Sci. Rep. Kanazawa Univ.
Vol.48, No.1,2, p.5-7, 2004

A representation of continuous additive functionals of zero energy

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(Received June 27, 2003)

Abstract: The notion of the continuous additive functionals of zero energy of symmetric Markov processes was introduced by M. Fukushima. There are two typical types of continuous additive functionals of zero energy. We shall decide the class of continuous additive functionals of zero energy of symmetric Markov processes which are represented by the sum of the above two typical additive functionals.

1. Introduction

Let X be a locally compact Hausdorff space with a countable basis and m be a positive Radon measure on $(X, \mathcal{B}(X))$ with supp(m)=X, where $\mathcal{B}(X)$ is the Borel σ-algebra of X. $L^2(X,m)$ is the space of all square m-integrable real-valued functions with the usual inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$. We consider a C_0-regular Dirichlet space $(\mathcal{E},\mathcal{F})$ on $L^2(X,m)$. We consider a m-symmetric Hunt process $\mathbf{M} = (\Omega, \mathcal{M}, X_t, P_x)$ associated with the Dirichlet space $(\mathcal{E},\mathcal{F})$. Let us discuss additive functionals of m-symmetric Hunt process $\mathbf{M} = (X_t, P_x)$. The energy $e(A)$ of an additive functional A of $\mathbf{M}$ is defined by

$$e(A) = \lim_{t \to +0} \frac{1}{2t} E_m[A_t^2] \quad \text{whenever the limit exists,}$$

where $E_m[\cdot]$ denotes the integration with respect to the measure $P_x(\cdot)dm(x)$. M. Fukushima [1] showed that for $u \in \mathcal{F}$, there exists a unique martingale additive functional $M^{[u]}$ of finite energy and a unique continuous additive functional $N^{[u]}$ of zero energy such that

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^{[u]} + N_t^{[u]},$$

where $\tilde{u}$ is a quasi-continuous version of u. We may consider that this decomposition is a sort of an extension of Itô formula. $N^{[u]}$ $(u \in \mathcal{F})$ are typical continuous additive functionals of zero energy (We call these continuous additive functionals of zero energy are type-II in this paper.). Another typical continuous additive functionals of zero energy are the following

$$\int_0^t u(X_s)ds \qquad u \in L^2(X,m)$$

(We call these continuous additive functionals of zero energy are type-I in this paper.).

In this paper we shall decide a class of continuous additive functionals of zero energy of the Hunt process $\mathbf{M}$ which are represented by the sum of continuous additive functionals of type-I and type-II. The local representation problem of continuous additive functionals of zero energy is also treated by T. Yamada[2].

2. Result

Let $\{T_t\}$ be a semigroup on $L^2(X,m)$ associated with the Dirichlet space $(\mathcal{E},\mathcal{F})$ and define S_t $(t>0)$ by

$$S_t = \int_0^t T_s ds.$$

Then we can obtain the following lemma in a functional analytic way.

Lemma. *Let $C_t \in L^2(X,m)$ $(t>0)$. Suppose that $\{C_t\}$ satisfies*

$$C_{t+s} = C_t + T_t C_s \qquad (s,t>0),$$
$$\lim_{t\to 0} \|C_t\| = 0.$$

Then there exists a unique $u \in L^2(X,m)$ such that

$$C_t = T_t u - u - S_t u \qquad (t>0).$$

Moreover, it holds that $u \in \mathcal{F}$ if and only if $\int_{0+} (C_t, C_t)/t^2 dt$ is finite.

We can obtain the following theorem concerning the representation of continuous additive functionals of zero energy of the m-symmetric Hunt process $\mathbf{M}$ by using the above lemma.

Theorem. *Let N be a continuous additive functionals of zero energy of the m-symmetric Hunt process $\mathbf{M}$ satisfying that*

$$\int_{0+} \frac{(C_t, C_t)}{t^2} dt < \infty$$

where $C_t(x) = E_x[N_t]$ and $E_x[\cdot]$ denotes the integration with respect to the probability measure P_x. Then there exists a unique $u \in \mathcal{F}$ such that

$$N_t = N_t^{[u]} - \int_0^t u(X_s)ds.$$

Corollary. *Let N be a continuous additive functional of zero energy of the conservative m-symmetric Hunt process $\mathbf{M}$. Then there exists a unique $u \in \mathcal{F}$ such that*

$$N_t = N_t^{[u]} - \int_0^t u(X_s)ds.$$

Remark. The integrable condition in Theorem is necessary in the case that the Hunt process $\mathbf{M}$ is not conservative.

References

[1] M. Fukushima, Dirichlet Forms and Markov Process, North-Holland and Kodansha, Amsterdam and Tokyo, 1980.

[2] T. Yamada, *On some representations concerning the stochastic integrals,* Probab. Math. Statist., **4** (1984), 153–166.