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Representation Modules and the Augumentation Ideal of a Finite group

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Abstract. For a finite group G and a G-module A, the structure of $I^nA/I^{n+1}A$ is studied, where I is the augumentation ideal of the group ring $Z[G]$.

1. Let G be a finite group and A be a G-module. Denote by I the augumentation ideal of the group ring $Z[G]$ of G with respect to the ring of integers. Put $G_o= G/[G, G]$, where $[G, G]$ stands for the commutator subgroup of G. Denote by $G_o^{\otimes n}$ the tensor product $G_o\otimes\cdots\otimes G_o$ of n-copies of G_o. Then $I^nA/I^{n+1}A$ is isomorphic to a factor group of $G_o^{\otimes n}\otimes(A/IA)$. This is shown in [1] together with other cohomological relationship by means of class field theory, when G is the Galois group of a Galois extension K of an algebraic number field k and A is a congruent ideal class group of K.

The purpose of the present paper is to get a simple proof of the purely group theoretical part of the above result.

2. Let I be as above the augumentation ideal of the group ring $Z[G]$

LEMMA 1. *We have the following congruences in $Z[G]$ for any σ and τ of G:*

(i) $\sigma\tau-1 \equiv (\sigma-1)+(\tau-1) \quad \text{mod. } I^2$

(ii) $\tau(\sigma-1) \equiv \sigma-1 \quad \text{mod. } I^2$

(iii) $\sigma\tau\sigma^{-1}\tau^{-1}-1 \equiv 0 \quad \text{mod. } I^2$

PROOF. (i) and (ii) are implied from $(\sigma-1)(\tau-1) = (\sigma\tau-1)-(\sigma-1)-(\tau-1)$ and $\tau(\sigma-1)-(\sigma-1) = (\tau-1)(\sigma-1)$. Moreover (i) implies $(\sigma-1)+(\sigma^{-1}-1) \equiv \sigma\sigma^{-1}-1 \equiv 0$ mod. I^2. Hence $\sigma^{-1}-1 \equiv -(\sigma-1)$ mod. I^2. This implies (iii) by using (i) again.

LEMMA 2. *Let φ be a mapping of $G_o\otimes(A/IA)$ to IA/I^2A defined by*

$$\varphi(\bar{\sigma}\otimes \alpha) \equiv (\sigma-1)\, a \text{ mod. } I^2A,$$

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where $\bar{\sigma}$ and α are elements of G_o and A/IA represented by σ of G and a of A respectively. Then φ induces a surjective homomorphism.

PROOF. Let $\bar{\sigma}_1=\bar{\sigma}_2$. Then there exists an element ρ of $[G, G]$ such that $\sigma_2=\sigma_1\rho$, and we see that $\sigma_2-1\equiv(\sigma_1-1)+(\rho-1)\equiv\sigma_1-1 \bmod. I^2$ by (i) and (iii) of Lemma 1. Moreover (i) of Lemma 1 implies that $(\sigma^n-1)a\equiv n(\sigma-1)a \bmod. I^2A$ for any integer n. Hence φ is well defined, and it is clear that φ induces a surjective homomorphism.

3. We denote by G^n the direct product of n-copies of G, and denote by $C^n(G, A)$ the group of cochains of G of dimension n with A as its coeffient module. For $f\in C^n(G, A)$, let

$$\Delta^n f = \sum(\tau_1-1)\cdots(\tau_n-1)f(\tau_1,\cdots,\tau_n),$$

where the sum is taken over all $(\tau_1,\cdots,\tau_n)$ of G^n. Denote by $R^n(G, A)$ the subgroup of $C^n(G, A)$ consisted of f such that $\Delta^n f=0$. Let μ be a homomorphism of $C^n(G, A)$ to $G_o^{\otimes n}\otimes(A/IA)$ defined by

$$\mu(f) = \sum\bar{\tau}_1\otimes\cdots\otimes\bar{\tau}_n\otimes\bar{f}(\tau_1,\cdots,\tau_n),$$

where $\bar{f}(\tau_1,\cdots,\tau_n)$ stands for the element of A/IA represented by $f(\tau_1,\cdots,\tau_n)$ and the sum is taken over all $(\tau_1,\cdots,\tau_n)$ of G^n.

THEOREM. *Notation being as above, we have*

$$I^nA/I^{n+1}A \cong \frac{G_o^{\otimes n}\otimes(A/IA)}{\mu R^n(G, A)}.$$

PROOF. Applying Lemma 2 repeatedly, we have a surjective homomorphism $\varphi^{(n)}$ of $G_o^{\otimes n}\otimes(A/IA)$ to $I^nA/I^{n+1}A$, which is defined by

$$\varphi^{(n)}(\bar{\sigma}_1\otimes\cdots\otimes\bar{\sigma}_n\otimes\alpha)\equiv(\sigma_1-1)\cdots(\sigma_n-1)a \mod. I^{n+1}A,$$

where $\bar{\sigma}_i$ and α are elements of G_o and A/IA represented by $\sigma_i\in G$ and $a\in A$ respectively ($i=1,\cdots,n$). Let us determine Ker $\varphi^{(n)}$. Put $(\bar{\sigma})=(\bar{\sigma}_1,\cdots,\bar{\sigma}_n)$, which runs over all elements of G_o^n, and let representatives σ_i in G of each class of G_o be fixed once for all. Now let $\alpha_{(\bar{\sigma})}$ be elements of A/IA such that $\varphi(\sum_{(\bar{\sigma})}\bar{\sigma}_1\otimes\cdots\otimes\bar{\sigma}_n\otimes\alpha_{(\bar{\sigma})})=0$. Then $\sum_{(\bar{\sigma})}(\sigma_1-1)\cdots(\sigma_n-1)a_{(\bar{\sigma})}\equiv 0$ *mod.* $I^{n+1}A$, where $a_{(\bar{\sigma})}$ are representatives of $\alpha_{(\bar{\sigma})}$ in A. There exist elements $b_{(\rho)}$ of A such that

$$\sum_{(\bar{\sigma})}(\sigma_1-1)\cdots(\sigma_n-1)a_{(\bar{\sigma})} = \sum_{(\rho)}(\rho_1-1)\cdots(\rho_{n+1}-1)b_{(\rho)},$$

where $(\rho)=(\rho_1,\cdots,\rho_{n+1})\in G^{n+1}$. Put $(\tau)=(\tau_1,\cdots,\tau_n)\in G^n$, and let $b_{(\rho)}^{(\tau)}=\sum_{\rho_{n+1}\in G}(\rho_{n+1}-1)b_{(\rho)}$, where $b(\rho)$ runs over all (ρ) such that $(\rho_1, \cdots, \rho_n) = (\tau_1, \cdots, \tau_n)$. Put $f(\tau_1, \cdots, \tau_n) = a_{(\bar{\sigma})} - b^{(\tau)}_{(\rho)}$ or $-b^{(\tau)}_{(\rho)}$ according as $(\tau_1, \cdots, \tau_n) = (\sigma_1, \cdots, \sigma_n)$ for some $(\bar{\sigma})$ or not. Then $f \in R^n(G, A)$. Moreover we have $\mu(f) = \Sigma \bar{\tau}_1 \otimes \cdots \otimes \bar{\tau}_n \otimes \bar{f}(\tau_1, \cdots, \tau_n) = \sum_{(\bar{\sigma})} \bar{\sigma}_1 \otimes \cdots \otimes \bar{\sigma}_n \otimes \alpha_{(\bar{\sigma})}$, because $b^{(\tau)}_{(\rho)} \in IA$. Now we have Ker $\varphi^{(n)} \subset \mu R^{(n)}(G, A)$. The opposite inclusion is clear by definition.

References

[1] Y. Furuta, On nilpotent factors of congruent ideal class groups of Galois extensions, Nagoya Math. J., **62** (1976), 13-28.