpha} \partial \bar{z}_i^{\beta}} \right)$$

The following definition was first given by Nakano [7].

DEFINITION 1. A holomorphic vector bundle is said to be *positive* (or *negative*) in the sense of Nakano, if there exists a fibre metric h such that the (mn, mn)-matrix

$$H_i=(H_{i\bar{\nu}\mu\alpha\bar{\beta}}), \qquad H_{i\bar{\nu}\mu\alpha\bar{\beta}}=\sum_{\lambda=1}^{m} h_{i\lambda\bar{\nu}}\Theta^{\lambda}_{i\,\mu\alpha\bar{\beta}}$$

is positive definite (or negative definite) everywhere.

In particular, for the case $m=1$, E is a holomorphic line bundle on X. Then E is represented by the transition functions $\{h_{ij}\}$ with respect to some open covering $\mathfrak{U}=\{U_i\}_{i\in I}$ of X. E is said to be *positive* (or ⟨*semi*⟩ *negative*) if there exists a metric $\{a_i\}_{i\in I}$ with $a_i>0$ on U_i for any $i\in I$ and $|h_{ij}|^2=a_i^{-1}\cdot a_j$ on $U_i\cap U_j$ such that the Levi-form $L(-\log a_i)$ is positive definite (or negative ⟨semi⟩ definite). We denote by $E^{-1}=E^*$ the dual bundle of E. If holomorphic line bundle F is positive (or negative), its dual line bundle F^* is negative (or positive). Now, we define some pseudoconvexity and refer to its vanishing theorem and finiteness theorem by Nakano and Ohsawa [8], [9], [11].

DEFINITION 2. A complex manifold X is said to be *weakly* 1-*complete* with respect to Φ, if (1) Φ is a pseudoconvex C^∞-function on X, (2) $\{z\in X\mid \Phi(z)<c\}$ is relatively compact or empty for each $c\in\boldsymbol{R}$.

When D is a domain on X, we define similarly as above weak 1-completeness of D.

REMARK. By definition, weakly 1-complete manifold is paracompact and any closed submanifold of weakly 1-complete manifold is weakly 1-complete.

Nakano established the following theorem.

NAKANO'S VANISHING THEOREM. *Let X be a weakly* 1-*complete manifold with respect to Φ and B be a positive line bundle on X, then*

$$H^p(X,\ \Omega^q(B))=0 \qquad \textit{for } p+q>n\ (n=\dim_{C}X)\ .$$

REMARK. First, Nakano proved that there holds $H^p(X_c,\ \Omega^q(B))=0$ for $p+q>n$ for any $c\in\boldsymbol{R}$, where $X_c=\{z\in X\mid \Phi(z)<c\}$, but in [9] he succeeded in its globalization.

Later, T. Ohsawa showed the following finiteness theorem which was conjectured by Nakano.

Let X be as above and B be a positive outside a compact subset A of X, then

$$\dim_{C}H^p(X,\ \Omega^n(B))<+\infty \ \text{ for } p\geqq 1$$

and

$$H^p(X,\Omega^n(B))\cong H^p(X_c,\Omega^n(B)) \ \text{ for any } c>0 \text{ and } p\geqq 1.$$

REMARK. A holomorphic line bundle is said to be positive (or negative) on a subset $Y\subset X$, if there exists a metric $\{a_i\}_{i\in I}$ on B for a suitable covering $\mathfrak{U}=\{U_i\}_{i\in I}$ of X such that the Levi form $L(-\log a_i)$ is positive definite (or negative definite) on $U_i\cap Y$

for any $i \in I$.

§ 1. Weak 1-completeness and holomorphical convexity.

In this section we prove the following

THEOREM 1. *Let X be a weakly 1-complete manifold with respect to Φ whose canonical line bundle K_X is negative on $X-A$, where A is a compact subset of X.*

(α) *Let z_0 be a point on ∂X_c for some $c \in \mathbf{R}$, where $X_c = \{z \in X \mid \Phi(z) < c\}$, Φ is strongly pseudoconvex at z_0* and $z_0 \notin A$. *Then there exists a holomorphic function f on X_c such that* $\lim_{\substack{z_n \to z_0 \\ z_n \in X_c}} f(z_n) = \infty$.

(β) *Let X satisfy the following conditions:*

(i) $c_R(K_X) \sim 0$ *(or strongly* $H_2(X, \mathbf{Z})=0$*) in* $H^2(X, \mathbf{R})$,

(ii) $b_1(X) = 2 \dim_C H^1(X, \mathcal{O})$,

where $c_R(K_X)$ *is the real* (1, 1)*-form represented by* $(2\pi\sqrt{-1})^{-1} \cdot (-\partial\bar{\partial} \log a_i)$ *for a metric* $\{a_i\}_{i \in I}$ *on* K_X *and* $b_1(X)$ *is the first Betti number of* X.
Then X is holomorphically convex with maximal compact analytic subset M.

PROOF of (α). Since X is weakly 1-complete with respect to Φ, there exists $c \in \mathbf{R}$ such that $A \subset \{z \in X \mid \Phi(z) \leqq c\}$ and $\partial\{z \in X \mid \Phi(z) \leqq c\}$ is smooth by Sard's theorem. We define $\lambda: \mathbf{R} \to \mathbf{R}$ as follows.

$$\lambda(t) = \begin{cases} 0 & \text{if } t \leqq c \\ \exp\left[-\dfrac{1}{(t-c)^2} + t - c\right] & \text{if } t > c \end{cases}$$

Then $\lambda(\Phi)$ is a pseudoconvex C^∞-function vanishing in a neighborhood of A and $\partial\{z \in X \mid \lambda(\Phi(z)) = 0\}$ is smooth, so we replace Φ by $\Psi = \lambda(\Phi)$. Then X is weakly 1-complete with respect to Ψ *). After now, we discuss about Ψ. Let z_0 be a point on ∂X_d for some $d > 0$ where Ψ is C^∞-strongly pseudoconvex at z_0. By Satz 1.4 in [3], there exists a neighborhood U of z_0 and a holomorphic function f on U such that $\{z \in U \mid f(z)=0\} \cap \{z \in X \mid \Psi(z) \leqq d\} = \{z_0\}$. Choose $d' > 0$ so near to $d > 0$ that $\{z \in U \mid f(z)=0\} \cap \partial U \cap X_{d'} = \phi$. Let $U' \Subset U$ be an open set containing z_0 with $\{z \in U \mid f(z)=0\} \cap (U-U') \cap X_{d'} = \phi$, then $1/f(z)$ is holomorphic in $U - \overline{U'}$. The functions $1/[f(z)]^r$ in U, 0 in $X_{d'} - \overline{U'}$ define a first Cousin datum in $X_{d'}$ for open covering $\{X_{d'} - \overline{U'}, U\}$ and every integer $r \in \mathbf{N}$. Let h_r be a corresponding element in $H^1(X_{d'}, \mathcal{O})$ for $r=1, 2, \cdots$. By finiteness theorem, $k = \dim_C H^1(X_{d'}, \mathcal{O}) = \dim_C H^1(X, \mathcal{O})$ is finite for $\mathcal{O} \cong \Omega^n(K_X^*)$, hence there are constants $c_1, \ldots, c_p$ ($p \leqq k+1$), $c_p \neq 0$ such that $\sum_{r=1}^{p} c_r h_r = 0$ in $H^1(X_{d'}, \mathcal{O})$. Since each h_r is a first Cousin datum and the first Cousin data for the same covering are additively closed, this implies that there exists a

*) This proof is due to Ohsawa ([11], Proposition 2.2).

meromorphic function g in $X_{d'}$ such that $g-\sum_{r=1}^{p} c_r/f^r$ is holomorphic in U, while g is holomorphic in $X_{d'}-\overline{U'}$. The restriction of g to X_d is a holomorphic function in X_d such that $\lim_{\substack{z_n \to z_0 \\ z_n \in X_d}} g(z_n)=\infty$. q.e.d.

Before we prove (β), we refer to the holomorphic line bundle and the associated real (1, 1)-form. Let $\mathscr{H}$ be the sheaf of germs of pluriharmonic functions on a paracompact complex manifold X. Then we consider the following

(1) $0 \longrightarrow \boldsymbol{R} \xrightarrow{i} \mathcal{O} \xrightarrow{j} \mathscr{H} \longrightarrow 0$,

where j is defined by $j(f)=\mathrm{Re}\, f$ for every $f \in \mathcal{O}$.

(2) $0 \longrightarrow \boldsymbol{Z} \longrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0$.

(3) $0 \longrightarrow \mathscr{H} \longrightarrow \mathcal{E}_R^\infty \longrightarrow \partial\bar\partial \mathcal{E}_R^\infty \longrightarrow 0$,

where $\mathcal{E}_R^\infty$ is the sheaf of germs of real valued C^∞-functions on X.

(4) Sheaf homomorphism $\rho: \mathcal{O}^* \to \mathscr{H}$, defined by $\rho(f)=(2\pi\sqrt{-1})^{-1} \log|f|$.

(5) Characteristic homomorphism $c_1: H^1(X, \mathcal{O}^*) \to H^2(X, \boldsymbol{Z})$ induced by (2) and homomorphism $i^*: H^2(X, \boldsymbol{Z}) \to H^2(X, \boldsymbol{R})$ induced by the sheaf homomorphism i: $\boldsymbol{Z} \to \boldsymbol{R}$, where $\boldsymbol{R}$ is the sheaf of germs of real constants. Put $c_R=i^* \circ c_1: H^1(X, \mathcal{O}^*) \to H^2(X, \boldsymbol{R})$.

Then, by (4) we obtain the homomorphism $\rho^*: H^1(X, \mathcal{O}^*) \to H^1(X, \mathscr{H})$. Now each cohomology class F has a representation $\{f_{ij}\}$ for a suitable covering, we can write $\rho^*(F)=((2\pi\sqrt{-1})^{-1}\log|f_{ij}|)=((4\pi\sqrt{-1})^{-1}(\log a_j-\log a_i))$, where $\{a_i\}$ is a metric of F for $\mathfrak{U}=\{U_i\}_{i\in I}$ such that $|f_{ij}|^2=a_i^{-1}\cdot a_j$ on every $U_i \cap U_j$, and moreover, from (3) real (1, 1)-form $-(4\pi\sqrt{-1})^{-1}\partial\bar\partial \log a_i$ on X corresponds to $\rho^*(F)$ by the isomorphism $H^1(X, \mathscr{H}) \cong \dfrac{\Gamma(X, \partial\bar\partial\mathcal{E}_R^\infty)}{\partial\bar\partial\Gamma(X, \mathcal{E}_R^\infty)}$. From (1) we obtain the homomorphism $\delta^*: H^1(X, \mathscr{H}) \to H^2(X, \boldsymbol{R})$, then the image of $\rho^*(F)$ by δ^* is represented by $((2\pi\sqrt{-1})^{-1}(\log f_{ij}+\log f_{jk}+\log f_{ki}))$.

Therefore, we obtain the following commutative diagram:

$$\begin{array}{ccc}
H^1(X, \mathcal{O}^*) & \xrightarrow{\quad\rho^*\quad} & H^1(X, \mathscr{H}) \\
 & {\scriptstyle c_R}\searrow \quad \circlearrowleft \quad \swarrow{\scriptstyle \delta^*} & \\
 & H^2(X, \boldsymbol{R}) &
\end{array}$$

Then, by de Rham isomorphism, real (1, 1)-form $-(2\pi\sqrt{-1})^{-1}\partial\bar{\partial}\log a_i$ on X corresponds to $c_R(F)$. We denote it by γ_F. From the exact cohomology sequence $H^1(X, \mathcal{O}) \xrightarrow{j^*} H^1(X, \mathcal{H}) \xrightarrow{\delta^*} H^2(X, \boldsymbol{R})$ induced by (1), we obtain

(*) $j^*(H^1(X, \mathcal{O}))=0 \rightleftarrows$ The homomorphism δ^* is injective.

Therefore, when (*) holds, for some $F \in H^1(X, \mathcal{O}^*)$ $c_R(F)\sim 0$ implies $4\pi\sqrt{-1}\,\rho^*(F)\sim 0$ in $H^1(X, \mathcal{H})$ i.e. by (3), there exists a real valued C^∞-function f on X such that $\partial\bar{\partial}f=4\pi\cdot\sqrt{-1}\,\gamma_F$. In particular, if F is a positive (or negative) holomorphic line bundle, $c_R(F)\sim 0$ implies that there exists a strongly pseudoconvex C^∞-function f (resp. $-f$). But this is impossible in compact case of X, for γ_F is a generator of $H^2(X, \boldsymbol{R})$*). When we consider the condition which $c_R(F)$ is cohomologous zero in $H^2(X, \boldsymbol{R})$ for any holomorphic line bundle F, we can assume $H_2(X, \boldsymbol{Z})=0$, where $H_2(X, \boldsymbol{Z})$ is 2nd singular homology group of X in the coefficient $\boldsymbol{Z}$. It is reason why (1) in paracompact Hausdorff space, Čech (co-) homology theory and singular (co-) homology theory are equivalent, and (2) by universal coefficient theorem $H_p(X, \boldsymbol{R})\cong H_p(X, \boldsymbol{Z})\otimes_Z \boldsymbol{R}$, $H^p(X, \boldsymbol{R})\cong \mathrm{Hom}_R(H_p(X, \boldsymbol{R}), \boldsymbol{R})$ for $0\leqq p\leqq \dim_R X$. Still, the above assumption is essential when X is a noncompact complex manifold.

Proof of (β). By finiteness theorem, the condition (ii) has the meaning. By the assumption, there exists a metric $\{a_i\}_{i\in I}$ on K_X for some open covering $\mathfrak{U}=\{U_i\}_{i\in I}$ of X such that the Levi-form $L(-\log a_i)$ is negative definite on $U_i\cap(X-A)$ for any $i\in I$. Since X is paracompact, we may assume that $J=\{i\in I \mid U_i\cap A\neq\phi\}$ is finite set and we can take $\mathfrak{U}=\{U_i\}_{i\in I}$ such that each $\bar{U}_i$ is compact in X for every $i\in I$. Put $c_{ij}(z)=\log a_j(z)-\log a_i(z)$ on $z\in U_i\cap U_j$, then cocycle $c=(c_{ij})\in H^1(X, \mathcal{H})$ is equal to $4\pi\sqrt{-1}\,\rho^*(K_X)$ by the above discussion. Consider the segment of exact cohomology sequence, $H^1(X, \boldsymbol{R}) \xrightarrow{i^*} H^1(X, \mathcal{O}) \xrightarrow{j^*} H^1(X, \mathcal{H}) \xrightarrow{\delta^*} H^2(X, \boldsymbol{R})$, then by the assumption (ii) and universal coefficient theorem, i^* is isomorphic as real finitely dimensional vector space, so the homomorphism δ^* is injective. Therefore by the assumption (i) $4\pi\sqrt{-1}\,\rho^*(K_X)=c$ is cohomologous zero in $H^1(X, \mathcal{H})$. Then, there exists a 1-cochain $(c_i)_{i\in I}\in C^0(\mathfrak{U}, \mathcal{H})$ such that $c_{ij}=c_j-c_i$ on $U_i\cap U_j$ for every $i\neq j$. Put $\phi(z)=\log a_i(z)-c_i(z)$ for $z\in U_i$, then ϕ is a global real valued C^∞-function on X and ϕ is C^∞-strongly pseudoconvex on $X-K$ where $K=\bigcup_{i\in J}\bar{U}_i$ is a compact subset of X. Let $\lambda(t)$ be a real valued C^∞-function of $t\in\boldsymbol{R}$ such that $\lambda(t)>0$, $\lambda'(t)>0$ and $\lambda''(t)>0$ for every $t\in\boldsymbol{R}$.

Put $\Psi=\Phi+\lambda(\phi)$, then we may assume $\Phi\geqq 0$ on X, it is easily verified that $\{z\in X \mid \Psi(z)<c\}$ is relatively compact in X for every $c>0$ and Ψ is C^∞-strongly pseudoconvex on $X-K$ by $\partial\bar{\partial}\Psi=\partial\bar{\partial}\Phi+\lambda'(t)\,\partial\bar{\partial}\phi+\lambda''(t)\,\partial\phi\wedge\bar{\partial}\phi$. As same as the proof of (α), we can replace Ψ by $\tilde{\Psi}$ on X such that $\tilde{\Psi}\equiv 0$ in a neighborhood W of K $(A\subset K)$, $\partial\{z\in X \mid \tilde{\Psi}(z)=0\}$ is smooth, $\tilde{\Psi}$ is C^∞-strongly pseudoconvex on $X-\bar{W}$ and

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*) See [6], Theorem 1.4 at p.88.

moreover $\{z \in X \mid \widetilde{\Psi}(z) < c\}$ is relatively compact in X for every $c > 0$. Therefore $\overline{W}$ is compact in X. Since $\widetilde{\Psi}$ is C^∞-pseudoconvex on X and C^∞-strongly pseudoconvex on $\partial\{z \in X \mid \widetilde{\Psi}(z) \leqq c\}$, from ($\alpha$) $X_c = \{z \in X \mid \widetilde{\Psi}(z) < c\}$ is holomorphically convex for every $c > 0$ and $X = \bigcup_{c>0} X_c$. Hence X is holomorphically convex. Let (π, S) be a holomorphic reduction of X, then each fibre $\pi^{-1}(w)$, $w \in S$ is a compact connected analytic subspace in X. Put $M = \{z \in X \mid z$ is not an isolated point in $\pi^{-1}\circ\pi(z)\}$, then we remark that M is an analytic subset in X. If $M \subset X - \overline{W}$, $\widetilde{\Psi}$ is constant on each component of M, if $M \cap (X - \overline{W}) \neq \phi$, also $\widetilde{\Psi}$ is constant on each component of $M \cap (X - \overline{W})$, these are contradictory to the strong pseudoconvexity of $\widetilde{\Psi}$ on $X - \overline{W}$. Therefore $M \subset \overline{W}$. Since $\overline{W}$ is compact, M is a compact analytic subset, so M is written as follows ; $M = \bigcup_{i=1}^{p} M_i$ where each M_i is a compact connected and nowhere discrete analytic subset ($1 \leqq i \leqq p$). And moreover M is the maximal compact analytic set in X (For a detail, see[3]). Put $\pi(M_i) = w_i \in S$ ($1 \leqq i \leqq p$). Since each fibre $\pi^{-1}(w)$ is a compact connected analytic subspace, by the maximality of M, $\pi^{-1}(w)$ is one point for every $w \in S - \{w_i\}_{i=1}^{p}$. Therefore $\pi : X - M \to S - \{w_i\}_{i=1}^{p}$ is biholomorphic, then in relation to the finiteness theorem, it holds that $\dim_C H^p(X, \mathscr{S}) < +\infty$ for $p > 0$ and any coherent analytic sheaf $\mathscr{S}$ on X (See [10]). q.e.d.

§ 2. Weak 1-completeness and strong pseudoconvexity.

In this section, we prove the following

THEOREM 2. *Let X be a complex manifold whose canonical line bundle K_X is negative and let D, A be a domain on X and a closed submanifold of X with codimension $r < n = \dim_C X$ respectively.*

(α) (i) *D is weakly 1-complete with respect to Φ,*
(ii) *$H_2(D, Z) = 0$,*

then D is Stein.

(β) (i) *A is weakly 1-complete with respect to Φ,*
(ii) *Normal bundle N_A of A is negative in the sense of Nakano,*
(iii) *$H_2(A, Z) = 0$,*

then A is Stein.

In particular, for case of $r=1$, we have only to require that line bundle N_A is semi-negative.

PROOF of (α). Put $K_D = K_X|_D$, where $K_X|_D$ is the restriction of K_X onto D. By vanishing theorem and (ii), we obtain $H^p(D, \mathcal{O}) \cong H^p(D, \Omega^n(K_D^*)) = 0$ for $p > 0$ and $c_R(K_D) \sim 0$. From the previous consideration, there exists a strongly pseudoconvex C^∞-function φ on D. Let $\lambda(t)$ be a real valued C^∞-function of $t \in \boldsymbol{R}$ such that $\lambda(t) > 0$, $\lambda'(t) > 0$, $\lambda''(t) > 0$ for every $t \in \boldsymbol{R}$.

Put $\psi(z) = \Phi(z) + \lambda(\varphi(z))$ for $z \in D$, where we may assume $\Phi(z) \geqq 0$ for every $z \in D$, then Ψ is a C^∞-function on D and it is easily verified that (i) Ψ is a strongly pseudoconvex C^∞-function on D (ii) $D_c = \{z \in D \mid \Psi(z) < c\}$ is relatively compact in D for every $c > 0$.

From Theorem 1 (α) (In this case, by using vanishing theorem: $H^1(D_c, \mathcal{O})=0$ for any $c > 0$, we obtain the same result with Theorem 1 (α)), every D_c is holomorphically convex and has no positive dimensional compact analytic subset, hence every D_c is Stein. Now $D=\bigcup_{c>0} D_c$, $D_c \Subset D_{c'}$ for $c' > c > 0$, so D is Stein.

PROOF of (β). Let $\{U_i\}_{i\in I}$ be a defining covering of A and let $(z_i, \sigma_i)=(z_i^1, \ldots, z_i^{n-r}, \sigma_i^1, \ldots, \sigma_i^r)$ be the local coordinates in U_i such that $\sigma_i^k=0$ $(1 \leqq k \leqq r)$ are local equations of A in U_i for every $i \in I$. Then N_A is represented by the covering $\{V_i\}_{i\in I}$ where $V_i=U_i\cap A$ for every $i \in I$.

Let $\pi: N_A \to A$ be the projection of N_A and $(z_i, \zeta_i)=(z_i^1, \ldots, z_i^{n-r}, \zeta_i^1, \ldots, \zeta_i^r)$ be the local coordinates in $\pi^{-1}(V_i)\cong V_i\times C^r$ for every $i \in I$. By assumption, there exists a hermitian metric $\{h_i=(h_{i\lambda\bar\mu})\}_{i\in I}$ on N_A for this covering such that $(r(n-r), r(n-r))$-matrix $H_i=(H_{i\bar\nu\mu\alpha\bar\beta})$ is negative definite everywhere. Put $h(z_i, \zeta_i)={}^t\zeta_i h_i(z_i)\bar\zeta_i$ on $\pi^{-1}(V_i)$, then h is a metric function on N_A along the fibre with constant zero on O_A, where O_A is the zero section of A in N_A and $O_A\cong A$ biholomorphic. The Levi form $L(h)$ with respect to (z_i, ζ_i) is written as follows (where the subscript i is omitted for simplicity).

$$L(h)=-\sum_{\alpha,\beta}\sum_{\mu,\nu} H_{\bar\nu\mu\alpha\bar\beta}\,\zeta^\mu\bar\zeta^\nu\, dz^\alpha d\bar z^\beta$$

$$+\sum_{\tau,\rho} h^{\bar\tau\rho}\Big(\sum_{\alpha,\lambda}\frac{\partial h_{\lambda\bar\tau}}{\partial z^\alpha}\zeta^\lambda dz^\alpha+\sum_\lambda h_{\lambda\bar\tau}d\zeta^\lambda\Big)\cdot\Big(\sum_{\beta,\mu}\frac{\partial h_{\rho\bar\mu}}{\partial \bar z^\beta}\bar\zeta^\mu d\bar z^\beta+\sum_\mu h_{\rho\bar\mu}d\bar\zeta^\mu\Big)^{*)}$$

By the negativity of $(H_{\bar\nu\mu\alpha\bar\beta})$ and the positivity of $h_i=(h_{i\lambda\bar\mu})$, $L(h)$ is positive definite on N_A-O_A. Put $\Psi=\Phi\circ\pi+h$ on N_A, then its Levi form $L(\Psi)=L(\Phi\circ\pi)+L(h)$ with respect to (z_i, ζ_i) is positive semi-definite on N_A and clearly $\{y \in N_A \mid \Phi(y) < c\}$ is relatively compact in N_A for every $c \in \boldsymbol{R}$. Therefore N_A is weakly 1-complete with respect to Ψ. Remarking $K_{N_A}=\pi^*(K_A)\otimes\pi^*(\det N_A)^{-1}$ and $K_A=K_X\mid_A\otimes\det N_A$, where K_A and K_{N_A} are canonical line bundles of A and N_A respectively, $K_X\mid_A$ is the restriction of K_X onto A, hence we obtain $K_{N_A}=\pi^*(K_X\mid_A)$. Now K_X is represented by $\{K_{X_{ij}}\}_{i,j\in I}$ for $\mathfrak{U}=\{U_i\}_{i\in I}$. Since K_X is negative, there exists a metric $\{a_i\}_{i\in I}$ for $\mathfrak{U}=\{U_i\}_{i\in I}$ such that the Levi form $L(-\log a_i)$ is negative definite everywhere. Put $k_{ij}(z_j)=K_{X_{ij}}(z_j, 0)$, $a_i(z_i)=a_i(z_i, 0)$ on every $V_i\cap V_j$ and V_i respectively, then $K_X\mid_A$ is represented by $\{k_{ij}(z_j)\}_{i,j\in I}$ and its metric is given by $\{a_i(z_i)\}_{i\in I}$. In particular, in case $r=1$ if holomorphic line bundle is semi-negative, since $K_A=K_X\mid_A\otimes N_A$, we may assume that K_A is negative by the definition of the negativity of the line bundle. From Theorem 2 (α), A is Stein. In case $r>1$, put $A_i(z_i, \zeta_i)=\pi^*(a_i(z_i))\cdot\exp h(z_i, \zeta_i)$ for every $i \in I$, then since $K_{N_A}=\pi^*(K_X\mid_A)$, $\{A_i\}_{i\in I}$ is a metric on K_{N_A} for the covering $\{\pi^{-1}(V_i)\}_{i\in I}$. The Levi form $L(-\log A_i)$ with respect to (z_i, ζ_i) is written $L(-\log\pi^*(a_i))-L(h)$, so

*) For a detail, see [13].

$L(-\log A_i)$ is negative definite everywhere. Therefore K_{N_A} is a negative holomorphic line bundle. We define a continuous function H from $N_A \times I$ onto N_A as follows, where $I=[0, 1]$ is the interval.

$$H(w, t)=(z_i(w), (1-t)\zeta_i(w))$$

$$=(z_i^1(w), \dots, z_i^{n-r}(w), (1-t)\zeta_i^1(w), \dots, (1-t)\zeta_i^r(w)).$$

Since transition functions for the local coordinates (ζ_i) are matrix functions, H is a well-defined continuous function on $N_A \times I$. It is clear that $H(w, 0)=$identity on N_A, $H(w, 1)=\pi$ where $\pi: N_A \to A$ is the projection. Therefore A is strongly deformation retract of N_A, so A and N_A are homotopy equivalent, it holds that $H_p(A, Z) \cong H_p(N_A, Z)$ for $0 \leqq p \leqq \dim_R A$ as the singular homology group. By the assumption, $H_2(N_A, Z)=0$. Finally, the normal bundle N_A satisfies the assumption of Theorem 2 (α), hence N_A is Stein. Since the zero section O_A is closed in N_A and $O_A \cong A$ biholomorphic, A is Stein. q.e.d.

REMARK 1. In Theorem 2 (α), for case $D=X$, there exists a complex manifold which satisfies weakly 1-completeness, the negativity of canonical line bundle and is not Stein. For example $C^m \times P^n$ is so, where P^n is n-dimensional complex projective space. (In this case, if we put $M=C^m \times P^n$, the canonical line bundle K_M is $\pi^*(K_{P^n})$ where $\pi: M \longrightarrow P^n$ is projection and K_{P^n} is the canonical line bundle of P^n. Since the canonical line bundle of complex projective space is negative, using strongly pseudoconvex C^∞-function $d(z)=\sum_{i=1}^m z_i\bar{z}_i$, we can choose a metric on K_M such that K_M is negative).

In passing, 2nd homology group of $C^m \times P^n$ does not vanish; $H_2(C^m \times P^n, Z) \cong Z$ and in relation to Therem 1 (β), $H_1(C^m \times P^n, Z)=0$ and $H^1(C^m \times P^n, \mathcal{O})=0$ by vanishing theorem.

REMARK 2. In Theorem 2, $H^2(D, Z)=0$ implies $H_2(D, Z)=0$, but converses are not always true, for $H^2(D, Z) \cong H_2(D, Z) \oplus \mathrm{Ext}\,(H_1(D, Z), Z)$ by universal coefficient theorem. If $H_1(D, Z)$ has non-torsion groups, $H_2(D, Z)=0$ implies $H^2(D, Z)=0$.

REMARK 3. As mentioned in Remark 1, the canonical line bundle of complex projective space is negative, but by Suzuki's results [12], any weakly 1-complete domain on P^n is Stein. There exists a complex manifold which has the negative canonical line bundle and does not satisfy the assumption of his theorem. For example $C^m \times P^n$ is so.

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