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Young-type integrals with respect to measurable processes

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Abstract For positive real numbers p and q satisfying $1/p+1/q>1$, it is known that if $f(u)$ and $g(u)$ have finite mean variations of orders p and q respectively, then an integral $\int_s^t f(u)dg(u)$ exists in the Riemann sense. The present paper extends this Stieltjes integration theory, discussed by L. C. Young, to the case where $f(u)$ and $g(u)$ are measurable processes. Moreover, path-by-path piecewise-linear functions are constructed via Riemann-Stieltjes sums for measurable processes, and convergence theorems on such functions are derived.

Key words and Phrases. Young-type integrals, Stieltjes integration, measurable proces
sses
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1 Introduction

For real-valued functions $f(u)$ and $g(u)$ defined on a closed interval $[s,t]$, f is said to be Stieltjes integrable in the Riemann sense with respect to g if the pair (f,g) satisfies the following: there exists a real value A such that for any $\varepsilon>0$, there exists $\delta>0$ for which every finite partition $\Delta=\{s=t_0<t_1<\cdots<t_n=t\}$ of $[s,t]$ with $|\Delta|\le\delta$ along with real numbers $\xi_i\in[t_{i-1},t_i]$ $(1\le i\le n)$ satisfies

$$\left|A-\sum_{i=1}^{n} f(\xi_i)(g(t_i)-g(t_{i-1}))\right|\le\varepsilon,$$

where $|\Delta|$ denotes the mesh of the partition Δ. The constant A is denoted by $\int_s^t f(u)dg(u)$ and called the Riemann-Stieltjes integral of f with respect to g over the interval $[s,t]$.

A function $g(u)$ of bounded variation on $[s,t]$ is associated with a real Borel measure on $[s,t]$, and the Riemann-Stieltjes integral $\int_s^t f(u)dg(u)$ is treated in the framework of measure theory. However, the measure-theoretic argument cannot be applied when $g(u)$

has unbounded variation. L.C.Young [4] discusses Stieltjes integrability for functions of unbounded variation. For $1 < p < \infty$, define

$$V_p(f) := \sup_{\Delta} \left(\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^p \right)^{\frac{1}{p}},$$

where $\Delta = \{s = t_0 < t_1 < \cdots < t_n = t\}$ and the supremum on the right hand side is taken over all finite partitions of $[s,t]$. f is said to be of bounded variation of order p if $V_p(f)$ is finite (See [1]). Suppose that real positive numbers p and q satisfy the relation $1/p + 1/q > 1$ and that f and g are of bounded variation of orders p and q, respectively. L.C.Young [4] states that if f and g have no common discontinuities, then f is Stieltjes integrable in the Riemann sense with respect to g.

Throughout this paper, a measure space $(\Omega, \mathscr{F}, P)$ and an interval $[0,T]$ $(0 < T < \infty)$ are fixed. For an $\mathscr{F}$-measurable real-valued function X defined on $(\Omega, \mathscr{F}, P)$, the notation $E[X]$ is used to denote the integral $\int_\Omega X dP$, whenever it exists, and is occasionally referred to as the expectation of X. A measurable process on $[0,T] \times \Omega$ is a measurable function with respect to $\mathscr{B}([0,T]) \times \mathscr{F}$. The aim of the present paper is to extend the Stieltjes integrals discussed by L.C.Young to integrals with respect to a pair (X,Y) of measurable processes, to yield *Young-type integrals*.

Let p, q, α, β be positive real numbers satisfying $1/p + 1/q = 1,\ \alpha, \beta \le 1$ and $\alpha + \beta < 2$. Assuming that $X_u \in L^p(\Omega, \mathscr{F}, P)$ and $Y_u \in L^q(\Omega, \mathscr{F}, P)$ for $u \in [0,T]$, the notion of mean variations of orders (p, α) and $((p,\alpha);(q,\beta))$, denoted by $V_p^{\alpha}(X;[s,t])$ and $V_{p,q}^{\alpha,\beta}(X,Y;[s,t])$ respectively, is introduced in Section 3. Namely, we define

$$V_p^{\alpha}(X;[s,t]) := \sup_{\Delta} \left(\sum_{k=1}^{n} E[|X_{t_k} - X_{t_{k-1}}|^p]^{\alpha} \right)^{\frac{1}{\alpha p}};$$

$$V_{p,q}^{\alpha,\beta}(X,Y;[s,t]) := \sup_{\Delta} \left(\sum_{k=1}^{n} E[|X_{t_k} - X_{t_{k-1}}|^p]^{\alpha} \right)^{\frac{1}{\alpha p}} \left(\sum_{k=1}^{n} E[|Y_{t_k} - Y_{t_{k-1}}|^q]^{\beta} \right)^{\frac{1}{\beta q}},$$

where $\Delta = \{s = t_0 < t_1 < \cdots < t_n = t\}$. One example of a measurable process with finite mean variation of order $(2, \alpha)$ is an additive functional of energy zero of a symmetric Markov processes. Also considered in Section 3 are Riemann-Stieltjes approximating sums:

$$F_{\Delta}(X,Y) := \sum_{\ell=1}^{n} X_{t_\ell}(Y_{t_\ell} - Y_{t_{\ell-1}}) \text{ and } \mathrm{F}_{\Delta}^{\xi}(X,Y) = \sum_{k=1}^{n} X_{\xi_k}\left(Y_{t_k} - Y_{t_{k-1}}\right),$$

where $\xi_k \in [t_{k-1}, t_k]$ $(1 \le k \le n)$. Using inequalities obtained in Section 2, which are extensions of Young's inequalities ([4]) from a measure-theoretic viewpoint, some important estimates on Riemann-Stieltjes approximating sums for measurable processes are derived in Section 3. These estimates are essential in obtaining the main results of this paper (Theorems A and B).

Consider the following conditions:

(A.1) $V_p^{\alpha}(X;[0,T]) < +\infty,\ V_q^{\beta}(Y;[0,T]) < +\infty.$
(A.2) At least one of the functions $(u,v) \mapsto E[|Y_u - Y_v|^q]$ and $(u,v) \mapsto E[|X_u - X_v|^p]$ is jointly continuous on $[0,T]\times[0,T]$.
(A.3) The function $(u,v) \mapsto E[|X_u - X_v|^p]$ is jointly continuous on $[0,T]\times[0,T]$, $\sup_{0\le u\le T}|X_u|$ and $\sup_{\substack{0\le v,u\le T\\ |u-v|\le\delta}}|Y_u - Y_v|$ are $\mathscr{F}-$measurable, and

$$E\left[\sup_{0\le u\le T}\left|X_u\right|^p\right] < \infty, \quad \lim_{\delta\to 0} E\left[\sup_{\substack{0\le u,v\le T\\ |u-v|\le\delta}}\left|Y_u - Y_v\right|^q\right] = 0.$$

Theorem A in Section 4 establishes that under certain conditions on measurable processes, Riemann-Stieltjes approximating sums over a given interval $[0,t]$ converge in the L^1-norm as the mesh of the partition for the sum tends to zero. This result is stated in Nakao [3], but without a proof there.

Theorem A. *Let p,q,α,β be positive real numbers satisfying $1/p+1/q=1$, $\alpha,\beta\le 1$, $\alpha+\beta<2$. Let $X=(X_u), Y=(Y_u)$ be measurable processes on $[0,T]\times\Omega$ such that $X_u\in L^p(\Omega,\mathscr{F},P)$, $Y_u\in L^q(\Omega,\mathscr{F},P)$ $(0\le u\le T)$. Suppose that the pair (X,Y) satisfies conditions* (A.1) *and* (A.2)*. Then for any $t\in(0,T]$, there exists a unique $\mathscr{F}$-measurable, integrable real-valued function H depending on t for which the following holds: for any $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ such that*

$$|\Delta|\le\delta \implies E[|H - \mathrm{F}_{\Delta}^{\xi}(X,Y)|]\le\varepsilon,$$

where $\Delta=\{0=t_0<\cdots<t_n=t\}$ is a finite partition of the interval $[0,t]$ and $\xi=\{\xi_k\}_{k=1}^n$ with $\xi_k\in[t_{k-1},t_k]$.

In Section 5, a path-by-path piecewise-linear process, denoted by $\mathrm{F}_{\Delta}^{\xi}(X,Y)(\Delta,t)$, is constructed via Riemann-Stieltjes approximating sums. Theorem B shows that under stronger conditions on measurable processes than those assumed in Theorem A, such piecewise-linear processes converge uniformly in L^1 as the mesh of the partition goes to zero.

Theorem B. *Let p,q,α,β be positive real numbers satisfying $1/p+1/q=1$, $\alpha<1$, $\beta=1$. Let $X=(X_u), Y=(Y_u)$ be measurable processes on $[0,T]\times\Omega$ such that $X_u\in L^p(\Omega,\mathscr{F},P)$, $Y_u\in L^q(\Omega,\mathscr{F},P)$ $(0\le u\le T)$. Suppose that the pair (X,Y) satisfies conditions* (A.1) *and* (A.3)*. Then there exists a unique measurable process $I(=I_t)$ on $[0,T]\times\Omega$ for which the following hold:*

(1) *$I.(\omega)$ is continuous on $[0,T]$ for each $\omega \in \Omega$ and* $E\left[\sup_{0\le t\le T}\left|I_t\right|\right] < \infty$.

(2) *For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$|\Delta| \le \delta \implies E\left[\sup_{0\le t\le T}\left|\mathrm{F}_{\Delta}^{\xi}(X,Y)(\Delta;t) - I_t\right|\right] \le \varepsilon,$$

where $\Delta = \{0 = t_0 < \cdots < t_n = t\}$ is a finite partition of the interval $[0,t]$ and $\xi = \{\xi_k\}_{k=1}^n$ with $\xi_k \in [t_{k-1}, t_k]$.

Furthermore, it is established that the limiting continuous processes in Theorem B can be regarded as a measurable process whose value at a given time t coincides with the limiting integrable function (depending on t) obtained in Theorem A. Construction of integrals with respect to additive functionals of energy zero, with the help of the theory of Dirichlet spaces, is provided in Nakao [2].

2 Extension of Young's inequalities

In this section we discuss the Young's inequalities appearing in [4] in terms of measurable functions. We need the following *condition* on positive real numbers p, q, α, β:

$$1/p + 1/q = 1,\ \alpha, \beta \le 1,\ \alpha + \beta < 2. \tag{2.1}$$

Lemma 2.1. *Let p, q, α, β be positive real numbers satisfying condition (2.1). Let $X_1, X_2, \ldots, X_n \in L^p(\Omega, \mathscr{F}, P)$, $Y_1, Y_2, \ldots, Y_n \in L^q(\Omega, \mathscr{F}, P)$. Then there exists a positive integer k $(1 \le k \le n)$ such that*

$$E[|X_k Y_k|] \le \left(\frac{1}{n}\right)^{\frac{1}{\alpha p} + \frac{1}{\beta q}} \left\{\sum_{i=1}^n E[|X_i|^p]^\alpha\right\}^{\frac{1}{\alpha p}} \times \left\{\sum_{i=1}^n E[|Y_i|^q]^\beta\right\}^{\frac{1}{\beta q}}. \tag{2.2}$$

Proof. Take a positive integer k $(1 \le k \le n)$ for which $E[|X_k Y_k|] = \min_{1\le j\le n} E[|X_j Y_j|]$. Using Hölder's inequality and the well-known inequality on arithmetic and geometric means,

$$E[|X_k Y_k|] \le \left\{E[|X_1|^p]^{\frac{1}{p}} \cdots E[|X_n|^p]^{\frac{1}{p}} E[|Y_1|^q]^{\frac{1}{q}} \cdots E[|Y_n|^q]^{\frac{1}{q}}\right\}^{\frac{1}{n}}$$

$$= \left(\{E[|X_1|^p]^\alpha \cdots E[|X_n|^p]^\alpha\}^{\frac{1}{n}}\right)^{\frac{1}{\alpha p}} \left(\left\{E[|Y_1|^q]^\beta \cdots E[|Y_n|^q]^\beta\right\}^{\frac{1}{n}}\right)^{\frac{1}{\beta q}}$$

$$\le \left(\frac{E[|X_1|^p]^\alpha + \cdots + E[|X_n|^p]^\alpha}{n}\right)^{\frac{1}{\alpha p}} \left(\frac{E[|Y_1|^q]^\beta + \cdots + E[|Y_n|^q]^\beta}{n}\right)^{\frac{1}{\beta q}},$$

thereby completing the proof. □

Let Q denote an operation which replaces some [,]'s by [+]'s in a family $\mathbf{X} := (X_1, X_2, \ldots, X_n)$ of measurable functions. For convenience, we write

$$Q\mathbf{X} = \mathbf{X}^{(\mathbf{1})} = \left(X_1^{(1)}, X_2^{(1)}, \ldots, X_m^{(1)}\right).$$

Note that $m \le n$ and each $X_j^{(1)}$ is a sum of some consecutive X_i's.

Definition 2.2. For positive real numers p, q, α, β satisfying condition (2.1), the $((p,\alpha);(q,\beta))$*-th mean variation* of a pair $(\mathbf{X}, \mathbf{Y}) = \big((X_1, X_2, \ldots, X_n),\ (Y_1, Y_2, \ldots, Y_n)\big)$ of two families of measurable functions is defined by

$$V_{p,q}^{\alpha,\beta}(\mathbf{X}, \mathbf{Y}) := \max_{Q} \left(\sum_{j=1}^{m} E[|X_j^{(1)}|^p]^\alpha\right)^{\frac{1}{\alpha p}} \left(\sum_{j=1}^{m} E[|Y_j^{(1)}|^q]^\beta\right)^{\frac{1}{\beta q}}$$

where Q on the right-hand side runs over all operations stated above.

Lemma 2.1 yields the following important inequality on mean variations.

Lemma 2.3. *Let p, q, α, β be positive real numbers satisfying condition (2.1). Let $X_1, X_2, \ldots, X_n \in L^p(\Omega, \mathscr{F}, P)$, $Y_1, Y_2, \ldots, Y_n \in L^q(\Omega, \mathscr{F}, P)$. Then*

$$E\left[\left|\sum_{1 \le i \le i' \le n} X_i Y_{i'}\right|\right] \le \left\{1 + \zeta\left(\frac{1}{\alpha p} + \frac{1}{\beta q}\right)\right\} V_{p,q}^{\alpha,\beta}(\mathbf{X}, \mathbf{Y}), \tag{2.3}$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$.

Proof. It is obvious that $1/(\alpha p) + 1/(\beta q) > 1$. By Lemma 2.1 applied to $(X_2, X_3, \ldots, X_n)$ and $(Y_1, Y_2, \ldots, Y_{n-1})$, there exists a positive integer d $(1 \le d \le n-1)$ such that

$$E[|X_{d+1} Y_d|] \le \left(\frac{1}{n-1}\right)^{\frac{1}{\alpha p} + \frac{1}{\beta q}} \left\{\sum_{r=1}^{n-1} E[|X_{r+1}|^p]^\alpha\right\}^{\frac{1}{\alpha p}} \left\{\sum_{r=1}^{n-1} E[|Y_r|^q]^\beta\right\}^{\frac{1}{\beta q}}.$$

Consider the operation Q_d which replaces the d-th [,] of $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)$ with [+]. Then the two families

$$Q_d \mathbf{X} = \mathbf{X}^{(\mathbf{1})} = \left(X_1^{(1)}, X_2^{(1)}, \ldots, X_{n-1}^{(1)}\right),\ Q_d \mathbf{Y} = \mathbf{Y}^{(\mathbf{1})} = \left(Y_1^{(1)}, Y_2^{(1)}, \ldots, Y_{n-1}^{(1)}\right)$$

satisfy

$$X_j^{(1)}, Y_j^{(1)} = \begin{cases} X_j, Y_j & (1 \le j \le d-1) \\ X_d + X_{d+1}, Y_d + Y_{d+1} & (j = d) \\ X_{j+1}, Y_{j+1} & (d+1 \le j \le n-1), \end{cases}$$

from which it follows that

$$\sum_{1\le j\le n-1}(X_1^{(1)}+\cdots+X_j^{(1)})Y_j^{(1)}=X_{d+1}Y_d+\sum_{1\le i\le n}(X_1+\cdots+X_i)Y_i.$$

Hence,

$$\left|\sum_{1\le i\le i'\le n}X_iY_{i'}\right|\le|X_{d+1}Y_d|+\left|\sum_{1\le j\le j'\le n-1}X_j^{(1)}Y_{j'}^{(1)}\right|.$$

Taking expectations on both sides,

$$\begin{aligned}
&E\left[\left|\sum_{1\le i\le i'\le n}X_iY_{i'}\right|\right]\\
&\le E\left[|X_{d+1}Y_d|\,\right]+E\left[\left|\sum_{1\le j\le j'\le n-1}X_j^{(1)}Y_{j'}^{(1)}\right|\right]\\
&\le(n-1)^{-(\frac{1}{\alpha p}+\frac{1}{\beta q})}\left(\sum_{r=1}^{n-1}E[|X_{r+1}|^p]^\alpha\right)^{\frac{1}{\alpha p}}\left(\sum_{r=1}^{n-1}E[|Y_r|^q]^\beta\right)^{\frac{1}{\beta q}}\\
&\quad+E\left[\left|\sum_{1\le j\le j'\le n-1}X_j^{(1)}Y_{j'}^{(1)}\right|\right]\\
&\le(n-1)^{-(\frac{1}{\alpha p}+\frac{1}{\beta q})}V_{p,q}^{\alpha,\beta}(\mathbf{X},\mathbf{Y})+E\left[\left|\sum_{1\le j\le j'\le n-1}X_j^{(1)}Y_{j'}^{(1)}\right|\right].
\end{aligned}$$

Next, apply Lemma 2.1 to $\left(X_2^{(1)},X_3^{(1)},\dots,X_{n-1}^{(1)}\right)$ and $\left(Y_1^{(1)},Y_2^{(1)},\dots,Y_{n-2}^{(1)}\right)$, and take a positive integer e $(1\le e\le n-2)$ for which

$$E\left[\left|X_{e+1}^{(1)}Y_e^{(1)}\right|\right]\le\left(\frac{1}{n-2}\right)^{\frac{1}{\alpha p}+\frac{1}{\beta q}}\left\{\sum_{s=1}^{n-2}E[|X_{s+1}^{(1)}|^p]^\alpha\right\}^{\frac{1}{\alpha p}}\left\{\sum_{s=1}^{n-2}E[|Y_s^{(1)}|^q]^\beta\right\}^{\frac{1}{\beta q}}.$$

Consider the operation Q_e which replaces the e-th [,] of $\mathbf{X}^{(\mathbf{1})}=\left(X_1^{(1)},X_2^{(1)},\dots,X_{n-1}^{(1)}\right)$ and $\mathbf{Y}^{(\mathbf{1})}=\left(Y_1^{(1)},Y_2^{(1)},\dots,Y_{n-1}^{(1)}\right)$ with [+]. Then the two families

$$Q_e\mathbf{X}^{(\mathbf{1})}=\mathbf{X}^{(\mathbf{2})}=\left(X_1^{(2)},X_2^{(2)},\dots,X_{n-2}^{(2)}\right),\ Q_e\mathbf{Y}^{(\mathbf{1})}=\mathbf{Y}^{(\mathbf{2})}=\left(Y_1^{(2)},Y_2^{(2)},\dots,Y_{n-2}^{(2)}\right)$$

satisfy

$$X_k^{(2)},Y_k^{(2)}=\begin{cases}X_k^{(1)},Y_k^{(1)} & (1\le k\le e-1)\\ X_e^{(1)}+X_{e+1}^{(1)},Y_e^{(1)}+Y_{e+1}^{(1)} & (k=e)\\ X_{k+1}^{(1)},Y_{k+1}^{(1)} & (e+1\le k\le n-2).\end{cases}$$

Hence,

$$\sum_{1\le k\le n-2}(X_1^{(2)}+\cdots+X_k^{(2)})Y_k^{(2)} = X_{e+1}^{(1)}Y_e^{(1)} + \sum_{1\le j\le n-1}(X_1^{(1)}+\cdots+X_j^{(1)})Y_j^{(1)}.$$

Therefore,

$$\left|\sum_{1\le j\le j'\le n-1} X_j^{(1)}Y_{j'}^{(1)}\right| \le \left|X_{e+1}^{(1)}Y_e^{(1)}\right| + \left|\sum_{1\le k\le k'\le n-2} X_k^{(2)}Y_{k'}^{(2)}\right|.$$

Taking expectations,

$$\begin{aligned}
&E\left[\left|\sum_{1\le j\le j'\le n-1} X_j^{(1)}Y_{j'}^{(1)}\right|\right]\\
&\le (n-2)^{-(\frac{1}{\alpha p}+\frac{1}{\beta q})}\, V_{p,q}^{\alpha,\beta}\left(\mathbf{X}^{(1)},\mathbf{Y}^{(1)}\right) + E\left[\left|\sum_{1\le k\le k'\le n-2} X_k^{(2)}Y_{k'}^{(2)}\right|\right]\\
&\le (n-2)^{-(\frac{1}{\alpha p}+\frac{1}{\beta q})}\, V_{p,q}^{\alpha,\beta}(\mathbf{X},\mathbf{Y}) + E\left[\left|\sum_{1\le k\le k'\le n-2} X_k^{(2)}Y_{k'}^{(2)}\right|\right].
\end{aligned}$$

Repetition of the same procedure leads to the desired inequality:

$$\begin{aligned}
E\left[\left|\sum_{1\le i\le i'\le n} X_iY_{i'}\right|\right] &\le \left\{(n-1)^{-(\frac{1}{\alpha p}+\frac{1}{\beta q})} + (n-2)^{-(\frac{1}{\alpha p}+\frac{1}{\beta q})} + \cdots + 1\right\}\\
&\quad \times V_{p,q}^{\alpha,\beta}(\mathbf{X},\mathbf{Y}) + E\left[\left|X_1^{(n-1)}Y_1^{(n-1)}\right|\right]\\
&\le \left\{1+\zeta\left(\frac{1}{\alpha p}+\frac{1}{\beta q}\right)\right\} V_{p,q}^{\alpha,\beta}(\mathbf{X},\mathbf{Y}),
\end{aligned}$$

where $X_1^{(n-1)} = X_1+X_2+\cdots+X_n$ and $Y_1^{(n-1)} = Y_1+Y_2+\cdots+Y_n$. □

3 Inequalities on mean variations of measurable processes

In this section we establish several inequalities which are employed in deriving Theorems A and B in the subsequent sections. A *measurable process* $X=(X_u)=X(u,\omega)(u\in[0,T])$ is a real-valued function defined on $[0,T]\times\Omega$ which is $\mathscr{B}([0,T])\times\mathscr{F}$-measurable.

Definition 3.1. Let $0<\alpha\le 1,\ 1<p<\infty$. Let $X=(X_u)$ be a measurable process with $X_u\in L^p(\Omega,\mathscr{F},P)$ $(u\in[0,T])$. The *mean variation of X over an interval* $[s,t]\,(\subset[0,T])$ *of order* (p,α) is defined by

$$V_p^{\alpha}(X;[s,t]) := \sup_{\Delta}\left(\sum_{k=1}^{n} E[|X_{t_k}-X_{t_{k-1}}|^p]^{\alpha}\right)^{\frac{1}{\alpha p}},$$

where the supremum is taken over all finite partitions $\Delta = \{s = t_0 < t_1 < \cdots < t_n = t\}$ of the interval $[s,t]$.

Definition 3.2. Let p,q,α,β be positive real numbers satisfying condition (2.1). Let $X = (X_u)$ and $Y = (Y_u)$ be measurable processes with $X_u \in L^p(\Omega,\mathscr{F},P)$, $Y_u \in L^q(\Omega,\mathscr{F},P)$ $(u \in [0,T])$. The *mean variation of the pair* (X,Y) *over an interval* $[s,t]\,(\subset [0,T])$ *of order* $((p,\alpha);(q,\beta))$ is defined by

$$V_{p,q}^{\alpha,\beta}(X,Y;[s,t]) := \sup_{\Delta} \left(\sum_{k=1}^{n} E[|X_{t_k} - X_{t_{k-1}}|^p]^{\alpha} \right)^{\frac{1}{\alpha p}} \left(\sum_{k=1}^{n} E[|Y_{t_k} - Y_{t_{k-1}}|^q]^{\beta} \right)^{\frac{1}{\beta q}},$$

where the supremum is taken over all finite partitions $\Delta = \{s = t_0 < t_1 < \cdots < t_n = t\}$ of the interval $[s,t]$.

Remark 3.3. By the above definition, the following inequality holds:

$$V_{p,q}^{\alpha,\beta}(X,Y;[s,t]) \leq V_p^{\alpha}(X;[s,t])V_q^{\beta}(Y;[s,t]).$$

Moreover, for $\alpha' > \alpha$,

$$V_p^{\alpha}(X;[s,t]) < \infty \Longrightarrow V_p^{\alpha'}(X;[s,t]) < \infty.$$

The next lemma provides a basic inequality on mean variations of measurable processes.

Lemma 3.4. *Let p,q,α,β be positive real numbers satisfying condition (2.1). Let $X = (X_u)$ and $Y = (Y_u)$ be measurable processes with $X_u \in L^p(\Omega,\mathscr{F},P)$ and $Y_u \in L^q(\Omega,\mathscr{F},P)$ $(u \in [0,T])$. Then for any real numbers $0 \leq s < t < r \leq T$,*

$$V_{p,q}^{\alpha,\beta}(X,Y;[s,t]) + V_{p,q}^{\alpha,\beta}(X,Y;[t,r]) \leq V_{p,q}^{\alpha,\beta}(X,Y;[s,r]). \tag{3.1}$$

Proof. Fix a finite partition $\Delta = \{s = t_0 < t_1 < \cdots < t_m = t < t_{m+1} < \cdots < t_n = r\}$ of the interval $[s,r]$. Noting $1/\alpha, 1/\beta \geq 1$ and using the Hölder's inequality,

$$\left(\sum_{\ell=1}^{m} E[|X_{t_\ell} - X_{t_{\ell-1}}|^p]^{\alpha} \right)^{\frac{1}{\alpha p}} \left(\sum_{\ell=1}^{m} E[|Y_{t_\ell} - Y_{t_{\ell-1}}|^q]^{\beta} \right)^{\frac{1}{\beta q}}$$
$$+ \left(\sum_{\ell=m+1}^{n} E[|X_{t_\ell} - X_{t_{\ell-1}}|^p]^{\alpha} \right)^{\frac{1}{\alpha p}} \left(\sum_{\ell=m+1}^{n} E[|Y_{t_\ell} - Y_{t_{\ell-1}}|^q]^{\beta} \right)^{\frac{1}{\beta q}}$$

$$\leq \left\{ \left(\sum_{\ell=1}^{m} E[|X_{t_\ell} - X_{t_{\ell-1}}|^p]^\alpha \right)^{\frac{1}{\alpha}} + \left(\sum_{\ell=m+1}^{n} E[|X_{t_\ell} - X_{t_{\ell-1}}|^p]^\alpha \right)^{\frac{1}{\alpha}} \right\}^{\frac{1}{p}}$$
$$\times \left\{ \left(\sum_{\ell=1}^{m} E[|Y_{t_\ell} - Y_{t_{\ell-1}}|^q]^\beta \right)^{\frac{1}{\beta}} + \left(\sum_{\ell=m+1}^{n} E[|Y_{t_\ell} - Y_{t_{\ell-1}}|^q]^\beta \right)^{\frac{1}{\beta}} \right\}^{\frac{1}{q}}.$$
$$\leq \left(\sum_{\ell=1}^{n} E[|X_{t_\ell} - X_{t_{\ell-1}}|^p]^\alpha \right)^{\frac{1}{\alpha p}} \left(\sum_{\ell=1}^{n} E[|Y_{t_\ell} - Y_{t_{\ell-1}}|^q]^\beta \right)^{\frac{1}{\beta q}}$$
$$\leq V_{p,q}^{\alpha,\beta}(X,Y;[s,r]).$$

The desired result follows immediately upon taking the supremum over Δ. □

The *Riemann-Stieltjes approximating sum of a pair* (X,Y) *of measurable processes over a partition* $\Delta = \{s = t_0 < t_1 < \cdots < t_n = t\}$ *of* $[s,t]$ is defined to be

$$F_\Delta(X,Y) := \sum_{\ell=1}^{n} X_{t_\ell}(Y_{t_\ell} - Y_{t_{\ell-1}}) = \sum_{1\leq \ell \leq m \leq n} \Delta_\ell X \Delta_m Y + X_s(Y_t - Y_s), \tag{3.2}$$

where $\Delta_\ell X = X_{t_\ell} - X_{t_{\ell-1}}$ and $\Delta_m Y = Y_{t_m} - Y_{t_{m-1}}$. (3.2) can be rewritten as

$$F_\Delta(X,Y) = X_t(Y_t - Y_s) + \sum_{1\leq m \leq \ell \leq n} (-\Delta_\ell X \Delta_m Y) + \sum_{\ell=1}^{n} \Delta_\ell X \Delta_\ell Y. \tag{3.3}$$

Applying Lemma 2.3 to $\mathbf{X} = (\Delta_1 X, \Delta_2 X, \ldots, \Delta_n X)$ and $\mathbf{Y} = (\Delta_1 Y, \Delta_2 Y, \ldots, \Delta_n Y)$,

$$E\left[\Big| \sum_{1\leq \ell \leq m \leq n} \Delta_\ell X \Delta_m Y \Big| \right] \leq \left\{ 1 + \zeta\left(\frac{1}{\alpha p} + \frac{1}{\beta q} \right) \right\} V_{p,q}^{\alpha,\beta}(\mathbf{X},\mathbf{Y}).$$

By the obvious inequality $V_{p,q}^{\alpha,\beta}(\mathbf{X},\mathbf{Y}) \leq V_{p,q}^{\alpha,\beta}(X,Y;[s,t])$,

$$E\left[\Big| \sum_{1\leq \ell \leq m \leq n} \Delta_\ell X \Delta_m Y \Big| \right] \leq \left\{ 1 + \zeta\left(\frac{1}{\alpha p} + \frac{1}{\beta q} \right) \right\} V_{p,q}^{\alpha,\beta}(X,Y;[s,t]). \tag{3.4}$$

Therefore, it follows from (3.2) that

$$E[|F_\Delta(X,Y) - X_s(Y_t - Y_s)|] \leq \left\{ 1 + \zeta\left(\frac{1}{\alpha p} + \frac{1}{\beta q} \right) \right\} V_{p,q}^{\alpha,\beta}(X,Y;[s,t]). \tag{3.5}$$

Similarly, (3.3) and (3.4) together with the Hölder's inequality yield

$$E[|F_\Delta(X,Y) - X_t(Y_t - Y_s)|] \leq \left\{ 2 + \zeta\left(\frac{1}{\alpha p} + \frac{1}{\beta q} \right) \right\} V_{p,q}^{\alpha,\beta}(X,Y;[s,t]). \tag{3.6}$$

Lemma 3.5 generalizes the inequalities (3.5) and (3.6).

Lemma 3.5. *Let p,q,α,β be positive real numbers satisfying condition (2.1). Let $X=(X_u)$ and $Y=(Y_u)$ be measurable processes with $X_u\in L^p(\Omega,\mathscr{F},P)$ and $Y_u\in L^q(\Omega,\mathscr{F},P)$ $(u\in[0,T])$. Let $\Delta=\{s=t_0<t_1<\cdots<t_j=\xi<t_{j+1}<\cdots<t_n=t\}$ be a finite partition of an interval $[s,t]$. Then*

$$E\left[\left|F_\Delta(X,Y)-X_\xi(Y_t-Y_s)\right|\right]\le\left\{2+\zeta(\frac{1}{\alpha p}+\frac{1}{\beta q})\right\}V_{p,q}^{\alpha,\beta}(X,Y;[s,t]). \tag{3.7}$$

Proof. By the equalities (3.2) and (3.3),

$$\begin{aligned}&|F_\Delta(X,Y)-X_\xi(Y_t-Y_s)| \qquad (3.8)\\ &\le\left|\sum_{\ell=1}^{j}X_{t_\ell}(Y_{t_\ell}-Y_{t_{\ell-1}})-X_\xi(Y_\xi-Y_s)\right|+\left|\sum_{\ell=j+1}^{n}X_{t_\ell}(Y_{t_\ell}-Y_{t_{\ell-1}})-X_\xi(Y_t-Y_\xi)\right|\\ &\le\left|\sum_{1\le m\le\ell\le j}\Delta_\ell X\Delta_m Y\right|+\left|\sum_{\ell=1}^{j}\Delta_\ell X\Delta_\ell Y\right|+\left|\sum_{j+1\le\ell\le m\le n}\Delta_\ell X\Delta_m Y\right|.\end{aligned}$$

The Hölder's inequality yields

$$E\left[\left|\sum_{\ell=1}^{j}\Delta_\ell X\Delta_\ell Y\right|\right]\le V_{p,q}^{\alpha,\beta}(X,Y;[s,\xi]).$$

Hence, taking expectations in (3.8) and using Lemma 3.4 and the inequality (3.4),

$$\begin{aligned}&E\left[\left|F_\Delta(X,Y)-X_\xi(Y_t-Y_s)\right|\right]\\ &\le\left\{1+\zeta(\frac{1}{\alpha p}+\frac{1}{\beta q})\right\}\left\{V_{p,q}^{\alpha,\beta}(X,Y;[s,\xi])+V_{p,q}^{\alpha,\beta}(X,Y;[\xi,t])\right\}+V_{p,q}^{\alpha,\beta}(X,Y;[s,\xi])\\ &\le\left\{2+\zeta(\frac{1}{\alpha p}+\frac{1}{\beta q})\right\}V_{p,q}^{\alpha,\beta}(X,Y;[s,t]),\end{aligned}$$

which completes the proof. □

Given a finite partition $\Delta=\{s=t_0<t_1<\cdots<t_n=t\}$ and real numbers $\xi=\{\xi_k\}_{k=1}^n$ such that $t_{k-1}\le\xi_k\le t_k$ $(1\le k\le n)$, ξ is said to *accompany the partition* Δ. Define

$$\mathrm{F}_\Delta^\xi(X,Y):=\sum_{k=1}^{n}X_{\xi_k}\left(Y_{t_k}-Y_{t_{k-1}}\right).$$

Lemma 3.6. *Let p,q,α,β be positive real numbers satisfying condition (2.1). Let $X=(X_u)$ and $Y=(Y_u)$ be measurable processes with $X_u\in L^p(\Omega,\mathscr{F},P)$ and $Y_u\in L^q(\Omega,\mathscr{F},P)$ $(u\in[0,T])$. Let $\Delta=\{s=t_0<t_1<\cdots<t_n=t\}$ be a finite partition of an interval*

$[s,t]$ which is accompanied by real numbers $\xi = \{\xi_k\}_{k=1}^n$. Let $\widehat{\Delta}$ be the finite partition constructed by adding ξ_k's to Δ. Let $\widetilde{\Delta}$ be an arbitrary refinement of $\widehat{\Delta}$. Then

$$\begin{aligned} E\left[\left|F_{\widetilde{\Delta}}(X,Y) - F_{\Delta}^{\xi}(X,Y)\right|\right] &\leq \left\{2+\zeta\left(\frac{1}{\alpha p}+\frac{1}{\beta q}\right)\right\}\sum_k V_{p,q}^{\alpha,\beta}(X,Y;[t_{k-1},t_k]) \\ &\leq \left\{2+\zeta\left(\frac{1}{\alpha p}+\frac{1}{\beta q}\right)\right\} V_{p,q}^{\alpha,\beta}(X,Y;[s,t]). \end{aligned} \tag{3.9}$$

Proof. Using the inequality (3.7) to the interval $[t_{k-1},t_k]$,

$$\begin{aligned} &E\left[\left|F_{\widetilde{\Delta}}(X,Y)\big|_{[t_{k-1},t_k]} - X_{\xi_k}(Y_{t_k}-Y_{t_{k-1}})\right|\right] \\ &\leq \left\{2+\zeta\left(\frac{1}{\alpha p}+\frac{1}{\beta q}\right)\right\} V_{p,q}^{\alpha,\beta}(X,Y;[t_{k-1},t_k]), \end{aligned}$$

where $F_{\widetilde{\Delta}}(X,Y)\big|_{[t_{k-1},t_k]}$ denotes the restriction of $F_{\widetilde{\Delta}}(X,Y)$ to the interval $[t_{k-1},t_k]$. Summing over k and applying Lemma 3.4, the desired inequality (3.9) follows. □

An important corollary of Lemma 3.6 is the following estimate on the Riemann-Stieltjes approximating sums with respect to two partitions.

Corollary 3.7. *Let p,q,α,β be positive real numbers satisfying condition (2.1). Let $\Delta = \{s = t_0 < t_1 < \cdots < t_n = t\}$ and $\Delta' = \{s = t'_0 < t_1 < \cdots < t'_{n'} = t\}$ be two partitions of the same interval $[s,t]$ accompanied by real numbers $\xi = \{\xi_k\}_{k=1}^n$ and $\xi' = \{\xi'_\ell\}_{\ell=1}^{n'}$, respectively. Then*

$$\begin{aligned} &E\left[\left|F_{\Delta}^{\xi}(X,Y) - F_{\Delta'}^{\xi'}(X,Y)\right|\right] \\ &\leq \left\{2+\zeta\left(\frac{1}{\alpha p}+\frac{1}{\beta q}\right)\right\} \\ &\quad \times \left(\sum_k V_{p,q}^{\alpha,\beta}(X,Y;[t_{k-1},t_k]) + \sum_\ell V_{p,q}^{\alpha,\beta}\left(X,Y;[t'_{\ell-1},t'_\ell]\right)\right) \\ &\leq 2\left\{2+\zeta\left(\frac{1}{\alpha p}+\frac{1}{\beta q}\right)\right\} V_{p,q}^{\alpha,\beta}(X,Y;[s,t]). \end{aligned} \tag{3.10}$$

4 Young-type integrals constructed via measurable processes and their convergence theorems

In this section we discusses the existence of Young-type integrals with respect to measurable processes, followed by consideration of convergence results of such integrals.

First, let p,q be positive real numbers satisfying $1 < p,q < \infty$. Let $X = (X_u) = X(u,\omega), Y = (Y_u) = Y(u,\omega)$ be measurable processes defined on $[0,T] \times \Omega$ such that $X_u \in L^p(\Omega, \mathscr{F}, P), Y_u \in L^q(\Omega, \mathscr{F}, P)$ $(u \in [0,T])$. For convenience, set

$$\begin{aligned}
&\sigma(u,v) := E[|X_u - X_v|^p] \;\; (u,v \in [0,T]),\\
&\gamma(u,v) := E[|Y_u - Y_v|^q] \;\; (u,v \in [0,T]),\\
&\mathrm{Osc}\;\sigma(\delta) := \sup_{\substack{|u-v|<\delta\\ 0\le u,v\le T}} \sigma(u,v) \;\; (0 < \delta \le T)\\
&\mathrm{Osc}\;\gamma(\delta) := \sup_{\substack{|u-v|<\delta\\ 0\le u,v\le T}} \gamma(u,v) \;\; (0 < \delta \le T)
\end{aligned}$$

Lemma 4.1. *Let $\alpha > 0$ and $1 < p < \infty$. Suppose that $X = (X_u)$ is a measurable process on $[0,T] \times \Omega$ with $V_p^\alpha(X;[0,T]) < \infty$ and that $\sigma(u,v)$ is continuous on $[0,T] \times [0,T]$. Then for any $0 \le s < t \le T$ and $\alpha' > \alpha$,*

$$V_p^{\alpha'}(X;[s,t]) \le V_p^{\alpha}(X;[s,t])^{\frac{\alpha}{\alpha'}}\, \mathrm{Osc}\;\sigma(t-s)^{\frac{\alpha'-\alpha}{\alpha' p}}. \tag{4.1}$$

Proof. Any finite partition $\Delta = \{s = t_0 < t_1 < \cdots < t_n = t\}$ of the interval $[s,t]$ satisfies the inequality

$$\sum_k E\left[\left|X_{t_k} - X_{t_{k-1}}\right|^p\right]^{\alpha'} \le \mathrm{Osc}\;\sigma(t-s)^{(\alpha'-\alpha)} \sum_k E\left[\left|X_{t_k} - X_{t_{k-1}}\right|^p\right]^{\alpha},$$

which yields

$$\left(\sum_k E\left[\left|X_{t_k} - X_{t_{k-1}}\right|^p\right]^{\alpha'}\right)^{\frac{1}{\alpha' p}} \le \mathrm{Osc}\;\sigma(t-s)^{\frac{\alpha'-\alpha}{\alpha' p}}\, V_p^{\alpha}(X;[s,t])^{\frac{\alpha}{\alpha'}}.$$

The inequality (4.1) follows immediately upon taking the supremum over Δ. □

A similar discussion establishes the following:

Lemma 4.2. *Let $\beta > 0$ and $1 < q < \infty$. Suppose that $Y = (Y_u)$ is a measurable process on $[0,T] \times \Omega$ with $V_q^\beta(Y;[0,T]) < \infty$ and that $\gamma(u,v)$ is continuous on $[0,T] \times [0,T]$. Then for any $0 \le s < t \le T$ and $\beta' > \beta$,*

$$V_q^{\beta'}(Y;[s,t]) \le V_q^{\beta}(Y;[s,t])^{\frac{\beta}{\beta'}}\, \mathrm{Osc}\;\gamma(t-s)^{\frac{\beta'-\beta}{\beta' q}}. \tag{4.2}$$

The next lemma states that, given a finite partition $\Delta = \{s = t_0 < t_1 < \cdots < t_n = t\}$ of an interval $[s,t]$, the sum of the mean variations over the intervals $[t_{k-1}, t_k]$ is dominated by the mean variation over the whole interval $[s,t]$.

Lemma 4.3. *Let p,q,α,β be positive real numbers satisfying condition (2.1). Let α',β' be real numbers such that $\alpha'>\alpha,\ \beta'>\beta,\ 1/(\alpha' p)+1/(\beta q)>1,\ 1/(\alpha p)+1/(\beta' q)>1$. Suppose that $X=(X_u), Y=(Y_u)$ are measurable processes on $[0,T]\times\Omega$ such that $X_u\in L^p(\Omega,\mathscr{F},P), Y_u\in L^q(\Omega,\mathscr{F},P)$ $(0\le u\le T)$. Then for any $0\le s=t_0<t_1<\cdots<t_n=t\le T$,*

$$\sum_{k=1}^{n} V_p^{\alpha}(X;[t_{k-1},t_k])^{\frac{\alpha}{\alpha'}}\, V_q^{\beta}(Y;[t_{k-1},t_k]) \le V_p^{\alpha}(X;[s,t])^{\frac{\alpha}{\alpha'}}\, V_q^{\beta}(Y;[s,t]), \tag{4.3}$$

$$\sum_{k=1}^{n} V_p^{\alpha}(X;[t_{k-1},t_k])\, V_q^{\beta}(Y;[t_{k-1},t_k])^{\frac{\beta}{\beta'}} \le V_p^{\alpha}(X;[s,t])\, V_q^{\beta}(Y;[s,t])^{\frac{\beta}{\beta'}}. \tag{4.4}$$

Proof. By the Hölder's inequality,

$$\begin{aligned}
&\sum_{k=1}^{n} V_p^{\alpha}(X;[t_{k-1},t_k])^{\frac{\alpha}{\alpha'}}\, V_q^{\beta}(Y;[t_{k-1},t_k])\\
&\quad=\sum_{k=1}^{n}\left\{V_p^{\alpha}(X;[t_{k-1},t_k])^{\alpha p}\right\}^{\frac{1}{\alpha' p}}\left\{V_q^{\beta}(Y;[t_{k-1},t_k])^{\beta q}\right\}^{\frac{1}{\beta q}}\\
&\quad\le\left(\sum_{k=1}^{n} V_p^{\alpha}(X;[t_{k-1},t_k])^{\alpha p}\right)^{\frac{1}{\alpha' p}}\left(\sum_{k=1}^{n} V_q^{\beta}(Y;[t_{k-1},t_k])^{\beta q}\right)^{\frac{1}{\beta q}}\\
&\quad\le V_p^{\alpha}(X;[s,t])^{\frac{\alpha}{\alpha'}}\, V_q^{\beta}(Y;[s,t]),
\end{aligned}$$

yielding (4.3). A similar argument yields the inequality (4.4). □

Let p,q,α,β,X,Y be as in Lemma 4.3. Consider the following *conditions* (A.1) and (A.2):

(A.1) $V_p^{\alpha}(X;[0,T])<\infty$ and $V_q^{\beta}(Y;[0,T])<\infty$.

(A.2) At least one of $\sigma(u,v)$ and $\gamma(u,v)$ is jointly continuous on $[0,T]\times[0,T]$.

The next theorem establishes the Cauchy condition for Riemann-Stieltjes approximating sums, which plays an essential role in defining Young-type integrals with respect to measurable processes.

Theorem 4.4. *Let p,q,α,β be positive real numbers satisfying condition (2.1). Let $X=(X_u), Y=(Y_u)$ be measurable processes on $[0,T]\times\Omega$ such that $X_u\in L^p(\Omega,\mathscr{F},P)$, $Y_u\in L^q(\Omega,\mathscr{F},P)$ $(0\le u\le T)$. Suppose that the pair (X,Y) satisfies conditions* (A.1) *and* (A.2)*. Fix $0<t\le T$. Then for any $\varepsilon>0$, there exists $\delta=\delta(\varepsilon)>0$ for which*

$$|\Delta|,|\Delta'|\le\delta \implies E\left[\left|\mathrm{F}_{\Delta}^{\xi}(X,Y)-\mathrm{F}_{\Delta'}^{\xi'}(X,Y)\right|\right]\le\varepsilon, \tag{4.5}$$

where Δ and Δ' are finite partitions of the interval $[0,t]$ which are accompanied by real numbers ξ and ξ', respectively.

Remark 4.5. The positive real number $\delta = \delta(\varepsilon)$ appearing in Theorem 4.4 is determined by $V_p^{\alpha}(X;[0,t])$, $V_q^{\beta}(Y;[0,t])$ along with Osc $\sigma(\delta)$ if $\sigma(u,v)$ is jointly continuous, or along with Osc $\gamma(\delta)$ if $\gamma(u,v)$ is jointly continuous.

Proof. Let $\Delta = \{0 = t_0 < \cdots < t_n = t\}$ and $\Delta' = \{0 = t'_0 < \cdots < t'_{n'} = t\}$. Noting condition (A.2), assume that $\gamma(u,v)$ is jointly continuous $[0,T]\times[0,T]$. Take β' satisfying $\beta' > \beta$ and $1/(\alpha p) + 1/(\beta' q) > 1$. Since p, q, α, β' satisfy the condition (2.1), recalling Remark 3.3 and using the inequality (3.10),

$$
\begin{aligned}
&E\left[\left|\mathrm{F}^{\xi}_{\Delta}(X,Y) - \mathrm{F}^{\xi'}_{\Delta'}(X,Y)\right|\right] \\
&\le \left\{2 + \zeta(\frac{1}{\alpha p} + \frac{1}{\beta' q})\right\}\left(\sum_k V^{\alpha,\beta'}_{p,q}(X,Y;[t_{k-1},t_k]) + \sum_{\ell} V^{\alpha,\beta'}_{p,q}\left(X,Y;[t'_{\ell-1},t'_{\ell}]\right)\right) \\
&\le \left\{2 + \zeta(\frac{1}{\alpha p} + \frac{1}{\beta' q})\right\}\left(\sum_k V^{\alpha}_{p}(X;[t_{k-1},t_k])\, V^{\beta'}_{q}(Y;[t_{k-1},t_k])\right. \\
&\qquad\qquad\left. + \sum_{\ell} V^{\alpha}_{p}\left(X;[t'_{\ell-1},t'_{\ell}]\right) V^{\beta'}_{q}\left(Y;[t'_{\ell-1},t'_{\ell}]\right)\right).
\end{aligned}
$$

Hence, (4.2) and (4.4) together yield

$$
\begin{aligned}
&E\left[\left|\mathrm{F}^{\xi}_{\Delta}(X,Y) - \mathrm{F}^{\xi'}_{\Delta'}(X,Y)\right|\right] \\
&\le \left\{2 + \zeta(\frac{1}{\alpha p} + \frac{1}{\beta' q})\right\}\left(\sum_k V^{\alpha}_{p}(X;[t_{k-1},t_k])\, V^{\beta}_{q}(Y;[t_{k-1},t_k])^{\frac{\beta}{\beta'}}\,\mathrm{Osc}\,\gamma(|\Delta|)^{\frac{\beta'-\beta}{\beta' q}}\right. \\
&\qquad\qquad\left. + \sum_{\ell} V^{\alpha}_{p}\left(X;[t'_{\ell-1},t'_{\ell}]\right) V^{\beta}_{q}\left(Y;[t'_{\ell-1},t'_{\ell}]\right)^{\frac{\beta}{\beta'}}\,\mathrm{Osc}\,\gamma(|\Delta'|)^{\frac{\beta'-\beta}{\beta' q}}\right) \\
&\le 2\left\{2 + \zeta(\frac{1}{\alpha p} + \frac{1}{\beta' q})\right\} \times \mathrm{Osc}\,\gamma(|\Delta| \vee |\Delta'|)^{\frac{\beta'-\beta}{\beta' q}}\, V^{\alpha}_{p}(X;[0,t])\, V^{\beta}_{q}(Y;[0,t])^{\frac{\beta}{\beta'}}.
\end{aligned}
$$

Since $\gamma(u,v)$ is jointly continuous, Osc $\gamma(|\Delta| \vee |\Delta'|) \longrightarrow 0$ as $|\Delta|$, $|\Delta'| \to 0$. Therefore, the desired result follows. When $\sigma(u,v)$ is jointly continuous, a similar discussion with the help of the inequalities (4.1) and (4.3) yields the same result. □

We are now ready to establish one of the main theorems of this paper, which states that a Young-type integral can be defined as a limit of Riemann-Stieltjes approximating sums.

Theorem A. *Let p, q, α, β be positive real numbers satisfying condition (2.1). Let $X = (X_u), Y = (Y_u)$ be measurable processes on $[0,T]\times\Omega$ such that $X_u \in L^p(\Omega,\mathscr{F},P)$, $Y_u \in$*

$L^q(\Omega,\mathscr{F},P)$ $(0 \le u \le T)$. *Suppose that the pair* (X,Y) *satisfies conditions* (A.1) *and* (A.2). *Then for each fixed* $t \in (0,T]$, *there exists a unique* $\mathscr{F}$*-measurable, integrable real-valued function* H *depending on* t *for which the following holds: for* $\varepsilon > 0$ *and* $\delta = \delta(\varepsilon)$ *appearing in Theorem* 4.4,

$$|\Delta| \le \delta \implies E[|H - \mathrm{F}_{\Delta}^{\xi}(X,Y)|] \le \varepsilon, \tag{4.6}$$

where Δ *is a finite partition of the interval* $[0,t]$ *which is accompanied by real number* ξ.

Proof. Let $\left\{\Delta^{(n)}\right\}_{n=1}^{\infty}$ be a sequence of finite partitions of $[0,t]$ with $\lim_{n\to\infty}\left|\Delta^{(n)}\right| = 0$. Let $H_n := \mathrm{F}_{\Delta^{(n)}}(X,Y)$ $(n \in \mathbf{N})$, the Riemann-Stieltjes approximating sum of the pair (X,Y) over $\Delta^{(n)}$. Then $\{H_n\}_{n=1}^{\infty}$ forms a Cauchy sequence in $L^1(\Omega,\mathscr{F},P)$ due to Theorem 4.4. Since $L^1(\Omega,\mathscr{F},P)$ is complete, there exists $H \in L^1(\Omega,\mathscr{F},P)$ for which $\lim_{n\to\infty} E\left[|H_n - H|\right] = 0$. We only need to show that H satisfies (4.6). If (4.6) failed, then there would be a positive real number ε_0 for which one can find a sequence $\left\{\widetilde{\Delta}^{(m)}\right\}$ of finite partitions of $[0,t]$, each accompanied by real numbers $\widetilde{\xi}^{(m)}$, such that $\lim_{m\to\infty}\left|\widetilde{\Delta}^{(m)}\right| = 0$ and $E\left[\left|H - \mathrm{F}_{\widetilde{\Delta}^{(m)}}^{\widetilde{\xi}^{(m)}}(X,Y)\right|\right] > \varepsilon_0$. If $\left|\Delta^{(n)}\right| \le \delta(\frac{\varepsilon_0}{2})$ and $\left|\widetilde{\Delta}^{(m)}\right| \le \delta(\frac{\varepsilon_0}{2})$, then by Theorem 4.4,

$$E\left[\left|\mathrm{F}_{\Delta^{(n)}}(X,Y) - \mathrm{F}_{\widetilde{\Delta}^{(m)}}^{\widetilde{\xi}^{(m)}}(X,Y)\right|\right] \le \frac{\varepsilon_0}{2}.$$

Take a sufficiently large n so that $E\left[|H_n - H|\right] \le \frac{\varepsilon_0}{2}$. Then we have

$$\begin{aligned}
\varepsilon_0 &< E\left[\left|H - \mathrm{F}_{\widetilde{\Delta}^{(m)}}^{\widetilde{\xi}^{(m)}}(X,Y)\right|\right] \\
&\le E\left[|H - \mathrm{F}_{\Delta^{(n)}}(X,Y)|\right] + E\left[\left|\mathrm{F}_{\Delta^{(n)}}(X,Y) - \mathrm{F}_{\widetilde{\Delta}^{(m)}}^{\widetilde{\xi}^{(m)}}(X,Y)\right|\right] \\
&\le \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2} = \varepsilon_0,
\end{aligned}$$

a contradiction. Thus, H satisfies (4.6). □

Definition 4.6. The integrable function H depending on $t \in (0,T]$ in Theorem A is called the *Young-type integral of the pair* (X,Y) *of measurable processes over the interval* $[0,t]$ and is denoted by $H = \int_0^t X_u dY_u$. For $t = 0$, we set $\int_0^t X_u dY_u = 0$.

The remainder of this section is devoted to convergence results of sequences of Young-type integrals. Namely, if two sequences $\left\{X^n\right\}$ and $\left\{Y^n\right\}$ of measurable processes converge to measurable processes X and Y respectively, then each of the three sequences $\left\{\int_0^t X_u dY_u^n\right\}$, $\left\{\int_0^t X_u^n dY_u\right\}$ and $\left\{\int_0^t X_u^n dY_u^n\right\}$ is shown to converge under certain conditions.

Theorem 4.7. *Let p,q,α,β be positive real numbers satisfying condition (2.1). Let $X=(X_u), Y=(Y_u), Y^n=(Y_u^n)$ $(n\in\mathbf{N})$ be measurable processes on $[0,T]\times\Omega$ such that $X_u\in L^p(\Omega,\mathscr{F},P)$, $Y_u\in L^q(\Omega,\mathscr{F},P)$, $Y_u^n\in L^q(\Omega,\mathscr{F},P)$ $(0\le u\le T, n\in\mathbf{N})$. Suppose that each of the pairs (X,Y) and (X,Y^n) $(n\in\mathbf{N})$ satisfies conditions* (A.1) *and* (A.2). *In condition* (A.2)*, assume that $\sigma(u,v)$ is jointly continuous on $[0,T]\times[0,T]$. Fix $t\in(0,T]$. Suppose also that $E[|Y_u^n-Y_u|^q]\longrightarrow 0$ as $n\to\infty$ for each $u\in[0,t]$ and $\sup_{n\in\mathbf{N}} V_q^{\beta}(Y^n;[0,t])<\infty$. Then*

$$H^n=\int_0^t X_u dY_u^n \overset{n\to\infty}{\longrightarrow} H=\int_0^t X_u dY_u \quad in \quad L^1(\Omega,\mathscr{F},P). \tag{4.7}$$

Proof. Let $\varepsilon>0$. In the light of Remark 4.5, $\delta=\delta(\varepsilon)$ in Theorem 4.4 corresponding to the pairs (X,Y) and (X,Y^n) $(n\in\mathbf{N})$ can be taken uniformly, due to the assumption $\sup_{n\in\mathbf{N}} V_q^{\beta}(Y^n;[0,t])<\infty$. Take any sequence $\left\{\Delta^{(m)}\right\}_{m=1}^{\infty}$ of finite partitions of the interval $[0,t]$ for which $\lim_{m\to\infty}|\Delta^{(m)}|=0$. For each $m\in\mathbf{N}$ with $|\Delta^{(m)}|\le\delta(\varepsilon)$,

$$E\left[|H-\mathrm{F}_{\Delta^{(m)}}(X,Y)|\right]\le\varepsilon \quad \text{and} \quad E\left[|H^n-\mathrm{F}_{\Delta^{(m)}}(X,Y^n)|\right]\le\varepsilon.$$

By Hölder's inequality,

$$\begin{aligned}
&E\left[|\mathrm{F}_{\Delta^{(m)}}(X,Y)-\mathrm{F}_{\Delta^{(m)}}(X,Y^n)|\right]\\
&=E\left[\left|\sum_k X_{t_k^{(m)}}(Y_{t_k^{(m)}}-Y_{t_{k-1}^{(m)}})-\sum_k X_{t_k^{(m)}}(Y^n_{t_k^{(m)}}-Y^n_{t_{k-1}^{(m)}})\right|\right]\\
&\le\sum_k E\left[\left|X_{t_k^{(m)}}\right|^p\right]^{\frac{1}{p}} E\left[\left|(Y^n_{t_k^{(m)}}-Y_{t_k^{(m)}})-(Y^n_{t_{k-1}^{(m)}}-Y_{t_{k-1}^{(m)}})\right|^q\right]^{\frac{1}{q}}\\
&\longrightarrow 0 \quad \text{as} \quad n\to\infty,
\end{aligned}$$

where $\Delta^{(m)}=\{0=t_0^{(m)}<t_1^{(m)}<\cdots<t_{n(m)}^{(m)}=t\}$. Hence,

$$\begin{aligned}
\overline{\lim_{n\to\infty}}E\left[|H-H^n|\right]&\le\overline{\lim_{n\to\infty}}E\left[|H^n-\mathrm{F}_{\Delta^{(m)}}(X,Y^n)|\right]+\overline{\lim_{n\to\infty}}E\left[|H-\mathrm{F}_{\Delta^{(m)}}(X,Y)|\right]\\
&\quad+\overline{\lim_{n\to\infty}}E\left[|\mathrm{F}_{\Delta^{(m)}}(X,Y)-\mathrm{F}_{\Delta^{(m)}}(X,Y^n)|\right]\\
&\le 2\varepsilon.
\end{aligned}$$

Since ε is arbitrary, it follows that $\lim_{n\to\infty}E\left[|H-H^n|\right]=0$. □

We can obtain the following theorems in the same way.

Theorem 4.8. *Let p, q, α, β be positive real numbers satisfying condition (2.1). Let $X = (X_u), Y = (Y_u), X^n = (X^n_u)$ $(n \in \mathbf{N})$ be measurable processes on $[0,T] \times \Omega$ such that $X_u \in L^p(\Omega, \mathscr{F}, P)$, $Y_u \in L^q(\Omega, \mathscr{F}, P)$, $X^n_u \in L^p(\Omega, \mathscr{F}, P)$ $(0 \le u \le T, n \in \mathbf{N})$. Suppose that each of the pairs (X,Y) and (X^n, Y) $(n \in \mathbf{N})$ satisfies conditions* (A.1) *and* (A.2). *In condition* (A.2), *assume that $\gamma(u,v)$ is jointly continuous on $[0,T] \times [0,T]$. Fix $t \in (0,T]$. Suppose also that $E[|X^n_u - X_u|^p] \longrightarrow 0$ as $n \to \infty$ for each $u \in [0,t]$ and $\sup_{n \in \mathbf{N}} V^{\alpha}_p(X^n; [0,t]) < \infty$. Then*

$$H^n = \int_0^t X^n_u dY_u \overset{n \to \infty}{\longrightarrow} H = \int_0^t X_u dY_u \quad in \quad L^1(\Omega, \mathscr{F}, P). \tag{4.8}$$

Theorem 4.9. *Let p, q, α, β be positive real numbers satisfying condition (2.1). Let $X = (X_u), Y = (Y_u), X^n = (X^n_u), Y^n = (Y^n_u)$ $(n \in \mathbf{N})$ be measurable processes on $[0,T] \times \Omega$ such that $X_u, X^n_u \in L^p(\Omega, \mathscr{F}, P)$, $Y_u, Y^n_u \in L^q(\Omega, \mathscr{F}, P)$ $(0 \le u \le T,\ n \in \mathbf{N})$. Suppose that each of the pairs (X,Y) and (X^n, Y^n) $(n \in \mathbf{N})$ satisfies conditions* (A.1). *Assume either that $\sigma_n(u,v) := E[|X^n_u - X^n_v|^p]$ is jointly continuous for each $n \in \mathbf{N}$ and converges uniformly to $\sigma(u,v)$ on $[0,T] \times [0,T]$, or that $\gamma_n(u,v) := E[|Y^n_u - Y^n_v|^q]$ is jointly continuous for each $n \in \mathbf{N}$ and converges uniformly to $\gamma(u,v)$ on $[0,T] \times [0,T]$. Fix $t \in (0,T]$. Suppose also that $E[|X^n_u - X_u|^p] \longrightarrow 0$ and $E[|Y^n_u - Y_u|^q] \longrightarrow 0$ as $n \to \infty$ for each $u \in [0,t]$. Moreover, suppose that $\sup_{n \in \mathbf{N}} V^{\alpha}_p(X^n; [0,t]) < \infty$ and $\sup_{n \in \mathbf{N}} V^{\beta}_q(Y^n; [0,t]) < \infty$. Then*

$$H^n = \int_0^t X^n_u dY^n_u \overset{n \to \infty}{\longrightarrow} H = \int_0^t X_u dY_u \quad in \quad L^1(\Omega, \mathscr{F}, P). \tag{4.9}$$

Proof. For a finite partition $\Delta = \{0 = t_0 < t_1 < \cdots < t_m = t\}$ of $[0,t]$, Hölder's inequality yields

$$\begin{aligned}
&E\left[|\mathrm{F}_{\Delta}(X^n, Y^n) - \mathrm{F}_{\Delta}(X,Y)|\right] \\
&\le E\left[\left|\sum_k X^n_{t_k}\left(Y^n_{t_k} - Y^n_{t_{k-1}}\right) - \sum_k X_{t_k}\left(Y^n_{t_k} - Y^n_{t_{k-1}}\right)\right|\right] \\
&\quad + E\left[\left|\sum_k X_{t_k}\left(Y^n_{t_k} - Y^n_{t_{k-1}}\right) - \sum_k X_{t_k}(Y_{t_k} - Y_{t_{k-1}})\right|\right] \\
&\le \sum_k E\left[\left|X^n_{t_k} - X_{t_k}\right|^p\right]^{\frac{1}{p}} E\left[\left|Y^n_{t_k} - Y^n_{t_{k-1}}\right|^q\right]^{\frac{1}{q}} \\
&\quad + \sum_k E\left[|X_{t_k}|^p\right]^{\frac{1}{p}} E\left[\left|\left(Y^n_{t_k} - Y_{t_k}\right) - (Y^n_{t_{k-1}} - Y_{t_{k-1}})\right|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

The assumption on the functions $\sigma_n(u,v), \sigma(u,v), \gamma_n(u,v)$ and $\gamma(u,v)$ implies that for any $\varepsilon > 0$, there exists $\delta > 0$ such that either $\sup_n \mathrm{Osc}\ \sigma_n(\delta) \le \varepsilon$ or $\sup_n \mathrm{Osc}\ \gamma_n(\delta) \le \varepsilon$. The proof of Theorem 4.9 is carried out in a similar way to that of Theorem 4.7. □

5 Uniform convergence of approximating sequences for Young-type integrals

Theorem A in Section 4 guarantees the existence of a Young-type integral $\int_0^t X_u dY_u$ for each fixed $t \in (0,T]$. The aim of this section is to establish that the family of integrals $\left(\int_0^t X_u dY_u\right)_{t\in[0,T]}$ can be regarded as a measurable process defined on $[0,T]\times\Omega$ (Theorem B).

Suppose that measurable processes $X=(X_u)=X(u,\omega)$ and $Y=(Y_u)=Y(u,\omega)$ satisfy $X_u \in L^p(\Omega,\mathscr{F},P)$, $Y_u \in L^q(\Omega,\mathscr{F},P)$ $(u\in[0,T])$ as well as condition (A.1).We introduce the following additional *condition*:

(A.3) The function $\sigma(u,v)=E[|X_u-X_v|^p]$ is jointly continuous on $[0,T]\times[0,T]$, $\sup_{0\le u\le T}|X_u|$ and $\sup_{\substack{0\le v,u\le T\\|u-v|\le\delta}}|Y_u-Y_v|$ are $\mathscr{F}-$measurable, and

$$E\left[\sup_{0\le u\le T}\left|X_u\right|^p\right]<\infty,\quad \lim_{\delta\to 0}E\left[\sup_{\substack{0\le u,v\le T\\|u-v|\le\delta}}\left|Y_u-Y_v\right|^q\right]=0.$$

For a finite partition $\Delta=\{0=t_0<t_1<\cdots<t_n=T\}$ of the interval $[0,T]$ and real numbers $\xi=\{\xi_k\}_{k=1}^n$ accompanying Δ, i.e., $\xi_k\in[t_{k-1},t_k]$ $(1\le k\le n)$, set

$$\begin{aligned}
&\mathrm{F}_\Delta^\xi(X,Y)(0):=0,\\
&\mathrm{F}_\Delta^\xi(X,Y)(t_i):=\sum_{r=1}^{i}X_{\xi_r}(Y_{t_r}-Y_{t_{r-1}}) \qquad (1\le i\le n).
\end{aligned}$$

A piecewise-linear process $\mathrm{F}_\Delta^\xi(X,Y)(\Delta;t)$ is constructed via linear interpolation as follows:

Definition 5.1. Define $\mathrm{F}_\Delta^\xi(X,Y)(\Delta;t)$ $(t\in[0,T])$ by

$$\mathrm{F}_\Delta^\xi(X,Y)(\Delta;t)=\begin{cases}0 & \text{if } t=0,\\ \mathrm{F}_\Delta^\xi(X,Y)(t_{i-1}) & \\ \quad+\frac{t-t_{i-1}}{t_i-t_{i-1}}\left(\mathrm{F}_\Delta^\xi(X,Y)(t_i)-\mathrm{F}_\Delta^\xi(X,Y)(t_{i-1})\right) & \text{if } t_{i-1}<t<t_i,\\ \mathrm{F}_\Delta^\xi(X,Y)(t_i) & \text{if } t=t_i.\end{cases}$$

The following lemma is used to derive Theorem 5.3.

Lemma 5.2. *Let $q>1$. Let $Y=(Y_u)$ be a measurable process on $[0,T]\times\Omega$ such that $Y_u\in L^q(\Omega,\mathscr{F},P)$ $(0\le u\le T)$. Let τ,η be $\mathscr{F}$-measurable functions on Ω satisfying $0\le\tau(\omega)<\eta(\omega)\le T$ $(\omega\in\Omega)$. Define a measurable process $\overline{Y}$ by*

$$\overline{Y}(u,\omega)=\overline{Y}_u(\omega)=\begin{cases}Y(\tau(\omega),\omega) & \text{if } 0\le u<\tau(\omega),\\ Y(u,\omega) & \text{if } \tau(\omega)\le u\le\eta(\omega),\\ Y(\eta(\omega),\omega) & \text{if } \eta(\omega)<u\le T.\end{cases}$$

Then

$$V_q^1\left(\overline{Y};[0,T]\right) \le V_q^1\left(Y;[0,T]\right) + 2^{\frac{1}{q}} E\left[\sup_{\substack{0\le v,u\le T\\ |u-v|\le\delta}} |Y_u - Y_v|^q\right]^{\frac{1}{q}}, \tag{5.1}$$

where $\delta = \sup_{\omega\in\Omega} |\eta(\omega) - \tau(\omega)|$.

Proof. Let $\Delta = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ be an arbitrary finite partition of $[0,T]$. For a fixed $\omega \in \Omega$ such that $0 \le t_{i(\omega)-1} < \tau(\omega) \le t_{i(\omega)} < \cdots < t_{j(\omega)} < \eta(\omega) \le t_{j(\omega)+1} \le T$,

$$\sum_{r=1}^{n} \left|\overline{Y}_{t_r}(\omega) - \overline{Y}_{t_{r-1}}(\omega)\right|^q \le \left|\overline{Y}_{t_{i(\omega)}}(\omega) - \overline{Y}_{t_{i(\omega)-1}}(\omega)\right|^q + \sum_{r=i(\omega)+1}^{j(\omega)} \left|\overline{Y}_{t_r}(\omega) - \overline{Y}_{t_{r-1}}(\omega)\right|^q + \left|\overline{Y}_{t_{j(\omega)+1}}(\omega) - \overline{Y}_{t_{j(\omega)}}(\omega)\right|^q.$$

Note

$$\left|\overline{Y}_{t_{i(\omega)}}(\omega) - \overline{Y}_{t_{i(\omega)-1}}(\omega)\right| = \begin{cases} \left|Y_{t_{i(\omega)}}(\omega) - Y_{\tau(\omega)}(\omega)\right| & \text{if } t_{i(\omega)} \le \eta(\omega),\\ \left|Y_{\eta(\omega)}(\omega) - Y_{\tau(\omega)}(\omega)\right| & \text{if } t_{i(\omega)} > \eta(\omega),\end{cases}$$

$$\le \sup_{\substack{0\le u,v\le T\\ |u-v|\le\delta}} |Y_u(\omega) - Y_v(\omega)|.$$

A similar observation yields

$$\left|\overline{Y}_{t_{j(\omega)+1}}(\omega) - \overline{Y}_{t_{j(\omega)}}(\omega)\right| \le \sup_{\substack{0\le u,v\le T\\ |u-v|\le\delta}} |Y_u(\omega) - Y_v(\omega)|.$$

Hence,

$$\sum_{r=1}^{n} \left|\overline{Y}_{t_r}(\omega) - \overline{Y}_{t_{r-1}}(\omega)\right|^q \le \sum_{r=1}^{n} \left|Y_{t_r}(\omega) - Y_{t_{r-1}}(\omega)\right|^q + 2\sup_{\substack{0\le u,v\le T\\ |u-v|\le\delta}} |Y_u(\omega) - Y_v(\omega)|^q.$$

It can be easily checked that this inequality is valid even for ω outside the above-specified subset of Ω. Therefore, it follows that

$$\sum_{r=1}^{n} E\left[\left|\overline{Y}_{t_r}(\omega) - \overline{Y}_{t_{r-1}}(\omega)\right|^q\right] \le V_q^1(Y;[0,T])^q + 2E\left[\sup_{\substack{0\le v,u\le T\\ |u-v|\le\delta}} |Y_u(\omega) - Y_v(\omega)|^q\right].$$

The desired result follows immediately upon taking the supremum over Δ. □

Theorem 5.3. *Let p,q,α,β be positive real numbers satisfying (2.1) with $\beta = 1$. Let $X = (X_u)$, $Y = (Y_u)$ be measurable processes on $[0,T]\times\Omega$ such that $X_u \in L^p(\Omega,\mathscr{F},P)$, $Y_u \in L^q(\Omega,\mathscr{F},P)$ $(0\le u\le T)$. Suppose that the pair (X,Y) satisfies conditions* (A.1) *and* (A.3)*. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$|\Delta|,|\Delta'|\le\delta \implies E\left[\sup_{0\le t\le T}\left|\mathrm{F}^{\xi}_{\Delta}(X,Y)(\Delta;t)-\mathrm{F}^{\xi'}_{\Delta'}(X,Y)(\Delta';t)\right|\right]\le\varepsilon, \tag{5.2}$$

where Δ and Δ' are finite partitions of $[0,T]$, and ξ and ξ' accompany Δ and Δ' respectively.

Proof. We may assume that $\Delta' = \{0 = t'_0 < t'_1 < \cdots < t'_m = T\}$ is a refinement of $\Delta = \{0 = t_0 < t_1 < \cdots < t_n = T\}$. Define a process $\mathrm{F}^{\xi'}_{\Delta'}(X,Y)(\Delta;t)$ $(t\in[0,T])$ by

$$\mathrm{F}^{\xi'}_{\Delta'}(X,Y)(\Delta;t) = \begin{cases} 0 & \text{if } t=0,\\ \frac{t-t_{i-1}}{t_i-t_{i-1}}\left(\mathrm{F}^{\xi'}_{\Delta'}(X,Y)(t_i)-\mathrm{F}^{\xi'}_{\Delta'}(X,Y)(t_{i-1})\right) & \text{if } t_{i-1}<t<t_i,\\ \mathrm{F}^{\xi'}_{\Delta'}(X,Y)(t_i) & \text{if } t=t_i. \end{cases}$$

By the triangle inequality,

$$\begin{aligned} &E\left[\sup_{0\le t\le T}\left|\mathrm{F}^{\xi}_{\Delta}(X,Y)(\Delta;t)-\mathrm{F}^{\xi'}_{\Delta'}(X,Y)(\Delta';t)\right|\right]\\ &\le E\left[\sup_{0\le t\le T}\left|\mathrm{F}^{\xi}_{\Delta}(X,Y)(\Delta;t)-\mathrm{F}^{\xi'}_{\Delta'}(X,Y)(\Delta;t)\right|\right]\\ &\quad+E\left[\sup_{0\le t\le T}\left|\mathrm{F}^{\xi'}_{\Delta'}(X,Y)(\Delta;t)-\mathrm{F}^{\xi'}_{\Delta'}(X,Y)(\Delta';t)\right|\right]\\ &=: I_1+I_2. \end{aligned}$$

Writing $\xi=\{\xi_k\}$ and $\xi'=\{\xi'_j\}$,

$$\begin{aligned} I_1 &= E\left[\max_{0\le i\le n}\left|\mathrm{F}^{\xi}_{\Delta}(X,Y)(\Delta;t_i)-\mathrm{F}^{\xi'}_{\Delta'}(X,Y)(\Delta;t_i)\right|\right]\\ &\le \sum_{k=1}^{n}E\left[\left|X_{\xi_k}(Y_{t_k}-Y_{t_{k-1}})-\sum_{t_{k-1}<t'_j\le t_k}X_{\xi'_j}(Y_{t'_j}-Y_{t'_{j-1}})\right|\right]. \end{aligned}$$

Take a real number α' satisfying $\alpha'>\alpha$ and $1/(\alpha' p)+1/q>1$. Then for each $1\le k\le n$, the inequality (3.7) yields

$$\begin{aligned} &E\left[\left|X_{\xi_k}(Y_{t_k}-Y_{t_{k-1}})-\sum_{t_{k-1}<t'_j\le t_k}X_{\xi'_j}(Y_{t'_j}-Y_{t'_{j-1}})\right|\right]\\ &\qquad\le\left\{2+\zeta\left(\frac{1}{\alpha' p}+\frac{1}{q}\right)\right\}V^{\alpha',1}_{p,q}(X,Y;[t_{k-1},t_k]). \end{aligned}$$

Hence, using Remark 3.3 and Lemma 4.1 along with the inequality (4.3),

$$
\begin{aligned}
I_1 &\le \left\{2+\zeta\left(\frac{1}{\alpha' p}+\frac{1}{q}\right)\right\} \sum_{k=1}^{n} V_{p,q}^{\alpha',1}(X,Y;[t_{k-1},t_k]) \\
&\le \left\{2+\zeta\left(\frac{1}{\alpha' p}+\frac{1}{q}\right)\right\} \sum_{k=1}^{n} V_{p}^{\alpha'}(X;[t_{k-1},t_k])\, V_q^1(Y;[t_{k-1},t_k]) \\
&\le \left\{2+\zeta\left(\frac{1}{\alpha' p}+\frac{1}{q}\right)\right\} \sum_{k=1}^{n} V_{p}^{\alpha}(X;[t_{k-1},t_k])^{\frac{\alpha}{\alpha'}}\, V_q^1(Y;[t_{k-1},t_k])\,\mathrm{Osc}\,\sigma(|\Delta|)^{\frac{\alpha'-\alpha}{\alpha' p}} \\
&\le \left\{2+\zeta\left(\frac{1}{\alpha' p}+\frac{1}{q}\right)\right\} V_{p}^{\alpha}(X;[0,T])^{\frac{\alpha}{\alpha'}}\, V_q^1(Y;[0,T])\,\mathrm{Osc}\,\sigma(|\Delta|)^{\frac{\alpha'-\alpha}{\alpha' p}}.
\end{aligned}
$$

Next, we estimate

$$
I_2 = E\left[\sup_{0\le t\le T}\left|\mathrm{F}_{\Delta'}^{\xi'}(X,Y)(\Delta;t)-\mathrm{F}_{\Delta'}^{\xi'}(X,Y)(\Delta';t)\right|\right]. \tag{5.3}
$$

The supremum is attained at one of the division points $t'_0, t'_1, \cdots, t'_m$ of the partition Δ'. For each $\omega\in\Omega$, let $t'_{k(\omega)}$ be the smallest real number of these points attaining the supremum in (5.3). And take $0 < i(\omega) \le n$ for which $t_{i(\omega)-1} < t'_{k(\omega)} \le t_{i(\omega)}$. Define

$$
\widetilde{X}_t(\omega) = \begin{cases} X_{t_{i(\omega)-1}}(\omega) & \text{if } 0\le t\le t_{i(\omega)-1}, \\ X_t(\omega) & \text{if } t_{i(\omega)-1} < t < t_{i(\omega)}, \\ X_{t_{i(\omega)}}(\omega) & \text{if } t_{i(\omega)} \le t \le T, \end{cases}
$$

and

$$
\widehat{X}_t(\omega) = \begin{cases} \widetilde{X}_t(\omega) & \text{if } 0\le t < t'_{k(\omega)}, \\ X_{t'_{k(\omega)}}(\omega) & \text{if } t'_{k(\omega)} \le t \le T. \end{cases}
$$

Define $\widetilde{Y}_t(\omega)$ and $\widehat{Y}_t(\omega)$ in a similar way. $\widetilde{X}, \widetilde{Y}, \widehat{X}, \widehat{Y}$ are all measurable processes on $[0,T]\times\Omega$. To estimate (5.3), we first deal with the inside of the expectation sign on the right hand side:

$$
\begin{aligned}
&\sup_{0\le t\le T}\left|\mathrm{F}_{\Delta'}^{\xi'}(X,Y)(\Delta;t)-\mathrm{F}_{\Delta'}^{\xi'}(X,Y)(\Delta';t)\right| \\
&= \left|\mathrm{F}_{\Delta'}^{\xi'}(\widetilde{X},\widetilde{Y})(\Delta;t'_k)-\mathrm{F}_{\Delta'}^{\xi'}(\widehat{X},\widehat{Y})(\Delta';t'_k)\right| \\
&\le \left|\mathrm{F}_{\Delta'}^{\xi'}(\widetilde{X},\widetilde{Y})(\Delta;t'_k)-\mathrm{F}_{\Delta'}^{\xi'}(\widehat{X},\widehat{Y})(\Delta;t'_k)\right| \\
&\quad + \left|\mathrm{F}_{\Delta'}^{\xi'}(\widehat{X},\widehat{Y})(\Delta;t'_k)-\mathrm{F}_{\Delta'}^{\xi'}(\widehat{X},\widehat{Y})(\Delta';t'_k)\right| \\
&=: J_1 + J_2.
\end{aligned}
$$

Using the inequality $\frac{t'_k - t_{i-1}}{t_i - t_{i-1}} \leq 1$ and the triangle inequality,

$$\begin{aligned} J_1 &= \left| \frac{t'_k - t_{i-1}}{t_i - t_{i-1}} \sum_{t_{i-1} < t'_j \leq t_i} X_{\xi'_j} \left(Y_{t'_j} - Y_{t'_{j-1}} \right) - \frac{t'_k - t_{i-1}}{t_i - t_{i-1}} \sum_{t_{i-1} < t'_j \leq t_i} \widehat{X}_{\xi'_j} \left(\widehat{Y}_{t'_j} - \widehat{Y}_{t'_{j-1}} \right) \right| \\ &\leq \left| \sum_{t_{i-1} < t'_j \leq t_i} X_{\xi'_j} \left(Y_{t'_j} - Y_{t'_{j-1}} \right) - X_{t_i} \left(Y_{t_i} - Y_{t_{i-1}} \right) \right| \\ &\quad + \left| \sum_{t_{i-1} < t'_j \leq t_i} \widehat{X}_{\xi'_j} \left(\widehat{Y}_{t'_j} - \widehat{Y}_{t'_{j-1}} \right) - X_{t_i} \left(\widehat{Y}_{t_i} - \widehat{Y}_{t_{i-1}} \right) \right| \\ &\quad + \left| X_{t_i} \left(Y_{t_i} - Y_{t_{i-1}} \right) - X_{t_i} \left(\widehat{Y}_{t_i} - \widehat{Y}_{t_{i-1}} \right) \right| . \end{aligned}$$

By the definitions of $\widehat{X}$ and $\widehat{Y}$,

$$\sum_{t_{i-1} < t'_j \leq t_i} \widehat{X}_{\xi'_j} \left(\widehat{Y}_{t'_j} - \widehat{Y}_{t'_{j-1}} \right) = \sum_{t_{i-1} < t'_j \leq t_i} X_{\xi'_j} \left(\widehat{Y}_{t'_j} - \widehat{Y}_{t'_{j-1}} \right) .$$

Hence,

$$\begin{aligned} J_1 &\leq \left| \sum_{t_{i-1} < t'_j \leq t_i} X_{\xi'_j} \left(Y_{t'_j} - Y_{t'_{j-1}} \right) - X_{t_i} \left(Y_{t_i} - Y_{t_{i-1}} \right) \right| \\ &\quad + \left| \sum_{t_{i-1} < t'_j \leq t_i} X_{\xi'_j} \left(\widehat{Y}_{t'_j} - \widehat{Y}_{t'_{j-1}} \right) - X_{t_i} \left(\widehat{Y}_{t_i} - \widehat{Y}_{t_{i-1}} \right) \right| \\ &\quad + \left| X_{t_i} \left(Y_{t_i} - Y_{t_{i-1}} \right) - X_{t_i} \left(\widehat{Y}_{t_i} - \widehat{Y}_{t_{i-1}} \right) \right| \\ &\leq \sum_{\ell=1}^{n} \left| \sum_{t_{\ell-1} < t'_j \leq t_\ell} X_{\xi'_j} \left(Y_{t'_j} - Y_{t'_{j-1}} \right) - X_{t_\ell} \left(Y_{t_\ell} - Y_{t_{\ell-1}} \right) \right| \\ &\quad + \sum_{\ell=1}^{n} \left| \sum_{t_{\ell-1} < t'_j \leq t_\ell} X_{\xi'_j} \left(\widehat{Y}_{t'_j} - \widehat{Y}_{t'_{j-1}} \right) - X_{t_\ell} \left(Y_{t_\ell} - Y_{t_{\ell-1}} \right) \right| \\ &\quad + \sup_{0 \leq u \leq T} |X_u| \sup_{\substack{0 \leq u, v \leq T \\ |u-v| \leq \delta}} |Y_u - Y_v| . \end{aligned}$$

Taking expectations on both sides, and then using the inequalities (3.6) and (4.3), Remark 3.3 and Lemma 4.1 and 5.2,

$$
\begin{aligned}
E[J_1] \le\ & E\left[\sum_{\ell=1}^{n}\left|\sum_{t_{\ell-1}<t'_j\le t_\ell} X_{\xi'_j}\left(Y_{t'_j}-Y_{t'_{j-1}}\right)-X_{t_\ell}\left(Y_{t_\ell}-Y_{t_{\ell-1}}\right)\right|\right] \\
& +E\left[\sum_{\ell=1}^{n}\left|\sum_{t_{\ell-1}<t'_j\le t_\ell} X_{\xi'_j}\left(\widehat{Y}_{t'_j}-\widehat{Y}_{t'_{j-1}}\right)-X_{t_\ell}\left(Y_{t_\ell}-Y_{t_{\ell-1}}\right)\right|\right] \\
& +E\left[\sup_{0\le u\le T}|X_u|^p\right]^{\frac{1}{p}} \quad E\left[\sup_{\substack{0\le u,v\le T\\ |u-v|\le\delta}}|Y_u-Y_v|^q\right]^{\frac{1}{q}} \\
\le\ & \left\{2+\zeta\left(\frac{1}{\alpha' p}+\frac{1}{q}\right)\right\} V_p^{\alpha}(X;[0,T])^{\frac{\alpha}{\alpha'}}\, V_q^1(Y;[0,T])\,\mathrm{Osc}\ \sigma(|\Delta|)^{\frac{\alpha'-\alpha}{\alpha' p}} \\
& +\left\{2+\zeta\left(\frac{1}{\alpha' p}+\frac{1}{q}\right)\right\} V_p^{\alpha}(X;[0,T])^{\frac{\alpha}{\alpha'}}\,\mathrm{Osc}\ \sigma(|\Delta|)^{\frac{\alpha'-\alpha}{\alpha' p}} \\
& \times\left(V_q^1(Y;[0,T])+2^{\frac{1}{q}}E\left[\sup_{\substack{0\le u,v\le T\\ |u-v|\le\delta}}|Y_u-Y_v|^q\right]^{\frac{1}{q}}\right) \\
& +E\left[\sup_{0\le u\le T}|X_u|^p\right]^{\frac{1}{p}} \quad E\left[\sup_{\substack{0\le u,v\le T\\ |u-v|\le\delta}}|Y_u-Y_v|^q\right]^{\frac{1}{q}}.
\end{aligned}
$$

On the other hand, using the inequality $\frac{t_i-t'_k}{t_i-t_{i-1}} \le 1$ along with the equality

$$
\sum_{t_{i-1}<t'_j\le t'_k} \widehat{X}_{\xi'_j}\left(\widehat{Y}_{t'_j}-\widehat{Y}_{t'_{j-1}}\right)=\sum_{t_{i-1}<t'_j\le t_i} \widehat{X}_{\xi'_j}\left(\widehat{Y}_{t'_j}-\widehat{Y}_{t'_{j-1}}\right)=\sum_{t_{i-1}<t'_j\le t_i} X_{\xi'_j}\left(\widehat{Y}_{t'_j}-\widehat{Y}_{t'_{j-1}}\right),
$$

J_2 can be estimated as follow:

$$
\begin{aligned}
J_2 \le\ & \left|\sum_{t_{i-1}<t'_j\le t_i} X_{\xi'_j}\left(\widehat{Y}_{t'_j}-\widehat{Y}_{t'_{j-1}}\right)\right| \\
\le\ & \left|\sum_{t_{i-1}<t'_j\le t_i} X_{\xi'_j}\left(\widehat{Y}_{t'_j}-\widehat{Y}_{t'_{j-1}}\right)-X_{t_i}\left(\widehat{Y}_{t_i}-\widehat{Y}_{t_{i-1}}\right)\right|+\left|X_{t_i}\left(\widehat{Y}_{t_i}-\widehat{Y}_{t_{i-1}}\right)\right| \\
\le\ & \sum_{\ell=1}^{n}\left|\sum_{t_{\ell-1}<t'_j\le t_\ell} X_{\xi'_j}\left(\widehat{Y}_{t'_j}-\widehat{Y}_{t'_{j-1}}\right)-X_{t_\ell}\left(Y_{t_\ell}-Y_{t_{\ell-1}}\right)\right| \\
& +\sup_{0\le u\le T}|X_u| \sup_{\substack{0\le u,v\le T\\ |u-v|\le\delta}}|Y_u-Y_v|.
\end{aligned}
$$

Taking expectations,

$$E[J_2] \leq \left\{ 2+\zeta\left(\frac{1}{\alpha' p}+\frac{1}{q}\right) \right\} V_p^{\alpha}(X;[0,T])^{\frac{\alpha}{\alpha'}} \operatorname{Osc} \sigma(|\Delta|)^{\frac{\alpha'-\alpha}{\alpha' p}}$$
$$\times \left(V_q^1(Y;[0,T]) + 2^{\frac{1}{q}} E\left[\sup_{\substack{0\leq u,v\leq T \\ |u-v|\leq\delta}} |Y_u - Y_v|^q \right]^{\frac{1}{q}} \right)$$
$$+ E\left[\sup_{0\leq u\leq T} |X_u|^p \right]^{\frac{1}{p}} \; E\left[\sup_{\substack{0\leq u,v\leq T \\ |u-v|\leq\delta}} |Y_u - Y_v|^q \right]^{\frac{1}{q}}.$$

The result now follows by putting together all of the estimates obtained above. □

Lemma 5.4. *Let* $\{Z^{(n)}\}_{n=1}^{\infty}$ *be a sequence of measurable processes on* $[0,T]\times\Omega$ *such that* $Z_{\cdot}^{(n)}(\omega)$ *is continuous on* $[0,T]$ *for each* $n\in\mathbf{N}$ *and* $\omega\in\Omega$. *Suppose that* $E\left[\sup_{0\leq t\leq T}\left|Z_t^{(n)}\right|\right]<\infty$ *for all* $n\in\mathbf{N}$ *and*

$$\lim_{n,m\to\infty} E\left[\sup_{0\leq t\leq T}\left|Z_t^{(n)}-Z_t^{(m)}\right|\right]=0.$$

Then there exists a unique measurable process $I(=I_t)$ *on* $[0,T]\times\Omega$ *such that* $I_{\cdot}(\omega)$ *is continuous on* $[0,T]$ *for each* $\omega\in\Omega$, $E\left[\sup_{0\leq t\leq T}|I_t|\right]<\infty$, *and*

$$\lim_{n\to\infty} E\left[\sup_{0\leq t\leq T}\left|Z_t^{(n)}-I_t\right|\right]=0.$$

Remark 5.5. For the sequence $\{Z^{(n)}\}_{n=1}^{\infty}$ in Lemma 5.4, there exists a subsequence $\{Z^{(n_k)}\}$ which converges to I uniformly on $[0,T]$ almost everywhere.

Proof. Since $\{Z^{(n)}\}_{n=1}^{\infty}$ is uniformly Cauchy on $[0,T]$ with respect to the L^1-norm, one can find a subsequence $\{n_k\}_{k=1}^{\infty}$ for which

$$E\left[\sup_{0\leq t\leq T}\left|Z_t^{(n_k)}-Z_t^{(n_\ell)}\right|\right]\leq\frac{1}{2^{2k}} \tag{5.4}$$

for all $\ell\geq k$. By the Chebyshev's inequality,

$$P\left(\sup_{0\leq t\leq T}\left|Z_t^{(n_k)}-Z_t^{(n_\ell)}\right|>\frac{1}{2^k}\right)\leq 2^k E\left[\sup_{0\leq t\leq T}\left|Z_t^{(n_k)}-Z_t^{(n_\ell)}\right|\right]$$
$$\leq 2^k\cdot\frac{1}{2^{2k}}=\frac{1}{2^k}$$

for all $\ell \geq k$. For each $k \in \mathbf{N}$, set

$$\Omega_k = \left\{ \sup_{0\leq t\leq T} \left| Z_t^{(n_k)} - Z_t^{(n_{k+1})} \right| > \frac{1}{2^k} \right\},$$

then $\sum_k P(\Omega_k) \leq \sum_k \frac{1}{2^k} = 1 < \infty$. Hence, the Borel-Cantelli Lemma yields $P\left(\overline{\lim_{k\to\infty}} \Omega_k \right) = 0$. Then $\left\{ Z_t^{(n_k)}(\omega) \right\}_{k=1}^{\infty}$ converges uniformly for each $\omega \in \underline{\lim}_{k\to\infty} \Omega_k^c$.

Now, define

$$I_t(\omega) := \begin{cases} \lim_{k\to\infty} Z_t^{(n_k)}(\omega) & \text{if } \omega \in \underline{\lim}_{n\to\infty} \Omega_n^c \text{ and } t \in [0,T], \\ 0 & \text{if } \omega \notin \underline{\lim}_{n\to\infty} \Omega_n^c \text{ and } t \in [0,T]. \end{cases}$$

Then $I(=I_t)$ is a measurable process on $[0,T]\times\Omega$ such that $I_\cdot(\omega)$ is continuous on $[0,T]$ for each $\omega \in \Omega$. Fix $k \in \mathbf{N}$. Then as $k \leq \ell \to \infty$, $\left| Z_t^{(n_k)} - Z_t^{(n_\ell)} \right| \longrightarrow \left| Z_t^{(n_k)} - I_t \right|$ uniformly on $[0,T]$ almost everywhere. By the dominated convergence theorem and the inequality (5.4),

$$E\left[\sup_{0\leq t\leq T} \left| Z_t^{(n_k)} - I_t \right| \right] = \lim_{\ell\to\infty} E\left[\sup_{0\leq t\leq T} \left| Z_t^{(n_k)} - Z_t^{(n_\ell)} \right| \right] \leq \frac{1}{2^{2k}},$$

which yields

$$\lim_{n\to\infty} E\left[\sup_{0\leq t\leq T} \left| Z_t^{(n)} - I_t \right| \right] = 0.$$

Moreover, by the triangle inequality,

$$E\left[\sup_{0\leq t\leq T} |I_t| \right] \leq E\left[\sup_{0\leq t\leq T} \left| Z_t^{(n_k)} - I_t \right| \right] + E\left[\sup_{0\leq t\leq T} \left| Z_t^{(n_k)} \right| \right],$$

which is finite by the above inequality as well as the assumption of this theorem. The uniqueness of I is obvious. □

Here is a main theorem of this section.

Theorem B. *Let p,q,α,β be positive real numbers satisfying condition (2.1) with $\beta = 1$. Let $X = (X_u), Y = (Y_u)$ be measurable processes on $[0,T]\times\Omega$ such that $X_u \in L^p(\Omega,\mathscr{F},P)$, $Y_u \in L^q(\Omega,\mathscr{F},P)$ $(0 \leq u \leq T)$. Suppose that the pair (X,Y) satisfies conditions* (A.1) *and* (A.3). *Then there exists a unique measurable process $I(=I_t)$ on $[0,T]\times\Omega$ for which the following hold:*

(1) *$I_\cdot(\omega)$ is continuous on $[0,T]$ for each $\omega\in\Omega$ and* $E\left[\sup_{0\le t\le T}\left|I_t\right|\right]<\infty$.

(2) *for any $\varepsilon>0$ and $\delta=\delta(\varepsilon)$ appearing in Theorem* (5.3) *such that*

$$|\Delta|\le\delta \implies E\left[\sup_{0\le t\le T}\left|\mathrm{F}^{\xi}_{\Delta}(X,Y)(\Delta;t)-I_t\right|\right]\le\varepsilon \tag{5.5}$$

where Δ is a finite partition of the interval $[0,T]$ and real number ξ accompany Δ.

Proof. Let $\{\Delta^{(m)}\}_{m=1}^{\infty}$ be a sequence of finite partitions of the time interval $[0,T]$ accompanied by $\xi^{(m)}$ with $\lim_{m\to\infty}\left|\Delta^{(m)}\right|=0$. Let $\varepsilon>0$. Then for $\delta=\delta(\varepsilon)$ appearing in Theorem 5.3,

$$\left|\Delta^{(n)}\right|,\left|\Delta^{(m)}\right|\le\delta \implies E\left[\sup_{0\le t\le T}\left|\mathrm{F}^{\xi^{(n)}}_{\Delta^{(n)}}(X,Y)(\Delta^{(n)};t)-\mathrm{F}^{\xi^{(m)}}_{\Delta^{(m)}}(X,Y)(\Delta^{(m)};t)\right|\right]\le\varepsilon.$$

Therefore, the sequence $\left\{\mathrm{F}^{\xi^{(m)}}_{\Delta^{(m)}}(X,Y)(\Delta^{(m)};t)\right\}_{m=1}^{\infty}$ satisfies the assumption of Lemma 5.4; hence, $I=(I_t)$ with the specified conditions uniquely exists. □

Definition 5.6. The measurable process $I=(I_t)$ appearing in Theorem B is called the *Young-type integral of the pair (X,Y) of measurable processes* and is denoted by $I_t=\int_0^t X_u dY_u$ for each $t\in[0,T]$.

Remark 5.7. For the sequence $\left\{\mathrm{F}^{\xi^{(m)}}_{\Delta^{(m)}}(X,Y)(\Delta^{(m)};t)\right\}_{m=1}^{\infty}$ in Theorem B, there exists a subsequence $\left\{\mathrm{F}^{\xi^{(m_k)}}_{\Delta^{(m_k)}}(X,Y)(\Delta^{(m_k)};t)\right\}_{k=1}^{\infty}$ which converges to I uniformly on $[0,T]$ almost everywhere.

Remark 5.8. Let $\Delta=\{0=t_0<\cdots<t_n=T\}$ be a finite partition of the interval $[0,T]$. For $t=t_\ell$, the equality

$$X_tY_t-X_0Y_0=\sum_k\left\{Y_{t_k}\left(X_{t_k}-X_{t_{k-1}}\right)\right\}+\sum_k\left\{X_{t_{k-1}}\left(Y_{t_k}-Y_{t_{k-1}}\right)\right\}$$

implies that under the same assumption of Theorem B, an integral $\int_0^t Y_u dX_u$ can also be defined.

We introduce the following *condition*:

(A.4) The function $\gamma(u,v)=E[|Y_u-Y_v|^q]$ is jointly continuous on $[0,T]\times[0,T]$, $\sup_{0\le u\le T}|Y_u|$ and $\sup_{\substack{0\le v,u\le T\\|u-v|\le\delta}}\left|X_u-X_v\right|$ are $\mathscr{F}-$measurable, and

$$E\left[\sup_{0\le u\le T}\left|Y_u\right|^q\right]<\infty,\ \ \lim_{\delta\to0}E\left[\sup_{\substack{0\le u,v\le T\\|u-v|\le\delta}}\left|X_u-X_v\right|^p\right]=0.$$

With this condition, the Young-type integral with respect to a pair (Y,X) of measurable processes is defined via Remark 5.9.

Remark 5.9. Let p,q,α,β be positive real numbers satisfying condition (2.1) with $\alpha = 1$. Let $X = (X_u), Y = (Y_u)$ be measurable processes on $[0,T]\times\Omega$ such that $X_u \in L^p(\Omega,\mathscr{F},P)$, $Y_u \in L^q(\Omega,\mathscr{F},P)$ $(0 \le u \le T)$. Suppose that the pair (X,Y) satisfies conditions (A.1) and (A.4). Then there exists a unique measurable process $I(=I_t)$ on $[0,T]\times\Omega$ for which the following hold:

(1) $I_\cdot(\omega)$ is continuous on $[0,T]$ for each $\omega\in\Omega$ and $E\left[\sup_{0\le t\le T}|I_t|\right] < \infty$.

(2) for any $\varepsilon > 0$ and $\delta = \delta(\varepsilon)$ appearing in Theorem (5.3),

$$|\Delta| \le \delta \implies E\left[\sup_{0\le t\le T}\left|\mathrm{F}^{\xi}_{\Delta}(Y,X)(\Delta;t) - I_t\right|\right] \le \varepsilon \tag{5.6}$$

where Δ is a finite partition of the interval $[0,T]$ and real number ξ accompanies Δ.

The next theorem establishes linearity of Young-type integrals, which is derived from the proofs of Theorems 5.3 and B.

Theorem 5.10. *Suppose that two pairs (X,Y) and (X',Y') both satisfy the assumption of Theorem* 5.3. *Let a,a' be real constants. Then the equalities*

$$\int_0^t (aX_u + a'X'_u)dY_u = a\int_0^t X_u dY_u + a'\int_0^t X'_u dY_u \quad (0\le t\le T),$$
$$\int_0^t X_u d(aY_u + a'Y'_u) = a\int_0^t X_u dY_u + a'\int_0^t X_u dY'_u \quad (0\le t\le T)$$

hold P-almost everywhere on Ω.

The Young-type integral $\int_0^t X_u dY_u$ with respect to a pair (X,Y) of measurable processes obtained in Theorem A of Section 4 arises for each fixed time t as the limiting integrable function of Riemann-Stieltjes approximating sums. On the other hand, Theorem B guarantees the existence of the family $\left(\int_0^t X_u dY_u\right)_{t\in[0,T]}$ of integrals which is a measurable process on $[0,T]\times\Omega$. The next theorem shows that it is reasonable to use the same terminology 'Young-type integral' for these two types of integrals.

Theorem 5.11. *Suppose that a pair (X,Y) of measurable processes satisfies the assumption of Theorem* B. *Fix $t\in(0,T]$. Write the Young-type integrals obtained in Theorems* A *and* B *as $H = \int_0^t X_u dY_u$ and $I_s = \int_0^s X_u dY_u$ $(0\le s\le T)$, respectively. Then $H = I_t$ P-almost everywhere on Ω.*

Proof. Let $\varepsilon > 0$ and let $\delta = \delta(\varepsilon)$ be the positive real number appearing in Theorem 5.3. Let Δ be a finite partition of $[0,t]$ with $|\Delta| \le \delta$. Let $\overline{\Delta} = \{0 = t_0 < t_1 < \cdots < t_n = t < t_{n+1} < \cdots < t_m = T\}$ be another partition of $[0,T]$ with $|\overline{\Delta}| \le \delta$ which coincides with Δ on the subinterval $[0,t]$. Let $\Delta' = \{0 = t'_0 < t'_1 < \cdots < t'_\ell = T\}$ be a finite partition of $[0,T]$ with $|\Delta'| \le \delta$. Then

$$\begin{aligned}
&E\left[\left|\sum_{i=1}^{n} X_{t_i}(Y_{t_i} - Y_{t_{i-1}}) - \mathrm{F}_{\Delta'}(X,Y)(\Delta';t)\right|\right] \\
&\le E\left[\sup_{0\le t\le T}\left|\mathrm{F}_{\overline{\Delta}}(X,Y)(\overline{\Delta};t) - \mathrm{F}_{\Delta'}(X,Y)(\Delta';t)\right|\right] \\
&\le \varepsilon.
\end{aligned}$$

If we set $\xi := \{t_i\}_{i=1}^m, \xi' := \{t'_j\}_{j=1}^\ell$, then $\mathrm{F}_{\overline{\Delta}}(X,Y)(\overline{\Delta};u) = \mathrm{F}^{\xi}_{\overline{\Delta}}(X,Y)(\overline{\Delta};u)$ and $\mathrm{F}_{\Delta'}(X,Y)(\Delta';u) = \mathrm{F}^{\xi'}_{\Delta'}(X,Y)(\Delta';u)$ $(0 \le u \le T)$. Therefore,

$$E\left[|H - I_t|\right] \le \varepsilon,$$

yielding the desired result. □

The next two theorems establish locality of the Young-type integral obtained in Theorem B. The results follow from the proofs of Theorems 5.3 and B.

Theorem 5.12. *Suppose that two pairs (X,Y) and (X',Y') of measurable processes both satisfy the assumption of Theorem* B. *Let $\tau : \Omega \longrightarrow [0,T]$ be an $\mathscr{F}$-measurable function such that*

$$X_u(\omega) = X'_u(\omega),\ Y_u(\omega) = Y'_u(\omega) \quad (0 \le u \le \tau(\omega))$$

for P-almost every $\omega \in \Omega$. Then

$$\int_0^t X_u(\omega)dY_u(\omega) = \int_0^t X'_u(\omega)dY'_u(\omega) \quad (0 \le t \le \tau(\omega))$$

for P-almost every $\omega \in \Omega$.

Theorem 5.13. *Suppose that two pairs (X,Y) and (X',Y') of measurable processes both satisfy the assumption of Theorem* B. *Let $\Omega' \in \mathscr{F}$ such that*

$$X_u(\omega) = X'_u(\omega),\ Y_u(\omega) = Y'_u(\omega) \quad (0 \le u \le T)$$

for P-almost every $\omega \in \Omega'$. Then

$$\int_0^t X_u(\omega)dY_u(\omega) = \int_0^t X'_u(\omega)dY'_u(\omega) \quad (0 \le t \le T)$$

for P-almost every $\omega \in \Omega'$.

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References

[1] M. BRENEAU, Variation totale d'une function. *Lecture Notes in Math.*, **413**, Springer-Verlag, 1974.

[2] S. NAKAO, Stochastic calculus for continuous additive functionals of zero energy, *Z. Wahr. verw. Geb.*, **68** (1985), 557–578.

[3] S. NAKAO, An extension of Stieltjes-Young integrals, *Sci. Rep. Kanazawa Univ.*, **48** (2004), 1–3.

[4] L.C. YOUNG, An inequality of the Hölder type, connected with Stieltjes integration, *Acta Math.*, **67** (1936), 251–282.