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Topological transitivity and sensitive dependence on initial conditions in discrete dynamical systems

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Abstract Let (X,d) be a complete separable metric space without isolated points. Let $\{f^n\}_{n\in\boldsymbol{N}^*}$ and $\{f^n\}_{n\in\boldsymbol{Z}}$ be the dynamical systems defined by a continuous map $f\colon X\to X$ and a homeomorphism $f\colon X\to X$ respectively, where $\boldsymbol{N}^*=\boldsymbol{N}\cup\{0\}$. We show that if the dynamical system $\{f^n\}_{n\in\boldsymbol{Z}}$ is topologically transitive, both $\{f^n\}_{n\in\boldsymbol{N}^*}$ and $\{f^{-n}\}_{n\in\boldsymbol{N}^*}$ are topologically transitive. Moreover, we show that if the dynamical system $\{f^n\}_{n\in\boldsymbol{Z}}$ is topologically transitive and has sensitive dependence on initial conditions, at least one of the dynamical system $\{f^n\}_{n\in\boldsymbol{N}^*}$ and $\{f^{-n}\}_{n\in\boldsymbol{N}^*}$ have sensitive dependence on initial conditions.

Key words and Phrases. Topological transitivity, sensitive dependence on initial conditions, dynamical systems, f-subinvariant functions, f-invariant functions
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1 Introduction

Let (X,d) be a metric space and f a continuous map from X to X. Let us consider the dynamical system $\{f^n\}_{n\in\boldsymbol{N}^*}$ defined by f. Here f^n denotes the n times composite map of f, and the parameter n runs over the set of all nonnegative integers $\boldsymbol{N}^*=\boldsymbol{N}\cup\{0\}$. The dynamical system $\{f^n\}_{n\in\boldsymbol{N}^*}$ is said to be topologically transitive if for any nonempty open sets U and V there is an integer $k\geq 1$ such that $f^k(U)\cap V$ is nonempty. Also, the dynamical system $\{f^n\}_{n\in\boldsymbol{N}^*}$ is said to have sensitive dependence on initial conditions if there exists a constant $\delta>0$ such that, for any $x\in X$ and for any neighborhood U of x with $\sharp U\geq 2$, there exist a point $y\in U$ and an integer $n\geq 0$ satisfying $d(f^n(x),f^n(y))>\delta$, where $\sharp U$ is the number of elements of the set U.

These two properties play an important role in the theory of chaos. In fact, Devaney [4] defined the dynamical system $\{f^n\}_{n\in\boldsymbol{N}^*}$ to be *chaotic* if it is topologically transitive and has sensitive dependence on initial conditions, and the periodic points of f are dense in X. The pioneering work by Li and Yorke [5] asserts that the dynamical system defined by a continuous map f on an interval is chaotic in this sense if f has three periodic points. Furthermore, J. Banks et al. [2] showed that if a continuous map f is topologically transitive and has dense periodic points, then f has sensitive dependence on initial conditions.

If the given map f is a homeomorphism, these two properties are defined also for dynamical system $\{f^n\}_{n\in\boldsymbol{Z}}$, where the parameter n runs over the set of all integers $\boldsymbol{Z}$. However, the relationship between these properties for $\{f^n\}_{n\in\boldsymbol{N}^*}$ and $\{f^n\}_{n\in\boldsymbol{Z}}$ has not been well understood. The aim of this paper is to clarify this problem. We show that if the dynamical system $\{f^n\}_{n\in\boldsymbol{Z}}$ defined by a homeomorphism f is topologically transitive, then both $\{f^n\}_{n\in\boldsymbol{N}^*}$ and $\{f^{-n}\}_{n\in\boldsymbol{N}^*}$ are topologically transitive (Theorem 4.1). Moreover, we show that if the dynamical system $\{f^n\}_{n\in\boldsymbol{Z}}$ is topologically transitive and has sensitive dependence on initial conditions, then at least one of the dynamical systems $\{f^n\}_{n\in\boldsymbol{N}^*}$ and $\{f^{-n}\}_{n\in\boldsymbol{N}^*}$ has sensitive dependence on initial conditions (Theorem 4.2). The idea of their proofs is to give new characterizations of topological transitivity and sensitive dependence on initial conditions, which constitutes the main part of the paper.

The paper is organized as follows. In section 2, we introduce new characterizations for topological transitivity of $\{f^n\}_{n\in\boldsymbol{N}^*}$ and $\{f^n\}_{n\in\boldsymbol{Z}}$. They are based on the use of the class of upper semicontinuous and f-subinvariant (resp. f-invariant) functions, which we denote by Γ_s and Γ respectively (Definitions 2.2 and 2.4). We prove that the dynamical system $\{f^n\}_{n\in\boldsymbol{N}^*}$ is topologically transitive if and only if any $\gamma\in\Gamma_s$ has a minimum and the set $M_\gamma := \{x\in X \mid \gamma(x) = \min\gamma\}$ is dense in X, where $\min\gamma = \min_{y\in X}\gamma(y)$ (Theorem 2.2). We can also prove a characterization of the topological transitivity of the dynamical system $\{f^n\}_{n\in\boldsymbol{Z}}$ by using the class of functions Γ (Theorem 2.4).

In section 3, we discuss the sensitive dependence on initial conditions for $\{f^n\}_{n\in\boldsymbol{N}^*}$ and $\{f^{-n}\}_{n\in\boldsymbol{N}^*}$. We construct some functions $r_+\in\Gamma_s$ and $r\in\Gamma$, and use them to prove that $\delta>0$ in the definition of sensitive dependence on initial conditions can be taken to be dependent on $x\in X$ if both $\{f^n\}_{n\in\boldsymbol{N}^*}$ and $\{f^n\}_{n\in\boldsymbol{Z}}$ have sensitive dependence on initial conditions (Theorem 3.1, Theorem 3.2). In section 4, we prove the main theorems (Theorem 4.1 and Theorem 4.2). We hope to construct an example of a homeomorphism f on a complete separable metric space (X,d) without isolated points such that dynamical systems $\{f^n\}_{n\in\boldsymbol{N}^*}$ and $\{f^{-n}\}_{n\in\boldsymbol{Z}}$ are topologically transitive, but that either $\{f^n\}_{n\in\boldsymbol{N}^*}$ or $\{f^{-n}\}_{n\in\boldsymbol{N}^*}$ has sensitive dependence on initial conditions. It is a future problem to construct such examples.

2 Topological transitivity in dynamical systems

(1) Topological transitivity in discrete dynamical systems with the parameter $\boldsymbol{N}^* = \boldsymbol{N} \cup \{0\}$

Let (X,d) be a metric space and f a continuous map from X to X. In this section we consider the dynamical system $\{f^n\}_{n\in\boldsymbol{N}^*}$ defined by a continuous map f in (X,d), where the parameter runs over the set of all nonnegative integers $N^* = \boldsymbol{N} \cup \{0\}$. For $x \in X$, the set $\{x, f(x), f^2(x), \cdots\}$ is called a *positive orbit of f* and denoted by $O_+(f;x)$, furthermore the set $\{f(x), f^2(x), \cdots\}$ is denoted by $O'_+(f;x)$. Also, we denote by D_+ the set of points $x \in X$ for which $O_+(f;x)$ is dense in X.

Definition 2.1. Let (X,d) be a metric space and f a continuous map from X to X. The dynamical system $\{f^n\}_{n\in\boldsymbol{N}^*}$ is said to be *topologically transitive* if for any nonempty open sets U and V there is an integer $k \geq 1$ such that $f^k(U) \cap V$ is nonempty.

Theorem 2.1 ([1])**.** *Let (X,d) be a complete separable metric space and f a continuous map from X to X. The following three conditions are mutually equivalent.*

(1) *The dynamical system $\{f^n\}_{n\in\boldsymbol{N}^*}$ is topologically transitive.*
(2) *There is a point $x \in X$ such that the orbit $O'_+(f;x)$ is dense in X.*
(3) *The set $\{x \in X \mid \overline{O'_+(f;x)} = X\}$ is dense in X.*

In particular if X has no isolated points, then the above three conditions are also mutually equivalent to the following (4) or (5).

(4) *There is a point $x \in X$ such that the positive orbit $O_+(f;x)$ is dense in X.*
(5) *The set $\{x \in X \mid \overline{O_+(f;x)} = X\}$ is dense in X.*

If we assume only that X is a topological space, Theorem 2.1([1]) is proved by using Baire's category theorem ([6]), axiom of separation, second axiom of countability. Let X be a topological space and X_n $(n \in \boldsymbol{N})$ an arbitrary sequence of closed sets of X. The space X is called *Baire space* or X satisfies Baire's category theorem if at least one of the X_n has an inner point provided that $\bigcup_{n=1}^{\infty} X_n$ has an inner point. If (X,d) is a complete metric space, X is a Baire space and Baire's category theorem holds in X. Moreover if X is a Baire space and Y is an open subset of X, then Y is also a Baire space as a subspace ([3]). Theorem 2.1 is well known and characterizes the topological transitivity by the existence of an $x \in X$ such that the orbit $O_+(f;x)$ is dense in X. Theorem 2.2 below gives a new characterization of topological transitivity in dynamical systems.

Definition 2.2. Let (X,d) be a metric space, γ a function from X to $[0,\infty)$, and f a continuous map from X to X. γ is said to be *f-subinvariant* (resp. *f-superinvariant*) if

$\gamma(x) \leq \gamma(f(x))$ (resp. $\gamma(x) \geq \gamma(f(x))$) holds for any $x \in X$. The class of functions γ is defined as follows:

$$\Gamma_s \underset{\text{def}}{=} \{\gamma : X \to [0,\infty) \mid \gamma \text{ is } f\text{-subinvariant, upper semicontinuous}\}$$

By using this class of functions, we give a necessary and sufficient condition for the dynamical system $\{f^n\}_{n\in \boldsymbol{N}^*}$ to be topologically transitive.

Theorem 2.2. *Let (X,d) be a complete separable metric space without isolated points and f a continuous map from X to X. Then the following two conditions are mutually equivalent.*

(1) *The dynamical system $\{f^n\}_{n\in \boldsymbol{N}^*}$ is topologically transitive.*
(2) *For any $\gamma \in \Gamma_s$, it has a minimum and the set $M_\gamma := \{x \in X \mid \gamma(x) = \min \gamma\}$ is dense in X, where $\min \gamma = \min_{y\in X} \gamma(y)$.*

Moreover when the condition (1) or (2) holds, we find $D_+ = \bigcap_{\gamma\in\Gamma_s} M_\gamma$.

To show Theorem 2.2, we prepare three Lemmas: Lemma 2.1,Lemma 2.2,Lemma 2.3.

Lemma 2.1. *Let (X,d) be a complete metric space without isolated points. Assume that the dynamical system $\{f^n\}_{n\in \boldsymbol{N}^*}$ is topologically transitive. Then for any $\gamma \in \Gamma_s$, it has a minimum and the set $M_\gamma := \{x \in X \mid \gamma(x) = \min \gamma\}$ is dense in X. Furthermore the inclusion relation $D_+ \subset M_\gamma$ holds for any $\gamma \in \Gamma_s$.*

Proof. Set $m = \inf_{x\in X} \gamma(x)$. Obviously, $m < \infty$ and $m \geq 0$. Set

$$X_n = \left\{ x \in X \;\middle|\; \gamma(x) \geq m + \frac{1}{n} \right\} \qquad (n \geq 1).$$

X_n is a closed set. We assume that γ does not have a minimum. It is clear $X = \bigcup_{n=1}^{\infty} X_n$. From Baire's category theorem, there exists a nonempty open set O such that $O \subset X_{n_0}$ for some $n_0 \in \boldsymbol{N}$. Since there is an $x_0 \in X$ such that the orbit $O_+(f;x_0)$ is dense in X from Theorem 2.1, there is $l \geq 0$ satisfying $f^l(x_0) \in O$. Also, we have

$$m + \frac{1}{n_0} \leq \gamma(f^l(x_0)) \leq \gamma(f^k(x_0)) \qquad \text{for } k \geq l.$$

Because X has no isolated points, $\{f^l(x_0), f^{l+1}(x_0), \cdots\}$ is dense subset of X. From this and the upper semicontinuity of γ,

$$\gamma(x) \geq m + \frac{1}{n_0} \qquad (x \in X).$$

So we have $m \geq m + \frac{1}{n_0}$. This is a contradiction and therefore γ has a minimum. To show that M_γ is dense in X, it is sufficient to prove the inclusion relation $D_+ \subset M_\gamma$. We assume that there exists $x_1 \in D_+$ such that $x_1 \notin M_\gamma$. It is clear that $m < \gamma(x_1) \leq \gamma(f(x_1)) \leq \gamma(f^2(x_1)) \leq \cdots$. The denseness of $O_+(f;x_1)$ and the upper semicontinuity of γ yield $\gamma(x) \geq \gamma(x_1) > m$ for any $x \in X$. This is a contradiction to the fact that m is minimum of γ. Therefore we have that $D_+ \subset M_\gamma$. □

Lemma 2.2. *Let (X,d) be a complete separable metric space, f a continuous map from X to X, and $\{U_n\}$ a countable basis of X which is nonempty for $n \geq 1$. We define a set $F_m^{(n)}$ and a map $\kappa : X \to [0,\infty)$ as follows:*

$$\begin{aligned} F_m^{(n)} &:= f^{-m}(U_n^c) \qquad (m \geq 0\ ,\ n \geq 1), \\ \kappa(x) &:= \sum_{n=1}^{\infty} \frac{1}{3^n} \cdot \prod_{m=0}^{\infty} \mathbf{1}_{F_m^{(n)}}(x) \qquad (x \in X). \end{aligned}$$

Then $\kappa \in \Gamma_s$ and $M_\kappa = D_+$.

Proof. Since $F_m^{(n)}$ $(m \geq 0\ ,\ n \geq 1)$ is a closed set of X, its indicator function $\mathbf{1}_{F_m^{(n)}}(x) : X \to [0,1]$ is upper semicontinuous. Also

$$\prod_{m=0}^{\infty} \mathbf{1}_{F_m^{(n)}}(x) = \lim_{k\to\infty} \prod_{m=0}^{k} \mathbf{1}_{F_m^{(n)}}(x)$$

is upper semicontinuous function. Set a function $g_n(x)$ as follows:

$$g_n(x) := \prod_{m=0}^{\infty} \mathbf{1}_{F_m^{(n)}}(x).$$

We will show $\kappa(x)$ is upper semicontinuous. For any $x_0 \in X, \varepsilon > 0$, there is an $N \in \boldsymbol{N}$ such that $\sum_{n=N+1}^{\infty} \frac{1}{3^n} < \varepsilon$. Hence we have

$$\kappa(x) < \sum_{n=1}^{N} \frac{1}{3^n} \cdot g_n(x) + \varepsilon \qquad \text{for any } \ x \in X.$$

Because $\sum_{n=1}^{N} \frac{1}{3^n} \cdot g_n(x)$ is upper semicontinuous at x_0, there exists an open neighborhood U of x_0 satisfying the following inequality.

$$\sum_{n=1}^{N} \frac{1}{3^n} \cdot g_n(x) < \sum_{n=1}^{N} \frac{1}{3^n} \cdot g_n(x_0) + \varepsilon \ \text{ for } \ x \in U.$$

From this,

$$\begin{aligned} \kappa(x) &< \sum_{n=1}^{N} \frac{1}{3^n} \cdot g_n(x) + \varepsilon \\ &\leq \kappa(x_0) + 2\varepsilon \quad (x \in U). \end{aligned}$$

So we see that κ is upper semicontinuous. Obviously $\mathbf{1}_{F_m^{(n)}}(f(x)) = \mathbf{1}_{F_{m+1}^{(n)}}(x)$, thus

$$\begin{aligned}\kappa(f(x)) &= \sum_{n=1}^{\infty} \frac{1}{3^n} \cdot \prod_{m=0}^{\infty} \mathbf{1}_{F_m^{(n)}}(f(x)) \\ &\geq \sum_{n=1}^{\infty} \frac{1}{3^n} \cdot \prod_{m=0}^{\infty} \mathbf{1}_{F_m^{(n)}}(x) = \kappa(x).\end{aligned}$$

Therefore we have $\kappa \in \Gamma_s$. Moreover, the following equivalent relations hold.

$$\begin{aligned} x \in M_\kappa &\iff \kappa(x) = 0 \\ &\iff \sum_{n=1}^{\infty} \frac{1}{3^n} \cdot \prod_{m=0}^{\infty} \mathbf{1}_{F_m^{(n)}}(x) = 0 \\ &\iff \prod_{m=0}^{\infty} \mathbf{1}_{F_m^{(n)}}(x) = 0 \qquad \text{for } \ n \geq 1 \\ &\iff {}^{\forall} n \geq 1 \ , \ {}^{\exists} m = m(n) \geq 0 \ , \ \text{ s.t. } \ f^m(x) \in U_n \\ &\iff \overline{O_+(f;x)} = X \\ &\iff x \in D_+ \end{aligned}$$

Consequently, $M_\kappa = D_+$. □

Lemma 2.3. *Let (X,d) be a complete separable metric space without isolated points and f a continuous map from X to X. Assume that for any $\gamma \in \Gamma_s$, it has a minimum and the set $M_\gamma := \{x \in X \mid \gamma(x) = \min \gamma\}$ is dense in X. Then the dynamical system $\{f^n\}_{n\in \boldsymbol{N}^*}$ is topologically transitive.*

Proof. Let $\kappa : X \to [0,\infty)$ be a function defined on Lemma 2.2. Since we know $\kappa \in \Gamma_s$ from Lemma 2.2, κ has a minimum and the set $M_\kappa := \{x \in X \mid \kappa(x) = \min \kappa\}$ is dense in X. Put $\min \kappa = a$. Clearly $a \geq 0$. We assume $a > 0$. There is a unique $\alpha_1, \alpha_2, \cdots \in \{0,1\}$ such that $a = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}$. Put $l = \min\{n \in \boldsymbol{N} \mid \alpha_n = 1\}$. M_κ is contained by U_l^c, which implies $U_l \cap M_\kappa = \emptyset$. This contradicts to the denseness of M_κ in X. Therefore we have $a = 0$. From Lemma 2.2, $M_\kappa = D_+$. Since M_κ is dense in X, D_+ is dense in X. It follows from Theorem 2.1 that the dynamical system $\{f^n\}_{n\in \boldsymbol{N}^*}$ is topologically transitive. □

Proof of Theorem 2.2. By Lemma 2.1, (1) implies (2) and the inclusion relation $D_+ \subset \bigcap_{\gamma\in\Gamma_s} M_\gamma$ holds. Lemma 2.3 shows that (2) implies (1). Furthermore, since $\kappa \in \Gamma_s$ and $M_\kappa = D_+$, we have the inclusion $\bigcap_{\gamma\in\Gamma_s} M_\gamma \subset D_+$. □

(2) Topological transitivity in discrete dynamical system with the parameter $\boldsymbol{Z}$

Let (X,d) be a metric space and f a homeomorphism from X to X. We consider the case when the parameter runs over the set of all integers $\boldsymbol{Z}$. The set $\{\cdots f^{-2}(x), f^{-1}(x), x\}$ is called a *negative orbit of f*, and is denoted by $O_{-}(f;x)$.

Also $\{\cdots, f^{-2}(x), f^{-1}(x), x, f(x), f^{2}(x)\cdots\}$ is called an *orbit of f* and is denoted by $O(f;x)$. Obviously $O_{+}(f^{-1};x) = O_{-}(f;x)$. We denote by D and D_{-} the sets of points $x \in X$ for which $O(f;x)$ and $O_{-}(f;x)$ are dense in X respectively.

Definition 2.3. Let (X,d) be a metric space and f a homeomorphism from X to X. The dynamical system $\{f^n\}_{n\in \boldsymbol{Z}}$ is *topologically transitive* if for any nonempty open sets U and V there exists an integer $n \in \boldsymbol{Z}$ such that $f^n(U) \cap V$ is nonempty.

The following Theorem 2.3 is well known.

Theorem 2.3([1]). *Let (X,d) be a complete separable metric space and f a homeomorphism from X to X. Then the following three conditions are mutually equivalent.*

(1) *The dynamical system $\{f^n\}_{n\in \boldsymbol{Z}}$ is topologically transitive.*
(2) *There is a point $x \in X$ such that the orbit $O(f;x)$ is dense in X.*
(3) *The set $\{x \in X \mid \overline{O(f;x)} = X\}$ is dense in X.*

Definition 2.4. Let γ be a function from X to $[0,\infty)$. γ is said to be *f-invariant* if $\gamma(x) = \gamma(f(x))$ holds for any $x \in X$. We define the class Γ of functions γ as follows:

$$\Gamma \underset{\text{def}}{=} \{\gamma : X \to [0,\infty) \mid \gamma \text{ is } f\text{-invariant, upper semicontinuous}\}$$

Theorem 2.4. *Let (X,d) be a complete separable metric space and f a homeomorphism from X to X. Then the following two conditions are mutually equivalent.*

(1) *The dynamical system $\{f^n\}_{n\in \boldsymbol{Z}}$ is topologically transitive.*
(2) *For any $\gamma \in \Gamma$, it has a minimum and the set $M_\gamma := \{x \in X \mid \gamma(x) = \min \gamma\}$ is dense in X.*

Moreover if the condition (1) or (2) holds, then $D = \bigcap_{\gamma\in\Gamma} M_\gamma$ holds.

To show Theorem 2.4, we prepare three lemmas: Lemma 2.4, Lemma 2.5, Lemma 2.6.

Lemma 2.4. *Let (X,d) be a complete separable metric space and f a homeomorphism from X to X. Suppose that the dynamical system $\{f^n\}_{n\in\mathbf{Z}}$ is topologically transitive. Then for any $\gamma \in \Gamma$, it has a minimum and the set $M_\gamma := \{x \in X \mid \gamma(x) = \min\gamma\}$ is dense in X. Also, the inclusion $D \subset M_\gamma$ holds.*

Lemma 2.5. *Let (X,d) be a complete separable metric space, f a homeomorphism from X to X and $\{U_n\}$ a countable basis of X which is nonempty for any $n \geq 1$. We define a set $F_m^{(n)}$ and a map $\kappa : X \to [0,\infty)$ as follows:*

$$\begin{aligned} F_m^{(n)} &:= f^{-m}(U_n^c) \qquad (m \in \boldsymbol{Z}, n \in \boldsymbol{N}), \\ \kappa(x) &:= \sum_{n=1}^{\infty} \frac{1}{3^n} \cdot \prod_{m=-\infty}^{\infty} \mathbf{1}_{F_m^{(n)}}(x) \qquad (x \in X). \end{aligned}$$

Then $\kappa \in \Gamma$ and $M_\kappa = D$.

Lemma 2.6. *Let (X,d) be a complete separable metric space and f a homeomorphism from X to X. Suppose that for any $\gamma \in \Gamma$, it has a minimum and the set $M_\gamma := \{x \in X \mid \gamma(x) = \min\gamma\}$ is dense in X. Then the dynamical system $\{f^n\}_{n\in\mathbf{Z}}$ is topologically transitive.*

We can show Lemma 2.4, Lemma 2.5, Lemma 2.6 in the same way as Lemma 2.1, Lemma 2.2, Lemma 2.3.

Proof of Theorem 2.4. By Lemma 2.4, (1) implies (2) and the inclusion $D_+ \subset M_\gamma$ holds. By Lemma 2.6, (2) implies (1). Furthermore, since $\kappa \in \Gamma$ and $M_\kappa = D$, the inclusion $D \subset \bigcap_{\gamma\in\Gamma} M_\gamma$ holds. □

3 Sensitive dependence on initial conditions in dynamical systems

(1) Sensitive dependence on initial conditions with the parameter $\boldsymbol{N}^* = \boldsymbol{N} \cup \{0\}$

Let (X,d) be a metric space. Sensitive dependence on initial conditions is the property characterized by the metric d. In Theorem 3.1 we show that the constant $\delta > 0$ in the definition can be taken to be dependent on $x \in X$. First we introduce the definition of sensitive dependence on initial conditions.

Definition 3.1. Let (X,d) be a metric space, f a continuous map from X to X, and $\{f^n\}_{n\in \boldsymbol{N}^*}$ the dynamical system defined by the continuous map f. The dynamical system $\{f^n\}_{n\in \boldsymbol{N}^*}$ is said to have *sensitive dependence on initial conditions* if there exists a constant $\delta > 0$ such that for any $x \in X$ and any open neighborhood U of x with $\sharp U \geq 2$, there exist a point $y \in U$ and a nonnegative integer $n \geq 0$ satisfying $d(f^n(x), f^n(y)) > \delta$.

In what follows, we use the notation $a \wedge b$ and $a \vee b$ to denote $\min\{a,b\}$ and $\max\{a,b\}$ respectively.

Definition 3.2. For $x \in X$ and an $\varepsilon > 0$, the numbers $r_+(\varepsilon,x)$ and $r_+(x)$ are defined as follows:

$$\begin{aligned} r_+(\varepsilon,x) &\underset{\text{def}}{=} \sup_{n\geq 0} \sup_{y,y'\in B_\varepsilon(x)} d(f^n(y), d^n(y')) \wedge 1 \\ r_+(x) &\underset{\text{def}}{=} \lim_{\varepsilon\downarrow 0} r_+(\varepsilon,x) \\ &= \lim_{\varepsilon\downarrow 0}\left(\sup_{n\geq 0} \sup_{y,y'\in B_\varepsilon(x)} d(f^n(y), d^n(y')) \wedge 1 \right), \end{aligned}$$

where $B_\varepsilon(x)$ is the open ball at $x \in X$ with radius $\varepsilon > 0$.

Remark. If $x \in X$ is an isolated point, there is an $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(x) = \{x\}$. Hence $r_+(\varepsilon,x) = 0$ for any $0 < \varepsilon \leq \varepsilon_0$, so $r_+(x) = 0$.

Theorem 3.1. *Let (X,d) be a metric space and f a continuous map from X to X. Assume that the dynamical system $\{f^n\}_{n\in \boldsymbol{N}^*}$ is topologically transitive. Then the following two conditions are mutually equivalent.*

(1) *The dynamical system $\{f^n\}_{n\in \boldsymbol{N}^*}$ has sensitive dependence on initial conditions.*
(2) *There exists $\delta = \delta(x) > 0$ for any $x \in X$ satisfying the following conditions: for any open neighborhood U of x with $\sharp U \geq 2$, there are a nonnegative integer $n \geq 0$ and a point $y \in U$ such that $d(f^n(x), f^n(y)) > \delta(x)$.*

To prove this theorem, we give some Lemmas: Lemma 3.1, Lemma 3.2. Lemma 3.1 and Lemma 3.2 show that r_+ is the function which characterizes sensitive dependence on initial conditions.

Lemma 3.1. *Let (X,d) be a metric space and f a continuous map from X to X. Then r_+ is upper semicontinuous.*

Proof. For any given $x_0 \in X$ and any $\varepsilon > 0$, there exists $\varepsilon_0 > 0$ such that $r_+(\varepsilon_0, x_0) \leq r_+(x_0) + \varepsilon$. For any $x \in B_{\varepsilon_0}(x_0)$ there is an $\varepsilon_1 > 0$ satisfying $B_{\varepsilon_1}(x) \subset B_{\varepsilon_0}(x_0)$, hence, r_+ is upper semicontinuous. □

Lamma 3.2. *Let (X,d) be a metric space and f a continuous map from X to X. Then r_+ is f-subinvariant. Furthermore, if f is an open map, r_+ is f-invariant.*

Proof. For any $x \in X$ and any $\varepsilon > 0$, $f^{-1}(B_\varepsilon(f(x)))$ is an open neighborhood of x. So there exists an $\varepsilon_1 > 0$ satisfying $B_{\varepsilon_1}(x) \subset f^{-1}(B_\varepsilon(f(x)))$. The following inequalities hold for any $\varepsilon' > 0$ with $0 < \varepsilon' \leq \varepsilon_1$.

$$\begin{aligned}
r_+(x) &\leq r_+(\varepsilon', x) \\
&\leq 2\varepsilon' \vee \left(\sup_{n\geq 1} \sup_{y,y' \in B_{\varepsilon'}(x)} d(f^n(y), f^n(y')) \wedge 1 \right) \\
&\leq 2\varepsilon' \vee \left(\sup_{n\geq 1} \sup_{y,y' \in f^{-1}(B_\varepsilon(f(x)))} d(f^n(y), f^n(y')) \wedge 1 \right) \\
&\leq 2\varepsilon' \vee \left(\sup_{n\geq 0} \sup_{y,y' \in B_\varepsilon(f(x))} d(f^n(y), f^n(y')) \wedge 1 \right) \\
&= 2\varepsilon' \vee r_+(\varepsilon, f(x)).
\end{aligned}$$

Letting $\varepsilon' \downarrow 0$, we have $r_+(x) \leq r_+(\varepsilon, f(x))$. Next letting $\varepsilon \downarrow 0$, we have $r_+(x) \leq r_+(f(x))$. Thus r_+ is f-subinvariant. Suppose that f is an open map. Then, $f(B_\varepsilon(x))$ is an open neighborhood of $f(x)$ for any $x \in X$ and $\varepsilon > 0$. Since the relation

$$\sup_{y,y' \in f(B_\varepsilon(x))} d(f^n(y), f^n(y')) \wedge 1 = \sup_{z,z' \in B_\varepsilon(x)} d(f^{n+1}(z), f^{n+1}(z')) \wedge 1$$

holds for arbitrary $n \geq 0$, we have

$$\begin{aligned}
\sup_{n\geq 0} \sup_{y,y' \in f(B_\varepsilon(x))} d(f^n(y), f^n(y')) \wedge 1 &= \sup_{n\geq 1} \sup_{z,z' \in B_\varepsilon(x)} d(f^n(z), f^n(z')) \wedge 1 \\
&\leq \sup_{n\geq 0} \sup_{z,z' \in B_\varepsilon(x)} d(f^n(z), f^n(z')) \wedge 1.
\end{aligned}$$

Therefore

$$\begin{aligned}
r_+(f(x)) &\leq \sup_{n\geq 0} \sup_{y,y' \in f(B_\varepsilon(x))} d(f^n(y), f^n(y')) \wedge 1 \\
&\leq r_+(\varepsilon, x).
\end{aligned}$$

Let $\varepsilon \downarrow 0$. Then r_+ is f-invariant. □

Proof of Theorem 3.1. From triangle inequality, the inequalities

$$\begin{aligned}\sup_{y\in B_\varepsilon(x)} d(f^n(x),f^n(y))\wedge 1 &\le \sup_{y,y'\in B_\varepsilon(x)} d(f^n(y),f^n(y'))\wedge 1\\ &\le 2\sup_{y\in B_\varepsilon(x)} d(f^n(x),f^n(y))\wedge 1\end{aligned}$$

hold for any $n \ge 0$. From these inequalities, we can see that the conditions (1) and (2) of Theorem 3.1 are equivalent to the following $(1')$ and $(2')$ respectively.

$(1')$ There exists a constant $\delta' > 0$ satisfying $r_+(\varepsilon,x) > \delta'$ for any $x \in X$ and any $\varepsilon > 0$.

$(2')$ For any $x \in X$, there exists a constant $\delta'(x) > 0$ satisfying $r_+(\varepsilon,x) > \delta'(x)$ for any $\varepsilon > 0$.

Furthermore, the conditions $(1')$ and $(2')$ are equivalent to the following $(1'')$ and $(2'')$ respectively.

$(1'')$ There exists a $\delta'' > 0$ satisfying $r_+(x) > \delta''$ for any $x \in X$.

$(2'')$ $r_+(x) > 0$ for any $x \in X$.

Lemma 3.1 and Lemma 3.2 imply that r_+ is upper semicontinuous and f-subinvariant and hence $r_+ \in \Gamma_s$. Since r_+ has a minimum from Lemma 2.1, we see that $(2'')$ implies $(1'')$. The converse is trivial. □

(2) Sensitive dependence on initial conditions with the parameter $\boldsymbol{Z}$

We discuss sensitive dependence on initial conditions when the parameter of the dynamical system $\{f^n\}_{n\in\boldsymbol{Z}}$ runs over $\boldsymbol{Z}$. Theorem 3.2 below says that Theorem 3.1 also holds in the case when f is a homeomorphism. First we introduce the definition of sensitive dependence on initial conditions in the case when the parameter runs over $\boldsymbol{Z}$.

Definition 3.3. Let (X,d) be a metric space, f a homeomorphism from X to X, and $\{f^n\}_{n\in\boldsymbol{Z}}$ the dynamical system defined by f. The dynamical system $\{f^n\}_{n\in\boldsymbol{Z}}$ is said to have *sensitive dependence on initial conditions* if there exists a constant $\delta > 0$ such that for any $x \in X$ and any open neighborhood U of x with $\sharp U \ge 2$, there exist a point $y \in U$ and an integer $n \in \boldsymbol{Z}$ satisfying $d(f^n(x),f^n(y)) > \delta$.

Definition 3.4. For $x \in X$ and an $\varepsilon > 0$, the numbers $r(\varepsilon,x)$ and $r(x)$ are defined as follows:

$$\begin{aligned} r(\varepsilon,x) &\underset{\text{def}}{=} \sup_{n\in\mathbf{Z}} \sup_{y,y'\in B_\varepsilon(x)} d(f^n(y),f^n(y'))\wedge 1, \\ r(x) &\underset{\text{def}}{=} \lim_{\varepsilon\downarrow 0} r(\varepsilon,x) \\ &= \lim_{\varepsilon\downarrow 0}\left(\sup_{n\in\mathbf{Z}} \sup_{y,y'\in B_\varepsilon(x)} d(f^n(y),f^n(y'))\wedge 1\right). \end{aligned}$$

Theorem 3.2. *Let (X,d) be a metric space, f a homeomorphism from X to X, and $\{f^n\}_{n\in\mathbf{Z}}$ the dynamical system defined by f. Suppose that $\{f^n\}_{n\in\mathbf{Z}}$ is topologically transitive. Then the following two conditions are mutually equivalent.*

(1) *The dynamical system $\{f^n\}_{n\in\mathbf{Z}}$ has sensitive dependence on initial conditions.*
(2) *For any $x\in X$ there exists a constant $\delta=\delta(x)>0$ satisfying the following conditions: for any open neighborhood U of x with $\sharp U\geq 2$, there are an integer $n\in\mathbf{Z}$ and a point $y\in U$ such that $d(f^n(x),f^n(y))>\delta(x)$.*

To show Theorem 3.2, we will give some Lemmas: Lemma 3.3, Lemma 3.4.

Lemma 3.3. *Let (X,d) be a metric space and f a homeomorphism from X to X. Then $r(x)$ is upper semicontinuous.*

Lemma 3.3 is shown in the same way as Lemma 3.1.

Lemma 3.4. *Let (X,d) be a metric space and f a homeomorphism from X to X. Then $r(x)$ is f-invariant.*

Proof. Since f is a homeomorphism, $f^{-1}(B_\varepsilon(f(x)))$ is an open neighborhood of x for any $x\in X$ and an $\varepsilon>0$. There exists an $\varepsilon_1>0$ satisfying $B_{\varepsilon_1}(x)\subset f^{-1}(B_\varepsilon(f(x)))$. Then

$$\begin{aligned} r(x) &\leq r(\varepsilon_1,x) \\ &\leq \sup_{n\in\mathbf{Z}} \sup_{y,y'\in f^{-1}(B_\varepsilon(f(x)))} d(f^n(y),f^n(y'))\wedge 1 \\ &= \sup_{n\in\mathbf{Z}} \sup_{z,z'\in B_\varepsilon(f(x))} d(f^n(y),f^n(y'))\wedge 1 \\ &= r(\varepsilon,f(x)). \end{aligned}$$

Letting $\varepsilon\downarrow 0$, we see that $r(x)$ is f-subinvariant. We show the converse inequality. For any $x\in X$ and any $\varepsilon>0$ given, $f(B_\varepsilon(x))$ is an open neighborhood of $f(x)$ and there is

$\varepsilon' > 0$ satisfying $B_{\varepsilon'}(f(x)) \subset f(B_\varepsilon(x))$. Then

$$\begin{aligned} r(f(x)) &\leq r(\varepsilon', f(x)) \\ &\leq \sup_{n\in\mathbf{Z}} \sup_{y,y'\in B_{\varepsilon'}(f(B_\varepsilon(x)))} d(f^n(y), f^n(y')) \wedge 1 \\ &= \sup_{n\in\mathbf{Z}} \sup_{z,z'\in B_\varepsilon(x)} d(f^n(f^{-1}(z)), f^n(f^{-1}(z'))) \wedge 1 \\ &= r(\varepsilon, x). \end{aligned}$$

Hence $r(x)$ is f-invariant. □

Proof of Theorem 3.2. From triangle inequality, the inequalities

$$\begin{aligned} \sup_{y\in B_\varepsilon(x)} d(f^n(x), f^n(y)) \wedge 1 &\leq \sup_{y,y'\in B_\varepsilon(x)} d(f^n(y), f^n(y')) \wedge 1 \\ &\leq 2 \sup_{y\in B_\varepsilon(x)} d(f^n(x), f^n(y)) \wedge 1 \end{aligned}$$

hold for $n \in \mathbf{Z}$. By these inequalities, we see in the same way as in the proof of Theorem 3.1 that conditions (1) and (2) of Theorem 3.2 are equivalent to the following conditions $(1'')$ and $(2'')$ respectively.

$(1'')$ There is a $\delta'' > 0$ such that $r(x) > \delta''$ for any $x \in X$.
$(2'')$ $r(x) > 0$ for any $x \in X$.

Lemma 3.3 and Lemma 3.4 yield $r \in \Gamma$. Since r has a minimum from Lemma 2.4, $(2'')$ implies $(1'')$. The converse is trivial. □

4 Applications of new characterizations to topological transitivity

In this final section, we apply the new characterization of topological transitivity in the previous sections to prove the main results.

Theorem 4.1. *Let (X,d) be a complete separable metric space without isolated points and f a homeomorphism from X to X. Suppose that the dynamical system $\{f^n\}_{n\in\mathbf{Z}}$ defined by f is topologically transitive. Then both $\{f^n\}_{n\in\mathbf{N}^*}$ and $\{f^{-n}\}_{n\in\mathbf{N}^*}$ are topologically transitive. Furthermore, $D_+ \cap D_-$ is dense in X.*

To prove this theorem, we need the following Lemma 4.1 and Lemma 4.2.

Lemma 4.1. *Let (X,d) be a complete separable metric space without isolated points, f a homeomorphism from X to X, and γ_+,γ_- upper semicontinuous functions from X to $[0,\infty)$. Suppose that the dynamical system $\{f^n\}_{n\in\boldsymbol{Z}}$ is topologically transitive and that γ_+ is f-subinvariant, γ_- is f-superinvariant. If* $\inf\gamma_+ = \inf\gamma_- = 0$*, then $\gamma_+ \wedge \gamma_-$ has a minimum* 0.

Proof. $\inf(\gamma_+ \wedge \gamma_-) = 0$ is trivial from $\inf\gamma_+ = \inf\gamma_- = 0$. Let $l \in \boldsymbol{N}$. We set

$$A_l := \left\{ x \in X \;\middle|\; \gamma_+(x) \wedge \gamma_-(x) \geq \inf(\gamma_+ \wedge \gamma_-) + \frac{1}{l} \right\}.$$

A_l is closed set. Suppose that $\gamma_+ \wedge \gamma_-$ does not have a minimum. It is clear that X is represented as a countable union of A_l. From Baire's category theorem, there are a nonempty open set O and $l_0 \in \boldsymbol{N}$ with $O \subset A_{l_0}$. We now put $A := A_{l_0}$ and $a = \frac{1}{l_0}$. Since the dynamical system $\{f^n\}_{n\in\boldsymbol{Z}}$ is topologically transitive, Theorem 2.3 yields that D is dense in X. Let $x_0 \in D$. Since X has no isolated points, we can see that the set $\{n \in \boldsymbol{Z} \mid f^n(x_0) \in A\}$ is infinite by $\sharp\{n \in \boldsymbol{Z} \mid f^n(x_0) \in O\} = \infty$ and the inclusion $A \supset O$.

When the set $\{n \in \boldsymbol{Z} \mid f^n(x_0) \in A\}$ is unbounded from above, there is an $m \in \{n \in \boldsymbol{Z} \mid f^n(x_0) \in A\}$ satisfying $n \leq m$ for any integer $n \in \boldsymbol{Z}$. Thus $\gamma_-(f^n(x_0)) \geq a$ for $n \in \boldsymbol{Z}$. $a \leq \overline{\lim}\gamma_-(y_n) \leq \gamma_-(y)$ holds for any $y \in X$ because of the upper semicontinuity of γ and the denseness of $O(f;x_0)$. Therefore we found $a \leq \gamma_-(y)$ for any $y \in X$. This is a contradiction to $\inf\gamma_- = 0$. When the set $\{n \in \boldsymbol{Z} \mid f^n(x_0) \in A\}$ is unbounded from below, there is an $m \in \{n \in \boldsymbol{Z} \mid f^n(x_0) \in A\}$ satisfying $n \geq m$ for any integer $n \in \boldsymbol{Z}$, hence, we can see $a \leq \gamma_+(f^m(x_0)) \leq \gamma_+(f^n(x_0))$. We have that $\gamma_+(x) \geq a$ holds for any $x \in X$ in the same way. This contradicts to $\inf\gamma_+ = 0$. Therefore $\gamma_+ \wedge \gamma_-$ has a minimum 0. □

Lemma 4.2. *Let X be a complete separable metric space without isolated points and f a continuous map from X to X. Moreover let γ_+ , γ_- be upper semicontinuous functions from X to $[0,\infty)$ such that γ_+ is f-subinvariant and γ_- is f-superinvariant. Suppose that the dynamical system $\{f^n\}_{n\in\boldsymbol{N}^*}$ is topologically transitive. Then $\gamma_+ \vee \gamma_-$ has a minimum. Furthermore, if f is a homeomorphism from X to X, then* $\min(\gamma_+ \vee \gamma_-) = \min\gamma_+ \vee \min\gamma_-$ *holds.*

Proof. Put $a := \inf(\gamma_+ \vee \gamma_-)$, and

$$B_n := \left\{ x \in X \;\middle|\; \gamma_+ \vee \gamma_-(x) \geq a + \frac{1}{n} \right\}.$$

B_n is a closed set. Suppose that $\gamma_+ \vee \gamma_-$ does not have a minimum, then X is represented as a countable union of B_n. From Baire's category theorem, there exists an $n_0 \in \boldsymbol{N}$ such that B_{n_0} has an inner point. Thus there are a real number $b \in (a,\infty)$ and a nonempty open set O with $O \subset \{x \in X \mid \gamma_+(x) \vee \gamma_-(x) \geq b\}$. For any $x \in D_+$ there is a nonnegative number $n(x) \geq 0$, which depends on x, satisfying $f^{n(x)}(x) \in O$ from the definition of D_+. Hence we can see that $\gamma_+(f^{n(x)}(x)) \geq b$ or $\gamma_-(f^{n(x)}(x)) \geq b$.

First, we consider the case when there exists a point $x \in D_+$ satisfying $\gamma_+(f^{n(x)}(x)) \geq b$. Since γ_+ is f-subinvariant, $\gamma_+(f^m(x)) \geq b$ for $m \geq n(x)$. Since the set $\{f^m(x) \mid m \geq n(x)\}$ is dense in X and the previous inequality, $\gamma_+(y) \geq b$ for any $y \in X$. Thus we have $\gamma_+ \vee \gamma_-(y) \geq b$ for any $y \in X$. This contradicts to $b > a$. Secondly, we consider the case when $\gamma_-(f^{n(x)}(x)) \geq b$ holds for any $x \in D_+$. Since γ_- is f-superinvariant, the inequalities $b \leq \gamma_-(f^{n(x)}(x)) \leq \gamma_-(x)$ holds for any $x \in D_+$. Hence these inequalities and the definition of D_+ lead to the fact that the inequality $\gamma_-(y) \geq b$ holds for any $y \in X$. Thus $\gamma_+ \vee \gamma_-(y) \geq b$ for any $y \in X$. This also contradicts to $b > a$. Therefore $\gamma_+ \vee \gamma_-$ has a minimum.

We assume that f is a homeomorphism and put $a := \min(\gamma_+ \vee \gamma_-)$. Since the dynamical system $\{f^n\}_{n \in \boldsymbol{N}^*}$ is topologically transitive, the dynamical system $\{f^{-n}\}_{n \in \boldsymbol{N}^*}$ is also topologically transitive. Thus, γ_+ has a minimum from Lemma 2.1 and γ_- also has a minimum since γ_- is f^{-1}-subinvariant.

Put $a_+ := \min \gamma_+$ and $a_- := \min \gamma_-$. Suppose that $a > a_+ \vee a_-$. First, we consider the case when $a_+ \geq a_-$. Clearly $a > a_+$. From Lemma 2.1, the inclusion $D_+ \subset \{x \in X \mid \gamma_+(x) = a_+\}$ holds. Hence $\gamma_-(x) \geq a$ for any $x \in D_+$ and therefore we can see that $\gamma_-(y) \geq a$ for any $y \in X$. This contradicts to $a > a_+ \vee a_-$. Secondly, when the case $a_+ < a_-$ holds, we have a contradiction in the same way.

We have $a \leq a_+ \vee a_-$, the converse $a \geq a_+ \vee a_-$ is trivial, hence, $a = a_+ \vee a_-$. □

Proof of Theorem 4.1. Let $\{U_n\}$ be a countable basis of X which is nonempty for any integer $n \geq 1$. We define a set $F_m^{(n)}$ and maps $\kappa_+(x)$, $\kappa_-(x)$, $\kappa(x)$ $(x \in X)$ as follows.

$$\begin{aligned}
F_m^{(n)} &:= f^{-m}(U_n^c) \qquad (m \in \boldsymbol{Z}\ ,\ n \geq 1) \\
\kappa_+(x) &:= \sum_{n=1}^{\infty} \frac{1}{3^n} \cdot \prod_{m=0}^{\infty} \mathbf{1}_{F_m^{(n)}}(x) \qquad (x \in X) \\
\kappa_-(x) &:= \sum_{n=1}^{\infty} \frac{1}{3^n} \cdot \prod_{m=-\infty}^{0} \mathbf{1}_{F_m^{(n)}}(x) \qquad (x \in X) \\
\kappa(x) &:= \sum_{n=1}^{\infty} \frac{1}{3^n} \cdot \prod_{m=-\infty}^{\infty} \mathbf{1}_{F_m^{(n)}}(x) \qquad (x \in X)
\end{aligned}$$

By Lemma 2.5, we have $D = \{x \in X \mid \kappa(x) = 0\}$. From the inequalities

$$\begin{aligned}
\kappa_-(f(x)) &= \sum_{n=1}^{\infty} \frac{1}{3^n} \cdot \prod_{m=-\infty}^{0} \mathbf{1}_{F_m^{(n)}}(f(x)) \\
&\leq \sum_{n=1}^{\infty} \frac{1}{3^n} \cdot \prod_{m=-\infty}^{0} \mathbf{1}_{F_m^{(n)}}(x) = \kappa_-(x)
\end{aligned}$$

and Lemma 2.2, κ_+ and κ_- are f-subinvariant, f-superinvariant respectively and they are upper semicontinuous. Also,

$$\begin{aligned}\lim_{l\to\infty}\kappa_+(f^{-l}(x)) &= \lim_{l\to\infty}\sum_{n=1}^{\infty}\frac{1}{3^n}\cdot\prod_{m=0}^{\infty}\mathbf{1}_{F_m^{(n)}}(f^{-l}(x)) \\ &= \lim_{l\to\infty}\sum_{n=1}^{\infty}\frac{1}{3^n}\cdot\prod_{m=0}^{\infty}\mathbf{1}_{F_{m-l}^{(n)}}(x) \\ &= \lim_{l\to\infty}\sum_{n=1}^{\infty}\frac{1}{3^n}\cdot\prod_{p=-l}^{\infty}\mathbf{1}_{F_p^{(n)}}(x) \\ &= \sum_{n=1}^{\infty}\frac{1}{3^n}\cdot\prod_{p=-\infty}^{\infty}\mathbf{1}_{F_p^{(n)}}(x) \\ &= \kappa(x).\end{aligned}$$

Thus $\kappa(x)=\lim_{l\to\infty}\kappa_+(f^{-l}(x))=0$ holds for $x\in D$. From this, $\inf\kappa_+\le 0$ holds. Since $\kappa_+\ge 0$, we see that $\inf\kappa_+=0$. In the same way, we can show that $\inf\kappa_-=0$ and therefore $\kappa_+\wedge\kappa_-$ has a minimum 0 from Lemma 4.1. Hence there exists a point $\bar{x}\in X$ such that $\kappa_+(\bar{x})\wedge\kappa_-(\bar{x})=0$.

If $\kappa_+(\bar{x})=0$, the dynamical system $\{f^n\}_{n\in\boldsymbol{N}^*}$ is topologically transitive. Hence the dynamical system $\{f^{-n}\}_{n\in\boldsymbol{N}^*}$ is also topologically transitive from the definition.

If $\kappa_-(\bar{x})=0$, the dynamical system $\{f^{-n}\}_{n\in\boldsymbol{N}^*}$ is topologically transitive. Hence the dynamical system $\{f^n\}_{n\in\boldsymbol{N}^*}$ is also topologically transitive.

Also, since $D_+=\{x\in X\mid\kappa_+(x)=0\}$ and $D_-=\{x\in X\mid\kappa_-(x)=0\}$ are shown, $\min\kappa_+(x)=0$ and $\min\kappa_-(x)=0$. Since $\min(\kappa_+\vee\kappa_-)=\min\kappa_+\vee\min\kappa_-=0$ from Lemma 4.2, there exists a point $\widetilde{x}\in X$ such that $\kappa_+(\widetilde{x})=0$ and $\kappa_-(\widetilde{x})=0$. Hence there is a point $\widetilde{x}\in D_+\cap D_-$, which implies that $D_+\cap D_-$ is nonempty. Since $\widetilde{x}\in D_+$, $f^l(\widetilde{x})\in D_+$ holds for any integer $l\in\boldsymbol{Z}$. In the same way, since $\widetilde{x}\in D_-$, $f^l(\widetilde{x})\in D_-$ holds for any integer $l\in\boldsymbol{Z}$. Thus $f^l(\widetilde{x})\in D_+\cap D_-$ for any integer $l\in\boldsymbol{Z}$ and therefore we have that $D_+\cap D_-$ is dense in X. □

Theorem 4.2. *Let (X,d) be a complete separable metric space without isolated points and f a homeomorphism from X to X. Consider the dynamical system $\{f^n\}_{n\in\boldsymbol{Z}}$ defined by f. Suppose that $\{f^n\}_{n\in\boldsymbol{Z}}$ is topologically transitive and has sensitive dependence on initial conditions. Then at least one of the dynamical systems $\{f^n\}_{n\in\boldsymbol{N}^*}$ and $\{f^{-n}\}_{n\in\boldsymbol{N}^*}$ have sensitive dependence on initial conditions.*

This theorem is the main result in this paper. To show Theorem 4.2, we prepare three lemmas: Lemma 4.3, Lemma 4.4, Lemma 4.5.

Lemma 4.3. *Let (X,d) be a complete separable metric space without isolated points, f a continuous map from X to X, and $\{f^n\}_{n\in\boldsymbol{N}^*}$ the dynamical system defined by f. Suppose that $\{f^n\}_{n\in\boldsymbol{N}^*}$ is topologically transitive. Then the inclusion $D_+ \subset \{x\in X \mid r_+(x) = \min r_+\}$ holds. Moreover if $\min r_+ = 0$, then $D_+ = \{x \in X \mid r_+(x) = 0\}$ holds.*

Proof. We know r_+ is in Γ_s from Lemma 3.1, Lemma 3.2. Theorem 2.2 yields that $D_+ \subset \{x \in X \mid r_+(x) = \min r_+\}$. Suppose that r_+ has a minimum 0 at $x \in X$, then $r_+(x) = 0$. We will show $\{x \in X \mid r_+(x) = 0\} \subset D_+ := \{x \in X \mid \overline{O_+(f;x)} = X\}$. We assume $\overline{O_+(f;x)} \neq X$. There are a point $y \in X$ and a constant $\delta > 0$ such that $O_+(f;x) \cap B_\delta(y)$ is empty. The equality $r_+(x) = 0$ leads to the existense of an $\varepsilon > 0$ satisfying $r_+(\varepsilon,x) < \frac{\delta}{2}$. Since the dynamical system $\{f^n\}_{n\in\boldsymbol{N}^*}$ is topologically transitive, there are a point $x' \in B_\varepsilon(x)$ and a nonnegative integer $n \geq 0$ such that $f^n(x') \in B_{\frac{\delta}{2}}(y)$. The inequalities $d(f^n(x), f^n(x')) < \frac{\delta}{2}$ and $d(f^n(x'), f^n(y)) < \frac{\delta}{2}$ imply that $f^n(x) \in B_\delta(y)$. This contradicts to the fact that $O_+(f;x) \cap B_\delta(y)$ is empty and therefore we see the inclusion $\{x \in X \mid r_+(x) = 0\} \subset D_+$. □

Lemma 4.4. *Let (X,d) be a complete separable metric space without isolated points, f a continuous map from X to X, and $\{f^n\}_{n\in\boldsymbol{N}^*}$ the dynamical system defind by f. Suppose that $\{f^n\}_{n\in\boldsymbol{Z}}$ is topologically transitive. Then $\min r_+ = 0$ if and only if $\{f^n\}_{n\in\boldsymbol{N}^*}$ does not have sensitive dependence on initial conditions.*

This lemma follows from the definition of sensitive dependence on initial conditions.

Definition 4.1. For $x \in X$ and an $\varepsilon > 0$, we define the numbers $r_-(\varepsilon,x)$ and $r_-(x)$ as follows:

$$\begin{aligned} r_-(\varepsilon,x) &\underset{\text{def}}{=} \sup_{n\leq 0} \sup_{y,y'\in B_\varepsilon(x)} d(f^n(y), f^n(y')) \wedge 1 \\ &= \sup_{n\geq 0} \sup_{y,y'\in B_\varepsilon(x)} d((f^{-1})^n(y), (f^{-1})^n(y')) \wedge 1, \\ r_-(x) &\underset{\text{def}}{=} \lim_{\varepsilon\downarrow 0} r_-(\varepsilon,x). \end{aligned}$$

Lemma 4.5. *Let (X,d) be a complete separable metric space without isolated points, f a homeomorphism from X to X, and $\{f^n\}_{n\in\boldsymbol{Z}}$ the dynamical system defined by f. Suppose that $\{f^n\}_{n\in\boldsymbol{Z}}$ is topologically transitive and has sensitive dependence on initial conditions. Moreover assume $\min r_+ = 0$. Then $\min r_- > 0$ and*

$$D_- \subset D_+ = \{x \in X \mid r_+(x) = 0\} \subset \{x \in X \mid r_-(x) = \min r_-\}$$

hold, where $D_- = \{x \in X \mid \overline{O_-(f;x)} = X\}$.

Proof. Theorem 4.1 shows that the dynamical systems $\{f^n\}_{n\in\boldsymbol{N}^*}$ and $\{f^{-n}\}_{n\in\boldsymbol{N}^*}$ defined by continuous maps f and f^{-1} are topologically transitive. From Lemma 3.1 and Lemma 3.2, r_+ and r_- are upper semicontinuous and f-invariant. By applying Lemma 2.1 for continuous maps f and f^{-1}, we see that each r_+ and r_- has a minimum and that the relations $D_+ \subset \{x \in X \mid r_+(x) = 0\}$ and $D_- \subset \{x \in X \mid r_-(x) = \min r_-\}$ hold. Suppose that $\min r_- = 0$. From Theorem 4.1, the set $D_+ \cap D_-$ is nonempty. It is clear that $r_+(x) = 0$ and $r_-(x) = 0$ for $x \in D_+ \cap D_-$. This implies that $r(x) = 0$, which contradicts to the fact that $\{f^n\}_{n\in\boldsymbol{Z}}$ has sensitive dependence on initial conditions. Therefore we have $\min r_- > 0$.

Let $x \in D_-$. $r_+(x) = r_+(f^{-n}(x))$ holds for any integer $n \geq 1$. Since r_+ is upper semicontinuous and $O_-(f;x)$ is dense in X, $r_+(y) \geq r_+(x)$ holds for any $y \in X$. Hence we have $r_+(x) = \min r_+ = 0$ and therefore the relation $D_- \subset \{x \in X \mid r_+(x) = 0\} = D_+$ follows from Lemma 4.3. Moreover it follows from Lemma 2.1 that $D_+ \subset \{x \in X \mid r_-(x) = \min r_-\}$. □

Proof of Theorem 4.2. If $\min r_+ > 0$, the dynamical system $\{f^n\}_{n\in\boldsymbol{N}^*}$ has sensitive dependence on initial conditions from Lemma 4.4. Suppose $\min r_+ = 0$, then $\min r_- > 0$ by Lemma 4.5, hence the dynamical system $\{f^{-n}\}_{n\in\boldsymbol{N}^*}$ has sensitive dependence on initial conditions. □

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References

[1] N. Aoki and K. Hiraide, *Topological Theory of Dynamical Systems Recent Advances*, North-Holland,1994.

[2] J. Banks, J. Brooks, G. Cairns, G. Davis and P. Stacey, On Devaney's definition of Chaos, Amer. Math. Monthly **99**, (1992), 332-334.

[3] N. Bourbaki, *Elements of Mathematics General Topology, Chapter 5-10*, Springer-Verlag, 1974.

[4] Robert L. Devaney, *An Introduction to Chaotic Dynamical Systems*, Benjamin/Cummings Publishing, 1986.

[5] Tien-Yien Li and James A. Yorke, Period Three implies Chaos, Amer. Math. Monthly **82**, (1975), 985-992.

[6] K. Yosida, *Functional Analysis 5th ed.*, Springer-Verlag, Berlin–Heidelberg–New York, 1978.