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Distinguishing Siegel theta series of degree 4 for the 32-dimensional even unimodular extremal lattices.

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Abstract

In a previous paper we showed that if one particulat Fourier coefficient of the Siegel theta series of degree 4 for a 32-dimensional even unimodular extremal lattice is known then the other Fourier coefficients of the series are in principle determined. In this paper we choose the quaternary positive definite symmetric matrix $\mathfrak{T}_{40}$, and calculate the Fourier coefficient $a(\mathfrak{T}_{40}, \mathcal{L}_{32})$ of the Siegel theta series of degree 4 associated with the five even unimodular extremal lattices which come from the five binary self-dual extremal $[32, 16, 8]$ codes. As a result we can show that the five Siegel theta series of degree 4 associated with the five 32-dimensional even unimodular extremal lattices are distinct.

1 Introduction

In our previous paper [11] we described a method for computing the Fourier coefficients of Siegel theta series of degree up to 3 associated with the 32-dimensional even unimodular extremal lattices $\mathcal{L}_{32}$. The computation is independent of the lattices chosen in this family. Further we *almost* determined the Fourier coefficients of the Siegel theta series of degree 4 associated with a 32-dimensional even unimodular extremal lattice $\mathcal{L}_{32}$, in the sense that if one could determine one of the initial values of the Fourier coefficients, then the other values would follow from it automatically. In this article we compute the Fourier coefficient $a(\mathfrak{T}_{40}, \mathcal{L}_{32})$ for each of the five extremal even unimodular lattices $\mathcal{L}_{32}$ that are constructed from the five self-dual doubly even binary codes $\mathbf{C}$ of length 32. This settles one of the questions left unanswered in [11].

First we express $a(\mathfrak{T}_{40}, \mathcal{L}_{32})$ as the sum of 43 different types of partial sums, each of which is the number of quadruples of vectors $\langle \mathbf{x}_1, \cdots, \mathbf{x}_4 \rangle \in (\Lambda_4(\mathcal{L}_{32}))^4$ satisfying certain specified inner product relations. Here $(\Lambda_4(\mathcal{L}_{32}))^4$ is the direct product of 4 copies of the set of norm 4 vectors in $\mathcal{L}_{32}$. Next we show that apart from few special cases each partial sum can be transformed into the enumeration of the quadruples of code words $\langle \mathbf{u}_1, \cdots, \mathbf{u}_4 \rangle \in \mathbf{C}_8^4$ satisfying the specified intersection relations. Here $\mathbf{C}_8$ is the set of code words of weight 8 in the code $\mathbf{C}$. To make the counting explicit we need to know some terms of the multiple weight enumerators of genus up to 4 for the codes, or the intersection enumerators of the codes (defined below). In the last section we show that the Siegel theta series of degree 4

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associated with the five extremal even unimodular lattices are distinct. This result settles a question raised by Salvati Manni [14].

2 Coding Theoretic Preliminaries

2.1 Binary linear codes

Let $\mathbb{F}_2 = GF(2)$ be the field of 2 elements. Let $V = \mathbb{F}_2^n$ be the vector space of dimension n over $\mathbb{F}_2$. A linear $[n,k]$ code $\mathbf{C}$ is a vector subspace of V of dimension k. An element $\mathbf{x}$ in $\mathbf{C}$ is called a code word of $\mathbf{C}$. In V, the inner product, which is denoted by $\mathbf{x}\cdot\mathbf{y}$ for $\mathbf{x},\mathbf{y}$ in V, is defined as usual. Two codes are said to be equivalent if after a suitable change of coordinate positions the code words in the two codes coincide. The dual code $\mathbf{C}^{\perp}$ of $\mathbf{C}$ is defined by

$$\mathbf{C}^{\perp} = \{\mathbf{u} \in V \mid \mathbf{u}\cdot\mathbf{v} = \mathbf{0}, \quad \forall \mathbf{v} \in \mathbf{C}\}.$$

The code $\mathbf{C}$ is called self-orthogonal if it satisfies $\mathbf{C} \subseteq \mathbf{C}^{\perp}$, and self-dual if it satisfies $\mathbf{C} = \mathbf{C}^{\perp}$. Self-dual $[n,k]$ codes exist only if $n \equiv 0 \pmod 2$ and $k = \dfrac{n}{2}$.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a vector in V. The Hamming weight $wt(\mathbf{x})$ of the vector $\mathbf{x}$ is defined to be the number of i's such that $x_i \neq 0$. The Hamming distance $d(\mathbf{x},\mathbf{y})$ on V is defined by $d(\mathbf{x},\mathbf{y}) = wt(\mathbf{x}-\mathbf{y})$. The minimum distance $d(\mathbf{C})$ of a code $\mathbf{C}$ is defined by

$$\begin{aligned} d(\mathbf{C}) &= \mathrm{Min}_{\mathbf{x},\mathbf{y}\in\mathbf{C},\mathbf{x}\neq\mathbf{y}} d(\mathbf{x},\mathbf{y}) \\ &= \mathrm{Min}_{\mathbf{x}\in\mathbf{C},\mathbf{x}\neq\mathbf{0}} wt(\mathbf{x}). \end{aligned}$$

Let $\mathbf{C}$ be a self-dual binary $[n, \dfrac{n}{2}]$ code. The weight $wt(\mathbf{x})$ of each code word $\mathbf{x}$ in $\mathbf{C}$ is an even number. Further, if the weight of each code word $\mathbf{x}$ in $\mathbf{C}$ is divisible by 4, then the code is called a doubly even binary code. It is known that doubly even self-dual binary codes $\mathbf{C}$ exist only when the length n of $\mathbf{C}$ is a multiple of 8. If $\mathbf{C}$ is a binary doubly even self-dual code, it is known that (cf. [10])

$$d(\mathbf{C}) \leq 4\left[\frac{n}{24}\right] + 4.$$

A self-dual code $\mathbf{C}$ satisfying $d(\mathbf{C}) = 4\left[\frac{n}{24}\right] + 4$ is called an extremal binary doubly even self-dual code.

Let $\mathbf{C}$ be a self-dual doubly even code of length n. Let $\mathbf{u} = (u_1, u_2, \cdots, u_n), \mathbf{v} = (v_1, v_2, \cdots, v_n)$ be any pair of vectors in $\mathbb{F}_2^n$. Then the number of coordinates where $\mathbf{u}$ and $\mathbf{v}$ are both 1 is denoted by $\mathbf{u} * \mathbf{v}$. This is called the intersection number of $\mathbf{u}$ and $\mathbf{v}$, and $\mathbf{u} * \mathbf{u}$ is simply $wt(\mathbf{u})$. We have

(1) $$wt(\mathbf{u}+\mathbf{v}) = wt(\mathbf{u}) + wt(\mathbf{v}) - 2\mathbf{u} * \mathbf{v}.$$

For $\mathbf{u}$ and $\mathbf{v}$ as above we write $\mathbf{u} \subseteq \mathbf{v}$ if $u_i = 1$ implies $v_i = 1$ $(1 \leq i \leq n)$. For instance $(1,0,1,0,0) \subseteq (1,0,1,1,0)$. We use $\mathbf{u} \cap \mathbf{v}$ to denote the binary vector whose i-th coordinate value is 1 if and only if both $u_i = 1$ and $v_i = 1$ hold. For instance $(1,0,1,0,0) \cap (1,0,1,1,0) = (1,0,1,0,0)$.

We quote a result in [7] as a proposition.

Proposition 2.1. *Let* $\mathbf{C}$ *be any one of five binary extremal* $[32, 16, 8]$ *codes. Let* $\mathbf{C}_8$ *be the set of the code words of weight* 8 *in the code* $\mathbf{C}$ *and* $\mathbf{a}$ *any binary vector of length* 32 *which lies in the ambient space* $\mathbb{F}_2^{32} \supset \mathbf{C}$*. Then we have*
(i) $\sum_{\mathbf{u}\in\mathbf{C}_8}(\mathbf{u} * \mathbf{a}) = 155(\mathbf{a} * \mathbf{a})$,
(ii) $\sum_{\mathbf{u}\in\mathbf{C}_8}(\mathbf{u} * \mathbf{a})^2 = 35(\mathbf{a} * \mathbf{a})^2 + 120(\mathbf{a} * \mathbf{a})$,
(iii) $\sum_{\mathbf{u}\in\mathbf{C}_8}(\mathbf{u} * \mathbf{a})^3 = 7(\mathbf{a} * \mathbf{a})^3 + 84(\mathbf{a} * \mathbf{a})^2 + 64(\mathbf{a} * \mathbf{a})$,
(iv) $6\sum_{\mathbf{u}\in\mathbf{C}_8}(\mathbf{u} * \mathbf{a})^5 - (40 + 5(\mathbf{a} * \mathbf{a}))\sum_{\mathbf{u}\in\mathbf{C}_8}(\mathbf{u} * \mathbf{a})^4 = -[5(\mathbf{a} * \mathbf{a})^5 + 160(\mathbf{a} * \mathbf{a})^4 + 1400(\mathbf{a} * \mathbf{a})^3 + 4480(\mathbf{a} * \mathbf{a})^2]$.

As an easy consequence of Proposition 2.1 we have

Proposition 2.2. *Let* ν_k *be the cardinality of the set* $\{\mathbf{u} \in \mathbf{C}_8 \mid \mathbf{u} * \mathbf{v} = k\}$*, where* $\mathbf{v}$ *is a fixed code word in* $\mathbf{C}_8$ *and* $k = 0, 2, 4, 8$*. Then*

$$\nu_k = \begin{cases} 1 & \text{if} \quad k = 8, \\ 84 & \text{if} \quad k = 4, \\ 448 & \text{if} \quad k = 2, \\ 87 & \text{if} \quad k = 0. \end{cases}$$

Proof. Putting $\mathbf{a} = \mathbf{v}$ in Proposition 2.1 (i),(ii) we have

$$\begin{aligned} 8\nu_8 + 4\nu_4 + 2\nu_2 &= 1240, \\ 64\nu_8 + 16\nu_4 + 4\nu_2 &= 3200. \end{aligned}$$

Obviously $\nu_8 = 1$ and so $\nu_4 = 84, \nu_2 = 448$. ν_0 is determined by the condition $\nu_8+\nu_4+\nu_2+\nu_0 = 620$. □

2.2 Multiple Weight Enumerator

Let $\mathbf{C}$ be a doubly even self-dual binary code of length n, g be a positive integer and let α run over the set $\mathbb{F}_2^g$ of g-tuple vectors. The 2^g algebraically independent variables x_α over $\mathbb{C}$ are parameterized by $\alpha \in \mathbb{F}_2^g$. Let $\mathbf{u}_1 = (u_1^1, u_1^2, \cdots, u_1^n), \mathbf{u}_2 = (u_2^1, u_2^2, \cdots, u_2^n), \cdots, \mathbf{u}_g = (u_g^1, u_g^2, \cdots, u_g^n)$ be a g-tuple of code words of $\mathbf{C}$. For each $\alpha \in \mathbb{F}_2^g$ the generalized weight $wt_\alpha(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_g)$ is defined to be the number of coordinates j $(1 \le j \le n)$ such that the equation $\alpha = (u_1^j, u_2^j, \cdots, u_g^j)$. The multiple weight enumerator $\mathbf{W}_g(x_\alpha; \mathbf{C})$ of genus g for the code $\mathbf{C}$ is defined by

$$\mathbf{W}_g(x_\alpha; \mathbf{C}) = \sum_{(\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_g)\in\mathbf{C}^g} \prod_{\alpha\in\mathbb{F}_2^g} x_\alpha^{wt_\alpha(\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_g)}.$$

The multiple weight enumerator of genus 1 is the ordinary weight enumerator. Here we give the weight enumerator $W_{\mathbf{C}}(x, y)$ of a binary self-dual extremal $[32, 16, 8]$ code $\mathbf{C}$:

$$W_{\mathbf{C}}(x, y) = x^{32} + 620x^{24}y^8 + 13888x^{20}y^{12} + 36518x^{16}y^{16} + 13888x^{12}y^{20} + 620x^8y^{24} + y^{32},$$

which will be used later. A multiple weight enumerator of genus 2 is called a biweight enumerator, a multiple weight enumerator of genus 3 is called a triweight enumerator, and a multiple weight enumerator of genus 4 is called a quadriweight enumerator.

The subsets of words of a binary self-dual extremal $[32, 16, 8]$ code $\mathbf{C}$ of weight i $(i = 0, 8, 12, 16, 20, 24, 32)$ is denoted by $\mathbf{C}_i$.

2.3 Intersection Enumerator and Intersection Matrix

Let $\mathbf{C}$ be a binary linear code of length n. The intersection enumerator of genus g is defined by

$$\mathcal{I}_g(\mathbf{C}; X_1, \cdots, X_g, Y_{i,j}) = \sum_{\mathbf{u}_1, \cdots, \mathbf{u}_g \in \mathbf{C}} X_1^{wt(\mathbf{u}_1)} \cdots X_g^{wt(\mathbf{u}_g)} \prod_{1 \le i < j \le g} Y_{i,j}^{\mathbf{u}_i * \mathbf{u}_j},$$

where $X_1, \ldots, X_g, Y_{i,j} (1 \le i < j \le g)$ are algebraically independent variables over $\mathbb{C}$. This polynomial is analogous to the Siegel theta series in the sense that the polynomial is the generating function of the quantities $b(M, \mathbf{C}) = \#\{\langle \mathbf{u}_1, \cdots, \mathbf{u}_g \rangle \in \mathbf{C}^g | [[\mathbf{u}_1, \cdots, \mathbf{u}_g]] = M\}$ (this notation will be explained below), while the Siegel theta series is the generating function of the number of representations of quadratic forms of a certain order by another quadratic form of greater order. Let $M = (m_{i,j})$ be a $g \times g$ matrix defined by $m_{i,j} = \mathbf{u}_i * \mathbf{u}_j$, where $\mathbf{u}_1, \cdots, \mathbf{u}_g$ are the code words in $\mathbf{C}$. We denote M by $[[\mathbf{u}_1, \cdots, \mathbf{u}_g]]$ and call it the intersection matrix for the g-tuple $\langle \mathbf{u}_1, \cdots, \mathbf{u}_g \rangle$. In later tables we will represent this matrix M by its entries:

$$M = (\mathbf{u}_1 * \mathbf{u}_1, \cdots, \mathbf{u}_g * \mathbf{u}_g, \mathbf{u}_2 * \mathbf{u}_1, \mathbf{u}_3 * \mathbf{u}_1, \mathbf{u}_3 * \mathbf{u}_2, \cdots).$$

For instance

$$[[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]] = (\mathbf{u}_1 * \mathbf{u}_1, \mathbf{u}_2 * \mathbf{u}_2, \mathbf{u}_3 * \mathbf{u}_3, \mathbf{u}_2 * \mathbf{u}_1, \mathbf{u}_3 * \mathbf{u}_1, \mathbf{u}_3 * \mathbf{u}_2).$$

Let $\mathcal{M}_{\mathbf{C},g}$ be the set of all possible intersection matrices M of size g for a given code $\mathbf{C}$. Here we introduce a formal notation:

$$((M, X, Y)) = X_1^{m_{11}} \cdots X_g^{m_{gg}} \prod_{1 \le i < j \le g} Y_{i,j}^{m_{i,j}} \; M \in \mathcal{M}_{\mathbf{C},g}.$$

Then we can rewrite $\mathcal{I}_g(\mathbf{C}; X_1, \cdots, X_g, Y_{i,j})$ as

$$\begin{aligned} \mathcal{I}_g(\mathbf{C}; X_1, \cdots, X_g, Y_{i,j}) &= \sum_{\mathbf{u}_1, \cdots, \mathbf{u}_g \in \mathbf{C}} (([[\mathbf{u}_1, \cdots, \mathbf{u}_g]], X, Y)) \\ &= \sum_{M \in \mathcal{M}_{\mathbf{C},g}} b_g(M, \mathbf{C})((M, X, Y)), \end{aligned}$$

where

$$b_g(M, \mathbf{C}) = \sum_{\substack{\mathbf{u}_1, \cdots, \mathbf{u}_g \in \mathbf{C} \\ [[\mathbf{u}_1, \cdots, \mathbf{u}_g]] = M}} 1. \tag{2}$$

In this paper we will use the intersection polynomial of genus 4:

$$\mathcal{I}_4(\mathbf{C}; X_1, \cdots, X_4, Y_{i,j}) = \sum_{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4 \in \mathbf{C}} X_1^{wt(\mathbf{u}_1)} X_2^{wt(\mathbf{u}_2)} X_3^{wt(\mathbf{u}_3)} X_4^{wt(\mathbf{u}_4)} Y_{1,2}^{\mathbf{u}_1 * \mathbf{u}_2} Y_{1,3}^{\mathbf{u}_1 * \mathbf{u}_3} Y_{1,4}^{\mathbf{u}_1 * \mathbf{u}_4} Y_{2,3}^{\mathbf{u}_2 * \mathbf{u}_3} Y_{2,4}^{\mathbf{u}_2 * \mathbf{u}_4} Y_{3,4}^{\mathbf{u}_3 * \mathbf{u}_4}.$$

Here we may remark that the intersection polynomial $\mathcal{I}_g(\mathbf{C}; X_1, \cdots, X_g, Y_{i,j})$ can be obtained as a specialization of the multiple weight enumerator of genus g. For instance the exponents of the variables in the above polynomial are given explicitly by

$$\begin{aligned}
wt(\mathbf{u}_1) &= wt_{1111} + wt_{1110} + wt_{1101} + wt_{1100} + wt_{1011} + wt_{1010} + wt_{1001} + wt_{1000},\\
wt(\mathbf{u}_2) &= wt_{1111} + wt_{1110} + wt_{1101} + wt_{1100} + wt_{0111} + wt_{0110} + wt_{0101} + wt_{0100},\\
wt(\mathbf{u}_3) &= wt_{1111} + wt_{1110} + wt_{1011} + wt_{1010} + wt_{0111} + wt_{0110} + wt_{0011} + wt_{0010},\\
wt(\mathbf{u}_4) &= wt_{1111} + wt_{1101} + wt_{1011} + wt_{1001} + wt_{0111} + wt_{0101} + wt_{0011} + wt_{0001},\\
\mathbf{u}_1 * \mathbf{u}_2 &= wt_{1111} + wt_{1110} + wt_{1101} + wt_{1100},\\
\mathbf{u}_1 * \mathbf{u}_3 &= wt_{1111} + wt_{1110} + wt_{1011} + wt_{1010},\\
\mathbf{u}_2 * \mathbf{u}_3 &= wt_{1111} + wt_{1110} + wt_{0111} + wt_{0110},\\
\mathbf{u}_1 * \mathbf{u}_4 &= wt_{1111} + wt_{1101} + wt_{1011} + wt_{1001},\\
\mathbf{u}_2 * \mathbf{u}_4 &= wt_{1111} + wt_{1101} + wt_{0111} + wt_{0101},\\
\mathbf{u}_3 * \mathbf{u}_4 &= wt_{1111} + wt_{1011} + wt_{0111} + wt_{0011}.
\end{aligned}$$

We consider the transformations:

$$x_{hijk} = \prod_{m=1}^{4} X_m^{\iota(hijk)} \prod_{1\le i<j\le 4} Y_{i,j}^{\iota(hijk)},$$

where

$$\iota(hijk) = \begin{cases} 1 & \text{if } wt_{hijk} \text{ appears in the exponents of } X_m \text{ or } Y_{i,j},\\ 0 & \text{otherwise.}\end{cases}$$

For instance

$$\begin{aligned}
x_{1111} &= X_1X_2X_3X_4Y_{1,2}Y_{1,3}Y_{1,4}Y_{2,3}Y_{2,4}Y_{3,4},\\
x_{1101} &= X_1X_2X_4Y_{1,2}Y_{2,3}Y_{2,4}.
\end{aligned}$$

If we substitute these transformations into $\mathbf{W}_4(x_\alpha; \mathbf{C})$, we obtain $\mathcal{I}_4(\mathbf{C}; X_1, \cdots, X_4, Y_{i,j})$.

Note 1. *At a later stage (Section 5.2.7) we will use the quadriweight enumerator of the code. There we will represent binary quadruples of length four by decimal numbes: if $hijk \in \mathbb{F}_2^4$, $dec(hijk)$ is the decimal number $h2^3 + i2^2 + j \cdot 2 + k$, where h, i, j, k are viewed as integers. Accordingly we may use the variables $x_{dec(hijk)}$ instead of x_{hijk} and the weight $wt_{dec(hijk)}$ instead of wt_{hijk}. For instance we write x_{15} for x_{1111}, and wt_{15} for wt_{1111}.*

2.4 From Binary Codes to Lattices

Let C be a binary self-orthogonal $[n, k]$ code. We recall the construction B_2 for the lattices (c.f. Conway-Sloane [5], Chap.5). Let

$$\rho \,:\, \mathbb{Z}^n \to \mathbb{F}_2^n$$

denote the reduction modulo 2. Then

$$M(\mathbf{C}) = \frac{1}{\sqrt{2}} \left\{ \mathbf{x} = (x_1, x_2, \ldots, x_n) \in \rho^{-1}(\mathbf{C}) | \sum_{i=1}^{n} x_i \equiv 0 \pmod 4 \right\}$$

defines an even lattice. Suppose that $\mathbf{C}$ is a doubly even self-dual binary extremal $[n, n/2]$ code. The so-called doubling process is as follows.
Put

$$\gamma = \begin{cases} \frac{1}{\sqrt{8}}(1, \dots, 1, -3) & \text{if } n \equiv 8 \pmod{16}, \\ \frac{1}{\sqrt{8}}(1, \dots, 1, 1) & \text{if } n \equiv 0 \pmod{16}. \end{cases}$$

Then

$$\mathcal{N}(\mathbf{C}) = \mathcal{M}(\mathbf{C}) \cup (\gamma + \mathcal{M}(\mathbf{C}))$$

is an even unimodular extremal lattice of rank n for $n = 8, 16, 24, 32, 40$.

3 Even Unimodular Extremal 32-dimensional Lattices

By Conway and Pless [3] there are five non-equivalent binary self-dual extremal $[32, 16, 8]$ codes. We will compute the Fourier coefficient $a(\mathfrak{T}_{40}, \mathcal{L})$ for each of the five 32-dimensional even unimodular lattices that are constructed from the five doubly even self-dual extremal binary codes. The final result that the different lattices have different Siegel theta series of degree 4 is reached at the end of the paper. Here we clearly present those five codes. As references we quote four articles: Conway-Pless [3], Conway-Pless-Sloane [4], Koch [7], Rains-Sloane [13]. We rename these five codes as CP1:= r_{32}, CP2:= q_{32}, CP3:= f_4^8, CP4:= g_{16}^2 and CP5:= f_2^{16}. Note that our ordering is different from that of [4]. When $\mathbf{C}$ is a doubly even self-dual binary $[32, 16, 8]$ code and $L(\mathbf{C}) = \mathcal{N}(\mathbf{C})$ is the even unimodular extremal lattice constructed from $\mathbf{C}$ in the previous section, we put $\Lambda_{2k} = \{\mathbf{x} \in L \mid (\mathbf{x}, \mathbf{x}) = 2k\} \quad (k \geq 0)$. The cardinality of the set Λ_{2k} is denoted by $\mid \Lambda_{2k} \mid$. The following cardinalities are well-known:

$$\begin{aligned} \mid \Lambda_2 \mid &= 0, \\ \mid \Lambda_4 \mid &= 146880, \\ \mid \Lambda_6 \mid &= 64757760, \\ \mid \Lambda_8 \mid &= 4844836800. \end{aligned}$$

We are particularly interested in the set $\Lambda_4(L(\mathbf{C}))$. $\Lambda_4 = \Lambda_4(L(\mathbf{C}))$ is a union of six mutually disjoint subsets:

(3) $$\Lambda_4 = \Lambda_{4,1} \cup \Lambda_{4,2} \cup \Lambda_{4,3} \cup \Lambda_{4,4} \cup \Lambda_{4,5} \cup \Lambda_{4,6},$$

defined by

$$
\begin{array}{rcll}
\Lambda_{4,1} & = & \left\{ \frac{1}{\sqrt{2}}((\pm 2)^2, 0^{30}) \right\}, & \mid \Lambda_{4,1} \mid = 1984, \\
\Lambda_{4,2} & = & \left\{ \frac{1}{\sqrt{2}}((\pm 1)^8, 0^{24}) \right\}, & \mid \Lambda_{4,2} \mid = 79360, \\
\Lambda_{4,3} & = & \left\{ \pm\frac{1}{2\sqrt{2}}((-1)^8, 1^{24}) \right\}, & \mid \Lambda_{4,3} \mid = 1240, \\
\Lambda_{4,4} & = & \left\{ \pm\frac{1}{2\sqrt{2}}((-1)^{12}, 1^{20}) \right\}, & \mid \Lambda_{4,4} \mid = 27776, \\
\Lambda_{4,5} & = & \left\{ \frac{1}{2\sqrt{2}}((-1)^{16}, 1^{16}) \right\}, & \mid \Lambda_{4,5} \mid = 36518, \\
\Lambda_{4,6} & = & \left\{ \pm\frac{1}{2\sqrt{2}}(1^{32}) \right\}, & \mid \Lambda_{4,6} \mid = 2.
\end{array}
$$

We also check that

$$1984 + 79360 + 27776 + 36518 + 1240 + 2 = 146880.$$

Remark 1. *The non-zero coordinates of an element in $\Lambda_{4,2}$ correspond to a code word of weight 8 in the code, and the number of minus signs in an element in $\Lambda_{4,2}$ is even. The coordinates of the minus signs in a vector in $\Lambda_{4,3}$ correspond to a code word of weights 8 or 24 in the code. The coordinates of minus sign in a vector in $\Lambda_{4,4}$ correspond to a code word of weights 12 or 20 in the code. The coordinates of minus sign in a vector in $\Lambda_{4,5}$ correspond to a code word of weight 16 in the code.*

We describe the subsets $\Lambda_{4,j} (3 \leq j \leq 6)$ more precisely. We put

$$\mathbf{x}_0 = \frac{1}{2\sqrt{2}}(1^{32}) \in \Lambda_{4,6}.$$

Any element $\mathbf{x} \in \Lambda_{4,j} (3 \leq j \leq 6)$ may be written as

$$\mathbf{x} = \mathbf{x}_0 - \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}), \mathbf{u} \in \mathbf{C} \tag{4}$$

where $\rho^{\#}(\mathbf{u})$ is the unique element of the subset $\rho^{-1}(\mathbf{u}) \subset \mathbb{Z}^{32}$, whose non-zero coordinates are all 1. We call (4) the standard form of $\mathbf{x} \in \Lambda_{4,j} (3 \leq j \leq 6)$. For instance, if $\mathbf{x} = \frac{1}{2\sqrt{2}}((-1)^8, 1^{24}) \in \Lambda_{4,3}$ then the standard form of $\mathbf{x}$ is

$$\mathbf{x} = \mathbf{x}_0 - \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}), \mathbf{u} \in \mathbf{C}_8.$$

For $-\frac{1}{2\sqrt{2}}((-1)^8, 1^{24}) \in \Lambda_{4,3}$ we observe that

$$
\begin{array}{rcl}
-\frac{1}{2\sqrt{2}}((-1)^8, 1^{24}) & = & \frac{1}{2\sqrt{2}}(1^8, (-1)^{24}), \\
& = & \mathbf{x}_0 - \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}') \ \text{ with } \ \mathbf{u}' = (1^{32}) + \mathbf{u} \in \mathbf{C}_{24}.
\end{array}
$$

We write

$$\begin{aligned}\Lambda_{4,3}^{+} &= \left\{\mathbf{x}_0 - \tfrac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}) \mid \mathbf{u} \in \mathbf{C}_8\right\},\\ \Lambda_{4,3}^{-} &= \left\{\mathbf{x}_0 - \tfrac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}) \mid \mathbf{u} \in \mathbf{C}_{24}\right\}.\end{aligned}$$

Likewise we may write

$$\begin{aligned}\Lambda_{4,4}^{+} &= \left\{\mathbf{x}_0 - \tfrac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}) \mid \mathbf{u} \in \mathbf{C}_{12}\right\},\\ \Lambda_{4,4}^{-} &= \left\{\mathbf{x}_0 - \tfrac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}) \mid \mathbf{u} \in \mathbf{C}_{20}\right\}.\end{aligned}$$

Note that $\Lambda_{4,3} = \Lambda_{4,3}^{+} \cup \Lambda_{4,3}^{-}$, $\Lambda_{4,4} = \Lambda_{4,4}^{+} \cup \Lambda_{4,4}^{-}$ are disjoint decompositions of $\Lambda_{4,3}$ and $\Lambda_{4,4}$ respectively.

For $\Lambda_{4,5}$ we cannot make distinguishable subsets, since the complement $\mathbf{u}' = (1^{32}) + \mathbf{u}$ of $\mathbf{u} \in \mathbf{C}_{16}$ is also a code word in $\mathbf{C}_{16}$. We simply rewrite

$$\Lambda_{4,5} = \left\{\mathbf{x}_0 - \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}) \mid \mathbf{u} \in \mathbf{C}_{16}\right\}.$$

Next we consider the elements in $\Lambda_{4,2}$. A general element of $\Lambda_{4,2}$ takes the shape

$$\mathbf{x} = \frac{1}{\sqrt{2}}((\pm 1)^8 0^{24}), \tag{5}$$

where in the right-hand side of the above expression minus signs should appear an even number of times. Therefore we may rewrite $\mathbf{x}$ as

$$\mathbf{x} = \frac{1}{\sqrt{2}}(1^8 0^{24}) - \sqrt{2}\delta, \tag{6}$$

where the coordinate values of δ are 0 or 1, and the value 1 corresponds to a minus sign of the initial expression of $\mathbf{x}$. We write

$$\mathbf{x} = \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}) - \sqrt{2}\delta, \text{ with } \mathbf{u} \in \mathbf{C}_8 \text{ and } \rho(\delta) \subseteq \mathbf{u} \tag{7}$$

Here $\rho^{\#}(\mathbf{u})$ has the same meaning as before. We may call (7) the standard form for $\mathbf{x} \in \Lambda_{4,2}$. There is a special subset $\Lambda_{4,2}(\mathbf{u})$ of $\Lambda_{4,2}$ defined by

$$\Lambda_{4,2}(\mathbf{u}) = \left\{\frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}) - \sqrt{2}\delta \mid \rho(\delta) \subseteq \mathbf{u}, wt(\rho(\delta)) \equiv 0 \pmod 2\right\},$$

where $\mathbf{u} \in \mathbf{C}_8$ is fixed and δ runs over under the condition $\rho(\delta) \subseteq \mathbf{u}$. It is easy to see that the cardinality of the set $\Lambda_{4,2}(\mathbf{u})$ is 2^7.

We now give a proposition which connects the inner product of vectors in the subset Λ_4 with the intersection in $\mathbf{C}$:

Proposition 3.1. *We have*
(i) $(\rho^{\#}(\mathbf{u}), \rho^{\#}(\mathbf{v})) = \mathbf{u} * \mathbf{v}$ *for* $\mathbf{u}, \mathbf{v} \in \mathbf{C}$,
(ii) $(\rho^{\#}(\mathbf{u}), \mathbf{x}_0) = \frac{1}{2\sqrt{2}} wt(\mathbf{u})$,
(iii) $(\delta, \rho^{\#}(\mathbf{u})) = \rho(\delta) * \mathbf{u}$, *where* δ *is a vector whose entries are 0 or 1,*
(iv) $(\delta_1, \delta_2) = \rho(\delta_1) * \rho(\delta_2)$,

(v) *let* $\mathbf{x} = \mathbf{x}_0 - \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}), \mathbf{y} = \mathbf{x}_0 - \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{v}) \in \bigcup_{j=3}^{6} \Lambda_{4,j}$ *have their standard forms. Then*

$$(\mathbf{x}, \mathbf{y}) = 4 - \frac{1}{4}\left(wt(\mathbf{u}) + wt(\mathbf{v}) - 2(\mathbf{u} * \mathbf{v})\right). \tag{8}$$

Proof. Proof of (i). Both sides count the common 1's in the i-th coordinate for all $1 \leq i \leq 32$, where the left-hand side count is in $\mathbb{Z}$ and the right-hand side count is in $\mathbb{F}_2$.
Proof of (ii). This is obvious since $\mathbf{x}_0$ may be regarded as $\frac{1}{2\sqrt{2}}(\rho^{\#}((1^{32})))$ with $(1^{32}) \in \mathbf{C}_{32}$.
(iii) and (iv) are obvious.
Proof of (v). Using (i) and (ii) we compute that

$$\begin{aligned}(\mathbf{x}_0 - \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}), \mathbf{x}_0 - \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{v})) &= 4 - \frac{1}{\sqrt{2}}\left((\rho^{\#}(\mathbf{u}), \mathbf{x}_0) + (\rho^{\#}(\mathbf{v}), \mathbf{x}_0)\right) + \frac{1}{2}\left((\rho^{\#}(\mathbf{u}), \rho^{\#}(\mathbf{v})\right), \\ &= 4 - \frac{1}{4}(wt(\mathbf{u}) + wt(\mathbf{v}) - 2(\mathbf{u} * \mathbf{v})).\end{aligned}$$

□

In the next section we will be particularly concerned with the pairs of $\mathbf{x}, \mathbf{y} \in \Lambda_4$ satisfying ($\mathbf{x}$,$\mathbf{y}$)=2. We first treat the easier cases in which $(\mathbf{x}, \mathbf{y}) = 2$ does not hold.

Proposition 3.2. *If* $\mathbf{x} \in \Lambda_{4,1}$ *and* $\mathbf{y} \in \Lambda_{4,3} \cup \Lambda_{4,4} \cup \Lambda_{4,5} \cup \Lambda_{4,6}$ *then* $(\mathbf{x}, \mathbf{y}) \neq 2$ *holds.*

Proof. Noting the shape of $\mathbf{x} \in \Lambda_{4,1}$ we see that $|(\mathbf{x}, \mathbf{x}_0)| \leq 1$ and $|(\mathbf{x}, \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}))| \leq 2$, for all $\mathbf{u} \in \mathbf{C}$. We temporally put $a = (\mathbf{x}, \mathbf{x}_0)$ and $b = (\mathbf{x}, \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}))$. Since both $\mathbf{x}_0$ and $\frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}), \mathbf{u} \in \mathbf{C}$ have non-negative coordinate entries, there are only three cases for the sign distributions of the pair a, b: (i) $a = 1$ and $b \geq 0$, (ii) $a = 0$ and $b = 1, 0, -1$, (iii) $a = -1$ and $b \leq 0$. In these three cases we see that we have

$$|(\mathbf{x}, (\mathbf{x}_0 - \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u})))| \leq 1.$$

□

Proposition 3.3. *Let* $\mathbf{x} = \mathbf{x}_0$ *and* $\mathbf{y} \in \Lambda_{4,3}^{-} \cup \bigcup_{j=4,5} \Lambda_{4,j}$. *Then we have that* $(\mathbf{x}, \mathbf{y}) \neq 2$.

Proof. Let $\mathbf{y} = \mathbf{x}_0 - \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{v})$ be in its standard form. By Proposition 3.1 we have

$$(\mathbf{x}, \mathbf{y}) = 4 - \frac{1}{4}wt(\mathbf{v}).$$

Since $wt(\mathbf{v}) = 12, 16, 20, 24$, the conclusion follows. □

Remark 2. *When* $\mathbf{x} = -\mathbf{x}_0$ *then we take* $\mathbf{y} \in \Lambda_{4,3}^{+} \cup \bigcup_{j=4,5} \Lambda_{4,j}$ *and* $(\mathbf{x}, \mathbf{y}) \neq 2$.

In an atempt to find such pairs we examine the possible intersections between code words of various weights. Here we give a table.

Table 1. Table of intersections between code words.

$wt(\mathbf{u})$	$wt(\mathbf{v})$	values of $\mathbf{u} * \mathbf{v}$	$wt(\mathbf{u})$	$wt(\mathbf{v})$	values of $\mathbf{u} * \mathbf{v}$
8	8	0,2,4,8	16	16	0,4,6,8,10,12,16
8	12	0,2,4,6	16	20	2,6,8,10,12,14
8	16	0,2,4,6,8	16	24	4,8,10,12,14,16
8	20	2,4,6,8	16	32	16
8	24	0,4,6,8	20	20	4,8,10,12,14,16,20
8	32	8	20	24	6,10,12,14,16,18
12	12	0,2,4,6,8,12	20	32	20
12	16	2,4,6,8,10	24	24	8,12,14,16,18,20,24
12	20	0,4,6,8,10,12	24	32	24
12	24	2,6,8,10,12			
12	32	12			

Proposition 3.4. *Assume* $\mathbf{x}, \mathbf{y} \in \Lambda_{4,3} \cup \Lambda_{4,4} \cup \Lambda_{4,5}$ *are in standard form:*

$$\begin{aligned} \mathbf{x} &= \mathbf{x}_0 - \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}), \\ \mathbf{y} &= \mathbf{x}_0 - \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{v}), \end{aligned}$$

where $\mathbf{u}, \mathbf{v} \in \mathbf{C}$. *Then* $(\mathbf{x}, \mathbf{y}) = 2$ *holds if and only if* $wt(\mathbf{u} + \mathbf{v}) = 8$.

Proof. By Proposition 3.1 we know that

$$\begin{aligned} (\mathbf{x}, \mathbf{y}) &= 4 - \frac{1}{4}\left(wt(\mathbf{u}) + wt(\mathbf{v}) - 2(\mathbf{u} * \mathbf{v})\right) \\ &= 2. \end{aligned}$$

From this we get $wt(\mathbf{u}) + wt(\mathbf{v}) - 2(\mathbf{u} * \mathbf{v}) = 8$, the left-hand side of which is $wt(\mathbf{u} + \mathbf{v})$ by the identity (1). □

The following table gives the intersections of the code words $\mathbf{u}, \mathbf{v}$ which satisfy the condition $wt(\mathbf{u} + \mathbf{v}) = 8$ from Proposition 3.4.

Table 2. Table of intersections for which $wt(\mathbf{u} + \mathbf{v}) = 8$.

$wt(\mathbf{u}) \backslash wt(\mathbf{v})$	8	12	16	20	24
8	4	6	8		
12	6	8	10	12	
16	8	10	12	14	16
20		12	14	16	18
24			16	18	20

In the above table the blanks indicate that there are no pairs of code words satisfying the condition $wt(\mathbf{u} + \mathbf{v}) = 8$. As a result of Table 2 we give another table that tells us which pairs of the subsets of Λ_4 contain vectors $\mathbf{x}, \mathbf{y}$ satisfying $(\mathbf{x}, \mathbf{y}) = 2$.

Table 3. Table of subsets in which there are pairs of vectors
$\mathbf{x} \in A, \mathbf{y} \in B$ satisfying $(\mathbf{x}, \mathbf{y}) = 2$.

$A \backslash B$	$\Lambda_{4,3}^+$	$\Lambda_{4,4}^+$	$\Lambda_{4,5}$	$\Lambda_{4,4}^-$	$\Lambda_{4,3}^-$
$\Lambda_{4,3}^+$	y	y	y	n	n
$\Lambda_{4,4}^+$	y	y	y	y	n
$\Lambda_{4,5}$	y	y	y	y	y
$\Lambda_{4,4}^-$	n	y	y	y	y
$\Lambda_{4,3}^-$	n	n	y	y	y

In Table 3 y indicates that A and B contain vectors satisfying $(\mathbf{x}, \mathbf{y}) = 2$, and n indicates the opposite case. Conditions for vectors in $\Lambda_{4,2}$ and one of the $\Lambda_{4,j},\ j \neq 2$ will be discussed later.

4 A Formula for a Fourier Coefficient

Let

$$\mathfrak{T}_{40} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

be a 4 by 4 matrix. The Fourier coefficient $a(\mathfrak{T}_{40}, \mathcal{L})$ at the index $\mathfrak{T}_{40}$ of the Siegel theta series of degree 4 associated with an even unimodular 32-dimensional extremal lattice $\mathcal{L}$ is given by

$$a(\mathfrak{T}_{40}, \mathcal{L}) = \sum_{\substack{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4 \in \Lambda_4 \\ [\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4] = 2\mathfrak{T}_{40}}} 1, \tag{9}$$

where Λ_4 is the set of norm 4 vectors in $\mathcal{L}$. From now on we assume the lattice $\mathcal{L}$ is one of the five even unimodular 32-dimensional extremal lattices constructed from the five doubly even self-dual extremal binary codes. Using the decomposition (9) we obtain a more precise expression for $a(\mathfrak{T}_{40}, \mathcal{L})$:

$$a(\mathfrak{T}_{40}, \mathcal{L}) = \sum_{\substack{\langle k_1,k_2,k_3,k_4 \rangle \in \{1,2,3,4,5,6\}^4 \\ [\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4] = 2\mathfrak{T}_{40}}} \sum_{\mathbf{x}_1 \in \Lambda_{4,k_1}} \sum_{\mathbf{x}_2 \in \Lambda_{4,k_2}} \sum_{\mathbf{x}_3 \in \Lambda_{4,k_3}} \sum_{\mathbf{x}_4 \in \Lambda_{4,k_4}} 1. \tag{10}$$

We use an abbreviated notation: for a sequence $1 \leq i_1 \leq i_2 \leq i_3 \leq i_4 \leq 4$ and integers $1 \leq k_{i_j} \leq 6$ instead of using

$$\sum_{\mathbf{x}_1 \in \Lambda_{4,k_1}} \sum_{\mathbf{x}_2 \in \Lambda_{4,k_2}} \sum_{\mathbf{x}_3 \in \Lambda_{4,k_3}} \sum_{\mathbf{x}_4 \in \Lambda_{4,k_4}} 1,$$

where $\mathbf{x}_1, \cdots \mathbf{x}_4$ satisfy the restriction $[\mathbf{x}_{i_1}, \mathbf{x}_{i_2}, \mathbf{x}_{i_3}, \mathbf{x}_{i_4}] = 2\mathfrak{T}_{40}$, we write simply

$$\sum_{1;k_1} \sum_{2;k_2} \sum_{3;k_3} \sum_{4;k_4} 1.$$

When some of k's are identical, for instance $k_1 = k_2$, we use the notation:

$$\sum_{1,2;k_1} \sum_{3;k_3} \sum_{4;k_4} 1.$$

Lemma 4.1. *Let k'_1, k'_2, k'_3, k'_4 be any permutation of k_1, k_2, k_3, k_4. Then we have*

$$\sum_{1;k'_1}\sum_{2;k'_2}\sum_{3;k'_3}\sum_{4;k'_4} 1 = \sum_{1;k_1}\sum_{2;k_2}\sum_{3;k_3}\sum_{4;k_4} 1.$$

Proof. This follows from the symmetry of the matrix $\mathfrak{T}_{40}$. □

The step from (9) to (10) is formally quite similar to the multinomial expansion of $(\xi_1+\xi_2+\xi_3+\xi_4+\xi_5+\xi_6)^4$ in the six variables $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6$. Some of the partial sums vanish because of Propositions 3.2 and 3.3. We obtain the following expression for $a(\mathfrak{T}_{40}, \mathcal{L})$:

$$\begin{aligned}
&a(\mathfrak{T}_{40}, \mathcal{L}) = \\
&\textstyle\sum_{1,2,3,4;1} 1 + 6\sum_{1,2;2}\sum_{3,4;1} 1 + 4\sum_{1,2,3;2}\sum_{4;1} 1 + \sum_{1,2,3,4;3} 1 + \sum_{1,2,3,4;2} 1 + 12\sum_{1;2}\sum_{2,3;4}\sum_{4;5} 1 \\
&\textstyle+ \sum_{1,2,3,4;4} 1 + \sum_{1,2,3,4;5} 1 + 12\sum_{1;2}\sum_{2;3}\sum_{3,4;5} 1 + 12\sum_{1;2}\sum_{2,3;3}\sum_{4;4} 1 + 12\sum_{1;2}\sum_{2;4}\sum_{3,4;5} 1 \\
&\textstyle+12\sum_{1;3}\sum_{2,3;4}\sum_{4;5} 1 + 12\sum_{1;3}\sum_{2;4}\sum_{3,4;5} 1 + 4\sum_{1;2}\sum_{2,3,4;4} 1 + 4\sum_{1;2}\sum_{2,3,4;5} 1 \\
&\textstyle+6\sum_{1,2;2}\sum_{3,4;4} 1 + 6\sum_{1,2;2}\sum_{3,4;5} 1 + 4\sum_{1,2,3;2}\sum_{4;4} 1 + 4\sum_{1,2,3;2}\sum_{4;5} 1 + 6\sum_{1,2;2}\sum_{3,4;3} 1 \\
&\textstyle+4\sum_{1,2,3;1}\sum_{4;2} 1 + 4\sum_{1,2,3;2}\sum_{4;3} 1 + 12\sum_{1,2;2}\sum_{3;3}\sum_{4;4} 1 + 12\sum_{1,2;2}\sum_{3;3}\sum_{4;5} 1 \\
&\textstyle+12\sum_{1;2}\sum_{2,3;3}\sum_{4;5} 1 + 12\sum_{1;2}\sum_{2;3}\sum_{3,4;4} 1 + 4\sum_{1;2}\sum_{2,3,4;3} 1 + 4\sum_{1;3}\sum_{2,3,4;5} 1 \\
&\textstyle+4\sum_{1,2,3;3}\sum_{4;4} 1 + 4\sum_{1,2,3;3}\sum_{4;5} 1 + 6\sum_{1,2;3}\sum_{3,4;4} 1 + 6\sum_{1,2;3}\sum_{3,4;5} 1 + 4\sum_{1;4}\sum_{2,3,4;5} 1 \\
&\textstyle+4\sum_{1,2,3;4}\sum_{4;5} 1 + 6\sum_{1,2;4}\sum_{3,4;5} 1 + 24\sum_{1;2}\sum_{2;3}\sum_{3;4}\sum_{4;5} 1 + 12\sum_{1,2;2}\sum_{3;4}\sum_{4;5} 1 \\
&\textstyle+12\sum_{1,2;3}\sum_{3;4}\sum_{4;5} 1 + 4\sum_{1;3}\sum_{2,3,4;4} 1 + 4\sum_{1,2,3;2}\sum_{4;6} 1 + 4\sum_{1,2,3;3}\sum_{4;6} 1 \\
&\textstyle+12\sum_{1;2}\sum_{2,3;3}\sum_{4;6} 1 + 12\sum_{1,2;2}\sum_{3;3}\sum_{4;6} 1.
\end{aligned}$$

By reordering the partial sums we obtain

$$
\begin{aligned}
&a(\mathfrak{T}_{40}, \mathcal{L}) = \\
&\sum_{1,2,3,4;1} 1 + \sum_{1,2,3,4;2} 1 + \sum_{1,2,3,4;3} 1 + \sum_{1,2,3,4;4} 1 + \sum_{1,2,3,4;5} 1 \\
&+4 \sum_{1,2,3;1} \sum_{4;2} 1 + 4 \sum_{1,2,3;2} \sum_{4;3} 1 + 4 \sum_{1,2,3;2} \sum_{4;1} 1 + 4 \sum_{1;2} \sum_{2,3,4;4} 1 \\
&+4 \sum_{1;2} \sum_{2,3,4;5} 1 + 4 \sum_{1,2,3;2} \sum_{4;4} 1 + 4 \sum_{1,2,3;2} \sum_{4;5} 1 + 4 \sum_{1,2,3;4} \sum_{4;5} 1 \\
&+4 \sum_{1;3} \sum_{2,3,4;4} 1 + 4 \sum_{1,2,3;2} \sum_{4;6} 1 + 4 \sum_{1,2,3;3} \sum_{4;6} 1 + 4 \sum_{1;2} \sum_{2,3,4;3} 1 \\
&+4 \sum_{1;3} \sum_{2,3,4;5} 1 + 4 \sum_{1,2,3;3} \sum_{4;4} 1 + 4 \sum_{1,2,3;3} \sum_{4;5} 1 + 4 \sum_{1;4} \sum_{2,3,4;5} 1 \\
&+6 \sum_{1,2;2} \sum_{3,4;1} 1 + 6 \sum_{1,2;2} \sum_{3,4;4} 1 + 6 \sum_{1,2;2} \sum_{3,4;5} 1 + 6 \sum_{1,2;2} \sum_{3,4;3} 1 \\
&+6 \sum_{1,2;3} \sum_{3,4;4} 1 + 6 \sum_{1,2;3} \sum_{3,4;5} 1 + 6 \sum_{1,2;4} \sum_{3,4;5} 1 \\
&+12 \sum_{1;2} \sum_{2,3;4} \sum_{4;5} 1 + 12 \sum_{1;2} \sum_{2;3} \sum_{3,4;5} 1 + 12 \sum_{1;2} \sum_{2,3;3} \sum_{4;4} 1 + 12 \sum_{1;2} \sum_{2,3;3} \sum_{4;5} 1 \\
&+12 \sum_{1;2} \sum_{2;4} \sum_{3,4;5} 1 + 12 \sum_{1;3} \sum_{2,3;4} \sum_{4;5} 1 + 12 \sum_{1;3} \sum_{2;4} \sum_{3,4;5} 1 \\
&+12 \sum_{1,2;2} \sum_{3;3} \sum_{4;4} 1 + 12 \sum_{1,2;2} \sum_{3;3} \sum_{4;5} 1 + 12 \sum_{1;2} \sum_{2,3;3} \sum_{4;6} 1 \\
&+12 \sum_{1;2} \sum_{2;3} \sum_{3,4;4} 1 + 12 \sum_{1,2;2} \sum_{3;4} \sum_{4;5} 1 + 12 \sum_{1,2;3} \sum_{3;4} \sum_{4;5} 1 \\
&+12 \sum_{1,2;2} \sum_{3;3} \sum_{4;6} 1 \\
&+24 \sum_{1;2} \sum_{2;3} \sum_{3;4} \sum_{4;5} 1.
\end{aligned}
$$

At this point we introduce a further abbreviation:

$$\sigma(k_1, k_2, k_3, k_4) = \sum_{1;k_1} \sum_{2;k_2} \sum_{3;k_3} \sum_{4;k_4} 1,$$

which will save space later.

5 Computation of Individual Sums

5.1 A Description of the Overall Strategy

The partial sums above can be roughly grouped into seven clusters. We give tables which describe the individual partial sums in Subsubsections 5.2.1 to $\sim$ 5.2.7.

Using the expressions (5) and (7) we can transform the inner product relations among the vectors in $\Lambda_{4,2}$ and the vectors in $\Lambda_{4,i}$ $(2 \leq i \leq 5)$ into assertions about the intersections among code words in the associated binary code.

Lemma 5.1. *Let* $\mathbf{x} = \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}) - \sqrt{2}\delta \in \Lambda_{4,2}$ *be in the standard form* (7).
(i) *Let* $\mathbf{y} = \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{v}) - \sqrt{2}\delta_2 \in \Lambda_{4,2}$. *Then*

$$(\mathbf{x},\mathbf{y}) = 2 \iff 4 + 2\rho(\delta) * \mathbf{v} + 2\rho(\delta_2) * \mathbf{u} = \mathbf{u} * \mathbf{v} + 4\rho(\delta) * \rho(\delta_2).$$

(ii) *Let* $\mathbf{y} = \mathbf{x}_0 - \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{v}) \in \Lambda_{4,j} (3 \le j \le 5, \mathbf{v} \in \mathbf{C})$ *in the standard form of* (5). *Then*

$$(\mathbf{x},\mathbf{y}) = 2 \iff \rho(\delta) = \mathbf{u} \cap \mathbf{v}.$$

Proof. (i $\Longrightarrow$). We compute

$$\begin{aligned}(\mathbf{x},\mathbf{y}) &= (\frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}) - \sqrt{2}\delta, \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{v}) - \sqrt{2}\delta_2) \\ &= \frac{1}{2}(\rho^{\#}(\mathbf{u}), \rho^{\#}(\mathbf{v})) - (\delta, \rho^{\#}(\mathbf{v})) - (\delta_2, \rho^{\#}(\mathbf{u})) + 2(\delta, \delta_2) \\ &= \frac{1}{2}\mathbf{u} * \mathbf{v} - \rho(\delta) * \mathbf{v} - \rho(\delta_2) * \mathbf{u} \\ &= 2.\end{aligned}$$

From the last equality follows

$$4 + 2\rho(\delta) * \mathbf{v} + 2\rho(\delta_2) * \mathbf{u} = \mathbf{u} * \mathbf{v} + 4\rho(\delta) * \rho(\delta_2).$$

The converse direction of the proof is also true.

(ii $\Longrightarrow$). Using Proposition 3.1 we have

$$\begin{aligned}(\mathbf{x},\mathbf{y}) &= (\frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}) - \sqrt{2}\delta, \mathbf{x}_0 - \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{v}) \\ &= \frac{1}{\sqrt{2}}(\rho^{\#}(\mathbf{u}), \mathbf{x}_0) - \sqrt{2}(\delta, \mathbf{x}_0) - \frac{1}{2}(\rho^{\#}(\mathbf{u}), \rho^{\#}(\mathbf{v})) + (\delta, \rho^{\#}(\mathbf{v}) \\ &= 2 - \frac{1}{2}wt(\rho(\delta)) - \frac{1}{2}\mathbf{u} * \mathbf{v} + \rho(\delta) * \mathbf{v} \\ &= 2.\end{aligned}$$

From this we obtain

$$-wt(\rho(\delta)) - \mathbf{u} * \mathbf{v} + 2\rho(\delta) * \mathbf{v} = 0. \tag{11}$$

Since we know $\rho(\delta) * \mathbf{v} \le wt(\rho(\delta))$ and $\rho(\delta) * \mathbf{v} \le \mathbf{u} * \mathbf{v}$, from (11) we obtain

$$\rho(\delta) * \mathbf{v} = wt(\rho(\delta)) = \mathbf{u} * \mathbf{v}. \tag{12}$$

If the i-th coordinate of $\rho(\delta)$ is 1 and the i-th coordinate of $\mathbf{v}$ is 0, then $wt(\rho(\delta)) > \mathbf{v} * \rho(\delta)$, contrary to the equation (12). Therefore we must have $\rho(\delta) \subseteq \mathbf{v}$. Thus $\rho(\delta) \subseteq \mathbf{u} \cap \mathbf{v}$. If $\rho(\delta) \ne \mathbf{u} \cap \mathbf{v}$, then we would have $wt(\rho(\delta)) < \mathbf{u} * \mathbf{v}$, which also contradicts to (12). Thus we conclude that $\rho(\delta) = \mathbf{u} \cap \mathbf{v}$.

(ii $\Longleftarrow$). Suppose that $\rho(\delta) = \mathbf{u} \cap \mathbf{v}$.Then (12) holds, which implies (11). From the last condition we have $(\mathbf{x},\mathbf{y}) = 2$. □

Note 2. *The equation* $4 + 2\rho(\delta) * \mathbf{v} + 2\rho(\delta_2) * \mathbf{u} = \mathbf{u} * \mathbf{v} + 4\rho(\delta) * \rho(\delta_2)$ *in* Lemma 5.1 *imposes a restriction on the intersection between* $\mathbf{u}$ *and* $\mathbf{v}$ *when both are in* $\mathbf{C}_8$. *In fact* $\mathbf{u} * \mathbf{v} = 4$ *or* 8, *since* $\rho(\delta) \subseteq \mathbf{u}$ *and* $\rho(\delta_2) \subseteq \mathbf{v}$ *imply* $\rho(\delta) * \mathbf{v} \ge \rho(\delta) * \rho(\delta_2)$ *and* $\rho(\delta_2) * \mathbf{u} \ge \rho(\delta) * \rho(\delta_2)$. *This fact will be used throughout the paper without further comments.*

5.2 From Lattice To Code

5.2.1 The First Cluster

The first cluster consists of the four partial sums, the first three of which are computed by manipulation. The computation of case $(Q4)$ is the most difficult.

Table 4-1a. Lattice components and the corresponding code components

type	$l-$ *components*	$c-$ *components*	*codeword condition*
$(Q1)$	$\Lambda_{4,1}, \Lambda_{4,1}, \Lambda_{4,1}, \Lambda_{4,1}$	*empty*	*not specific*
$(Q2)$	$\Lambda_{4,1}, \Lambda_{4,1}, \Lambda_{4,1}, \Lambda_{4,2}$	$\mathbf{C}_8$	*not specific*
$(Q3)$	$\Lambda_{4,1}, \Lambda_{4,1}, \Lambda_{4,2}, \Lambda_{4,2}$	$\mathbf{C}_8, \mathbf{C}_8$	*connected with biweight enumerator*
$(Q4)$	$\Lambda_{4,1}, \Lambda_{4,2}, \Lambda_{4,2}, \Lambda_{4,2}$	$\mathbf{C}_8, \mathbf{C}_8, \mathbf{C}_8$	*connected with triweight enumerator*

Lemma 5.2. *Suppose that* $\rho(\delta_1) \subseteq \mathbf{u}_1, \rho(\delta_2) \subseteq \mathbf{u}_2$ *and* $\rho(\delta_1) * \mathbf{u}_2 + \rho(\delta_2) * \mathbf{u}_1 = 2\ \rho(\delta_1) * \rho(\delta_2)$. *Then*

$$\rho(\delta_1) * \mathbf{u}_2 = \rho(\delta_2) * \mathbf{u}_1 = \rho(\delta_1) * \rho(\delta_2).$$

Proof. By the first two assumptions we see that $\rho(\delta_1) * \rho(\delta_2) \leq \rho(\delta_1) * \mathbf{u}_2$ and $\rho(\delta_1) * \rho(\delta_2) \leq \rho(\delta_2) * \mathbf{u}_1$, therefore

$$0 = [\rho(\delta_1) * \mathbf{u}_2 - \rho(\delta_1) * \rho(\delta_2)] + [\rho(\delta_2) * \mathbf{u}_1 - \rho(\delta_1) * \rho(\delta_2)] \geq 0.$$

Thus one obtains

$$\rho(\delta_1) * \mathbf{u}_2 = \rho(\delta_2) * \mathbf{u}_1 = \rho(\delta_1) * \rho(\delta_2)$$

as desired. □

Any element of $\frac{1}{\sqrt{2}}\Lambda_{4,1}$ has norm 2 and defines a reflection with respect to the hyperplane that is orthogonal to the element. The group $\mathcal{G}_0$ generated by all such reflections acts on the set $\Lambda_{4,1} \cup \Lambda_{4,2}$. Since $\frac{1}{\sqrt{2}}\Lambda_{4,1}$ is a root system of type D_{32}, $\mathcal{G}_0$ acts transitively on the set $\Lambda_{4,2}(\mathbf{u})$, which is defined immediately following equation (7). To compute the sum $\sigma(2,2,2,1)$ first we seek the vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \Lambda_{4,2}$ which satisfy $(\mathbf{x}_i, \mathbf{x}_j) = 2,\ 1 \leq i < j \leq 3$ then we explore $\mathbf{x}_4 \in \Lambda_{4,1}$ satisfying $(\mathbf{x}_i, \mathbf{x}_4) = 2,\ 1 \leq i \leq 3$. With the group $\mathcal{G}_0$ in mind we may assume that the shape of $\mathbf{x}_1$ is of the form $\frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}_1),\ \mathbf{u}_1 \in \mathbf{C}_8$. We take

$$\mathbf{x}_2 = \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}_2) - \sqrt{2}\delta_2, \mathbf{x}_3 = \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}_3) - \sqrt{2}\delta_3,\ \mathbf{u}_2, \mathbf{u}_3 \in \mathbf{C}_8$$

in their standard form. By Proposition 5.1,(i) we have

$$4 + 2\rho(\delta_j) * \mathbf{u}_1 = \mathbf{u}_1 * \mathbf{u}_j\ j = 2, 3, \tag{13}$$

and

$$4 + 2\rho(\delta_2) * \mathbf{u}_3 + 2\rho(\delta_3) * \mathbf{u}_2 = \mathbf{u}_2 * \mathbf{u}_3 + 4\rho(\delta_2) * \rho(\delta_3). \tag{14}$$

Concerning the mutual relations among $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ we may distinguish three cases: (i) all three are equal, (ii) two of three are equal and the remaining one is different, (iii) all three are different.

First we treat the case (i). When $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{u}_3$, the conditions (13) and (14) become

$$(13)_1 \qquad \rho(\delta_j) * \mathbf{u}_1 = 2,\ j = 2, 3,$$

and

$$(14)_1 \qquad \rho(\delta_2) * \mathbf{u}_3 + \rho(\delta_3) * \mathbf{u}_2 = 2 + 2\rho(\delta_2) * \rho(\delta_3).$$

Using $(13)_1$ the last condition is transformed into

$$(14)_{11} \qquad \rho(\delta_2) * \rho(\delta_3) = 1.$$

By a simple combinatorial argument the number of possible pairs $\langle \rho(\delta_2), \rho(\delta_3) \rangle$ is $28 \cdot 12$. For each fixed pair $\langle \rho(\delta_2), \rho(\delta_3) \rangle$ there are $10 + 5$ vectors $\mathbf{x}_4 \in \Lambda_{4,1}$ satisfying $(\mathbf{x}_4, \mathbf{x}_i) = 2,\ 1 \le i \le 3$. Thus the contribution of this case to $(Q4)$ is $2^7 \cdot 620 \cdot 28 \cdot 12 \cdot 15 = 399974400$.

Next we treat the case (ii). (ii)-(i) When $\mathbf{u}_1 = \mathbf{u}_2 \neq \mathbf{u}_3$, then $\mathbf{u}_1 * \mathbf{u}_2 = 8, \mathbf{u}_1 * \mathbf{u}_3 = \mathbf{u}_2 * \mathbf{u}_3 = 4$ hold. With these equalities the conditions (13) and (14) can be rewritten as

$$(13)_2 \qquad \rho(\delta_2) * \mathbf{u}_1 = 2, \rho(\delta_3) * \mathbf{u}_1 = 0,$$

and

$$(14)_2 \qquad \rho(\delta_2) * \mathbf{u}_3 + \rho(\delta_3) * \mathbf{u}_2 = 2\rho(\delta_2) * \rho(\delta_3).$$

By Lemma 5.2 the condition $(14)_2$ becomes

$$(14)_{21} \qquad \rho(\delta_2) * \mathbf{u}_3 = \rho(\delta_3) * \mathbf{u}_2 = \rho(\delta_2) * \rho(\delta_3).$$

Combining the conditions $(13)_2$ and $(14)_{21}$ we have

$$(14)_{22} \qquad \rho(\delta_2) * \mathbf{u}_3 = 0.$$

The number of pairs $\langle \rho(\delta_2), \rho(\delta_3) \rangle$ which satisfy the conditions $(13)_2$ and $(14)_{22}$ is 48. For each fixed pair $\langle \rho(\delta_2), \rho(\delta_3) \rangle$ there are 6 vectors $\mathbf{x}_4 \in \Lambda_{4,1}$ satisfying $(\mathbf{x}_4, \mathbf{x}_i) = 2,\ 1 \le i \le 3$. Thus the contribution of this case to $(Q4)$ is $2^7 \cdot 620 \cdot 84 \cdot 48 \cdot 6$.

There are two remaining cases, namely (ii)-(ii) when $\mathbf{u}_1 = \mathbf{u}_3 \neq \mathbf{u}_2$ and (ii)-(iii) when $\mathbf{u}_2 = \mathbf{u}_3 \neq \mathbf{u}_1$. The case (ii)-(ii) is treated in the same way as case (ii)-(i), and the contribution to $(Q4)$ is the same. The case (ii)-(iii) is a little different. The main difference comes from the transformations of the conditions (13) and (14). In this case we obtain

$$(13)_3 \qquad \rho(\delta_j) * \mathbf{u}_1 = 0,\ j = 2, 3,$$

and

$$(14)_3 \qquad \rho(\delta_2) * \mathbf{u}_3 + \rho(\delta_3) * \mathbf{u}_2 = 2 + \rho(\delta_2) * \rho(\delta_3).$$

However, the count itself is the same as in the preceding two cases. The total contribution of the case (ii) is $2^7 \cdot 620 \cdot 84 \cdot 48 \cdot 6 \cdot 3 = 5759631360$.

The case (iii) when $\mathbf{u}_i \neq \mathbf{u}_j,\ 1 \le i < j \le 3$. Now conditions (13) and (14) become $(13)_3$ and $(14)_{21}$ respectively. After several trials we discover that the number of pairs $\langle \rho(\delta_2), \rho(\delta_3) \rangle$ may depend on the terms that come from the code words $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ satisfying

$\mathbf{u}_i * \mathbf{u}_j = 4,\ 1 \le i < j \le 3$ in the triweight enumerator of the code. We need a portion of the triweight enumerator for each of the five extremal binary self-dual $[32, 16, 8]$ codes, and we compute

$$\begin{aligned}
&\mathbf{W}_3(x_\alpha; \mathbf{C}_{\mathrm{CP1}})\\
&= \cdots + 260400 x_{111}^4 x_{100}^4 x_{010}^4 x_{001}^4 x_{000}^{16} + 1249920 x_{111}^2 x_{110}^2 x_{101}^2 x_{100}^2 x_{011}^2 x_{010}^2 x_{001}^2 x_{000}^{18}\\
&\quad +52080 x_{110}^4 x_{101}^4 x_{011}^4 x_{000}^{20} + \cdots,\\
&\mathbf{W}_3(x_\alpha; \mathbf{C}_{\mathrm{CP2}})\\
&= \cdots + 52080 x_{111}^4 x_{100}^4 x_{010}^4 x_{001}^4 x_{000}^{16} + 833280 x_{111}^3 x_{110} x_{101} x_{100}^3 x_{011} x_{010}^3 x_{001}^3 x_{000}^{17} +\\
&\quad +416640 x_{111}^2 x_{110}^2 x_{101}^2 x_{100}^2 x_{011}^2 x_{010}^2 x_{001}^2 x_{000}^{18} + 52080 x_{110}^4 x_{101}^4 x_{011}^4 x_{000}^{20} + \cdots,\\
&\mathbf{W}_3(x_\alpha; \mathbf{C}_{\mathrm{CP3}})\\
&= \cdots + 56112 x_{111}^4 x_{100}^4 x_{010}^4 x_{001}^4 x_{000}^{16} + 817152 x_{111}^3 x_{110} x_{101} x_{100}^3 x_{011} x_{010}^3 x_{001}^3 x_{000}^{17} +\\
&\quad +432768 x_{111}^2 x_{110}^2 x_{101}^2 x_{100}^2 x_{011}^2 x_{010}^2 x_{001}^2 x_{000}^{18} + 52080 x_{110}^4 x_{101}^4 x_{011}^4 x_{000}^{20} + \cdots,\\
&\mathbf{W}_3(x_\alpha; \mathbf{C}_{\mathrm{CP4}})\\
&= \cdots + 99120 x_{111}^4 x_{100}^4 x_{010}^4 x_{001}^4 x_{000}^{16} + 645120 x_{111}^3 x_{110} x_{101} x_{100}^3 x_{011} x_{010}^3 x_{001}^3 x_{000}^{17} +\\
&\quad +604800 x_{111}^2 x_{110}^2 x_{101}^2 x_{100}^2 x_{011}^2 x_{010}^2 x_{001}^2 x_{000}^{18} + 52080 x_{110}^4 x_{101}^4 x_{011}^4 x_{000}^{20} + \cdots,\\
&\mathbf{W}_3(x_\alpha; \mathbf{C}_{\mathrm{CP5}})\\
&= \cdots + 58800 x_{111}^4 x_{100}^4 x_{010}^4 x_{001}^4 x_{000}^{16} + 806400 x_{111}^3 x_{110} x_{101} x_{100}^3 x_{011} x_{010}^3 x_{001}^3 x_{000}^{17} +\\
&\quad +443520 x_{111}^2 x_{110}^2 x_{101}^2 x_{100}^2 x_{011}^2 x_{010}^2 x_{001}^2 x_{000}^{18} + 52080 x_{110}^4 x_{101}^4 x_{011}^4 x_{000}^{20} + \cdots.
\end{aligned}$$

For the term $x_{111}^4 x_{100}^4 x_{010}^4 x_{001}^4 x_{000}^{16}$ by a simple inspection we find that there arise 64 pairs $\langle \rho(\delta_2), \rho(\delta_3)\rangle$ satisfying the conditions $(13)_3$ and $(14)_{21}$. The configuration of non zero coordinates of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ that produce the term $x_{111}^4 x_{100}^4 x_{010}^4 x_{001}^4 x_{000}^{16}$ may be roughly represented (we ignore real coordinate positions) by

$$\begin{array}{ll}
\mathbf{u}_1 & 1111111100000000\\
\mathbf{u}_2 & 1111000011110000\\
\mathbf{u}_3 & 1111000000001111.
\end{array}$$

The term $x_{111}^3 x_{110} x_{101} x_{100}^3 x_{011} x_{010}^3 x_{001}^3 x_{000}^{17}$ comes from the code word configuration:

$$\begin{array}{ll}
\mathbf{u}_1 & 111111110000000\\
\mathbf{u}_2 & 111100001111000\\
\mathbf{u}_3 & 111010001000111,
\end{array}$$

and the term $x_{111}^2 x_{110}^2 x_{101}^2 x_{100}^2 x_{011}^2 x_{010}^2 x_{001}^2 x_{000}^{18}$ comes from the code word configuration:

$$\begin{array}{ll}
\mathbf{u}_1 & 11111111000000\\
\mathbf{u}_2 & 11110000111100\\
\mathbf{u}_3 & 11001100110011.
\end{array}$$

As to the last two types of code word configurations we count the pairs $\langle \rho(\delta_2), \rho(\delta_3)\rangle$ satisfying conditions $(13)_3$ and $(14)_{21}$ by a simple programming. We give the results as the following table.

Table 4-1b. Number of pairs $\langle \rho(\delta_2), \rho(\delta_3)\rangle$ that correspond to the term displayed at the leftmost column, and the number of admissible vectors in $\Lambda_{4,1}$.

$x_{111}^4 x_{100}^4 x_{010}^4 x_{001}^4 x_{000}^{16}$	64	6
$x_{111}^3 x_{110} x_{101} x_{100}^3 x_{011} x_{010}^3 x_{001}^3 x_{000}^{17}$	32	3
$x_{111}^2 x_{110}^2 x_{101}^2 x_{100}^2 x_{011}^2 x_{010}^2 x_{001}^2 x_{000}^{18}$	16	1
$x_{110}^4 x_{101}^4 x_{011}^4 x_{000}^{20}$	3	0

Finding the vectors $\mathbf{x}_4 \in \Lambda_{4,1}$ which satisfy $(\mathbf{x}_i, \mathbf{x}_4) = 2,\ 1 \leq i \leq 3$ is rather easy in each case. Two non zero coordinates of $\mathbf{x}_4$ should be part of the coordinates corresponding to x_{111}. We summarize the contribution of case (iii) as follows:

CP1 $2^7 \cdot (260400 \cdot 64 \cdot 6 + 1249920 \cdot 16) = 15359016960,$
CP2 $2^7 \cdot (52080 \cdot 64 \cdot 6 + 833280 \cdot 32 \cdot 3 + 416640 \cdot 16) = 13652459520,$
CP3 $2^7 \cdot (56112 \cdot 64 \cdot 6 + 817152 \cdot 32 \cdot 3 + 432768 \cdot 16) = 13685489664,$
CP4 $2^7 \cdot (99120 \cdot 64 \cdot 6 + 645120 \cdot 32 \cdot 3 + 604800 \cdot 16) = 14037811200,$
CP5 $2^7 \cdot (58800 \cdot 64 \cdot 6 + 806400 \cdot 32 \cdot 3 + 443520 \cdot 16) = 13707509760.$

5.2.2 The Second Cluster

The second cluster consists of the four partial sums $(Q5) - (Q8)$ which are connected with the terms of the triweight enumerator of the code.

When $\Lambda_{4,6}$ enters as a component, we take $\mathbf{x}_0 \in \Lambda_{4,6}$ and then by Proposition 3.2 $\Lambda_{4,1}, \Lambda_{4,3}^-, \Lambda_{4,4}, \Lambda_{4,5}$ cannot be components in a partial sum, hence only $\Lambda_{4,2}$ and $\Lambda_{4,3}^+$ could be the other components. In the case when $-\mathbf{x}_0 \in \Lambda_{4,2}$ then $\Lambda_{4,3}^-$ could be the other component. We confirm that all possible quadruples are listed in Table 4-2 below. Omitting the full details we simply state that

$$\sigma(2,2,2,6) = 2\, b(M_{24}, \mathbf{C}).$$

This equation is just for the case $(Q5)$ below. The remaining cases $(Q6) - (Q8)$ are exhibited in Table 4-2.
We summarize our present discussion in Table 4-2.

Table 4-2. Lattice components and the corresponding code components

type	$l-$ *components*	*codeword condition* $[[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]] = M_*$
$(Q5)$	$\Lambda_{4,2}, \Lambda_{4,2}, \Lambda_{4,2}, \Lambda_{4,6}$	$M_{24} = (8,8,8,4,4,4)$
$(Q6)$	$\Lambda_{4,2}, \Lambda_{4,2}, \Lambda_{4,3}, \Lambda_{4,6}$	$M_{24,a} = (8,8,8,4,0,0)$
$(Q7)$	$\Lambda_{4,2}, \Lambda_{4,3}, \Lambda_{4,3}, \Lambda_{4,6}$	$M_{24,b} = (8,8,8,0,0,4)$
$(Q8)$	$\Lambda_{4,3}, \Lambda_{4,3}, \Lambda_{4,3}, \Lambda_{4,6}$	$M_{24} = (8,8,8,4,4,4)$

5.2.3 The Third Cluster

A partial sum belonging to this cluster is explained as follows. Let $\mathbf{x}_1 \in \Lambda_{4,j_1}, \mathbf{x}_2 \in \Lambda_{4,j_2}, \mathbf{x}_3 \in \Lambda_{4,j_3}, \mathbf{x}_4 \in \Lambda_{4,j_4}$ $3 \leq j_1 \leq j_2 \leq j_3 \leq j_4 \leq 5$, and we express $\mathbf{x}_i = \mathbf{x}_0 - \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}_i)\ \mathbf{u}_i \in \mathbf{C}_i, i \in \{8, 12, 16, 20, 24\}$ in standard form. Then by Proposition 3.4 we know that

$$(\mathbf{x}_{i_1}, \mathbf{x}_{i_2}) = 2 \Longleftrightarrow wt(\mathbf{u}_{i_1} + \mathbf{u}_{i_2}) = 8.$$

There are fifteen quadruples $\langle \Lambda_{4,j_1}, \Lambda_{4,j_2}, \Lambda_{4,j_3}, \Lambda_{4,j_4} \rangle, 3 \leq j_1 \leq j_2 \leq j_3 \leq j_4 \leq 5$. Viewing Table 5, of these fifteen quadruples only 3 quadruples $\langle \Lambda_{4,4}, \Lambda_{4,4}, \Lambda_{4,4}, \Lambda_{4,4} \rangle$, $\langle \Lambda_{4,4}, \Lambda_{4,4}, \Lambda_{4,4}, \Lambda_{4,5} \rangle$, and $\langle \Lambda_{4,4}, \Lambda_{4,4}, \Lambda_{4,5}, \Lambda_{4,5} \rangle$ could admit finer divisions. Out of $\langle \Lambda_{4,4}, \Lambda_{4,4}, \Lambda_{4,4}, \Lambda_{4,4} \rangle$

there arise $\langle \Lambda_{4,4}^{+}, \Lambda_{4,4}^{+}, \Lambda_{4,4}^{+}, \Lambda_{4,4}^{+} \rangle$, $\langle \Lambda_{4,4}^{+}, \Lambda_{4,4}^{+}, \Lambda_{4,4}^{+}, \Lambda_{4,4}^{-} \rangle$, $\langle \Lambda_{4,4}^{+}, \Lambda_{4,4}^{+}, \Lambda_{4,4}^{-}, \Lambda_{4,4}^{-} \rangle$ and the same with opposite signs. Thus we conclude that

$$\begin{aligned}\sigma(4,4,4,4) &= 2\sigma(4^+,4^+,4^+,4^+) + 8\sigma(4^+,4^+,4^+,4^-) + 6\sigma(4^+,4^+,4^-,4^-) \\ &= 2b(M_{19,1},\mathbf{C}) + 8b(M_{19,2},\mathbf{C}) + 6b(M_{19,3},\mathbf{C}),\end{aligned}$$

where 4^+(*resp.* 4^-) indicates $\Lambda_{4,4}^{+}$(*resp.* $\Lambda_{4,4}^{-}$).
Likewise we have

$$\begin{aligned}\sigma(4,4,4,5) &= 2\sigma(4^+,4^+,4^+,5) + 6\sigma(4^+,4^+,4^-,5) \\ &= 2b(M_{20,1},\mathbf{C}) + 6b(M_{20,2},\mathbf{C}),\end{aligned}$$

and the other cases are summarized in Table 4-3 below.

Table 4-3. Lattice components and the corresponding code components

type	$l-components$	*codeword condition* $[[\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4]] = M_*$
$(Q9)$	$\Lambda_{4,3},\Lambda_{4,3},\Lambda_{4,3},\Lambda_{4,3}$	$M_9 = (8,8,8,8,4,4,4,4,4,4)$
$(Q10)$	$\Lambda_{4,3},\Lambda_{4,3},\Lambda_{4,3},\Lambda_{4,4}$	$M_{10} = (8,8,8,12,4,4,4,6,6,6)$
$(Q11)$	$\Lambda_{4,3},\Lambda_{4,3},\Lambda_{4,3},\Lambda_{4,5}$	$M_{11} = (8,8,8,16,4,4,4,8,8,8)$
$(Q12)$	$\Lambda_{4,3},\Lambda_{4,3},\Lambda_{4,4},\Lambda_{4,4}$	$M_{12} = (8,8,12,12,4,6,6,6,6,8)$
$(Q13)$	$\Lambda_{4,3},\Lambda_{4,3},\Lambda_{4,4},\Lambda_{4,5}$	$M_{13} = (8,8,12,16,4,6,6,8,8,10)$
$(Q14)$	$\Lambda_{4,3},\Lambda_{4,3},\Lambda_{4,5},\Lambda_{4,5}$	$M_{14} = (8,8,16,16,4,8,8,8,8,12)$
$(Q15)$	$\Lambda_{4,3},\Lambda_{4,4},\Lambda_{4,4},\Lambda_{4,4}$	$M_{15} = (8,12,12,12,6,6,8,6,8,8)$
$(Q16)$	$\Lambda_{4,3},\Lambda_{4,4},\Lambda_{4,4},\Lambda_{4,5}$	$M_{16} = (8,12,12,16,6,6,8,8,10,10)$
$(Q17)$	$\Lambda_{4,3},\Lambda_{4,4},\Lambda_{4,5},\Lambda_{4,5}$	$M_{17} = (8,12,16,16,6,8,10,8,10,12)$
$(Q18)$	$\Lambda_{4,3},\Lambda_{4,5},\Lambda_{4,5},\Lambda_{4,5}$	$M_{18} = (8,16,16,16,8,8,12,8,12,12)$
$(Q19)_1$	$\Lambda_{4,4}^{+},\Lambda_{4,4}^{+},\Lambda_{4,4}^{+},\Lambda_{4,4}^{+}$	$M_{19,1} = (12,12,12,12,8,8,8,8,8,8)$
$(Q19)_2$	$\Lambda_{4,4}^{+},\Lambda_{4,4}^{+},\Lambda_{4,4}^{+},\Lambda_{4,4}^{-}$	$M_{19,2} = (12,12,12,20,8,8,8,12,12,12)$
$(Q19)_3$	$\Lambda_{4,4}^{+},\Lambda_{4,4}^{+},\Lambda_{4,4}^{-},\Lambda_{4,4}^{-}$	$M_{19,3} = (12,12,20,20,8,12,12,12,12,16)$
$(Q20)_1$	$\Lambda_{4,4}^{+},\Lambda_{4,4}^{+},\Lambda_{4,4}^{+},\Lambda_{4,5}$	$M_{20.1} = (12,12,12,16,8,8,8,10,10,10)$
$(Q20)_2$	$\Lambda_{4,4}^{+},\Lambda_{4,4}^{+},\Lambda_{4,4}^{-},\Lambda_{4,5}$	$M_{20,2} = (12,12,20,16,8,12,12,10,10,14)$
$(Q21)_1$	$\Lambda_{4,4}^{+},\Lambda_{4,4}^{+},\Lambda_{4,5},\Lambda_{4,5}$	$M_{21,1} = (12,12,16,16,8,10,10,10,10,12)$
$(Q21)_2$	$\Lambda_{4,4}^{+},\Lambda_{4,4}^{-},\Lambda_{4,5},\Lambda_{4,5}$	$M_{21,2} = (12,20,16,16,12,10,14,10,14,12)$
$(Q22)$	$\Lambda_{4,4},\Lambda_{4,5},\Lambda_{4,5},\Lambda_{4,5}$	$M_{22} = (12,16,16,16,10,10,12,10,12,12)$
$(Q23)$	$\Lambda_{4,5},\Lambda_{4,5},\Lambda_{4,5},\Lambda_{4,5}$	$M_{23} = (16,16,16,16,12,12,12,12,12,12)$

5.2.4 The Fourth Cluster

When $\Lambda_{4,2}$ enters only once as a component in a partial sum, then we consider $\mathbf{x}_1 \in \Lambda_{4,2}, \mathbf{x}_2 \in \Lambda_{4,j_2}, \mathbf{x}_3 \in \Lambda_{4,j_3}, \mathbf{x}_4 \in \Lambda_{4,j_4}$ $3 \le j_2 \le j_3 \le j_4 \le 5$. One may take $\mathbf{x}_1 = \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}_1) - \sqrt{2}\delta \in \Lambda_{4,2}$ and $\mathbf{x}_i = \mathbf{x}_0 - \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}_i)$ to be in standard form. There are ten such quadruples. Out of these ten quadruples only two quadruples $\langle \Lambda_{4,2}, \Lambda_{4,4}, \Lambda_{4,4}, \Lambda_{4,4} \rangle$ and $\langle \Lambda_{4,2}, \Lambda_{4,4}, \Lambda_{4,4}, \Lambda_{4,5} \rangle$ could admit finer divisions, namely, the former into $\langle \Lambda_{4,2}, \Lambda_{4,4}^{+}, \Lambda_{4,4}^{+}, \Lambda_{4,4}^{+} \rangle$ and $\langle \Lambda_{4,2}, \Lambda_{4,4}^{+}, \Lambda_{4,4}^{+}, \Lambda_{4,4}^{-} \rangle$ and the same with the signs reversed, and the latter is into $\langle \Lambda_{4,2}, \Lambda_{4,4}^{+}, \Lambda_{4,4}^{+}, \Lambda_{4,5} \rangle$

and $\langle \Lambda_{4,2}, \Lambda_{4,4}^+, \Lambda_{4,4}^-, \Lambda_{4,5} \rangle$ and the same with the signs reversed. We treat the case $\mathbf{x}_1 \in \Lambda_{4,2}, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \in \Lambda_{4,4}^+$. By Lemma 5.1,(ii) it holds that

$$(\mathbf{x}_1, \mathbf{x}_i) = 2 \ (i = 2, 3, 4) \iff \rho(\delta) = \mathbf{u}_1 \cap \mathbf{u}_i.$$

We may write the last conditions as

$$(C1) \qquad \mathbf{u}_1 \cap \mathbf{u}_2 = \mathbf{u}_1 \cap \mathbf{u}_3 = \mathbf{u}_1 \cap \mathbf{u}_4.$$

We remark that the condition $(C1)$ is a common feature of the quadruple in this cluster, although the code words $\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ vary according to the quadruple. By Proposition 3.4 we know that

$$(\mathbf{x}_i, \mathbf{x}_j) = 2 \ (2 \le i < j \le 4) \iff wt(\mathbf{u}_i + \mathbf{u}_j) = 8.$$

Since $\mathbf{u}_i \ (i = 2, 3, 4) \in \mathbf{C}_{12}$ the last conditions are identical to

$$\mathbf{u}_i * \mathbf{u}_j = 8, 2 \le i < j \le 4.$$

If we use the intersection matrix of degree 3 described below, the conditions can be written as $[[\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4]] = M_{30,1}$ (c.f. Table 4-4).
For the quadruple $\langle \Lambda_{4,2}, \Lambda_{4,4}^+, \Lambda_{4,4}^+, \Lambda_{4,4}^- \rangle$, the inner product conditions are transformed to $(C1)$ and $[[\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4]] = M_{30,2}$. Extending the notation of (2) we write

$$\sum_{\substack{\mathbf{u}_1 \in \mathbf{C}_8, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4 \in \mathbf{C}_{12} \\ C(1), [[\mathbf{u}_2, \cdots, \mathbf{u}_4]] = M_{30,1}}} 1 = b(C(1), M_{30,1}, \mathbf{C}).$$

Then we have

$$\begin{aligned} \sigma(2,4,4,4) &= 2b(C(1), M_{30,1}, \mathbf{C}) + 6b(C(1), M_{30,2}, \mathbf{C}), \\ \sigma(2,4,4,5) &= 2b(C(1), M_{31,1}, \mathbf{C}) + 2b(C(1), M_{31,2}, \mathbf{C}). \end{aligned}$$

Table 4-4 summarizes this cluster.

Table 4-4. Lattice components and the corresponding code conditions.

type	*l − components*	*codeword condition* $C(1)$ *plus* $[[\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4]] = M_*$
$(Q24)$	$\Lambda_{4,2}, \Lambda_{4,3}, \Lambda_{4,3}, \Lambda_{4,3}$	$M_{24} = (8, 8, 8, 4, 4, 4)$
$(Q25)$	$\Lambda_{4,2}, \Lambda_{4,3}, \Lambda_{4,3}, \Lambda_{4,4}$	$M_{25} = (8, 8, 12, 4, 6, 6)$
$(Q26)$	$\Lambda_{4,2}, \Lambda_{4,3}, \Lambda_{4,3}, \Lambda_{4,5}$	$M_{26} = (8, 8, 16, 4, 8, 8)$
$(Q27)$	$\Lambda_{4,2}, \Lambda_{4,3}, \Lambda_{4,4}, \Lambda_{4,4}$	$M_{27} = (8, 12, 12, 6, 6, 8)$
$(Q28)$	$\Lambda_{4,2}, \Lambda_{4,3}, \Lambda_{4,4}, \Lambda_{4,5}$	$M_{28} = (8, 12, 16, 6, 8, 10)$
$(Q29)$	$\Lambda_{4,2}, \Lambda_{4,3}, \Lambda_{4,5}, \Lambda_{4,5}$	$M_{29} = (8, 16, 16, 8, 8, 12)$
$(Q30)_1$	$\Lambda_{4,2}, \Lambda_{4,4}^+, \Lambda_{4,4}^+, \Lambda_{4,4}^+$	$M_{30,1} = (12, 12, 12, 8, 8, 8)$
$(Q30)_2$	$\Lambda_{4,2}, \Lambda_{4,4}^+, \Lambda_{4,4}^+, \Lambda_{4,4}^-$	$M_{30,2} = (12, 12, 20, 8, 12, 12)$
$(Q31)_1$	$\Lambda_{4,2}, \Lambda_{4,4}^+, \Lambda_{4,4}^+, \Lambda_{4,5}$	$M_{31,1} = (12, 12, 16, 8, 10, 10)$
$(Q31)_2$	$\Lambda_{4,2}, \Lambda_{4,4}^+, \Lambda_{4,4}^-, \Lambda_{4,5}$	$M_{31,2} = (12, 20, 16, 12, 10, 14)$
$(Q32)$	$\Lambda_{4,2}, \Lambda_{4,4}, \Lambda_{4,5}, \Lambda_{4,5}$	$M_{32} = (12, 16, 16, 10, 10, 12)$
$(Q33)$	$\Lambda_{4,2}, \Lambda_{4,5}, \Lambda_{4,5}, \Lambda_{4,5}$	$M_{33} = (16, 16, 16, 12, 12, 12)$

5.2.5 The Fifth Cluster

For a partial sum in which exactly two vectors belong to $\Lambda_{4,2}$ we consider $\mathbf{x}_1, \mathbf{x}_2 \in \Lambda_{4,2}, \mathbf{x}_3 \in \Lambda_{4,j_3}, \mathbf{x}_4 \in \Lambda_{4,j_4}$ $3 \le j_3 \le j_4 \le 5$. There are six such quadruples. There is one singular quadruple $\langle \Lambda_{4,2}, \Lambda_{4,2}, \Lambda_{4,4}, \Lambda_{4,4} \rangle$, which is divided into $\langle \Lambda_{4,2}, \Lambda_{4,2}, \Lambda^+_{4,4}, \Lambda^+_{4,4} \rangle$, $\langle \Lambda_{4,2}, \Lambda_{4,2}, \Lambda^+_{4,4}, \Lambda^-_{4,4} \rangle$ and the same with signs reversed. We take $\mathbf{x}_1 = \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}_1) - \sqrt{2}\delta_1, \mathbf{x}_2 = \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}_2) - \sqrt{2}\delta_2 \in \Lambda_{4,2}$ and $\mathbf{x}_3 = \mathbf{x}_0 - \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}_3), \mathbf{x}_4 = \mathbf{x}_0 - \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}_4) \in \Lambda^+_{4,4}$ to be in standard form. By Lemma 5.1, (i) we know that

$$(\mathbf{x}_1, \mathbf{x}_2) = 2 \Longleftrightarrow 4 + 2\rho(\delta_1) * \mathbf{u}_2 + 2\rho(\delta_2) * \mathbf{u}_1 = \mathbf{u}_1 * \mathbf{u}_2 + 4\rho(\delta_1) * \rho(\delta_2).$$

By Lemma 5.1, (ii) we know that

$$(\mathbf{x}_i, \mathbf{x}_j) = 2 \ (i = 1, 2, j = 3, 4) \Longleftrightarrow \rho(\delta_i) = \mathbf{u}_i \cap \mathbf{u}_3 = \mathbf{u}_i \cap \mathbf{u}_4 \ (i = 1, 2).$$

Since $\rho(\delta_1) = \mathbf{u}_1 \cap \mathbf{u}_3 = \mathbf{u}_1 \cap \mathbf{u}_4$ and $\rho(\delta_2) = \mathbf{u}_2 \cap \mathbf{u}_3 = \mathbf{u}_2 \cap \mathbf{u}_4$, the three intersections $\rho(\delta_1) * \mathbf{u}_2, \rho(\delta_2) * \mathbf{u}_1, \rho(\delta_1) * \rho(\delta_2)$ count the same common coordinates that are 1's. Therefore $\rho(\delta_1) * \mathbf{u}_2 = \rho(\delta_2) * \mathbf{u}_1 = \rho(\delta_1) * \rho(\delta_2)$, and $\mathbf{u}_1 * \mathbf{u}_2 = 4$. As a summary we write

$$(C2) \qquad \mathbf{u}_1 \cap \mathbf{u}_3 = \mathbf{u}_1 \cap \mathbf{u}_4, \mathbf{u}_2 \cap \mathbf{u}_3 = \mathbf{u}_2 \cap \mathbf{u}_4, \mathbf{u}_1 * \mathbf{u}_2 = 4.$$

By Proposition 3.4 we have

$$(\mathbf{x}_3, \mathbf{x}_4) = 2 \Longleftrightarrow wt(\mathbf{u}_3 + \mathbf{u}_4) = 8.$$

We note that $(C2)$ and $wt(\mathbf{u}_3 + \mathbf{u}_4) = 8$ guarantee $[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4] = 2\mathfrak{T}_{40}$. Thus we have

$$\sum_{\substack{\mathbf{x}_1,\mathbf{x}_2 \in \Lambda_{4,2}, \mathbf{x}_3, \mathbf{x}_4 \in \Lambda^+_{4,4} \\ [\mathbf{x}_1, \cdots, \mathbf{x}_4] = 2\mathfrak{T}_{40}}} 1 = \sum_{\substack{\mathbf{u}_1, \mathbf{u}_2 \in \mathbf{C}_8, \mathbf{u}_3, \mathbf{u}_4 \in \mathbf{C}_{12} \\ C(2), [[\mathbf{u}_3, \mathbf{u}_4]] = M_{37,1}}} 1$$

To express the last term we use the notation

$$b(C(2), M_{37,1}) = \sum_{\substack{\mathbf{u}_1, \mathbf{u}_2 \in \mathbf{C}_8, \mathbf{u}_3, \mathbf{u}_4 \in \mathbf{C}_{12} \\ C(2), [[\mathbf{u}_3, \mathbf{u}_4]] = M_{37,1}}} 1,$$

then we have

$$\sigma(2, 2, 4^+, 4^-) = b(C(2), M_{37,1}, \mathbf{C}).$$

Table 4-5. Lattice components and the corresponding code components

type	*l − components*	*codeword condition $C(2)$ plus $[[\mathbf{u}_3, \mathbf{u}_4]] = M_*$*
$(Q34)$	$\Lambda_{4,2}, \Lambda_{4,2}, \Lambda_{4,3}, \Lambda_{4,3}$	$M_{34} = (8, 8, 4)$
$(Q35)$	$\Lambda_{4,2}, \Lambda_{4,2}, \Lambda_{4,3}, \Lambda_{4,4}$	$M_{35} = (8, 12, 6)$
$(Q36)$	$\Lambda_{4,2}, \Lambda_{4,2}, \Lambda_{4,3}, \Lambda_{4,5}$	$M_{36} = (8, 16, 8)$
$(Q37)_1$	$\Lambda_{4,2}, \Lambda_{4,2}, \Lambda^+_{4,4}, \Lambda^+_{4,4}$	$M_{37,1} = (12, 12, 8)$
$(Q37)_2$	$\Lambda_{4,2}, \Lambda_{4,2}, \Lambda^+_{4,4}, \Lambda^-_{4,4}$	$M_{37,2} = (12, 20, 12)$
$(Q38)$	$\Lambda_{4,2}, \Lambda_{4,2}, \Lambda_{4,4}, \Lambda_{4,5}$	$M_{38} = (12, 16, 10)$
$(Q39)$	$\Lambda_{4,2}, \Lambda_{4,2}, \Lambda_{4,5}, \Lambda_{4,5}$	$M_{39} = (16, 16, 12)$

5.2.6 The Sixth Cluster

When $\Lambda_{4,2}$ enters three times as a component in a partial sum, then we consider $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \Lambda_{4,2}, \mathbf{x}_4 \in \Lambda_{4,j_4},\ 3 \le j_4 \le 5$. We take $\mathbf{x}_1 = \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}_1) - \sqrt{2}\delta_1, \mathbf{x}_2 = \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}_2) - \sqrt{2}\delta_2, \mathbf{x}_3 = \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}_3) - \sqrt{2}\delta_3 \in \Lambda_{4,2}$ and $\mathbf{x}_4 = \mathbf{x}_0 - \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}_4), \in \Lambda_{4,j_4}$ to be in standard form. By Lemma 5.1, (i) we have

$(\mathbf{x}_{i_1}, \mathbf{x}_{i_2}) = 2 \iff$

$$(C3)_0 \qquad 4 + 2\rho(\delta_{i_1}) * \mathbf{u}_{i_2} + 2\rho(\delta_{i_2}) * \mathbf{u}_{i_1} = \mathbf{u}_{i_1} * \mathbf{u}_{i_2} + 4\rho(\delta_{i_1}) * \rho(\delta_{i_2})\ (1 \le i_1 < i_2 \le 3).$$

By Lemma 5.1, (ii) we have $(\mathbf{x}_i, \mathbf{x}_{j_4}) = 2 \iff$

$$(C3)_2 \qquad \rho(\delta_i) = \mathbf{u}_i \cap \mathbf{u}_4\ 1 \le i \le 3.$$

By using both conditions $(C3)_0, (C3)_2$ we can say that $\rho(\delta_{i_1}) * \mathbf{u}_{i_2} = \rho(\delta_{i_2}) * \mathbf{u}_{i_1} = \rho(\delta_{i_1}) * \rho(\delta_{i_2})$ and

$$(C3)_1 \qquad \mathbf{u}_1 * \mathbf{u}_2 = 4, \mathbf{u}_1 * \mathbf{u}_3 = 4, \mathbf{u}_2 * \mathbf{u}_3 = 4.$$

Conversely if assume $(C3)_1$ and $(C3)_2$ we see that $[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4] = 2\mathfrak{T}_{40}$ holds. Thus we have

$$\sum_{\substack{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3 \in \Lambda_{4,2}, \mathbf{x}_4 \in \Lambda_{4,j_4} \\ [\mathbf{x}_1,\cdots,\mathbf{x}_4] = 2\mathfrak{T}_{40}}} 1 \;=\; 2 \sum_{\substack{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3 \in \mathbf{C}_8, \mathbf{u}_4 \in \mathbf{C}_{k_4} \\ (C3)_1,(C3)_2}} 1 \qquad j_4 = 3,4,5,\ k_4 = 4j_4 - 4$$

We use a formal convention $b(C(3)_1, C(3)_2, \mathbf{C}_{k_4})$ to denote

$$\sum_{\substack{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3 \in \mathbf{C}_8, \mathbf{u}_4 \in \mathbf{C}_{k_4} \\ (C3)_1,(C3)_2}} 1.$$

Proposition 5.3. *We have*

$$b(C(3)_1, C(3)_2, \mathbf{C}_{k_4}) = b(M_{24}, \mathbf{C})|\mathbf{C}_{k_4}|.$$

Proof. We remark that the condition $C(3)_1$ does not impose any restriction on the code word $\mathbf{u}_4 \in \mathbf{C}_{k_4}$, but it determines $\rho(\delta_i)$ completely. Thus we see that

$$\begin{aligned} b(C(3)_1, C(3)_2, \mathbf{C}_{k_4}) &= \sum_{\substack{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3 \in \mathbf{C}_8, \mathbf{u}_4 \in \mathbf{C}_{k_4} \\ (C3)_1,(C3)_2}} 1 \\ &= |\mathbf{C}_{k_4}| \sum_{\substack{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3 \in \mathbf{C}_8 \\ [[\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3]] = M_{24}}} 1 \\ &= |\mathbf{C}_{k_4}| b(M_{24}, \mathbf{C}). \end{aligned}$$

□

Table 4-6. Lattice components and the corresponding code components

type	*l − components*	*codeword condition*
$(Q40)$	$\Lambda_{4,2}, \Lambda_{4,2}, \Lambda_{4,2}, \Lambda_{4,3}$	$C(3)_1, C(3)_2$
$(Q41)$	$\Lambda_{4,2}, \Lambda_{4,2}, \Lambda_{4,2}, \Lambda_{4,4}$	$C(3)_1, C(3)_2$
$(Q42)$	$\Lambda_{4,2}, \Lambda_{4,2}, \Lambda_{4,2}, \Lambda_{4,5}$	$C(3)_1, C(3)_2$

5.2.7 The Seventh Cluster

This cluster consists of the unique partial sum described in the table below.

Table 4-7a. Table of the lattice components.

type	*l − components*	*c − components*	*codeword condition*
$(Q43)$	$\Lambda_{4,2}, \Lambda_{4,2}, \Lambda_{4,2}, \Lambda_{4,2}$	$\mathbf{C}_8, \mathbf{C}_8, \mathbf{C}_8, \mathbf{C}_8$	*not specified*

In treating this partial sum $\sigma(2,2,2,2)$ we may assume that $\mathbf{x}_1 = \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}_1)$, since the group $\mathcal{G}_0$ introduced in Section 5.2.1 acts transitively on the set $\Lambda_{4,2}(\mathbf{u})$. We take the other elements in $\Lambda_{4,2}$ to be

$$\mathbf{x}_2 = \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}_2) - \sqrt{2}\delta_2, \mathbf{x}_3 = \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}_3) - \sqrt{2}\delta_3, \mathbf{x}_4 = \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}_4) - \sqrt{2}\delta_4.$$

Then by Lemma 5.1,(i) we get for $2 \le j \le 4$,

(15) $$(\mathbf{x}_1, \mathbf{x}_j) = 2 \Longleftrightarrow 4 + 2\rho(\delta_j) * \mathbf{u}_1 = \mathbf{u}_1 * \mathbf{u}_j,$$

and for $2 \le i < j \le 4$

(16) $$(\mathbf{x}_i, \mathbf{x}_j) = 2 \Longleftrightarrow 4 + 2\rho(\delta_i) * \mathbf{u}_j + 2\rho(\delta_j) * \mathbf{u}_i = \mathbf{u}_i * \mathbf{u}_j + 4\rho(\delta_i) * \rho(\delta_j).$$

It follows from the conditions (15) and (16) that

$$\mathbf{u}_i * \mathbf{u}_j = 4 \text{ or } 8, (1 \le i < j \le 4).$$

We may distinguish five cases (reminding us of the five partitions $4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1$):
(i) all $\mathbf{u}_i$ $(1 \le i \le 4)$ are equal,
(ii) three of $\mathbf{u}_i$ $(1 \le i \le 4)$ are equal and the remaining one is different,
(iii) there are two different pairs and the members of each pair are equal,
(iv) one pair with the equal code word and other two are different from each other and from the members of the pair,
(v) all code words $\mathbf{u}_i$ $(1 \le i \le 4)$ are different from each other.
We examine each case in detail.

(i) We take $\mathbf{u}_1 = \cdots = \mathbf{u}_4 \in \mathbf{C}_8$. The condition (15) is rewritten as

$(15)_1$ $$\rho(\delta_j) * \mathbf{u}_1 = 2 \quad (2 \le j \le 4).$$

The condition (16) is rewritten as

$(16)_1$ $$\rho(\delta_i) * \mathbf{u}_j + \rho(\delta_j) * \mathbf{u}_i = 2 + 2\rho(\delta_i) * \rho(\delta_j) \quad 2 \le i < j \le 4.$$

Combining $(15)_1$ and $(16)_1$ we have

(17) $$\rho(\delta_i) * \rho(\delta_j) = 1, \quad 2 \le i < j \le 4.$$

We can easily count the triples $\langle \rho(\delta_2), \rho(\delta_3), \rho(\delta_4) \rangle$ satisfying $(15)_1$ and (17). The number is $28 \cdot 12 \cdot 6 = 2016$. The contribution of this type of vectors in $\Lambda_{4,2}$ is $2^7 \cdot 620 \cdot 2016 = 159989760$, which is independent of the choice of the extremal binary codes.

(ii)-(i) When $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{u}_3 \neq \mathbf{u}_4$. Then one sees that $\mathbf{u}_i * \mathbf{u}_j = 8, \ (1 \leq i < j \leq 3), \mathbf{u}_i * \mathbf{u}_4 = 4 \ (1 \leq i \leq 3)$. With these the condition (15) implies

$$(15)_2 \qquad \rho(\delta_j) * \mathbf{u}_1 = 2 \quad (2 \leq j \leq 3), \rho(\delta_4) * \mathbf{u}_1 = 0.$$

From $(15)_2$ and (16) we get

$$(16)_2 \qquad \rho(\delta_2) * \rho(\delta_3) = 1.$$

For a fixed pair $\langle \mathbf{u}_1, \mathbf{u}_4 \rangle$ satisfying $\mathbf{u}_1 * \mathbf{u}_4 = 4$ the number of triples $\langle \rho(\delta_2), \rho(\delta_3), \rho(\delta_4) \rangle$ that satisfy $(15)_2, (16)_2$ is $6 \cdot 4 \cdot 8 = 192$. By Proposition 2.2 there are $620 \cdot 84$ pairs of $\langle \mathbf{u}_1, \mathbf{u}_4 \rangle$ with the condition $\mathbf{u}_i * \mathbf{u}_4 = 4$. Thus the number of quadruples $\langle \mathbf{x}_1, \cdots, \mathbf{x}_4 \rangle$ that come from this case is $2^7 \cdot 620 \cdot 84 \cdot 192 = 1279918080$. The remaining subcases (ii)-(ii)$\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{u}_4 \neq \mathbf{u}_3$,(ii)-(iii)$\mathbf{u}_1 = \mathbf{u}_3 = \mathbf{u}_4 \neq \mathbf{u}_2$ are counted in the same way. As to the last subcase (ii)-(iv)$\mathbf{u}_2 = \mathbf{u}_3 = \mathbf{u}_4 \neq \mathbf{u}_1$, the conditions are a little different from other subcases. From (15) one gets

$$(15)_3 \qquad \rho(\delta_j) * \mathbf{u}_1 = 0 \quad 2 \leq j \leq 4,$$

and from (16) one gets $(16)_1$. This time we write a program that counts the triples $\langle \rho(\delta_2), \rho(\delta_3), \rho(\delta_4) \rangle$ satisfying $(15)_3, (16)_1$ for a fixed pair $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle$. However the result is 192. Thus the concluding count is the same as in the previous three cases.

(iii)-(i) When $\mathbf{u}_1 = \mathbf{u}_2 \neq \mathbf{u}_3 = \mathbf{u}_4$, then $\mathbf{u}_1 * \mathbf{u}_2 = 8, \mathbf{u}_3 * \mathbf{u}_4 = 8, \mathbf{u}_i * \mathbf{u}_j = 4, (i = 1, 2, j = 3, 4)$. It follows from (15) and (16) that

$$(15)_4 \qquad \begin{cases} \rho(\delta_2) * \mathbf{u}_1 = 2, \\ \rho(\delta_j) * \mathbf{u}_1 = 0, \ (j = 3, 4) \end{cases}$$

and

$$(16)_3 \qquad \begin{cases} \rho(\delta_2) * \mathbf{u}_j + \rho(\delta_j) * \mathbf{u}_2 = 2\rho(\delta_2) * \rho(\delta_j), j = 3, 4, \\ \rho(\delta_3) * \mathbf{u}_4 + \rho(\delta_4) * \mathbf{u}_3 = 2 + \rho(\delta_3) * \rho(\delta_4) \end{cases}$$

The second condition in $(15)_4$ is stronger than the first condition in $(16)_3$. Actually we see that $\rho(\delta_2) * \mathbf{u}_3 = \rho(\delta_3) * \mathbf{u}_2 = \rho(\delta_2) * \rho(\delta_3) = 0$ and $\rho(\delta_2) * \mathbf{u}_4 = \rho(\delta_4) * \mathbf{u}_2 = \rho(\delta_2) * \rho(\delta_4) = 0$. We may count the triples $\langle \rho(\delta_2), \rho(\delta_3), \rho(\delta_4) \rangle$ satisfying the condition $(15)_4$ and the last condition of $(16)_3$. The number is $6 \cdot 48$. Thus the number of quadruples $\langle \mathbf{x}_1, \cdots, \mathbf{x}_4 \rangle$ that come from this case is $2^7 \cdot 620 \cdot 84 \cdot 6 \cdot 48 = 1919877120$.

The remaining subcases (iii)-(ii)$\mathbf{u}_1 = \mathbf{u}_3 \neq \mathbf{u}_2 = \mathbf{u}_4$ and (iii)-(iii) $\mathbf{u}_1 = \mathbf{u}_4 \neq \mathbf{u}_2 = \mathbf{u}_3$ lead to the same results as the subcase (iii)-(i).

(iv)-(i) When $\mathbf{u}_1 = \mathbf{u}_2, \mathbf{u}_i \neq \mathbf{u}_j, \ (2 \leq i < j \leq 4)$, then $\mathbf{u} * \mathbf{u}_2 = 8, \mathbf{u}_i * \mathbf{u}_j = 4, \ (2 \leq i < j \leq 4)$. Then the condition (15) reads

$$(15)_5 \qquad \rho(\delta_2) * \mathbf{u}_1 = 2, \rho(\delta_3) * \mathbf{u}_1 = 0, \rho(\delta_4) * \mathbf{u}_1 = 0.$$

From the first condition of $(15)_5$ we have

$$(18) \qquad wt(\rho(\delta_2)) = 2.$$

The condition (16) reads

$$(16)_4 \qquad \rho(\delta_i) * \mathbf{u}_j + \rho(\delta_j) * \mathbf{u}_i = 2\rho(\delta_i) * \rho(\delta_j) \quad 2 \leq i < j \leq 4.$$

By Lemma 5.2 we have

$$\rho(\delta_i) * \mathbf{u}_j = \rho(\delta_j) * \mathbf{u}_i = \rho(\delta_i) * \rho(\delta_j) \quad 2 \le i < j \le 4. \tag{19}$$

The last two conditions of $(15)_5$ yield

$$\rho(\delta_2) * \mathbf{u}_j = \rho(\delta_j) * \mathbf{u}_2 = \rho(\delta_2) * \rho(\delta_j) = 0 \quad 3 \le j \le 4. \tag{20}$$

Thus the most important conditions are

$$\rho(\delta_3) * \mathbf{u}_4 = \rho(\delta_4) * \mathbf{u}_3 = \rho(\delta_3) * \rho(\delta_4). \tag{21}$$

Here we may note that the conditions $(15)_5, (20)$ restrict the range of $\rho(\delta_2), \rho(\delta_3), \rho(\delta_4)$. The number of triples $\langle \rho(\delta_2), \rho(\delta_3), \rho(\delta_4) \rangle$ satisfying the conditions $(15)_5, (18), (20), (21)$ may depend on the mutual intersection scheme among the code words $\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$. This intersection scheme can be described by the terms in the triweight enumerator of the code $\mathbf{C}$. Table 4-7b below gives the number of the triples in question. For instance, when $wt_{111}(\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4) = 4, wt_{100}(\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4) = 4, wt_{010}(\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4) = 4, wt_{001}(\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4) = 4, wt_{000}(\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4) = 16$ and the values of other generalized weights are zero then the number of triples is computed to be 384.

Table 4-7b. Number of triples $\langle \rho(\delta_2), \rho(\delta_3), \rho(\delta_4) \rangle$.

monomial	*triples*
$x_{111}^4 x_{100}^4 x_{010}^4 x_{001}^4 x_{000}^{16}$	384
$x_{111}^3 x_{110} x_{101} x_{100}^3 x_{011} x_{010}^3 x_{001}^3 x_{000}^{17}$	96
$x_{111}^2 x_{110}^2 x_{101}^2 x_{100}^2 x_{011}^2 x_{010}^2 x_{001}^2 x_{000}^{18}$	16
$x_{110}^4 x_{101}^4 x_{011}^4 x_{000}^{20}$	0

As to the frequency of each term in the above table one can use the portion of the triweight enumerators for the five extremal codes given in Subsubsection 5.2.1. Recall that we first chose $\mathbf{x}_1 = \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}_1),\ \mathbf{u}_1 \in \mathbf{C}_8$ as a representative of $\Lambda_{4,2}$ under the action of the group $\mathcal{G}_0$. There are $2^7 = 128$ members in an orbit of $\mathbf{x}_1$. Thus the contribution of this case to the partial sum $\sigma(2,2,2,2)$ amounts to

$$\begin{cases} 2^7 \cdot (260400 \cdot 384 + 1249920 \cdot 16) & \text{if } \mathbf{C} = \text{CP1}, \\ 2^7 \cdot (52080 \cdot 384 + 833280 \cdot 96 + 416640 \cdot 16) & \text{if } \mathbf{C} = \text{CP2}, \\ 2^7 \cdot (56112 \cdot 384 + 817152 \cdot 96 + 432768 \cdot 16) & \text{if } \mathbf{C} = \text{CP3}, \\ 2^7 \cdot (99120 \cdot 384 + 645120 \cdot 96 + 604800 \cdot 16) & \text{if } \mathbf{C} = \text{CP4}, \\ 2^7 \cdot (58800 \cdot 384 + 806400 \cdot 96 + 443520 \cdot 16) & \text{if } \mathbf{C} = \text{CP5}. \end{cases}$$

The cases (iv)-(ii) $\mathbf{u}_1 = \mathbf{u}_3, \mathbf{u}_i \ne \mathbf{u}_j,\ (2 \le i < j \le 4)$ and (iv)-(iii) $\mathbf{u}_1 = \mathbf{u}_4, \mathbf{u}_i \ne \mathbf{u}_j,\ (2 \le i < j \le 4)$ lead to the same counts as case (iv)-(i).

For the case (iv)-(iv) $\mathbf{u}_2 = \mathbf{u}_3, \mathbf{u}_1 \ne \mathbf{u}_2, \mathbf{u}_1 \ne \mathbf{u}_4, \mathbf{u}_2 \ne \mathbf{u}_4$ the conditions derived from (15) and (16) are clearly different. From (15) we have

$$\rho(\delta_2) * \mathbf{u}_1 = 0, \rho(\delta_3) * \mathbf{u}_1 = 0, \rho(\delta_4) * \mathbf{u}_1 = 0, \tag{15_6}$$

and from (16) we have

$$\begin{cases} \rho(\delta_4) * \mathbf{u}_2 + \rho(\delta_2) * \mathbf{u}_4 = 2\rho(\delta_2) * \rho(\delta_4) \\ \rho(\delta_3) * \mathbf{u}_4 + \rho(\delta_4) * \mathbf{u}_3 = 2\rho(\delta_3) * \rho(\delta_4) \\ \rho(\delta_2) * \mathbf{u}_3 + \rho(\delta_3) * \mathbf{u}_2 = 2 + 2\rho(\delta_2) * \rho(\delta_3) \end{cases} \tag{16_5}$$

Again by Lemma 5.2 the first two of $(16)_5$ will be transformed into

$$(16)_6 \qquad \begin{cases} \rho(\delta_4) * \mathbf{u}_2 = \rho(\delta_2) * \mathbf{u}_4 = \rho(\delta_2) * \rho(\delta_4), \\ \rho(\delta_3) * \mathbf{u}_4 = \rho(\delta_4) * \mathbf{u}_3 = \rho(\delta_3) * \rho(\delta_4). \end{cases}$$

For a fixed quadruple $\langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4 \rangle$ which satisfies $\mathbf{u}_2 * \mathbf{u}_3 = 8, \mathbf{u}_1 * \mathbf{u}_2 = 4, \mathbf{u}_1 * \mathbf{u}_3 = 4, \mathbf{u}_1 * \mathbf{u}_4 = \mathbf{u}_2 * \mathbf{u}_4 = 4, \mathbf{u}_3 * \mathbf{u}_4 = 4$ we seek the number of triples $\langle \rho(\delta_2), \rho(\delta_3), \rho(\delta_4) \rangle$ that satisfy the conditions $(15)_6, (16)_6$ and the last one of $(16)_5$. The number depends on the intersection scheme among $\mathbf{u}_i$'s. We find that the numbers of triples that are given in Table 4-7b are also valid in this case. The remaining two cases (iv)-(v) $\mathbf{u}_2 = \mathbf{u}_4, \mathbf{u}_1 \neq \mathbf{u}_2, \mathbf{u}_1 \neq \mathbf{u}_3, \mathbf{u}_2 \neq \mathbf{u}_3$ and (iv)-(vi) $\mathbf{u}_3 = \mathbf{u}_4, \mathbf{u}_1 \neq \mathbf{u}_2, \mathbf{u}_1 \neq \mathbf{u}_3, \mathbf{u}_2 \neq \mathbf{u}_3$ lead to the same consequence as (iv)-(iv).

Table 4-7d. Table of code word sums

case	CP1	CP2	CP3	CP4	CP5
$(iv) - (i)$	119992320	106659840	106917888	109670400	107089920
total	92154101760	81914757120	82112937984	84226867200	82245058560

(v) When $\mathbf{u}_i \neq \mathbf{u}_j$ $1 \leq i < j \leq 4$ holds, we have

$$(18) \qquad \mathbf{u}_i * \mathbf{u}_j = 4, \ (1 \leq i < j \leq 4).$$

Also, from conditions (15) and (16), we have

$$(15)_5 \qquad \rho(\delta_j) * \mathbf{u}_1 = 0,$$

and

$$(16)_5 \qquad \rho(\delta_i) * \mathbf{u}_j + \rho(\delta_j) * \mathbf{u}_i = 2\rho(\delta_i) * \rho(\delta_j).$$

There are thirty two different terms which satisfy this condition in the quadriweight enumerators of doubly even self-dual binary [32, 16, 8] codes.

Table 4-7e. Terms satisfying the conditions (18) in the quadriweight enumerator.

τ_1	200202200220200	τ_{12}	201102111102112	τ_{23}	211010030112122
τ_2	002222002200002	τ_{13}	200220020220022	τ_{24}	211001121003122
τ_3	020220202020020	τ_{14}	200211111111022	τ_{25}	210110121012032
τ_4	022002202002200	τ_{15}	200202202002022	τ_{26}	201111021102023
τ_5	022020020220200	τ_{16}	121010121012121	τ_{27}	310000130013132
τ_6	111111111111111	τ_{17}	112011021102112	τ_{28}	301001030103123
τ_7	220000220022220	τ_{18}	111120021111022	τ_{29}	300110030112033
τ_8	210110120121121	τ_{19}	111111112002022	τ_{30}	300101121003033
τ_9	210101211012121	τ_{20}	211001120112211	τ_{31}	211010031003033
τ_{10}	202002020202202	τ_{21}	022020022002022	τ_{32}	400000040004044
τ_{11}	201111020211112	τ_{22}	300101120112122		

In the above table, the data $(\tau_i | t_{ij} : \ 1 \leq j \leq 15)$ indicates the monomial $\tau_i = \prod_{j=0}^{15} x_j^{t_{ij}}$, where the t_{ij} are decimal representations of generalized weights in descending orders and x_j's are the decimal representations of the associated variables as explained in Note 2.3. The value $t_{i,0}$ is omitted since it is determined by $\sum_{j=0}^{15} w_j = 32$. For example,

$$\begin{aligned} \tau_1 &= x_{15}^2 x_{12}^2 x_{10}^2 x_9^2 x_6^2 x_5^2 x_3^2 x_1^2 x_0^{18} \\ &= x_{1111}^2 x_{1100}^2 x_{1010}^2 x_{1001}^2 x_{0110}^2 x_{0101}^2 x_{0011}^2 x_{0001}^2 x_{0000}^{18} \end{aligned}$$

Table 4-7f. The number of possible triples $\langle \rho(\delta_2), \rho(\delta_3), \rho(\delta_4) \rangle$.

$\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6$	16
$\tau_7, \tau_8, \tau_9, \tau_{10}, \tau_{11}, \tau_{12}, \tau_{13}, \tau_{14}, \tau_{15}, \tau_{16}, \tau_{17}, \tau_{18}, \tau_{19}, \tau_{20}, \tau_{21}$	32
$\tau_{22}, \tau_{23}, \tau_{24}, \tau_{25}, \tau_{26}$	64
$\tau_{27}, \tau_{28}, \tau_{29}, \tau_{30}, \tau_{31}$	128
τ_{32}	512

By use of the computer we found:

$$\begin{aligned} &\mathbf{W}_4(x_\alpha; \mathbf{C}_{\text{CP1}}) \\ &= \cdots + 1249920(\tau_1 + \tau_2 + \tau_3 + \tau_4 + \tau_5 + \tau_7 + \tau_{10} + \tau_{13} + \tau_{15} + \tau_{21}) + 19998720\tau_6 + 1041600\tau_{32} + \cdots, \\ &\mathbf{W}_4(x_\alpha; \mathbf{C}_{\text{CP2}}) \\ &= \cdots + 416640(\tau_1 + \tau_2 + \tau_3 + \tau_4 + \tau_5) + 357120(\tau_8 + \tau_9 + \tau_{11} + \tau_{12} + \tau_{14} + \tau_{16} + \tau_{17} + \tau_{18} + \tau_{19} + \tau_{20}) \\ &\quad + 1190400(\tau_{22} + \tau_{23} + \tau_{24} + \tau_{25} + \tau_{26}) + 714240(\tau_{27} + \tau_{28} + \tau_{29} + \tau_{30} + \tau_{31}) + 29760\tau_{32} + \cdots, \\ &\mathbf{W}_4(x_\alpha; \mathbf{C}_{\text{CP3}}) \\ &= \cdots + 432768(\tau_1 + \tau_2 + \tau_3 + \tau_4 + \tau_5) + 387072(\tau_8 + \tau_9 + \tau_{11} + \tau_{12} + \tau_{14} + \tau_{16} + \tau_{17} + \tau_{18} + \tau_{19} + \tau_{20}) \\ &\quad + 1139712(\tau_{22} + \tau_{23} + \tau_{24} + \tau_{25} + \tau_{26}) + 709632(\tau_{27} + \tau_{28} + \tau_{29} + \tau_{30} + \tau_{31}) \\ &\quad + 56448(\tau_7 + \tau_{10} + \tau_{13} + \tau_{15} + \tau_{21}) + 47040\tau_{32} + \cdots, \\ &\mathbf{W}_4(x_\alpha; \mathbf{C}_{\text{CP4}}) \\ &= \cdots + 604800(\tau_1 + \tau_2 + \tau_3 + \tau_4 + \tau_5 + \tau_7 + \tau_{10} + \tau_{13} + \tau_{15} + \tau_{21}) + 645120(\tau_6 + \tau_8 + \tau_9 \\ &\quad + \tau_{11} + \tau_{12} + \tau_{14} + \tau_{16} + \tau_{17} + \tau_{18} + \tau_{19} + \tau_{20} + \tau_{22} + \tau_{23} + \tau_{24} \\ &\quad + \tau_{25} + \tau_{26} + \tau_{27} + \tau_{28} + \tau_{29} + \tau_{30} + \tau_{31}) + 235200\tau_{32} + \cdots, \\ &\mathbf{W}_4(x_\alpha; \mathbf{C}_{\text{CP5}}) \\ &= \cdots + 443520(\tau_1 + \tau_2 + \tau_3 + \tau_4 + \tau_5) + 645120\tau_6 + 40320(\tau_7 + \tau_{10} + \tau_{15} + \tau_{13} + \tau_{21}) \\ &\quad + 345600(\tau_8 + \tau_9 + \tau_{11} + \tau_{12} + \tau_{14} + \tau_{16} + \tau_{17} + \tau_{18} + \tau_{19} + \tau_{20}) \\ &\quad + 1152000(\tau_{22} + \tau_{23} + \tau_{24} + \tau_{25} + \tau_{26}) + 691200(\tau_{27} + \tau_{28} + \tau_{29} + \tau_{30} + \tau_{31}) + 62400\tau_{32} + \cdots. \end{aligned}$$

By using the data in Tables 4-7e, 4-7f and the shapes of the parts of quadriweight enumerators above we obtain

Table 4-7g. Table of the contribution of the case (v) to $(Q43)$ partial sum

CP1	$2^7 \cdot (1249920 \cdot 16 \cdot 5 + 1249920 \cdot 32 \cdot 5 + 19998720 \cdot 16 + 1041600 \cdot 512) = 147617218560$
CP2	$2^7 \cdot (416640 \cdot 16 \cdot 5 + 357120 \cdot 32 \cdot 10 + 1190400 \cdot 64 \cdot 5 + 714240 \cdot 128 \cdot 5$ $+29760 \cdot 512) = 128113704960$
CP3	$2^7 \cdot (432768 \cdot 16 \cdot 5 + 387072 \cdot 32 \cdot 10 + 1139712 \cdot 64 \cdot 5 + 709632 \cdot 128 \cdot 5$ $+56448 \cdot 32 \cdot 5 + 47040 \cdot 512) = 129340538880$
CP4	$2^7 \cdot (604800 \cdot 16 \cdot 5 + 604800 \cdot 32 \cdot 5 + 645120 \cdot 16 + 645120 \cdot 32 \cdot 10$ $+645120 \cdot 64 \cdot 5 + 645120 \cdot 128 \cdot 5 + 235200 \cdot 512) = 141011189760$
CP5	$2^7 \cdot (443520 \cdot 16 \cdot 5 + 645120 \cdot 16 + 40320 \cdot 32 \cdot 5 + 345600 \cdot 32 \cdot 10$ $+1152000 \cdot 64 \cdot 5 + 691200 \cdot 128 \cdot 5 + 62400 \cdot 512) = 128742850560$

5.3 Two Technical Lemmas

To reduce the run time of the computations we made use of the following technical lemma.

Lemma 5.4. (i) *If* $\mathbf{u}_1, \mathbf{u}_2 \in \mathbf{C}_8$ *and* $\mathbf{u}_3 \in \mathbf{C}_{12}$ *satisfy the conditions* $\mathbf{u}_1 * \mathbf{u}_2 = 4, \mathbf{u}_1 * \mathbf{u}_3 = 6, \mathbf{u}_2 * \mathbf{u}_3 = 6$*, then* $\mathbf{u}_1, \mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 + \mathbf{u}_3 \in \mathbf{C}_8$ *and*

$$\mathbf{u}_1 * (u_1 + \mathbf{u}_2) = 4, \mathbf{u}_1 * (u_1 + \mathbf{u}_3) = 2, (\mathbf{u}_1 + \mathbf{u}_2) * (u_1 + \mathbf{u}_3) = 4.$$

(ii) *If* $\mathbf{u}_1 \in \mathbf{C}_8$ *and* $\mathbf{u}_2, \mathbf{u}_3 \in \mathbf{C}_{12}$ *satisfy the conditions* $\mathbf{u}_1 * \mathbf{u}_2 = 6, \mathbf{u}_1 * \mathbf{u}_3 = 6, \mathbf{u}_2 * \mathbf{u}_3 = 8$*, then* $\mathbf{u}_1, \mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 + \mathbf{u}_3 \in \mathbf{C}_8$ *and*

$$\mathbf{u}_1 * (u_1 + \mathbf{u}_2) = 2, \mathbf{u}_1 * (u_1 + \mathbf{u}_3) = 2, (\mathbf{u}_1 + \mathbf{u}_2) * (u_1 + \mathbf{u}_3) = 4.$$

(iii) *If* $\mathbf{u}_1, \mathbf{u}_2 \in \mathbf{C}_8$ *and* $\mathbf{u}_3 \in \mathbf{C}_{16}$ *satisfy the conditions* $\mathbf{u}_1 * \mathbf{u}_2 = 4, \mathbf{u}_1 * \mathbf{u}_3 = 8, \mathbf{u}_2 * \mathbf{u}_3 = 8$*, then* $\mathbf{u}_1, \mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 + \mathbf{u}_3 \in \mathbf{C}_8$ *and*

$$\mathbf{u}_1 * (u_1 + \mathbf{u}_2) = 4, \mathbf{u}_1 * (u_1 + \mathbf{u}_3) = 0, (\mathbf{u}_1 + \mathbf{u}_2) * (u_1 + \mathbf{u}_3) = 4.$$

(iv) *If* $\mathbf{u}_1 \in \mathbf{C}_8, \mathbf{u}_2 \in \mathbf{C}_{12}$ *and* $\mathbf{u}_3 \in \mathbf{C}_{16}$ *satisfy the conditions* $\mathbf{u}_1 * \mathbf{u}_2 = 6, \mathbf{u}_1 * \mathbf{u}_3 = 8, \mathbf{u}_2 * \mathbf{u}_3 = 10$*, then* $\mathbf{u}_1, \mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 + \mathbf{u}_3 \in \mathbf{C}_8$ *and*

$$\mathbf{u}_1 * (u_1 + \mathbf{u}_2) = 2, \mathbf{u}_1 * (u_1 + \mathbf{u}_3) = 0, (\mathbf{u}_1 + \mathbf{u}_2) * (u_1 + \mathbf{u}_3) = 4.$$

(v) *If* $\mathbf{u}_1 \in \mathbf{C}_8$ *and* $\mathbf{u}_2, \mathbf{u}_3 \in \mathbf{C}_{16}$ *satisfy the conditions* $\mathbf{u}_1 * \mathbf{u}_2 = 8, \mathbf{u}_1 * \mathbf{u}_3 = 8, \mathbf{u}_2 * \mathbf{u}_3 = 12$*, then* $\mathbf{u}_1, \mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 + \mathbf{u}_3 \in \mathbf{C}_8$ *and*

$$\mathbf{u}_1 * (u_1 + \mathbf{u}_2) = 0, \mathbf{u}_1 * (u_1 + \mathbf{u}_3) = 0, (\mathbf{u}_1 + \mathbf{u}_2) * (u_1 + \mathbf{u}_3) = 4.$$

(vi) *If* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathbf{C}_{12}$ *satisfy the conditions* $\mathbf{u}_1 * \mathbf{u}_2 = 8, \mathbf{u}_1 * \mathbf{u}_3 = 8, \mathbf{u}_2 * \mathbf{u}_3 = 8$*, then* $\mathbf{u}_1 \in \mathbf{C}_{12}$ *and* $\mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 + \mathbf{u}_3 \in \mathbf{C}_8$ *and*

$$\mathbf{u}_1 * (u_1 + \mathbf{u}_2) = 4, \mathbf{u}_1 * (u_1 + \mathbf{u}_3) = 4, (\mathbf{u}_1 + \mathbf{u}_2) * (u_1 + \mathbf{u}_3) = 4.$$

(vii) *If* $\mathbf{u}_1, \mathbf{u}_2 \in \mathbf{C}_{12}$ *and* $\mathbf{u}_3 \in \mathbf{C}_{16}$ *satisfy the conditions* $\mathbf{u}_1 * \mathbf{u}_2 = 8, \mathbf{u}_1 * \mathbf{u}_3 = 10, \mathbf{u}_2 * \mathbf{u}_3 = 10$*, then* $\mathbf{u}_1 \in \mathbf{C}_{12}$ *and* $\mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 + \mathbf{u}_3 \in \mathbf{C}_8$ *and*

$$\mathbf{u}_1 * (u_1 + \mathbf{u}_2) = 4, \mathbf{u}_1 * (u_1 + \mathbf{u}_3) = 2, (\mathbf{u}_1 + \mathbf{u}_2) * (u_1 + \mathbf{u}_3) = 4.$$

(viii) *If* $\mathbf{u}_1 \in \mathbf{C}_{12}$ *and* $\mathbf{u}_2, \mathbf{u}_3 \in \mathbf{C}_{16}$ *satisfy the conditions* $\mathbf{u}_1 * \mathbf{u}_2 = 10, \mathbf{u}_1 * \mathbf{u}_3 = 10, \mathbf{u}_2 * \mathbf{u}_3 = 12$*, then* $\mathbf{u}_1 \in \mathbf{C}_{12}$ *and* $\mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 + \mathbf{u}_3 \in \mathbf{C}_8$ *and*

$$\mathbf{u}_1 * (u_1 + \mathbf{u}_2) = 2, \mathbf{u}_1 * (u_1 + \mathbf{u}_3) = 2, (\mathbf{u}_1 + \mathbf{u}_2) * (u_1 + \mathbf{u}_3) = 4.$$

(ix) *If* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathbf{C}_{16}$ *satisfy the conditions* $\mathbf{u}_1 * \mathbf{u}_2 = 12, \mathbf{u}_1 * \mathbf{u}_3 = 12, \mathbf{u}_2 * \mathbf{u}_3 = 12$*, then* $\mathbf{u}_1 \in \mathbf{C}_{16}$ *and* $\mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 + \mathbf{u}_3 \in \mathbf{C}_8$ *and*

$$\mathbf{u}_1 * (u_1 + \mathbf{u}_2) = 4, \mathbf{u}_1 * (u_1 + \mathbf{u}_3) = 4, (\mathbf{u}_1 + \mathbf{u}_2) * (u_1 + \mathbf{u}_3) = 4.$$

Remark 3. *Each of the above statements gives a bijection* $\langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \rangle \mapsto \langle \mathbf{u}_1, \mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 + \mathbf{u}_3 \rangle$ *between certain sets* $\mathcal{X}_1, \mathcal{X}_2$ *of triples of code words. For instance, in (i) one has*

$$\begin{aligned} \mathcal{X}_1 &= \{\langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \rangle \in \mathbf{C}_8 \times \mathbf{C}_8 \times \mathbf{C}_{12} | \mathbf{u}_1 * \mathbf{u}_2 = 4, \mathbf{u}_1 * \mathbf{u}_3 = 4, \mathbf{u}_2 * \mathbf{u}_3 = 4\}, \\ \mathcal{X}_2 &= \{\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle \in \mathbf{C}_8 \times \mathbf{C}_8 \times \mathbf{C}_8 | \mathbf{v}_1 * \mathbf{v}_2 = 4, \mathbf{v}_1 * \mathbf{v}_3 = 2, \mathbf{v}_2 * \mathbf{v}_3 = 4\}. \end{aligned}$$

The following lemma may be verified, for example, using Magma[1].

Lemma 5.5. *Let* $\mathcal{G}_i$ *be the group of automorphisms of the code* CPi $(1 \leq i \leq 5)$*. Then* $\mathcal{G}_i$ *acts on* $\mathbf{C}_8$*, the set of code words of weight* 8 *in each code* CPi*, and:*
(i) $\mathcal{G}_1$ *acts on* $\mathbf{C}_8$ *transitively,*
(ii) $\mathcal{G}_2$ *acts on* $\mathbf{C}_8$ *transitively,*
(iii) *when* $\mathcal{G}_3$ *acts on* $\mathbf{C}_8$ *, there are exactly three orbits* $\mathbf{C}_8(1), \mathbf{C}_8(2)$ *and* $\mathbf{C}_8(3)$*. The cardinality of* $\mathbf{C}_8(1)$ *is* 256*, the cardinality of* $\mathbf{C}_8(2)$ *is* 336 *and the cardinality of* $\mathbf{C}_8(3)$ *is* 28*,*
(iv) *when* $\mathcal{G}_4$ *acts on* $\mathbf{C}_8$ *, there are exactly two orbits* $\mathbf{C}_8(1)$ *and* $\mathbf{C}_8(2)$*. The cardinality of* $\mathbf{C}_8(1)$ *is* 560 *and the cardinality of* $\mathbf{C}_8(2)$ *is* 60*,*
(v) *when* $\mathcal{G}_5$ *acts on* $\mathbf{C}_8$ *, there are exactly three orbits* $\mathbf{C}_8(1), \mathbf{C}_8(2)$ *and* $\mathbf{C}_8(3)$*. The cardinality of* $\mathbf{C}_8(1)$ *is* 320*, the cardinality of* $\mathbf{C}_8(2)$ *is* 240 *and the cardinality of* $\mathbf{C}_8(3)$ *is* 60*.*

6 Determination of $a(\mathfrak{z}_{40}, \mathcal{L}_{32})$

Table 5. Table of code sum contribution for five extremal $[32, 16, 8]$ codes.

case	CP1	CP2	CP3	CP4	CP5	m_i
$(Q1)$	773283840	773283840	773283840	773283840	773283840	1
$(Q2)$	159989760	159989760	159989760	159989760	159989760	4
$(Q3)$	1546567680	1546567680	1546567680	1546567680	1546567680	6
$(Q4)_1$	266649600	266649600	266649600	266649600	266649600	4
$(Q4)_2$	5759631360	5759631360	5759631360	5759631360	5759631360	4
$(Q4)_3$	15359016960	13652459520	13685489664	14037811200	13707509760	4
$b(M_{24}, \mathbf{C})$	1562400	1354080	1358112	1401120	1360800	8
$b(M_{24,a}, \mathbf{C})$	1770720	1562400	1566432	1609440	1569120	24
$b(M_{24,b}, \mathbf{C})$	1770720	1562400	1566432	1609440	1569120	24
$b(M_{24}, \mathbf{C})$	1562400	1354080	1358112	1401120	1360800	8
$b(M_9, \mathbf{C})$	33539520	15207360	15610560	19830720	15798720	2
$b(M_{10}, \mathbf{C})$	9999360	18570240	18321408	15805440	18293760	8
$b(M_{11}, \mathbf{C})$	1770720	104160	168672	803040	157920	8
$b(M_{12}, \mathbf{C})$	59996160	55948800	56146944	58060800	56079360	12
$b(M_{13}, \mathbf{C})$	4999680	1249920	1290240	1774080	1370880	24
$b(M_{14}, \mathbf{C})$	1041600	29760	47040	235200	62400	12
$b(M_{15}, \mathbf{C})$	429972480	344739840	346278912	362880000	347489280	8
$b(M_{16}, \mathbf{C})$	59996160	54401280	54491136	55480320	54581760	24
$b(M_{17}, \mathbf{C})$	9999360	20117760	19977216	18385920	19791360	24
$b(M_{18}, \mathbf{C})$	42080640	20504640	20866944	24823680	21200640	8
$b(M_{19,1}, \mathbf{C})$	3317704320	3000641280	3006777984	3072236160	3010869120	2
$b(M_{19,2}, \mathbf{C})$	1249920	8451840	8367744	7378560	8219520	8
$b(M_{19,3}, \mathbf{C})$	1249920	357120	411264	927360	385920	6
$b(M_{20,1}, \mathbf{C})$	1838215680	1542222720	1548029952	1609843200	1551770880	8
$b(M_{20,2}, \mathbf{C})$	34997760	34462080	34384896	33707520	34479360	24
$b(M_{21,1}, \mathbf{C})$	1866547200	1618944000	1623745536	1674946560	1626931200	12
$b(M_{21,2}, \mathbf{C})$	379975680	299504640	301228032	319334400	302100480	12
$b(M_{22}, \mathbf{C})$	3918082560	3466623360	3475153920	3566492160	3481186560	8
$b(M_{23}, \mathbf{C})$	15409430400	13147908480	13192163712	13663413120	13220860800	1

Table 5. (continued)

case	CP1	CP2	CP3	CP4	CP5	m_i
$b(C(1), M_{24}, \mathbf{C})$	53121600	40652160	40891200	43444800	41054400	8
$b(C(1), M_{25}, \mathbf{C})$	84994560	87434880	87392256	86929920	87356160	24
$b(C(1), M_{26}, \mathbf{C})$	8853600	1592160	1730400	3208800	1826400	24
$b(C(1), M_{27}, \mathbf{C})$	623293440	525085440	526977024	547169280	528253440	24
$b(C(1), M_{28}, \mathbf{C})$	84994560	87434880	87392256	86929920	87356160	48
$b(C(1), M_{29}, \mathbf{C})$	60204480	46901760	47156928	49882560	47330880	24
$b(C(1), M_{30,1}, \mathbf{C})$	6332094720	5649340800	5662578432	5803741440	5671365120	8
$b(C(1), M_{30,2}, \mathbf{C})$	42497280	49937280	49797888	48303360	49697280	24
$b(C(1), M_{31,1}, \mathbf{C})$	4306391040	3749998080	3760748544	3875450880	3767946240	24
$b(C(1), M_{31,2}, \mathbf{C})$	509967360	425091840	426725376	444165120	427829760	24
$b(C(1), M_{32}, \mathbf{C})$	6997885440	6236803200	6251556864	6408890880	6261354240	24
$b(C(1), M_{33}, \mathbf{C})$	26440391040	23217799680	23280130944	23945066880	23321754240	4
$b(C(2), M_{34}, \mathbf{C})$	148740480	131241600	131580288	135192960	131806080	12
$b(C(2), M_{35}, \mathbf{C})$	793282560	699955200	701761536	721029120	702965760	24
$b(C(2), M_{36}, \mathbf{C})$	154052640	135928800	136279584	140021280	136513440	24
$b(C(2), M_{37,1}, \mathbf{C})$	11304276480	9974361600	10000101888	10274664960	10017262080	12
$b(C(2), M_{37,2}, \mathbf{C})$	594961920	524966400	526321152	540771840	527224320	12
$b(C(2), M_{38}, \mathbf{C})$	11899238400	10499328000	10526423040	10815436800	10544486400	24
$b(C(2), M_{39}, \mathbf{C})$	40556570880	35785209600	35877558528	36862613760	35939124480	6
$b(C(3)_1, C(3)_2, 8)$	968688000	839529600	842029440	868694400	843696000	8
$b(C(3)_1, C(3)_2, 12)$	21698611200	18805463040	18861459456	19458754560	18898790400	8
$b(C(3)_1, C(3)_2, 16)$	57055723200	49448293440	49595534016	51166100160	49693694400	4
$(Q43)_1$	159989760	159989760	159989760	159989760	159989760	1
$(Q43)_2$	5119672320	5119672320	5119672320	5119672320	5119672320	1
$(Q43)_3$	5759631360	5759631360	5759631360	5759631360	5759631360	1
$(Q43)_4$	92154101760	81914757120	82112937984	84226867200	82245058560	1
$(Q43)_5$	147617218560	128113704960	129340538880	141011189760	128742850560	1

In the above table the last column denotes the multiplicity of each code sum. Now we collect all the terms displayed above. To get the value $a(\mathfrak{T}_{40}, \mathcal{L}_{32}(\mathrm{CP1}))$ we use

$$a(\mathfrak{T}_{40}, \mathcal{L}_{32}(\mathrm{CP1})) = \sum_i b_i(1)m_i,$$

where b_i denotes the value of second column in the way: $a(\mathfrak{T}_{40}, \mathcal{L}_{32}(\mathrm{CP1})) = 773283840 \cdot 1 + 159989760 \cdot 4 + \cdots + 147617218560 \cdot 1$. In this way we obtain

$$\begin{aligned} a(\mathfrak{T}_{40}, \mathcal{L}_{32}(\mathrm{CP1})) &= 2019470745600, \\ a(\mathfrak{T}_{40}, \mathcal{L}_{32}(\mathrm{CP2})) &= 1778114764800, \\ a(\mathfrak{T}_{40}, \mathcal{L}_{32}(\mathrm{CP3})) &= 1783635517440, \\ a(\mathfrak{T}_{40}, \mathcal{L}_{32}(\mathrm{CP4})) &= 1841107968000, \\ a(\mathfrak{T}_{40}, \mathcal{L}_{32}(\mathrm{CP5})) &= 1785900441600. \end{aligned}$$

As a summary of our rather lengthy series of computations we have

Theorem 6.1. *For the five even unimodular 32-dimensional extremal lattices constructed from five extremal self-dual binary codes the Siegel theta series of degree 4, $\Theta_4(Z, \mathcal{L}_{32}(\mathrm{CP}i))$ ($1 \leq i \leq 5$), are all distinct.*

6.1 A Very Small Table of the Fourier Coefficients

Using the result in the previous subsection and Proposition 4.3 in [11] we obtain the following table.

Table 6. Fourier coefficients of the Siegel-theta series of degree 4 associated with the 32-dimensional even unimodular extremal lattices

D	reduced form	$\mathcal{L}_{32}$(CP1)	$\mathcal{L}_{32}$(CP2)
*64	(2,2,2,2,0,0,0,2,2,2)	337671244800	96315264000
*80	(2,2,2,2,2,0,0,2,0,2)	2019470745600	1778114764800
81	(2,2,2,2,1,1,1,2,2,-1)	2611032883200	2128320921600
84	(2,2,2,2,1,0,0,2,2,2)	2937411993600	3420123955200
96	(2,2,2,2,2,1,-1,0,0,2)	22030589952000	21065166028800
105	(2,2,2,2,2,1,0,0,1,2)	70497887846400	70980599808000
108	(2,2,2,2,2,1,-1,-1,1,-1)	101830282444800	103761130291200
112	(2,2,2,2,2,1,0,2,0,0)	171628786483200	169697938636800
116	(2,2,2,2,2,1,0,0,2,0)	269822273126400	272235832934400
120	(2,2,2,2,1,1,1,2,2,0)	434736975052800	433771551129600
121	(2,2,2,2,2,1,0,1,1,2)	490128172646400	485301053030400
125	(2,2,2,2,1,1,-1,-1,1,1)	761377188741120	758480916971520
*128	(2,2,2,2,0,0,0,2,2,0)	1032723244339200	1028378836684800
128	(2,2,2,2,2,1,0,0,0,2)	1032080685465600	1035942381158400
129	(2,2,2,2,1,1,1,1,2,2)	1151465501491200	1150982789529600
132	(2,2,2,2,2,1,-1,0,0,1)	1568578004582400	1569543428505600

D	$\mathcal{L}_{32}$(CP3)	$\mathcal{L}_{32}$(CP4)	$\mathcal{L}_{32}$(CP5)
*64	101836016640	159308467200	104100940800
*80	1783635517440	1841107968000	1785900441600
81	2139362426880	2254307328000	2143892275200
84	3409082449920	3294137548800	3404552601600
96	21087249039360	21317138841600	21096308736000
105	70969558302720	70854613401600	70965028454400
108	103716964270080	103257184665600	103698844876800
112	169742104657920	170201884262400	169760224051200
116	272180625408000	271605900902400	272157976166400
120	433793634140160	434023523942400	433802693836800
121	485411468083200	486560917094400	485456766566400
125	758547166003200	759236835409920	758574345093120
*128	1028478210232320	1029512714342400	1028518978867200
128	1035854049116160	1034934489907200	1035817810329600
129	1150993831034880	1151108775936000	1150998360883200
132	1569521345495040	1569291455692800	1569512285798400

In the above table the discriminants d_T marked by $*$ indicate that the quaternary quadratic forms have imprimitive coefficients.

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