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# REMARKS ON THE GAUSS IMAGES OF COMPLETE MINIMAL SURFACES IN EUCLIDEAN FOUR-SPACE

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*Dedicated to Professor Hiroo Naitoh on his 65th birthday*

ABSTRACT. We perform a systematic study of the image of the Gauss map for complete minimal surfaces in Euclidean four-space. In particular, we give a geometric interpretation of the maximal number of exceptional values of the Gauss map of a complete orientable minimal surface in Euclidean four-space. We also provide optimal results for the maximal number of exceptional values of the Gauss map of a complete minimal Lagrangian surface in the complex two-space and the generalized Gauss map of a complete nonorientable minimal surface in Euclidean four-space.

## 1. INTRODUCTION

The study of geometric aspects of value distribution theory of complex analytic mappings has achieved many important advances. One of the most brilliant results in the study is to give a geometric interpretation of the precise maximum for the number of exceptional values of a nonconstant holomorphic map from the complex plane  $\mathbf{C}$  to a closed Riemann surface  $\bar{\Sigma}_\gamma$  of genus  $\gamma$ . Here we call a value that a function or a map never attains an *exceptional value* of the function or map. In fact, Ahlfors [1] and Chern [6] proved that the least upper bound for the number of exceptional values of a nonconstant holomorphic map from  $\mathbf{C}$  to  $\bar{\Sigma}_\gamma$  coincides with the Euler characteristic of  $\bar{\Sigma}_\gamma$  by using Nevanlinna theory (see also [26, 33, 34, 36]). In particular, for a nonconstant meromorphic function on  $\mathbf{C}$ , the geometric interpretation of the maximal number 2 of exceptional values is the Euler characteristic of the Riemann sphere  $\bar{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$ . We remark that if the closed Riemann surface is of  $\gamma \geq 2$ , then such a map does not exist because the Euler characteristic is negative.

There exist several classes of immersed surfaces in 3-dimensional space forms whose Gauss maps have value-distribution-theoretical property. For instance, Fujimoto [11, Theorem I] proved that the Gauss map of a nonflat complete minimal surface in Euclidean 3-space  $\mathbf{R}^3$  can omit at most 4 values. The fourth author and Nakajo [25] obtained the

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maximal number of exceptional values of the Lagrangian Gauss map of a weakly complete improper affine front in the affine 3-space  $\mathbf{R}^3$  is 3, unless it is an elliptic paraboloid. We note that an improper affine front is also called an improper affine map in [30]. We here call it an improper affine front because Nakajo [32] and Umehara and Yamada [40] showed that an improper affine map is a front in  $\mathbf{R}^3$ . Moreover, we [22, 23] gave similar result for flat fronts in  $\mathbf{H}^3$ . In [21], we obtained a geometric interpretation for the maximal number of exceptional values of their Gauss maps. To be precise, we gave a curvature bound for the conformal metric  $ds^2 = (1 + |g|^2)^m |\omega|^2$  on an open Riemann surface  $\Sigma$ , where  $m$  is a positive integer,  $\omega$  is a holomorphic 1-form and  $g$  is a meromorphic function on  $\Sigma$  ([21, Theorem 2.1]) and, as a corollary of the theorem, proved that the precise maximal number of exceptional values of the nonconstant meromorphic function  $g$  on  $\Sigma$  with the complete conformal metric  $ds^2$  is  $m + 2$  ([21, Corollary 2.2 and Proposition 2.4]). We remark that the geometric meaning of the 2 in  $m + 2$  is the Euler characteristic of  $\overline{\mathbf{C}}$  ([21, Remark 2.3]). Since the induced metric from  $\mathbf{R}^3$  of a minimal surface is  $ds^2 = (1 + |g|^2)^2 |\omega|^2$  (i.e.,  $m = 2$ ), the maximal number of exceptional values of the Gauss map  $g$  of a nonflat complete minimal surface in  $\mathbf{R}^3$  is  $4 (= 2 + 2)$ . For the Lagrangian Gauss map  $\nu$  of a weakly complete improper affine front, because  $\nu$  is meromorphic,  $dG$  is holomorphic and the complete metric  $d\tau^2 = (1 + |\nu|^2) |dG|^2$  (i.e.,  $m = 1$ ), the maximal number of exceptional values of the Lagrangian Gauss map of a weakly complete improper affine front is  $3 (= 1 + 2)$ , unless it is an elliptic paraboloid.

On the other hand, Fujimoto [11, Theorem II] also obtained an optimal estimate for the number of exceptional values of the Gauss map of a nonflat complete (orientable) minimal surface in  $\mathbf{R}^4$ , and Hoffman and Osserman [16] gave a similar result for a nonflat algebraic minimal surface in  $\mathbf{R}^4$  (by algebraic minimal surface, we mean a complete minimal surface of finite total curvature). Recently, we [20] gave an effective estimate for the number of exceptional values of the Gauss map for a special class of complete minimal surfaces in  $\mathbf{R}^4$  that includes algebraic minimal surfaces (this class is called the pseudo-algebraic minimal surfaces. For the corresponding result in  $\mathbf{R}^3$ , see [24]). This also provided a geometric interpretation of the Fujimoto and Hoffman-Osserman results for this class, because the estimate is described in terms of geometric invariants. However, from [20], it was still not possible to understand a geometric interpretation for general class. Moreover there has been no unified explanation for the study of the image of the Gauss map of complete minimal surfaces in  $\mathbf{R}^4$  including nonorientable case.

The purpose of this paper is to perform a systematic study of the image of the Gauss map for complete minimal surfaces in  $\mathbf{R}^4$ . The paper is organized as follows: In Section 2, we give an optimal estimate for the size of the image of the holomorphic map  $G = (g_1, \dots, g_n): \Sigma \rightarrow (\overline{\mathbf{C}})^n := \underbrace{\overline{\mathbf{C}} \times \dots \times \overline{\mathbf{C}}}_n$  on an open Riemann surface  $\Sigma$  with the complete conformal metric  $ds^2 = \prod_{i=1}^n (1 + |g_i|^2)^{m_i} |\omega|^2$ , where  $\omega$  is a holomorphic 1-form on  $\Sigma$  and

each  $m_i$  ( $i = 1, \dots, n$ ) is a positive integer (Theorem 2.1 and Proposition 2.2). The result is a generalization of [21, Corollary 2.2]. In Section 3.1, applying the result, we give a geometric interpretation of the Fujimoto result [11, Theorem II] for the maximal number of exceptional values of the Gauss map  $G = (g_1, g_2)$  of a complete orientable minimal surface in  $\mathbf{R}^4$ , that is, the maximal number deeply depends on the induced metric from  $\mathbf{R}^4$  and the Euler characteristic of  $\overline{\mathbf{C}}$ . In Section 3.2, after reviewing basic facts, we give the maximal number of exceptional values of the nonconstant part of the Gauss map of a complete minimal Lagrangian surface in the complex two-space  $\mathbf{C}^2$  (Corollary 3.3). In Section 3.3, we study the value distribution of the generalized Gauss map of a complete nonorientable minimal surface in  $\mathbf{R}^4$ . Recently the study of complete nonorientable minimal surfaces has attracted a lot of attention (for example, see [4], [5], [28], [37], [38] and [39], for a good survey see [29]). In [14], the geometry and topology of complete nonorientable maximal surfaces with lightlike singularities in the Lorentz-Minkowski 3-space are studied. In this paper, we give an effective estimate for the maximal number of exceptional values of the generalized Gauss map of a complete nonorientable minimal surface in  $\mathbf{R}^4$  (Corollary 3.4). Moreover, by using the argument of López-Martín [27], we construct examples showing that the estimate is sharp (Proposition 3.5 and Remark 3.6).

## 2. MAIN THEOREM

We first state the main theorem of this paper.

**THEOREM 2.1.** *Let  $\Sigma$  be an open Riemann surface with the conformal metric*

$$(1) \quad ds^2 = \prod_{i=1}^n (1 + |g_i|^2)^{m_i} |\omega|^2,$$

where  $G = (g_1, \dots, g_n): \Sigma \rightarrow (\overline{\mathbf{C}})^n := \underbrace{\overline{\mathbf{C}} \times \dots \times \overline{\mathbf{C}}}_n$  is a holomorphic map,  $\omega$  is a holomorphic 1-form on  $\Sigma$  and each  $m_i$  ( $i = 1, \dots, n$ ) is a positive integer. Assume that  $g_{i_1}, \dots, g_{i_k}$  ( $1 \leq i_1 < \dots < i_k \leq n$ ) are nonconstant and the others are constant. If the metric  $ds^2$  is complete and each  $g_{i_l}$  ( $l = 1, \dots, k$ ) omits  $q_{i_l} > 2$  distinct values, then we have

$$(2) \quad \sum_{l=1}^k \frac{m_{i_l}}{q_{i_l} - 2} \geq 1.$$

We note that Theorem 2.1 also holds for the case where at least one of  $m_1, \dots, m_n$  is positive and the others are zeros. For instance, we assume that  $g := g_{i_1}$  is nonconstant and the others are constant. If  $m := m_{i_1}$  is a positive integer and the others are zeros, then the inequality (2) coincides with

$$\frac{m}{q - 2} \geq 1 \iff q \leq m + 2,$$

where  $q := q_{i_1}$ . The result corresponds with [21, Corollary 2.2]. Moreover if all  $m_i$  are zeros, then the metric  $ds^2 = |\omega|^2$  is flat and complete on  $\Sigma$ . We thus may assume that

each  $g_{i_l}$  is a nonconstant meromorphic function on  $\mathbf{C}$  because there exists a holomorphic universal covering map  $\pi: \mathbf{C} \rightarrow \Sigma$  and each  $g_{i_l}$  is replaced by  $g_{i_l} \circ \pi$ . By the little Picard theorem, we have each  $g_{i_l}$  can omit at most 2 distinct values. We remark that the geometric interpretation of the precise maximum 2 for the number of exceptional values of a nonconstant meromorphic function on  $\mathbf{C}$  is the Euler characteristic of the Riemann sphere  $\overline{\mathbf{C}}$  ([1], [6]).

The inequality (2) is optimal because there exist the following examples.

**PROPOSITION 2.2.** *Let  $\Sigma$  be the complex plane punctured at  $p - 1$  distinct points  $\alpha_1, \dots, \alpha_{p-1}$  or the universal cover of that punctured plane. We set*

$$\omega = \frac{dz}{\prod_{j=1}^{p-1} (z - \alpha_j)}$$

and the map  $G = (g_1, \dots, g_n)$  is given by

$$g_{i_1} = \dots = g_{i_k} = z \quad (1 \leq i_1 < \dots < i_k \leq n)$$

and the others are constant. Then all  $g_{i_l}$  ( $l = 1, \dots, k$ ) omit  $p$  distinct values  $\alpha_1, \dots, \alpha_{p-1}, \infty$  and the metric (1) is complete if and only if

$$p \leq 2 + \sum_{l=1}^k m_{i_l}.$$

In particular, there exist examples which satisfy the equality of (2).

**PROOF.** A divergent path  $\Gamma$  in  $\Sigma$  must tend to one of the points  $\alpha_1, \dots, \alpha_{p-1}$  or  $\infty$ . Thus we have

$$\int_{\Gamma} ds = \int_{\Gamma} \prod_{i=1}^n (1 + |g_i|^2)^{m_i/2} |\omega| = C \int_{\Gamma} \frac{\prod_{l=1}^k (1 + |z|^2)^{m_{i_l}/2}}{\prod_{j=1}^{p-1} |z - \alpha_j|} |dz| = \infty$$

when  $p \leq 2 + \sum_{l=1}^k m_{i_l}$ . Here  $C$  is some constant. Then the equality of (2) holds if and only if  $p = 2 + \sum_{l=1}^k m_{i_l}$ .  $\square$

Before proceeding to the proof of Theorem 2.1, we recall the notion of chordal distance between two distinct values in  $\overline{\mathbf{C}}$  and two function-theoretic lemmas. For two distinct values  $\alpha, \beta \in \overline{\mathbf{C}}$ , we set

$$|\alpha, \beta| := \frac{|\alpha - \beta|}{\sqrt{1 + |\alpha|^2} \sqrt{1 + |\beta|^2}}$$

if  $\alpha \neq \infty$  and  $\beta \neq \infty$ , and  $|\alpha, \infty| = |\infty, \alpha| := 1/\sqrt{1 + |\alpha|^2}$ . We note that, if we take  $v_1, v_2 \in \mathbf{S}^2$  with  $\alpha = \varpi(v_1)$  and  $\beta = \varpi(v_2)$ , we have  $|\alpha, \beta|$  is a half of the chordal distance between  $v_1$  and  $v_2$ , where  $\varpi$  denotes the stereographic projection of the 2-sphere  $\mathbf{S}^2$  onto  $\overline{\mathbf{C}}$ .

LEMMA 2.3. [13, (8.12) in page 136] *Let  $g$  be a nonconstant meromorphic function on  $\Delta_R = \{z \in \mathbf{C}; |z| < R\}$  ( $0 < R \leq +\infty$ ) which omits  $q$  values  $\alpha_1, \dots, \alpha_q$ . If  $q > 2$ , then for each positive  $\eta$  with  $\eta < (q-2)/q$ , there exists a positive constant  $C'$  depending on  $q$  and  $L := \min_{i < j} |\alpha_i, \alpha_j|$  such that*

$$(3) \quad \frac{|g'_z|}{(1 + |g|^2) \prod_{j=1}^q |g, \alpha_j|^{1-\eta}} \leq C' \frac{R}{R^2 - |z|^2}.$$

LEMMA 2.4. [12, Lemma 1.6.7] *Let  $d\sigma^2$  be a conformal flat-metric on an open Riemann surface  $\Sigma$ . Then, for each point  $p \in \Sigma$ , there exists a local diffeomorphism  $\Phi$  of a disk  $\Delta_R = \{z \in \mathbf{C}; |z| < R\}$  ( $0 < R \leq +\infty$ ) onto an open neighborhood of  $p$  with  $\Phi(0) = p$  such that  $\Phi$  is an isometry, that is, the pull-back  $\Phi^*(d\sigma^2)$  is equal to the standard Euclidean metric  $ds_E^2$  on  $\Delta_R$  and that, for a specific point  $a_0$  with  $|a_0| = 1$ , the  $\Phi$ -image  $\Gamma_{a_0}$  of the curve  $L_{a_0} = \{w := a_0 s; 0 < s < R\}$  is divergent in  $\Sigma$ .*

*Proof of Theorem 2.1.* Assume that each  $g_{i_l}$  ( $l = 1, \dots, k$ ) omits  $q_{i_l}$  distinct values,  $\alpha_{i_l}^1, \dots, \alpha_{i_l}^{q_{i_l}}$ . After a suitable Möbius transformation for each  $g_{i_l}$ , we may assume that  $\alpha_{i_l}^1 = \dots = \alpha_{i_l}^{q_{i_l}} = \infty$ . Suppose that each  $q_{i_l} > 2$  and

$$(4) \quad \sum_{l=1}^k \frac{m_{i_l}}{q_{i_l} - 2} < 1.$$

Then, by (4), we ultimately suppose that  $q_{i_l} > m_{i_l} + 2$  for each  $i_l$  ( $l = 1, \dots, k$ ). Taking some positive number  $\eta$  with

$$(5) \quad 0 < \eta < \frac{q_{i_l} - 2 - m_{i_l}}{q_{i_l}}$$

for each  $i_l$  ( $l = 1, \dots, k$ ). We set

$$\lambda_{i_l} := \frac{m_{i_l}}{q_{i_l} - 2 - q_{i_l}\eta} \quad (l = 1, \dots, k).$$

For a sufficiently small number  $\eta$ , we have

$$(6) \quad \Lambda := \sum_{l=1}^k \lambda_{i_l} = \sum_{l=1}^k \frac{m_{i_l}}{q_{i_l} - 2 - q_{i_l}\eta} < 1$$

and

$$(7) \quad \frac{\lambda_{i_l}}{1 - \Lambda} > 1 \quad (l = 1, \dots, k).$$

Then we define a new metric

$$(8) \quad d\sigma^2 = |\hat{\omega}_z|^{\frac{2}{1-\Lambda}} \prod_{l=1}^k \left( \frac{1}{|g'_{i_l}|} \prod_{j=1}^{q_{i_l}-1} \left( \frac{|g_{i_l} - \alpha_j^l|}{\sqrt{1 + |\alpha_j^l|^2}} \right)^{1-\eta} \right)^{\frac{2\lambda_{i_l}}{1-\Lambda}} |dz|^2$$

on  $\Sigma' = \{p \in \Sigma; g'_{i_l} \neq 0 \text{ for each } l\}$ , where  $\omega = \hat{\omega}_z dz$  and  $g'_{i_l} = dg_{i_l}/dz$ . Take a point  $p \in \Sigma'$ . Since  $d\sigma^2$  is flat, by Lemma 2.4, there exists an isometry  $\Phi$  satisfying  $\Phi(0) = p$

from a disk  $\Delta_R = \{z \in \mathbf{C}; |z| < R\}$  ( $0 < R \leq +\infty$ ) with the standard Euclidean metric  $ds_E^2$  onto an open neighborhood of  $p \in \Sigma'$  with the metric  $d\sigma^2$ , such that, for a specific point  $a_0$  with  $|a_0| = 1$ , the  $\Phi$ -image  $\Gamma_{a_0}$  of the curve  $L_{a_0} = \{w = a_0 s; 0 < s < R\}$  is divergent in  $\Sigma'$ . For brevity, we denote  $g_{i_l} \circ \Phi$  on  $\Delta_R$  by  $g_{i_l}$  in the following. By Lemma 2.3, for each  $i_l$ , we get

$$(9) \quad R \leq C'_{i_l} \frac{1 + |g_{i_l}(0)|^2}{|g'_{i_l}(0)|} \prod_{j=1}^{q_{i_l}} |g_{i_l}(0), \alpha_j^l|^{1-\eta} < +\infty,$$

that is, the radius  $R$  is finite. Hence

$$L_{d\sigma}(\Gamma_{a_0}) = \int_{\Gamma_{a_0}} d\sigma = R < +\infty,$$

where  $L_{d\sigma}(\Gamma_{a_0})$  denotes the length of  $\Gamma_{a_0}$  with respect to the metric  $d\sigma^2$ .

Now we prove that  $\Gamma_{a_0}$  is divergent in  $\Sigma$ . Indeed, if not, then  $\Gamma_{a_0}$  must tend to a point  $p_0 \in \Sigma \setminus \Sigma'$ , where  $g'_{i_l}(p_0) = 0$  for some  $i_l$ . Taking a local complex coordinate  $\zeta := g'_{i_l}$  in a neighborhood of  $p_0$  with  $\zeta(p_0) = 0$ , we can write the metric  $d\sigma^2$  as

$$d\sigma^2 = |\zeta|^{-2\lambda_{i_l}/(1-\Lambda)} w |d\zeta|^2,$$

with some positive function  $w$ . Since  $\lambda_{i_l}/(1-\Lambda) > 1$ , we have

$$R = \int_{\Gamma_{a_0}} d\sigma > \tilde{C} \int_{\Gamma_{a_0}} \frac{|d\zeta|}{|\zeta|^{\lambda_{i_l}/(1-\Lambda)}} = +\infty.$$

Moreover, in the same way, if there exists a subset  $\{l_1, \dots, l_m\}$  in  $\{1, \dots, k\}$  such that each  $g_{i_{l_j}}$  ( $j = 1, \dots, m$ ) have a zero at  $p_0$ , we also get  $R = +\infty$  because

$$\sum_{s=1}^m \frac{\lambda_{i_{l_s}}}{1-\Lambda} > 1.$$

These contradict that  $R$  is finite.

Since  $\Phi^*d\sigma^2 = |dz|^2$ , we have by (8)

$$|\hat{\omega}_z| = \prod_{l=1}^k \left( |g'_{i_l}| \prod_{j=1}^{q_{i_l}-1} \left( \frac{\sqrt{1 + |\alpha_j^l|^2}}{|g_{i_l} - \alpha_j^l|} \right)^{1-\eta} \right)^{\lambda_{i_l}}.$$

By Lemma 2.3, we have

$$\begin{aligned} \Phi^*ds &= |\hat{\omega}_z| \prod_{i=1}^n (1 + |g_i|^2)^{m_i/2} |dz| \\ &\leq C_1 \left( \prod_{l=1}^k |g'_{i_l}| (1 + |g_{i_l}|^2)^{m_{i_l}/2\lambda_{i_l}} \prod_{j=1}^{q_{i_l}-1} \left( \frac{\sqrt{1 + |\alpha_j^l|^2}}{|g_{i_l} - \alpha_j^l|} \right)^{1-\eta} \right)^{\lambda_{i_l}} |dz| \\ &= C_1 \prod_{l=1}^k \left( \frac{|g'_{i_l}|}{(1 + |g_{i_l}|^2) \prod_{j=1}^{q_{i_l}} |g_{i_l}, \alpha_j^l|^{1-\eta}} \right)^{\lambda_{i_l}} |dz| \leq C_2 \left( \frac{R}{R^2 - |z|^2} \right)^\Lambda |dz|. \end{aligned}$$

Now we consider the geodesic distance  $d(p)$  with the respect to the metric  $ds^2$  from each point  $p \in \Sigma$  to the boundary of  $\Sigma$ . Then we have

$$d(p) \leq \int_{\Gamma_{a_0}} ds = \int_{L_{a_0}} \Phi^* ds \leq C_2 \int_{L_{a_0}} \left( \frac{R}{R^2 - |z|^2} \right)^\Lambda |dz| \leq C_2 \frac{R^{1-\Lambda}}{1-\Lambda} < +\infty$$

because  $0 < \Lambda < 1$ . This contradicts the assumption that the metric  $ds^2$  is complete.  $\square$

### 3. APPLICATIONS

**3.1. Gauss images of complete orientable minimal surfaces in  $\mathbf{R}^4$ .** We first recall some basic facts of minimal surfaces in  $\mathbf{R}^4$ . Details can be found, for example, [7, 9, 16, 17, 35]. Let  $X = (x^1, x^2, x^3, x^4): \Sigma \rightarrow \mathbf{R}^4$  be an oriented minimal surface in  $\mathbf{R}^4$ . By associating a local complex coordinate  $z = u + \sqrt{-1}v$  with each positive isothermal coordinate system  $(u, v)$ ,  $\Sigma$  is considered as a Riemann surface whose conformal metric is the induced metric  $ds^2$  from  $\mathbf{R}^4$ . Then

$$(10) \quad \Delta_{ds^2} X = 0$$

holds, that is, each coordinate function  $x^i$  is harmonic. With respect to the local coordinate  $z$  of the surface, (10) is given by

$$\bar{\partial} \partial X = 0,$$

where  $\partial = (\partial/\partial u - \sqrt{-1}\partial/\partial v)/2$ ,  $\bar{\partial} = (\partial/\partial u + \sqrt{-1}\partial/\partial v)/2$ . Hence each  $\phi_i := \partial x^i dz$  ( $i = 1, 2, 3, 4$ ) is a holomorphic 1-form on  $\Sigma$ . If we set

$$\omega = \phi_1 - \sqrt{-1}\phi_2, \quad g_1 = \frac{\phi_3 + \sqrt{-1}\phi_4}{\phi_1 - \sqrt{-1}\phi_2}, \quad g_2 = \frac{-\phi_3 + \sqrt{-1}\phi_4}{\phi_1 - \sqrt{-1}\phi_2},$$

then  $\omega$  is a holomorphic 1-form and  $g_1$  and  $g_2$  are meromorphic functions on  $\Sigma$ . Moreover the holomorphic map  $G := (g_1, g_2): \Sigma \rightarrow \overline{\mathbf{C}} \times \overline{\mathbf{C}}$  coincides with the Gauss map of  $X(\Sigma)$ . We remark that the Gauss map of  $X(\Sigma)$  in  $\mathbf{R}^4$  is the map from each point of  $\Sigma$  to its oriented tangent plane, the set of all oriented (tangent) planes in  $\mathbf{R}^4$  is naturally identified with the quadric

$$\mathbf{Q}^2(\mathbf{C}) = \{[w^1 : w^2 : w^3 : w^4] \in \mathbf{P}^3(\mathbf{C}); (w^1)^2 + \cdots + (w^4)^2 = 0\}$$

in  $\mathbf{P}^3(\mathbf{C})$ , and the quadric  $\mathbf{Q}^2(\mathbf{C})$  is biholomorphic to the product of the Riemann spheres  $\overline{\mathbf{C}} \times \overline{\mathbf{C}}$ . Furthermore, the induced metric from  $\mathbf{R}^4$  is given by

$$(11) \quad ds^2 = (1 + |g_1|^2)(1 + |g_2|^2)|\omega|^2.$$

Applying Theorem 2.1 to the metric  $ds^2$ , we can get the Fujimoto theorem for the Gauss map of complete orientable minimal surfaces in  $\mathbf{R}^4$ .

**THEOREM 3.1.** [11, Theorem II] *Let  $X: \Sigma \rightarrow \mathbf{R}^4$  be a complete orientable nonflat minimal surface and  $G = (g_1, g_2): \Sigma \rightarrow \overline{\mathbf{C}} \times \overline{\mathbf{C}}$  the Gauss map of  $X(\Sigma)$ .*



(i) Assume that  $g_1$  and  $g_2$  are both nonconstant and omit  $q_1$  and  $q_2$  distinct values respectively. If  $q_1 > 2$  and  $q_2 > 2$ , then we have

$$(12) \quad \frac{1}{q_1 - 2} + \frac{1}{q_2 - 2} \geq 1.$$

(ii) If either  $g_1$  or  $g_2$ , say  $g_2$ , is constant, then  $g_1$  can omit at most 3 distinct values.

PROOF. We first show (i). Since  $g_1$  and  $g_2$  are both nonconstant and  $m_1 = m_2 = 1$  from (11), we can prove the inequality (12) by Theorem 2.1. Next we show (ii). If we set that  $g_1$  omits  $q_1$  values, then we obtain

$$\frac{1}{q_1 - 2} \geq 1$$

from Theorem 2.1 because  $m_1 = 1$ . Thus we have  $q_1 \leq 3$ .  $\square$

Hence we reveal that the Fujimoto theorem depends on the orders of the factors  $(1 + |g_1|^2)$  and  $(1 + |g_2|^2)$  in the induced metric from  $\mathbf{R}^4$  and the Euler characteristic of the Riemann sphere  $\overline{\mathbf{C}}$ .

**3.2. Gauss images of complete minimal Lagrangian surfaces in  $\mathbf{C}^2$ .** There exists a complex representation for a minimal Lagrangian surface  $\Sigma (\subset \mathbf{C}^2)$  in terms of holomorphic data. On the representation for the surface  $\Sigma$ , Chen-Morvan [8] proved that there exists an explicit correspondence in  $\mathbf{C}^2$  between minimal Lagrangian surfaces and holomorphic curves with a nondegenerate condition. Indeed, this correspondence is given by exchanging the orthogonal complex structure  $J$  in  $\mathbf{C}^2$  to another one on  $\mathbf{R}^4 = \mathbf{C}^2$ . For the complete case, this result can also be proved from [31, Theorem II] and the well-known fact [15] that any minimal Lagrangian submanifold in  $\mathbf{C}^n$  is stable. More generally, Hélein-Romon [18, 19] and the first author [2, 3] proved that every Lagrangian surface  $\Sigma$  in  $\mathbf{C}^2$ , not necessarily minimal, is represented in terms of a plus spinor (or a minus spinor) of the  $\text{spin}^{\mathbf{C}}$  bundle  $(\underline{\mathbf{C}}_{\Sigma} \oplus \underline{\mathbf{C}}_{\Sigma}) \oplus (K_{\Sigma}^{-1} \oplus K_{\Sigma})$  satisfying the Dirac equation with potential (see [3, Section 1] for more details). Here,  $\underline{\mathbf{C}}_{\Sigma}$  and  $K_{\Sigma}$  denote respectively the trivial complex line bundle and the canonical complex line bundle of  $\Sigma$ . Note that the representation in terms of plus spinors in  $\Gamma(\underline{\mathbf{C}}_{\Sigma} \oplus \underline{\mathbf{C}}_{\Sigma}) = \Gamma(\Sigma \times \mathbf{C}^2)$  given by the first author is a natural generalization of the one given by Chen-Morvan. Here we remark that the Lagrangian angle of any minimal Lagrangian surface is constant. Combining these results, we get the following:

**THEOREM 3.2.** ([8], [2, 3]) *Let  $\Sigma$  be a Riemann surface with an isothermal coordinate  $z = u + \sqrt{-1}v$  around each point. Let  $F = (F_1, F_2): \Sigma \rightarrow \mathbf{C}^2$  be a holomorphic map satisfying  $|S_1|^2 + |S_2|^2 \neq 0$  everywhere on  $\Sigma$ , where  $S_1 := (F_2)'_z = dF_2/dz$  and  $S_2 := -(F_1)'_z = -dF_1/dz$ . Then*

$$(13) \quad f = \frac{1}{\sqrt{2}} e^{\sqrt{-1}\beta/2} (F_1 - \sqrt{-1}F_2, F_2 + \sqrt{-1}F_1)$$

is a minimal Lagrangian conformal immersion from  $\Sigma$  to  $\mathbf{C}^2$  with constant Lagrangian angle  $\beta \in \mathbf{R}/2\pi\mathbf{Z}$ . The induced metric  $ds^2$  on  $\Sigma$  by  $f$  and its Gaussian curvature  $K_{ds^2}$  are respectively given by

$$(14) \quad ds^2 = (|S_1|^2 + |S_2|^2)|dz|^2, \quad K_{ds^2} = -2 \frac{|S_1(S_2)_z - S_2(S_1)_z|}{(|S_1|^2 + |S_2|^2)^3}.$$

Conversely, every minimal Lagrangian immersion  $f: M \rightarrow \mathbf{C}^2$  with constant Lagrangian angle  $\beta$  is congruent with the one constructed as above.

Set a meromorphic function  $g := -S_2/S_1$ . Then

$$G := (g, e^{\sqrt{-1}\beta}): \Sigma \rightarrow \overline{\mathbf{C}} \times \overline{\mathbf{C}}$$

can be regarded as the Gauss map of  $F(\Sigma)$  in  $\mathbf{R}^4 = \mathbf{C}^2$  (cf. [16, 17]). Thus we get the following result.

**COROLLARY 3.3.** *The first component  $g$  of the Gauss map of a complete minimal Lagrangian surface in  $\mathbf{C}^2$  which is not a Lagrangian plane can omit at most 3 values.*

**PROOF.** We assume that  $g$  omits  $q$  distinct values and set a holomorphic 1-form  $\omega := S_1 dz$  on  $\Sigma$ . In terms of the data  $(\omega, g)$  of  $\Sigma$ , the induced metric can be rewritten as  $ds^2 = (1 + |g|^2)|\omega|^2$ , that is,  $m_1 = 1$  and  $m_2 = 0$ . For this case, the first component  $g$  of the Gauss map is nonconstant and the second one is constant. From Theorem 2.1, we obtain  $q \leq 1 + 2 = 3$ .  $\square$

### 3.3. Generalized Gauss images of complete nonorientable minimal surfaces in $\mathbf{R}^4$ .

We first summarize some basic facts of nonorientable minimal surfaces in  $\mathbf{R}^4$ . For more details, we refer the reader to [10] and [29]. Let  $\widehat{X}: \widehat{\Sigma} \rightarrow \mathbf{R}^4$  be a conformal minimal immersion of a nonorientable Riemann surface  $\widehat{\Sigma}$  in  $\mathbf{R}^4$ . If we consider the orientable conformal double cover  $\pi: \Sigma \rightarrow \widehat{\Sigma}$ , then the composition  $X := \widehat{X} \circ \pi: \Sigma \rightarrow \mathbf{R}^4$  is a conformal minimal immersion of the orientable Riemann surface  $\Sigma$  in  $\mathbf{R}^4$ . Let  $I: \Sigma \rightarrow \Sigma$  denote the antiholomorphic order two deck transformation associated to the orientable cover  $\pi: \Sigma \rightarrow \widehat{\Sigma}$ , then  $I^*(\phi_j) = \bar{\phi}_j$  ( $j = 1, \dots, 4$ ) or equivalently,

$$(15) \quad g_1 \circ I = -\frac{1}{\bar{g}_1}, \quad g_2 \circ I = -\frac{1}{\bar{g}_2}, \quad I^*\omega = \overline{g_1 g_2 \omega}.$$

Conversely, if  $(g_1, g_2, \omega)$  is the Weierstrass data of an orientable minimal surface  $X: \Sigma \rightarrow \mathbf{R}^4$  and  $I$  is an antiholomorphic involution without fixed points in  $\Sigma$  satisfying (15), then the unique map  $\widehat{X}: \widehat{\Sigma} = \Sigma/\langle I \rangle \rightarrow \mathbf{R}^4$  satisfying that  $X = \widehat{X} \circ \pi$  is a nonorientable minimal surface in  $\mathbf{R}^4$ .

The fact that  $g_k \circ I = -(\bar{g}_k)^{-1}$  ( $k = 1, 2$ ) implies the existence of a map  $\hat{g}_k: \widehat{\Sigma} \rightarrow \mathbf{RP}^2$  satisfying  $\hat{g}_k \circ \pi = \pi_0 \circ g_k$ , where  $\pi_0: \overline{\mathbf{C}} \rightarrow \mathbf{RP}^2 \equiv \overline{\mathbf{C}}/\langle I_0 \rangle$  is the natural projection and  $I_0 := -(\bar{z})^{-1}$  is the antipodal map of  $\overline{\mathbf{C}}$ . We call the map  $\widehat{G} = (\hat{g}_1, \hat{g}_2): \widehat{\Sigma} \rightarrow \mathbf{RP}^2 \times \mathbf{RP}^2$

the *generalized Gauss map* of  $\widehat{X}(\widehat{\Sigma})$ . Applying Theorem 3.1 to the generalized Gauss map, we get the following:

**COROLLARY 3.4.** *Let  $\widehat{X}: \widehat{\Sigma} \rightarrow \mathbf{R}^4$  be a nonflat complete nonorientable minimal surface and  $\widehat{G} = (\widehat{g}_1, \widehat{g}_2)$  the generalized Gauss map of  $\widehat{X}(\widehat{\Sigma})$ .*

- (i) *Assume that  $\widehat{g}_1$  and  $\widehat{g}_2$  are both nonconstant and omit  $q_1$  and  $q_2$  distinct points in  $\mathbf{RP}^2$  respectively. If  $q_1 > 1$  and  $q_2 > 1$ , then*

$$(16) \quad \frac{1}{q_1 - 1} + \frac{1}{q_2 - 1} \geq 2.$$

- (ii) *If either  $\widehat{g}_1$  or  $\widehat{g}_2$ , say  $\widehat{g}_2$ , is constant, then  $\widehat{g}_1$  can omit at most 1 point in  $\mathbf{RP}^2$ .*

The inequality (16) is optimal because there exist the following examples.

**PROPOSITION 3.5.** *There exist nonflat complete nonorientable minimal surfaces in  $\mathbf{R}^4$  each of which components  $\widehat{g}_i$  ( $i = 1, 2$ ) of the generalized Gauss map  $\widehat{G} = (\widehat{g}_1, \widehat{g}_2)$  is nonconstant and omits 2 distinct points in  $\mathbf{RP}^2$ .*

**PROOF.** We take 2 distinct points  $\alpha, \beta$  in  $\mathbf{C} \setminus \{0\}$  and assume that  $\alpha \neq -(\bar{\beta})^{-1}$ . Let  $\Sigma$  be the complex plane punctured at 4 distinct points  $\alpha, \beta, -(\bar{\alpha})^{-1}, -(\bar{\beta})^{-1}$ . We set

$$\check{g}_1 = z, \quad \check{g}_2 = z, \quad \check{\omega} = \frac{dz}{(z - \alpha)(z - \beta)(\bar{\alpha}z + 1)(\bar{\beta}z + 1)}$$

on  $\Sigma$ . If we define  $\check{I}: \Sigma \rightarrow \Sigma$ ,  $\check{I}(z) = -(\bar{z})^{-1}$ , then  $\check{I}$  is an antiholomorphic involution without fixed points and the following inequalities hold:

$$(17) \quad \check{g}_1 \circ \check{I} = -\frac{1}{\bar{g}_1}, \quad \check{g}_2 \circ \check{I} = -\frac{1}{\bar{g}_2}, \quad \check{I}^* \check{\omega} = \overline{\check{g}_1 \check{g}_2 \check{\omega}}.$$

Thus if we set

$$\check{\phi}_1 = \frac{1}{2}(1 + \check{g}_1 \check{g}_2) \check{\omega}, \quad \check{\phi}_2 = \frac{\sqrt{-1}}{2}(1 - \check{g}_1 \check{g}_2) \check{\omega}, \quad \check{\phi}_3 = \frac{1}{2}(\check{g}_1 - \check{g}_2) \check{\omega}, \quad \check{\phi}_4 = -\frac{\sqrt{-1}}{2}(\check{g}_1 + \check{g}_2) \check{\omega},$$

then we easily show that  $\check{I}^* \check{\phi}_i = \bar{\phi}_i$  ( $i = 1, \dots, 4$ ). Moreover, these holomorphic 1-forms satisfy that  $\sum_{i=1}^4 \check{\phi}_i^2 \equiv 0$  and  $\sum_{i=1}^4 |\check{\phi}_i|^2$  is a complete conformal metric on  $\Sigma$ .

Let  $\widetilde{\Sigma}$  be a universal cover surface of  $\Sigma$ . By the uniformization theorem, we may assume that  $\widetilde{\Sigma}$  is the unit disk  $\mathbf{D}$ . Let  $\pi: \mathbf{D} \rightarrow \Sigma$  be the conformal universal covering map and  $\widetilde{I}$  a lift of  $\check{I}$  to  $\mathbf{D}$ . If we set  $\widetilde{\phi}_i := \pi^*(\check{\phi}_i)$ , then  $\widetilde{I}^*(\widetilde{\phi}_i) = \bar{\phi}_i$  ( $i = 1, \dots, 4$ ). Since  $\widetilde{I}$  is an antiholomorphic involution on  $\Sigma$  without fixed points,  $\widetilde{I}^{2k+1}$  ( $k \in \mathbf{Z}$ ) is also an antiholomorphic transformation on  $\mathbf{D}$  without fixed points. From the argument of the proof of Lemma 1 in [27],  $\widetilde{I}^{2k}$  ( $k \in \mathbf{Z} \setminus \{0\}$ ) has no fixed points on  $\mathbf{D}$ ,  $\langle \widetilde{I}^2 \rangle \simeq \mathbf{Z}$ , and  $\mathbf{D}/\langle \widetilde{I}^2 \rangle$  is biholomorphic to the annulus  $A(R) = \{z \in \mathbf{C}; R^{-1} < |z| < R\}$  for a suitable  $R > 1$ . Since  $(\widetilde{I}^2)^*(\widetilde{\phi}_i) = \widetilde{\phi}_i$ , each holomorphic 1-form  $\widetilde{\phi}_i$  ( $i = 1, \dots, 4$ ) can be induced on the quotient  $\mathbf{D}/\langle \widetilde{I}^2 \rangle$ . The corresponding holomorphic 1-forms on  $\mathbf{D}/\langle \widetilde{I}^2 \rangle$  are denoted by  $\phi_1, \phi_2, \phi_3$  and  $\phi_4$ , and obviously satisfy that  $\sum_{i=1}^4 \phi_i^2 \equiv 0$ ,  $ds^2 := \sum_{i=1}^4 |\phi_i|^2$  is a

complete conformal metric on  $\mathbf{D}/\langle \tilde{I}^2 \rangle \simeq A(R)$  and  $I^*(\phi_i) = \bar{\phi}_i$  ( $i = 1, \dots, 4$ ), where  $I: A(R) \rightarrow A(R)$  induced by  $\tilde{I}$ . Then it holds that  $I(z) = -(\bar{z})^{-1}$  on  $A(R)$ . Moreover the two meromorphic functions

$$g_1 = \frac{\phi_3 + \sqrt{-1}\phi_4}{\phi_1 - \sqrt{-1}\phi_2} \quad \text{and} \quad g_2 = \frac{-\phi_3 + \sqrt{-1}\phi_4}{\phi_1 - \sqrt{-1}\phi_2}$$

on  $A(R)$  omit 4 points  $\alpha, \beta, -(\bar{\alpha})^{-1}$  and  $-(\bar{\beta})^{-1}$  in  $\bar{\mathbf{C}}$ .

Let  $f: \bar{\mathbf{C}} \rightarrow \bar{\mathbf{C}}$  be a rational function given in Lemma 2 in [27], that is, the function  $f$  satisfies the following three conditions:

- (a) The only poles of  $f$  are 0 and  $\infty$ ,
- (b)  $f \circ I_0 = \bar{f}$ ,
- (c)  $f$  has no zeros on the circle  $\{z; |z| = 1\}$ .

Set  $\phi_j = (\varphi_j/z)dz$  ( $j = 1, \dots, 4$ ) and write the Laurent series expansion of  $\varphi_j$  as

$$\varphi_j(z) = a_0^j + \sum_{n>0} (a_n^j z^n + (-1)^{n+1} \bar{a}_n^j z^{-n}), \quad a_0^j \in \sqrt{-1}\mathbf{R}.$$

We easily check that the Laurent series expansion of  $f$  is written as

$$f(z) = \sum_{n=1}^m (b_n z^n + (-1)^n \bar{b}_n z^{-n}),$$

where  $m \in \mathbf{Z}_+$ . Let  $k$  be an odd positive number with  $k > m$ . Then it holds that

$$(18) \quad \text{Res}_{z=0} \left( \left[ \sum_{n>0} (a_n^j z^{kn} + (-1)^{n+1} \bar{a}_n^j z^{-kn}) \right] f(z) \frac{dz}{z} \right) = 0, \quad j = 1, \dots, 4.$$

Furthermore, by the virtue of the property for  $f(z)$ , we have

$$(19) \quad \text{Res}_{z=0} \left( a_0^j f(z) \frac{dz}{z} \right) = 0, \quad j = 1, \dots, 4.$$

We consider the covering  $T_k: A(R^{1/k}) \rightarrow A(R)$ ,  $T_k(z) = z^k$  and define the holomorphic 1-forms  $\psi_j$  ( $j = 1, \dots, 4$ ) on  $A(R^{1/k})$  as follows:

$$\psi_j := f(z) T_k^*(\phi_j) = k f(z) \varphi_j(z^k) \frac{dz}{z}.$$

From (18) and (19), we deduce that each  $\int_1^z \psi_j$  is well-defined on  $A(R^{1/k})$ . Moreover  $\sum_{j=1}^4 \psi_j^2 \equiv 0$  holds. Since  $k$  is odd, we have

$$(20) \quad I^*(\psi_j) = \bar{\psi}_j, \quad j = 1, \dots, 4,$$

where  $I: A(R^{1/k}) \rightarrow A(R^{1/k})$  is the lift of the previous involution in  $A(R)$ . Indeed,  $I$  is represented as  $I(z) = -(\bar{z})^{-1}$  here. We note that  $\lim_{k \rightarrow \infty} R^{1/k} = 1$  and the zeros of  $f$  are not on the circle  $\{z; |z| = 1\}$ . Thus we take  $k$  large enough, we can assume that  $f$  never

vanishes on the closure of  $A(R^{1/k})$ . Furthermore, since the only poles of  $f$  are 0 and  $\infty$ , there exists some real number  $c > 1$  such that

$$\frac{1}{c} < |f(z)| < c,$$

for any  $z \in A(R^{1/k})$ . Hence  $\sum_{j=1}^4 |\psi_j|^2 \neq 0$ , and if we define  $ds_0^2 = \sum_{j=1}^4 |\psi_j|^2$ , then we have

$$\frac{1}{c^2} T_k^*(ds^2) \leq ds_0^2 \leq c^2 T_k^*(ds^2).$$

Since  $ds^2$  is complete, the metric  $T_k^*(ds^2)$  and  $ds_0^2$  are also complete.

Therefore we obtain the conformal minimal immersion

$$X: A(R^{1/k}) \rightarrow \mathbf{R}^4, \quad X(z) = \operatorname{Re} \int_1^z (\psi_1, \psi_2, \psi_3, \psi_4)$$

and the induced metric  $ds_0^2$  is complete and each component of the Gauss map  $g_i \circ T_k$  ( $i = 1, 2$ ) omits 4 points in  $\overline{\mathbf{C}}$ . From (20), the immersion  $X$  induces a minimal immersion from the Möbius strip  $A(R^{1/k})/\langle I \rangle$  to  $\mathbf{R}^4$ , and each component of the generalized Gauss map omits 2 points in  $\mathbf{RP}^2$ .  $\square$

REMARK 3.6. From a similar argument of the proof, we can show that there exist nonflat complete nonorientable minimal surfaces in  $\mathbf{R}^4$  one of which components of the generalized Gauss map is nonconstant and omits 1 point in  $\mathbf{RP}^2$  and the other is constant.

Finally, we deal with value distribution of the generalized Gauss map of complete nonorientable minimal surfaces in  $\mathbf{R}^4$  of finite total curvature. Applying [17, Theorem 6.9] (see also [20, Theorem 3.2]) to the generalized Gauss map, we get the following:

PROPOSITION 3.7. *Let  $\widehat{X}: \widehat{\Sigma} \rightarrow \mathbf{R}^4$  be a nonflat complete nonorientable minimal surface of finite total curvature and  $\widehat{G} = (\widehat{g}_1, \widehat{g}_2)$  the generalized Gauss map of  $\widehat{X}(\widehat{\Sigma})$ .*

- (i) *Assume that  $\widehat{g}_1$  and  $\widehat{g}_2$  are both nonconstant. Then at least one of them can omit at most 1 point in  $\mathbf{RP}^2$ .*
- (ii) *If either  $\widehat{g}_1$  or  $\widehat{g}_2$ , say  $\widehat{g}_2$ , is constant, then  $\widehat{g}_1$  can omit at most 1 point in  $\mathbf{RP}^2$ .*

However we do not know whether Proposition 3.7 is optimal or not.

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