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Extension of Noether's Theorem to Non-local Field Theory

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Abstract : Noether's theorem is extended to a non-local field theory. As a model, a Schroedinger field interacting through a potential acting with a finite range is studied. It is shown that an invariance of a Lagrangian under a continuous transformation does not necessarily mean a local conservation of the Noether's current because of an apperance of an additional source due to the non-locality of the interaction. Some examples are shown to explain the extended Noethor's theorem.

1. Introduction

A non-perturbative approach has attracted much interest in many branches of field theories[1)~7)]. In particular, local conservation laws for various physical quantities related to a transformation properties, known as the Noether's theorem, play important role[3,5,6)]. The theorem states that there exists a four-current $j_\mu(x)$ ($\mu=0,1,2,3$) associated with a continuous transformation of the field. It satisfies a continuity equation

$$\Sigma \partial_\mu j_\mu \ (x) = q\ (x), \tag{1-1}$$

where x ($=x_0,\ x_1,\ x_2,\ x_3$) is a set of space-time coordinates, and $q\ (x)$ is a source of the current. If an action integral for the field is kept invariant under the transformation the source vanishes identically, thus we have a local conservation law associated with the transformation.

The proof for the theorem has been made thus far for a local field under a primary interest to apply to a relativistic field theory. However, in non-relativistic fields which are used to apply to condensed matter physics we are faced to a field described by a non-local Lagrangian with an interaction through a finite range potential, e.g., a coulomb potential between charged particles. In the non-local field the invariance of the action under a transformation dose not necessarily mean the local conservation of the related four-current because an additional non-local source appears. Therefore, it is worth wile to examine the validity of the Noether's theorem and extend it to the non-local field theory.

In the following, a general transformation formula for the non-local field is derived (§ 2), and some examples of the conserved and non-conserved quantities related with the special transformations are shown. In §4 discussions and conclusions are given.

2. General formula for non-local field

Let us consider a (complex) field described by a Lagrangian density

$$L = L_0 + L_i, \tag{2-1}$$

where L_0 is a term bilinear in the field $\psi_s(x)$ and its complex conjugate $\psi_s^*(x)$ ($s=1,2,\cdots$, n), and L_i is a nonlinear and non-local interaction. To be concrete, we are interested specifically in a Schroedinger field described as (a sum over s is omitted)

$$L_0 = \frac{i}{2}\{\psi^*(x)\partial_0\psi(x) - (\partial_0\psi^*(x))\psi(x)\}$$
$$-\frac{1}{2m}\sum_i \partial_i\psi^*(x)\partial_i\psi(x) + V(x)\psi^*(x)\psi(x), \tag{2-2}$$

and

$$L_i = \int d^4y W(\psi(x),\psi(y);x,y), \tag{2-3}$$

where

$$W(\psi(x),\psi(y);x,y) = U(x\text{-}y)\psi^*(x)\psi^*(y)\psi(y)\psi(x), \tag{2-4}$$

and

$$U(x\text{-}y) = U(\mathbf{x}\text{-}\mathbf{y})\,\delta(x_0\text{-}y_0), \tag{2-5}$$

is an instantaneously propergating potential. The integral in Eq.(2-3) is taken over an infinite space-time volume. Note that the Lagrangian depends explicitly on the space-time coordinates through the external potential $V(x)$ and the interaction potential $U(x\text{-}y)$.

Now, let us consider local variations of the field and the space-time coordinates as

$$\psi(x) \to \psi(x') = \psi(x) + \theta\delta\psi(x), \tag{2-6}$$
$$x\mu \to x'\mu = x\mu + \theta\delta x\mu, \tag{2-7}$$

where θ is an infinitesimal parameter. The variation $\delta\psi(x)$ is supposed to be given by the linear combination of the field components and/or their derivatives. Because the field derivatives are involved in L_0 only, an action principle leads to the Eular's equation of a form as

$$\sum_\mu \partial\mu \frac{\partial L_0}{\partial(\partial\mu\psi(x))} = \frac{\partial L_0}{\partial\psi(x)} + 2\int d^4y \frac{\partial W}{\partial\psi(x)}, \tag{2-8}$$

Equation (2-8) is identical to the Schroedinger equation,

$$i\partial_0\psi(x) = -\frac{1}{2m}\sum_i \partial_i\, 2\, \psi(x) + \int d^4y U(x\text{-}y)\psi^*(x)\psi(y)\psi(x), \tag{2-9}$$

where the sum over i is taken over the spatial components of the coordinate.

We consider a variation δS of the action due to the variation given by Eqs.(2-6) and (2-7),

$$\delta S = \int \Omega d^4x\,[J_0\{L_0(\psi'(x'),\partial'\mu\psi'(y');x')$$
$$+\int d^4y J(y)\{W(\psi'(x'),\psi'(y');x',y')$$
$$-L_0(\psi(x),\partial\mu\psi(x);x)$$
$$-\int d^4y W(\psi(x),\psi(y);x,y)\}]$$
$$=\theta\int \Omega d^4x\delta L, \tag{2-10}$$

where Ω is an arbitrary space-time volume, and J is the Jacobian for the transformation (2-7).

Calculation of the r.h.s. of Eq.(2-10) is done by a straightforward extension of a usual procedure to prove the Noether's theorem for the local theory. It gives

$$\delta S=\theta\int\ \Omega^4x\ [\frac{\partial L_0}{\partial\psi(x)}\delta\psi(x)$$
$$+\sum_\mu\frac{\partial L_0}{\partial(\partial\psi(x))}\ \{\ \partial v\delta\psi\ (x)-\sum_v(\ \partial v\delta x\mu\ )\ \partial v\psi\ (x)$$
$$+\sum_\mu\frac{\partial L_0}{\partial x\mu}+\Sigma(\ \partial\mu\delta x\mu\ )L_0]$$
$$+\theta\int\ \Omega d^4\int\ d^4y\ [\frac{\partial W}{\partial\psi(x)}\delta\psi(x)+\frac{\partial W}{\partial\psi(x)}\delta\psi(y)$$
$$+\sum_\mu\ \{\frac{\partial W}{\partial x\mu}\delta x\mu+\frac{\partial W}{\partial y\mu}\ \delta y\mu\ \}$$
$$+\sum_\mu\ \{(\ \partial\mu\delta x\mu\ )W+(\ \partial^y\mu\delta y\mu\ )W\ \}]\ , \tag{2-11}$$

up to the first order terms in the parameter θ. The last term in the integrals in the r.h.s. of Eq.(2-11) emerged from the Jacobian for the transformation by Eq.(2-7). The derivatives with respect to the field mean such that

$$\frac{\partial L_0}{\partial\psi(x)}=\sum_s\frac{\partial L_0}{\partial\psi_s(x)},\ \text{ets.} \tag{2-12}$$

Using Eq.(2-8) and integrating partially we rewrite Eq.(2-11) as

$$\delta S=\theta\int\ \Omega d^4x\ [\sum_\mu \partial\mu j\mu\ (x)$$
$$+\int\ d^4y\ \{\frac{\partial W}{\partial\psi(y)}\delta^0\psi(y)-\frac{\partial W}{\partial\psi(x)}\delta^0\psi(x)\}]\ , \tag{2-13}$$

where

$$\delta^0\psi(x)=\psi'(x)-\psi(x)$$
$$=\delta\psi(x)-\sum_v\delta xv\partial v\psi\ (x), \tag{2-14}$$

is a Lie derivative of the field, and

$$j\mu(x)=\frac{\partial L}{\partial(\partial\mu\psi(x)\ )}\delta^0\psi(x)+\delta x\mu L\,. \tag{2-15}$$

The r.h.s. of Eq.(2-15) is formally same to the Noether's current for the local theory[7].

Comparing Eq.(2-10) and Eq.(2-13) and considering an arbitrariness of the volume Ω of the integration, we arrive at the continuity equation as Eq.(1-1), the source of the current being expressed as

$$q(x)\ =\delta L+q_{\text{nl}}(x), \tag{2-16}$$

where

$$q_{\text{nl}}(x)=\int\ d^4y\ \{\frac{\partial W}{\partial\psi(x)}\delta^0\psi(x)-\frac{\partial W}{\partial\psi(y)}\delta^0\psi(y)\}\ , \tag{2-17}$$

is a source due to the non-local interaction. Note that the r.h.s. of Eq.(2-17) vanishes, and the results reduces to the formula known in the local case, if a local instantaneous potential

$$U(x\text{-}y)\ =U\delta^4(x\text{-}y), \tag{2-18}$$

is considered. In the local case the invariance of the action under the transformation leads to the local conservation law with vanishing source. However, in the non-local case the invariance does not necessarily mean the local conservation of the Noether current. Physically speaking, this failure of the local conservation is related to the fact that the term $L\delta x\mu$ in the r.f.s. of Eq.(2-15) involves a flow caused by a force exerted by a field at a distance. As shown in the next section the transformation can be classified into two classes with respect to this view point, namely, according to whether the transformation induces the force by an environmental field at a distance or not. The first class is the locally conserving, and the second is the locally non-conserving.

In spite of the fact above mentioned, it should be noted that the spatially integrated 0-th component of the current,

$J_0 = \int d^3x j_0(x)$

remains to be a constant of motion if the action is invariant under the transformation in the non-local case just as in the local case.

3. Examples

1) Phase transformation

The Lagrangian given by Eqs.(2-1)~(2-3) has an internal symmetry with respect to a global phase tranformation as

$$\psi(x) \to \psi'(x) = \exp(i\theta)\psi(x). \quad (3\text{-}1)$$

The Lie derivative is given by

$$\delta^0 \psi(x) = -i\psi(x). \quad (3\text{-}2)$$

In this case the non-local source vanishes, and the conserving Noether current is given by

$$j_0(x) = \psi^+(x)\psi(x), \quad (3\text{-}3)$$

$$j_i(x) = \frac{1}{2mi} \{\psi^*(x)\partial_i\psi(x) - (\partial_i\psi^*(x))\psi(x)\psi(x)\}, \quad (3\text{-}4)$$

as is well known as a conservation law of a particle number.

If an electromagnetic field is applied to a charged field the space-time derivatives in the Lagrangian (2-2) is modified as

$$\begin{aligned} &\partial_0 \to \partial_0 + ieU, \\ &\partial_i \to \partial_i - i(e/c)A_i, \end{aligned} \quad (3\text{-}5)$$

where U and A_i are scalar and vector potentials for the electromagnetic field. The phase transformation modifies the spatial component of the Noether's current given by Eq. (3-4) as

$$\begin{aligned} j_i = \frac{1}{2mi} \{&\psi^*(x)(\partial_i - i\frac{e}{c}A_i)\psi(x) \\ &-()\partial_i + i\frac{e}{c}A_i)\psi^*(x))\psi(x). \end{aligned} \quad (3\text{-}6)$$

It is easy to show that a gauge invariance of the theory requires that

$$j_0 = -\partial L/\partial(eU), \quad (3\text{-}7)$$

$j_i = c\partial L/\partial(eA_i)$.

Equation (3-7) gives in fact Eqs.(3-3) and (3-6).

2) Translation in space

Consider a translation in a spatial direction j as

$$x_i \to x_i' = x_i + \theta\delta_{ij} \tag{3-8}$$

Then,

$$\delta_i^0 = -\partial_i\psi(x), \tag{3-9}$$

which leads to the Noether's current

$$T_{0j}(x) = -(i/2)\ \{\psi^*(x)\partial_j\psi(x) - (\partial_j\psi^*(x))\psi(x), \tag{3-10}$$

$$T_{ij}(x) = (1/2m)\ \{\partial_i\psi^*(x)\partial_j\psi(x) + (\partial_j\psi^*(x)\} + L\delta_{ij}, \tag{3-11}$$

where a conventional tensor notation is used. T_{0j} is known as a momentum density and T_{ij} as a stress tensor of the field.

It is noted that the current given by Eqs.(3-10) and (3-11) does not conserve locally because the non-local source in the present case

$$q_{\text{nl}}(x) = (1/2)\int\ d^4yU(x\text{-}y)\ \{\psi^*(y)\partial_j(\psi^*(x)\psi(x))\psi(y) - \psi^*(x)\partial_j^y(\psi^*(y)\psi(y))\psi(x)\}\ , \tag{3-12}$$

does not vanish if the potential acts with a finite range. The spatial translation induces a work exerted by a field at a distance to the field at a point under consideration, which gives rise to the source for the momentum flow.

3) Translation in time

A translation in time

$$t \to t' = t - \theta, \tag{3-13}$$

induces a field variation

$$\delta^0\psi(x) = \partial_0\psi(x). \tag{3-14}$$

It generates a current

$$T_{00}(x) = (i/2)\ \{\psi^*(x)\partial_0\psi(x) - (\partial_0\psi^*(x))\psi(x)\} - L = H(x), \tag{3-15}$$

$$T_{i0}(x) = (1/2m)\ \{\partial_0\psi^*(x)\partial_i\psi(x) + (\partial_i\psi^*(x)\partial_0\psi(x)\}\ , \tag{3-16}$$

where $H(x)$ is the Hamiltonian density of the field. Equation (3-15) shows that the generator of the time displacement is given by the total Hamiltonian $H = \int\ d^3xH(x)$, as it should be. As the example 2) the non-local source does not vanish and is given by

$$q_{\text{nl}}(x) = (1/2)\int\ d^4yU(x\text{-}y)\ \{\psi^*(x)\partial_0^y(\psi^*(y)\psi(y))\psi(x) - \psi^*(y)\partial_0(\psi^*(x)\psi(x))\psi(y). \tag{3-17}$$

4) Spatial rotation

According to a transformation property for a spatial rotation the fields are classified into a scalar, spinor, vector and so on. For concreteness, we consider a spinor field with two components $\alpha(\alpha=1,2)$. For an infinitesimal rotation $-\theta$ about an axis in a direction $\mathbf{n}$ the coordinate is transformed as

$$\delta x_i = -\sum_{jk} \varepsilon_{ijk} x_j n_k, \tag{3-18}$$

and the Lie derivative is given by

$$\delta^0 \psi\alpha(x) = -\sum_{\beta} \sum_{k} n_k \ \{l_k \delta\alpha\beta + (s_k)\alpha\beta\} \ \psi\beta(x), \tag{3-19}$$

where ε_{ijk} is the Levi-Civita's tensor, sk is a component of a spin angular momentum , and

$$l_k = -(i/2) \sum_{ij} \varepsilon_{ijk}(x_i \partial_j) - x_j \partial_i), \tag{3-20}$$

is a component of an orbital angular momentum operator. The Noether's current is a set of a total angular momentum and its flow as,

$$g_0(x) = (1/2) \sum_k n_k \ [\sum_\alpha \ \{\psi\alpha^*(x) l_k \psi\alpha(x) - (l_k^+ \psi\alpha^*(x)\ ) \psi\alpha(x)\}$$

$$+ \sum_{\alpha\beta} \psi\alpha^*(x) \ (s_k)\alpha\beta\psi\beta(x), \tag{3-21}$$

$$g_i(x) = -(1/2mi) \sum_k n_k \sum_\alpha \ \{(l_k^+ \psi\alpha(x)\ ) \partial_i \psi\alpha(x)$$

$$+ (\partial_i \psi\alpha^*(x)\ ) l_k \psi\alpha(x)\}$$

$$+ \sum_{\alpha\beta} (s_k)\alpha\beta(\psi\alpha^*(x) \partial_i \psi\beta(x) - (\partial_i \psi\alpha^*(x)\ ) \psi\beta(x)]$$

$$-\sum_{jk} \varepsilon_{ijk} x_j n_k L(x), \tag{3-22}$$

It is frequently useful to consider separately the spin part of Eqs.(3-21) and (3-22). Provided the interaction is invariant under the rotation in the spin space, the spin part of the current is conserved locally by itself. In this case the non-local source vanishes as in the example 1).

5) Galilean transformation

A Galilean transformation to a reference system with a relative velocity$u\mathbf{n}$,

$$\mathbf{r} \to \mathbf{r}' = \mathbf{r} - u\mathbf{n}t, \tag{3-23}$$

$$t \to t' = t, \tag{3-24}$$

induces a field variation

$$\psi(x) \to \psi'(x') = \exp(ig(x)\ ) \psi(x), \tag{3-25}$$

where,

$$g(x) = -mu\mathbf{n} \cdot \mathbf{r} + (m/2) u^2 t. \tag{3-26}$$

The Lie derivative is given by

$$\delta^0 \psi(x) = -i \sum_i n_i (x_i + it\partial_i) \psi(x). \tag{3-27}$$

The Noether's current is

$$G_0(x)=(1/2)\sum_i n_i \ \{\psi^*(x)\ (mx_i+it\partial_i)\psi(x)$$
$$+(\ (mx_i-it\partial_i)\psi^*(x)\ )\psi(x)\} \qquad (3\text{-}28)$$
$$G_i(x)=(i/2)\mathbf{n}\cdot\mathbf{r}\ \{\psi^*(x)\ \partial_i\psi(x)-(\partial_i\psi^*(x)\ )\psi(x)$$
$$-(1/2m)\ t\sum_j n_j\ \{(\partial_i\psi^*(x)\ )\partial_j\psi(x)$$
$$+(\partial_j\psi^*(x)\ )\partial_i\psi(x)\}-tun_iL$$
$$=\mathbf{n}\cdot\mathbf{r}T_{0i}-t\sum n_jT_{ij}, \qquad (3\text{-}29)$$

The non-local source does not vanish as given by

$$q_{\text{nl}}=(1/2)t\sum_i n_i\int\ d^4yU(x\text{-}y)\ \{\psi^*(y)\partial_i\psi^*(x)\psi(x)\psi(y)$$
$$-\psi^*(x)\partial_i^y\psi^*(y)\psi(y)\psi(x)\}\ . \qquad (3\text{-}30)$$

As an application of the Galilean transformation we derive a theorem of a center-of-mass motion. To prove this we put as

$$X_i(t)=\int\ d^3xx_i\psi^*(x)\psi(x), \qquad (3\text{-}31)$$
$$P_i(t)=-(i/2)\int\ d^3x\ \{\psi^*(x)\partial_i\psi(x)-(\partial_i\psi^*(x)\ )\psi(x)\}\ . \qquad (3\text{-}32)$$

Note that

$$G_0(t)=\int\ d^3xG_0(x)$$
$$=\mathbf{n}\cdot\{m\mathbf{X}(t)-t\mathbf{P}(t)\}\ . \qquad (3\text{-}33)$$

If the Lagrangian is Galilei-invariant, $G_0(t)=G_0$ and $\mathbf{P}(t)=\mathbf{P}$ are constants of motion. This leads to the theorem of the center-of-mass motion as

$$\mathrm{m}\frac{\partial}{\partial t}\times(t)=\mathbf{P}. \qquad (3\text{-}34)$$

4. Conclusion

We have shown the Noether's theorem for the non-local field with some concrete examples. The continuity equations derived in the above examples are same as those obtained by a canonical formalism[9].

Although we have treated only the classical field, the formulae above derived are easily carried over to a quantized field, and also to the Matubara formalism at a finite temperature by a substitution of the real time t by an imaginary oneit.

It is demonstrated in the examles that the Noether's currents are classified into two groups according to the conserving property. (The phase transformation (example 1) and the spin-space rotation (example 4) fall into the first group where the non-local interacton does not generate the non-local source and the continuity of the Noether's currents holds exactly as in the case of the local interaction. On the other hand the space and time translation and the rotation (exmples 2,3 and 4) belong to the second group where the non-local interaction generates the additional non-local source. However, for the latter case it is able to construct an approximately conserving current provided a variation of the field can be considered to be sufficiently slow as compared with the scale of a range of the interaction. This fact provides a basis to derive a hydrodynamic theory for the system described by the non-local field [9,10].

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