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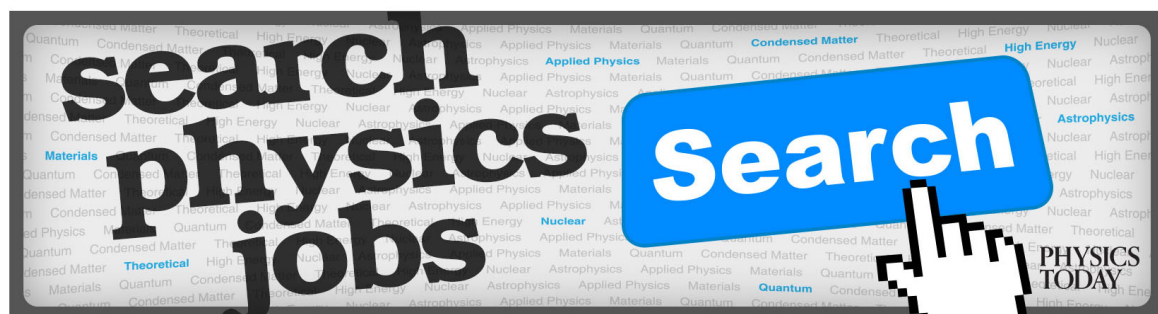
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# Irregular conformal blocks, with an application to the fifth and fourth Painlevé equations

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We develop the theory of irregular conformal blocks of the Virasoro algebra. In previous studies, expansions of irregular conformal blocks at regular singular points were obtained as degeneration limits of regular conformal blocks; however, such expansions at irregular singular points were not clearly understood. This is because precise definitions of irregular vertex operators had not been provided previously. In this paper, we present precise definitions of irregular vertex operators of two types and we prove that one of our vertex operators exists uniquely. Then, we define irregular conformal blocks with at most two irregular singular points as expectation values of given irregular vertex operators. Our definitions provide an understanding of expansions of irregular conformal blocks and enable us to obtain expansions at irregular singular points. As an application, we propose conjectural formulas of series expansions of the tau functions of the fifth and fourth Painlevé equations, using expansions of irregular conformal blocks at an irregular singular point. © 2015 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4937760>]

## I. INTRODUCTION

Conformal blocks are building blocks of the correlation functions of two-dimensional conformal field theory.<sup>4</sup> They are defined as the expectation values,

$$\langle \Phi(z_N) \cdots \Phi(z_1), \rangle$$

of the vertex operators  $\Phi(z_i)$  on the Verma modules of the Virasoro algebra satisfying the commutation relations

$$[L_n, \Phi(z_i)] = z_i^n \left( z_i \frac{\partial}{\partial z_i} + (n+1)\Delta_i \right) \Phi(z_i),$$

where  $L_n$  ( $n \in \mathbb{Z}$ ) are generators of the Virasoro algebra, and the complex parameters  $\Delta_i$  ( $i = 1, \dots, N$ ) are conformal dimensions. The conformal blocks can be viewed as special functions with Virasoro symmetry.

Conformal blocks can be calculated from the definition. They are formal power series in  $z_i/z_{i+1}$  ( $i = 1, \dots, N-1$ ) and are believed to be absolutely convergent in  $|z_1| < \cdots < |z_N|$ . As a special case, the 4-point conformal block with one null vector condition becomes the hypergeometric series. In the general case, however, their explicit series expansions had not been determined. Recently, Alday, Gaiotto, and Tachikawa proposed that the Virasoro conformal blocks correspond to the instanton part of the Nekrasov partition functions of a particular class of four-dimensional supersymmetric gauge theories,<sup>2</sup> and their proposed correspondence was proved in Ref. 1. Given this so-called AGT correspondence, we have explicit series expansions of the conformal blocks.

Conformal blocks with null vector conditions satisfy partial differential systems with regular singularities, so-called Belavin-Polyakov-Zamolodchikov (BPZ) equations, whereas the general

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conformal blocks whose central charge  $c = 1$  appears in series expansions of the tau functions of Garnier systems,<sup>14</sup> which describe isomonodromy deformations of  $2 \times 2$  Fuchsian systems. Therefore, conformal blocks can be viewed as special functions with regular singularities.

Irregular versions of conformal blocks have been studied in relation to four-dimensional supersymmetric gauge theories and quantum Painlevé equations. Note that canonical quantization of the sixth Painlevé equation is the BPZ equation for five points. Particular irregular vertex operators of Wess-Zumino-Novikov-Witten conformal field theory were presented by free field realizations, and a recursion rule for constructing integral representations of irregular conformal blocks was provided in Ref. 18. These realizations are also applicable to free field realizations of irregular vertex operators and irregular conformal blocks in Virasoro conformal field theory. Hence, in a special case, we have integral formulas of irregular conformal blocks.

Another method of obtaining irregular conformal blocks, initiated by Gaiotto<sup>10</sup> and developed in Refs. 6 and 11 is to take pairings of irregular vectors  $|\Lambda\rangle$  and  $\langle\Lambda'|\$  embedded in a Verma module and a dual Verma module, respectively, such that for non-negative integers  $r, s$ , and tuples  $\Lambda = (\Lambda_r, \Lambda_{r+1}, \dots, \Lambda_{2r})$ ,  $\Lambda' = (\Lambda'_s, \Lambda'_{s+1}, \dots, \Lambda'_{2s})$ ,

$$\begin{aligned} L_n|\Lambda\rangle &= \Lambda_n|\Lambda\rangle \quad (n = r, \dots, 2r), \quad L_n|\Lambda\rangle = 0 \quad (n > 2r), \\ \langle\Lambda'|L_n &= \Lambda'_n\langle\Lambda'| \quad (n = -s, \dots, -2s), \quad \langle\Lambda'|L_n = 0 \quad (n < -2s). \end{aligned} \quad (1.1)$$

The existence of irregular vectors in a Verma module of the Virasoro algebra of the  $r = 1$  case was verified in Refs. 17 and 25. For the  $r > 1$  case, the construction of irregular vectors was performed in Ref. 8 and it was revealed that the first-order irregular vectors is uniquely determined by condition (1.1), up to a scalar. However, the higher-order irregular vectors contain an infinite number of parameters, and thus they are not unique.

On the other hand, as the confluent hypergeometric series is obtained from the hypergeometric series by confluence, irregular conformal blocks can be derived from ordinary conformal blocks using particular delicate limiting procedures.<sup>3,11,20,7</sup> We emphasize that some properties of irregular conformal blocks might be the limits of regular conformal blocks; however, it might be easier to derive the results directly than to justify the limiting procedures, as is true for the hypergeometric function and the confluent hypergeometric function.

In this study, we provide precise definitions of irregular vertex operators of two types and we prove that one of our vertex operators exists uniquely. Then, we directly define irregular conformal blocks with at most two irregular singular points, as expectation values of the newly introduced vertex operators. Recall that in the regular case, the vertex operators are uniquely determined from the commutation relations and three conformal dimensions, and the conformal blocks are defined as expectation values of the vertex operators.

Our definitions provide an understanding of series expansions of irregular conformal blocks and enable us to obtain series expansions of irregular conformal blocks at irregular singular points.

As an application, we present conjectural formulas of series expansions of the tau functions of the fifth and fourth Painlevé equations, using our irregular conformal blocks, following a recent remarkable discovery by Gamayun, Iorgov, and Lisovyy<sup>12</sup> that the tau function of the sixth Painlevé equation is expressed as a series expansion in terms of a 4-point conformal block. By taking scaling limits, short-distance expansions of the tau functions of  $P_V$ ,  $P_{III}$  in terms of irregular conformal blocks were obtained in Ref. 13. Furthermore, it was conjectured in Ref. 15 that a long-distance expansion of the tau function of a special case of the third Painlevé equation  $P_{III}(D_8)$ <sup>21</sup> is also a series expansion in terms of some function determined from the differential equation  $P_{III}(D_8)$ .

In the  $P_V$  case, the tau function is expected to be a series expansion in terms of 3-point irregular conformal blocks with two regular singular points and one irregular singular point of rank 1. Naively, this is expressed as

$$\langle\Delta|\Phi(z)|\Lambda\rangle,$$

where  $\langle\Delta|$  is the highest weight vector of the dual Verma module  $V_{\Delta}^*$ ,  $\Phi(z)$  is a usual vertex operator, and  $|\Lambda\rangle$  is the irregular vector of rank 1 with  $\Lambda = (\Lambda_1, \Lambda_2)$ . Our definitions of irregular conformal

blocks distinguish between

$$\left(\langle \Delta | \Phi_{\Delta, \Delta_1}^{*, \Delta_2}(z) \rangle \cdot |\Lambda \rangle\right) \tag{1.2}$$

and

$$\langle \Delta | \cdot \left( \Phi_{\Lambda', \Lambda}^{\Delta_2}(z) | \Lambda \rangle \right), \tag{1.3}$$

where  $\Phi_{\Delta, \Delta_1}^{*, \Delta_2}(z): V_{\Delta}^* \rightarrow V_{\Delta_1}^*$  is the usual vertex operator,  $\Phi_{\Lambda', \Lambda}^{\Delta_2}(z): V_{\Lambda}^{[1]} \rightarrow V_{\Lambda'}^{[1]}$  is the newly introduced rank 0 vertex operator from a rank 1 Verma module to another rank 1 Verma module such that  $\Lambda' = (\Lambda'_1, \Lambda_2)$ . Expectation values (1.2) and (1.3) are computed by the pairings  $V_{\Delta_1}^* \times V_{\Lambda}^{[1]} \rightarrow \mathbb{C}$ ,  $V_{\Delta}^* \times V_{\Lambda'}^{[1]} \rightarrow \mathbb{C}$ , respectively. Irregular conformal block (1.2) is the expansion at a regular singular point and, in fact, is equal to the one derived by the collision limit of the 4-point conformal block and the building block of the short-distance expansion of the fifth Painlevé tau function obtained in Ref. 13. Using irregular conformal block (1.3), the expansion at an irregular singular point, we present a conjectural long-distance expansion of the fifth Painlevé tau function.

In the  $P_{IV}$  case, as a building block of a series expansion of the tau function, we use the 2-point irregular conformal blocks with one regular singular point and one irregular singular point of rank 2. This is expressed as follows:

$$\langle 0 | \cdot \left( \Phi_{\Lambda', \Lambda}^{\Delta_2}(z) | \Lambda \rangle \right),$$

where  $\Phi_{\Lambda', \Lambda}^{\Delta_2}(z): V_{\Lambda}^{[2]} \rightarrow V_{\Lambda'}^{[2]}$  is the newly introduced rank 0 vertex operator from a rank 2 Verma module to another rank 2 Verma module,  $\Lambda = (\Lambda_2, \Lambda_3, \Lambda_4)$  and  $\Lambda' = (\Lambda'_2, \Lambda_3, \Lambda_4)$ . The expectation value is computed by the pairing  $V_0^* \times V_{\Lambda'}^{[2]} \rightarrow \mathbb{C}$ .

The remainder of this paper is organized as follows. In Section II, we introduce a rank  $r$  Verma module of the Virasoro algebra as a special Whittaker module of that algebra. Then, generalizing the confluent primary field introduced in Ref. 18, we present definitions of irregular vertex operators of two types, and we prove that one of our vertex operators exists uniquely. In Section III, we define irregular conformal blocks with at most two irregular singular points, using the newly introduced irregular vertex operators in Sec. II. In Section IV, we review the series expansions of the tau functions of the Painlevé equations. Then, we propose series expansions of the tau functions of  $P_V$  and  $P_{IV}$  in terms of our irregular conformal blocks.

## II. IRREGULAR VERTEX OPERATORS

### A. Modules

The Virasoro algebra,

$$\text{Vir} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}C,$$

is the Lie algebra satisfying the following commutation relations:

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}C,$$

$$[\text{Vir}, C] = 0,$$

where  $\delta_{i,j}$  stands for Kronecker's delta.

Let  $r$  be a non-negative integer and  $\text{Vir}_r$  the subalgebra generated by  $C$  and  $L_n$  ( $n \in \mathbb{Z}_{\geq r}$ ). For a complex number  $c$  and an  $(r + 1)$ -tuple of parameters  $\Lambda = (\Lambda_r, \Lambda_{r+1}, \dots, \Lambda_{2r}) \in \mathbb{C}^r \times \mathbb{C}^\times$ , let  $\mathbb{C}|\Lambda\rangle$  be the one-dimensional  $\text{Vir}_r$ -module defined by

$$C|\Lambda\rangle = c|\Lambda\rangle,$$

$$L_n|\Lambda\rangle = \Lambda_n|\Lambda\rangle \quad (n = r, r + 1, \dots, 2r),$$

$$L_n|\Lambda\rangle = 0 \quad (n > 2r).$$

*Definition 2.1.* A rank  $r$  Verma module  $V_{\Lambda}^{[r]}$  with parameters  $\Lambda$  is the induced module

$$V_{\Lambda}^{[r]} = \text{Ind}_{\text{Vir}_r}^{\text{Vir}} \mathbb{C}|\Lambda\rangle (= U(\text{Vir}) \otimes_{\text{Vir}_r} \mathbb{C}|\Lambda\rangle).$$

Denote the dual module of  $V_{\Lambda}^{[r]}$  by  $V_{\Lambda}^{*,[r]}$  such that

$$\begin{aligned} \langle \Lambda | C &= c \langle \Lambda |, \\ \langle \Lambda | L_n &= \Lambda_n \langle \Lambda | \quad (n = -r, -r - 1, \dots, -2r), \\ \langle \Lambda | L_n &= 0 \quad (n < -2r). \end{aligned}$$

The module  $V_{\Lambda}^{*,[r]}$  is spanned by linearly independent vectors of the form

$$\langle \Lambda | L_{i_1} \cdots L_{i_k} \quad (-r < i_1 \leq \dots \leq i_k).$$

*Remark 2.2.* When  $r = 0$ , a rank 0 Verma module  $V_{\Lambda}^{[0]}$  is a standard Verma module of the Virasoro algebra. It is usual to denote  $V_{\Lambda}^{[0]}$  as  $V_{\Lambda}$  in conformal field theory. In the case of  $r \geq 1$ , a rank  $r$  Verma module is equal to the universal Whittaker module discussed in Refs. 23, 16, and 8. It has been shown that the universal Whittaker modules are irreducible, if  $\Lambda_{2r} \neq 0$  or  $\Lambda_{2r-1} \neq 0$ .<sup>16,8</sup> Hence, a rank  $r$  Verma module  $V_{\Lambda}^{[r]}$  for  $r \geq 1$  is irreducible.

A bilinear pairing  $\langle \cdot | \cdot \rangle: V_{\Lambda}^{*,[0]} \times V_{\Lambda}^{[1]} \rightarrow \mathbb{C}$  is uniquely defined by

$$\begin{aligned} \langle \Delta | \cdot | \Lambda \rangle &= 1, \\ \langle u | L_n \cdot | v \rangle &= \langle u | \cdot L_n | v \rangle \equiv \langle u | L_n | v \rangle, \end{aligned}$$

where  $\langle u | \in V_{\Lambda}^{*,[0]}$  and  $|v\rangle \in V_{\Lambda}^{[1]}$ . Denote by  $\bar{V}_0^{*,[0]}$  an irreducible highest weight module. A bilinear pairing  $\langle \cdot | \cdot \rangle: \bar{V}_0^{*,[0]} \times V_{\Lambda}^{[2]} \rightarrow \mathbb{C}$  is uniquely defined by

$$\begin{aligned} \langle 0 | \cdot | \Lambda \rangle &= 1, \\ \langle u | L_n \cdot | v \rangle &= \langle u | \cdot L_n | v \rangle \equiv \langle u | L_n | v \rangle, \end{aligned}$$

where  $\langle u | \in \bar{V}_0^{*,[0]}$  and  $|v\rangle \in V_{\Lambda}^{[2]}$ , because  $\langle 0 | L_1 = 0$ . The bilinear pairings on  $V_{\Lambda}^{*,[1]} \times V_{\Lambda}^{[0]} \rightarrow \mathbb{C}$  and  $V_{\Lambda}^{*,[2]} \times \bar{V}_0^{[0]} \rightarrow \mathbb{C}$  are defined in the same manner.

For cases other than the above, it is possible to define a bilinear pairing using the Heisenberg algebra and its Fock spaces. Let

$$\mathcal{H} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} a_n,$$

be the Lie algebra with commutation relations

$$[a_m, a_n] = m \delta_{m+n, 0}.$$

We call this the Heisenberg algebra.

*Definition 2.3.* A (bosonic) rank  $r$  Fock space of  $\mathcal{H}$  is a representation  $\mathcal{F}_{\lambda}^{[r]}$  such that for a given tuple of parameters  $\lambda = (\lambda_0, \dots, \lambda_r) \in \mathbb{C}^r \times \mathbb{C}^{\times}$ ,

$$a_n |\lambda\rangle = \lambda_n |\lambda\rangle \quad (n = 0, 1, \dots, r), \quad a_n |\lambda\rangle = 0 \quad (n > r)$$

and  $\mathcal{F}_{\lambda}^{[r]}$  is the linear span of linearly independent vectors of the form

$$a_{-i_k} \cdots a_{-i_1} |\lambda\rangle \quad (0 < i_1 \leq \dots \leq i_k).$$

*Definition 2.4.* A rank  $r$  dual Fock space of  $\mathcal{H}$  is a representation  $\mathcal{F}_{\lambda}^{*,[r]}$  such that for a given tuple of parameters  $\lambda = (\lambda_0, \dots, \lambda_r) \in \mathbb{C}^r \times \mathbb{C}^{\times}$ ,

$$\langle \lambda | a_n = \lambda_n \langle \lambda | \quad (n = 0, -1, \dots, -r), \quad \langle \lambda | a_n = 0 \quad (n < -r)$$

and  $\mathcal{F}_\lambda^{*,[r]}$  is the linear span of linearly independent vectors of the form

$$\langle \lambda | a_{i_1} \cdots a_{i_k} \quad (0 < i_1 \leq \cdots \leq i_k).$$

For  $\lambda = (\lambda_0, \dots, \lambda_r) \in \mathbb{C}^r \times \mathbb{C}^\times$  and  $\mu = (\lambda_0, \mu_1, \dots, \mu_s) \in \mathbb{C}^s \times \mathbb{C}^\times$ , a bilinear pairing  $\langle \cdot \rangle: \mathcal{F}_\mu^{*,[s]} \times \mathcal{F}_\lambda^{[r]} \rightarrow \mathbb{C}$  is defined by

$$\begin{aligned} \langle \mu | \cdot | \lambda \rangle &= 1, \\ \langle u | a_n \cdot | v \rangle &= \langle u | \cdot a_n | v \rangle \equiv \langle u | a_n | v \rangle, \end{aligned}$$

where  $\langle u | \in \mathcal{F}_\mu^{*,[s]}$  and  $|v\rangle \in \mathcal{F}_\lambda^{[r]}$ . Note that the first elements of  $\lambda$  and  $\mu$  must be the same.

On a rank  $r$  (dual) Fock space, we can define the structure of a Vir-module by the following formulas:

$$L_n = \frac{\epsilon}{2} a_{n/2}^2 + \sum_{m > -n/2} a_{-m} a_{m+n} - (n+1)\rho a_n, \tag{2.1}$$

$$C = 1 - 12\rho^2, \tag{2.2}$$

where  $\epsilon = 0$  if  $n$  is odd and  $\epsilon = 1$  if  $n$  is even.

*Proposition 2.5.* For  $|\lambda\rangle \in \mathcal{F}_\lambda$ , the operators  $L_n$  ( $n \geq r$ ) defined as (2.1) act on  $|\lambda\rangle$  as follows:

$$L_n |\lambda\rangle = 0 \quad (n > 2r), \quad L_n |\lambda\rangle = \left( \frac{1}{2} \sum_{m=0}^n \lambda_m \lambda_{n-m} - \delta_{n,r} (n+1)\rho \lambda_n \right) |\lambda\rangle \quad (n = r, r+1, \dots, 2r),$$

where  $\lambda_i = 0$  for  $i < 0$  or  $i > r$ .

The operators  $L_n$  ( $n \leq -r$ ) defined as in (2.1) act on  $\langle \lambda | \in \mathcal{F}_\lambda^*$  similarly. As a result of this proposition, it is possible to define a bilinear pairing  $\langle \cdot \rangle: V_\Lambda^{*,[s]} \times V_\Lambda^{[r]} \rightarrow \mathbb{C}$ .

### B. Vertex operators

The free boson field  $\varphi(z)$  is defined by

$$\varphi(z) = q + a_0 \log(z) - \sum_{n \neq 0} \frac{a_n}{n} z^{-n},$$

where  $[a_m, q] = \delta_{m,0}$ . The commutation relations of the Heisenberg algebra imply the operator product expansion at  $|z| > |w|$

$$\varphi(z)\varphi(w) = \log(z-w) + : \varphi(z)\varphi(w) : .$$

Here, the normal order is defined by

$$: a_m a_n := \begin{cases} a_m a_n & (n \geq 0), \\ a_n a_m & (n < 0), \end{cases} \quad : q a_n := \begin{cases} q a_n & (n \geq 0), \\ a_n q & (n < 0). \end{cases}$$

Define  $T(z)$  by  $T(z) = : \varphi'(z)^2 / 2 : + \rho \varphi^{(2)}(z)$ . Then, the coefficients  $L_n$  of  $T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$  coincide with (2.1).

The confluent primary field introduced in Ref. 18 is defined as

$$\Phi_\lambda^{[r]}(z) = : \exp \left( \sum_{n=0}^r \frac{\lambda_n}{n!} \varphi^{(n)}(z) \right) :,$$

for an  $(r+1)$ -tuple of parameters  $\lambda = (\lambda_0, \dots, \lambda_r) \in \mathbb{C}^r \times \mathbb{C}^\times$ .

*Proposition 2.6 (Ref. 18).* If  $|z| > |w|$ , then we have

$$T(z)\Phi_\lambda^{[r]}(w) = \frac{1}{2} \left( \sum_{i=0}^r \frac{\lambda_i}{(z-w)^{i+1}} \right)^2 \Phi_\lambda^{[r]}(w) + \frac{1}{z-w} \partial_w \Phi_\lambda^{[r]}(w) + \sum_{i=1}^r \frac{1}{(z-w)^{i+1}} D_{i-1} \Phi_\lambda^{[r]}(w) - \rho \sum_{i=0}^r \frac{(i+1)\lambda_i}{(z-w)^{i+2}} \Phi_\lambda^{[r]}(w) + : T(z)\Phi_\lambda^{[r]}(w) :,$$

where  $D_k = \sum_{p=1}^{r-k} p \lambda_{p+k} \partial / \partial \lambda_p$ .

The above operator product expansion corresponds to the following commutation relations.

*Corollary 2.7.*

$$[L_n, \Phi_\lambda^{[r]}(w)] = w^{n+1} \partial_w \Phi_\lambda^{[r]}(w) + \sum_{i=0}^{r-1} \binom{n+1}{i+1} w^{n-i} D_i \Phi_\lambda^{[r]}(w) + \frac{1}{2} \sum_{i,j=0}^r \lambda_i \lambda_j \binom{n+1}{i+j+1} w^{n-i-j} \Phi_\lambda^{[r]}(w) - \rho \sum_{i=0}^r \binom{n+1}{i+1} (i+1) \lambda_i w^{n-i} \Phi_\lambda^{[r]}(w).$$

*Proposition 2.8.* For  $\lambda = (\lambda_0, \dots, \lambda_r) \in \mathbb{C}^r \times \mathbb{C}^\times$  and  $\mu = (\mu_0, \dots, \mu_s) \in \mathbb{C}^s \times \mathbb{C}^\times$ , the confluent primary field  $\Phi_\lambda^{[r]}(z)$  acts on  $|\mu\rangle = \exp(\sum_{n=1}^s \mu_n a_{-n} / n + \mu_0 q) \in \mathcal{F}_\mu$  as follows:

$$\Phi_\lambda^{[r]}(z)|\mu\rangle = z^{\lambda_0 \mu_0} \exp \left( \sum_{m=0}^r \sum_{n=0}^s \frac{(-1)^{m-1}}{m!} (n+1)_{m-1} \lambda_m \mu_n z^{-n-m} \right) \times \left( \exp \left( \sum_{m=1}^r \frac{\lambda_m}{m} a_{-m} + \sum_{n=1}^s \frac{\mu_n}{n} a_{-n} + (\mu_0 + \lambda_0) q \right) + O(z) \right),$$

where  $(a)_m = a(a+1) \cdots (a+m-1)$ .

Given these facts, we define general vertex operators as follows.

*Definition 2.9.* The rank  $r$  vertex operator  $\Phi_{\Delta, \Lambda}^{[r], \lambda}(z): V_\Delta^{[0]} \rightarrow V_\Lambda^{[r]}$  with  $\lambda = (\lambda_0, \dots, \lambda_r)$ ,  $(\lambda_r \neq 0)$ ,  $\Lambda = (\Lambda_r, \dots, \Lambda_{2r})$  ( $\Lambda_{2r} \neq 0$ ) is defined by

$$[L_n, \Phi_{\Delta, \Lambda}^{[r], \lambda}(z)] = z^{n+1} \partial_z \Phi_{\Delta, \Lambda}^{[r], \lambda}(z) + \sum_{i=0}^{r-1} \binom{n+1}{i+1} z^{n-i} D_i \Phi_{\Delta, \Lambda}^{[r], \lambda}(z) + \frac{1}{2} \sum_{i,j=0}^r \lambda_i \lambda_j \binom{n+1}{i+j+1} z^{n-i-j} \Phi_{\Delta, \Lambda}^{[r], \lambda}(z) - \rho \sum_{i=0}^r \binom{n+1}{i+1} (i+1) \lambda_i z^{n-i} \Phi_{\Delta, \Lambda}^{[r], \lambda}(z), \quad (2.3)$$

where  $D_k = \sum_{p=1}^{r-k} p \lambda_{p+k} \frac{\partial}{\partial \lambda_p}$  and

$$\Phi_{\Delta, \Lambda}^{[r], \lambda}(z)|\Delta\rangle = z^\alpha \exp \left( \sum_{n=0}^r \frac{\beta_n}{z^n} \right) \sum_{m=0}^\infty v_m z^m, \quad (2.4)$$

where  $v_0 = |\Delta\rangle$ ,  $v_m \in V_\Lambda^{[r]}$  ( $m \geq 1$ ).

*Definition 2.10.* For non-negative  $r$ , define a rank 0 vertex operator  $\Phi_{\Lambda', \Lambda}^\Delta(z): V_{\Lambda'}^{[r]} \rightarrow V_\Lambda^{[r]}$  by

$$[L_n, \Phi_{\Lambda', \Lambda}^\Delta(z)] = z^n \left( z \frac{\partial}{\partial z} + (n+1)\Delta \right) \Phi_{\Lambda', \Lambda}^\Delta(z), \quad (2.5)$$

$$\Phi_{\Lambda', \Lambda}^\Delta(z)|\Lambda\rangle = z^\alpha \exp \left( \sum_{n=0}^r \frac{\beta_n}{z^n} \right) \sum_{m=0}^\infty v_m z^m, \quad (2.6)$$

where  $v_0 = |\Lambda'\rangle$ ,  $v_m \in V_{\Lambda'}^{[r]}$  ( $m \geq 1$ ).



When  $r = 0$ , the rank 0 vertex operator is known to exist uniquely if the Verma module  $V_{\Lambda'}^{[0]}$  is irreducible. The dual rank  $r$  vertex operator  $\Phi_{\Delta, \Lambda}^{*, [r], \lambda}(z): V_{\Delta}^{*, [0]} \rightarrow V_{\Lambda}^{*, [r]}$  and the dual rank 0 vertex operator  $\Phi_{\Lambda, \Lambda'}^{*, \Delta}(z): V_{\Lambda}^{*, [r]} \rightarrow V_{\Lambda'}^{*, [r]}$  are defined in the same manner. In the general case, it seems that a vertex operator  $\Phi_{\Lambda, \Lambda'}^{[p], \lambda}(z): V_{\Lambda}^{[r]} \rightarrow V_{\Lambda'}^{[s]}$  can be defined by commutation relations (2.3) and

$$\Phi_{\Lambda, \Lambda'}^{[p], \lambda}(z)|\Lambda\rangle = z^{\alpha} \exp\left(\sum_{n=0}^{p+r} \frac{\beta_n}{z^n}\right)(|\Lambda'\rangle + O(z)), \tag{2.7}$$

where  $s = \max(r, p)$ . However, when  $r = s = 1$ , it is observed that vertex operators satisfying (2.3) and (2.7) exist but are not unique. Hence, if we want to have uniqueness, we must add additional conditions.

*Conjecture 2.11.* The rank  $r$  vertex operator  $\Phi_{\Delta, \Lambda}^{[r], \lambda}(z): V_{\Delta}^{[0]} \rightarrow V_{\Lambda}^{[r]}$  exists and is uniquely determined by the given parameters  $\Delta, \lambda, a$  with  $\beta_r = a\lambda_r, \alpha = \alpha(\lambda_0, a, \Delta), \beta_i(\lambda_0, \dots, \lambda_r, a, \Delta)$  and

$$\Lambda_n = \frac{1}{2} \sum_{i=0}^r \lambda_i \lambda_{n-i} + \delta_{n,r} ((-1)^{r+1} r \beta_r - (r+1) \rho \lambda_r) \quad (n = r, \dots, 2r),$$

$$D_i(\beta_k) = (-1)^i (k+i) \beta_{k+i} \quad (k = 0, 1, \dots, r).$$

We have checked Conjecture 2.11 for  $r = 1, 2, 3$ , that is, in this case, the first  $v_1, \dots, v_{10}$  of (2.4) are uniquely determined.

For a partition  $\lambda = (\lambda_1, \dots, \lambda_n)$  ( $\lambda_i \geq \lambda_{i+1}$ ), set  $L_{\lambda} = L_{-\lambda_1+r} \cdots L_{-\lambda_n+r}$ . From Theorem 2.4 in Ref. 8, the vectors  $L_{\lambda}|\Lambda'\rangle$  where  $\lambda$  runs over the set of all partitions, form a basis of  $V_{\Lambda'}^{[r]}$ . Define the degree of  $L_{\lambda}|\Lambda'\rangle$  by  $\deg(L_{\lambda}|\Lambda'\rangle) = |\lambda| = \sum_{i=1}^n \lambda_i$  and for  $v = \sum_{\lambda} c_{\lambda} L_{\lambda}|\Lambda'\rangle$ ,  $\deg(v) = \max\{\deg(L_{\lambda}|\Lambda'\rangle)\}$ . Set

$$U_m = \{v \in V_{\Lambda'}^{[r]} \mid 0 \leq \deg(v) \leq m\}.$$

A basis of  $U_m$  consists of  $L_{\lambda}|\Lambda'\rangle$  ( $0 \leq |\lambda| \leq m$ ) and hence  $\dim(U_m) = 1 + \sum_{i=1}^m p(i)$ , where  $p(i)$  is the number of all partitions of  $i$ .

**Theorem 2.12.** For any positive integer  $r$ , the rank 0 vertex operator  $\Phi_{\Delta, \Lambda}^{\Delta}(z): V_{\Lambda}^{[r]} \rightarrow V_{\Lambda'}^{[r]}$  exists and is uniquely determined by the given parameters  $\Lambda, \Delta, \beta_r$ . In particular,

$$\Lambda'_n = \Lambda_n - \delta_{n,r} r \beta_r \quad (n = r, \dots, 2r),$$

$\alpha$  and  $\beta_n$  for  $n = 1, \dots, r-1$  are polynomials in  $\Delta, \beta_r, \Lambda_r, \dots, \Lambda_{2r}, \Lambda_{2r}^{-1}$ . Moreover,  $v_m \in U_m$  and the coefficients  $c_{\lambda}$  of the vectors  $L_{\lambda}|\Lambda'\rangle$  in  $v_m$  are uniquely determined as polynomials in  $\Delta, \beta_r, \Lambda_r, \dots, \Lambda_{2r}, \Lambda_{2r}^{-1}$ .

We see that  $\Lambda_{2r} = 0$  is the only singular point of the rank 0 vertex operator. This is the reason we set  $\Lambda_{2r} \neq 0$  in the definition of the rank  $r$  Verma module. For the case of  $\Lambda_{2r} = 0$  and  $\Lambda_{2r-1} \neq 0$ , the universal Whittaker module is still irreducible.<sup>16,8</sup> We expect from observations that such an irreducible Whittaker module describes irregular singularities of rank  $r/2$ .

The examples of vertex operators are provided in Appendix. Let us examine the rank  $r$  vertex operator and prove Theorem 2.12 in Subsections II C and II D.

**C. On a rank  $r$  vertex operator from a Verma module to an irregular Verma module**

Notice that the commutation relations (2.3) of the rank  $r$  vertex operator and condition (2.4) imply

$$\begin{aligned}
 L_n v_m &= \delta_{n,0} \Delta v_m + (\alpha + m - n - (n + 1)\rho\lambda_0 + (n + 1)D_0)v_{m-n} \\
 &+ \sum_{i=1}^{r-1} \binom{n+1}{i+1} D_i v_{m-n+i} - \sum_{i=1}^r \left( i\beta_i + (i+1)\rho\lambda_i \binom{n+1}{i+1} \right) v_{m-n+i} \\
 &+ \sum_{i,j=1}^r \binom{n+1}{i} D_{i-1}(\beta_j) v_{m-n-1+i+j} + \frac{1}{2} \sum_{i,j=0}^r \lambda_i \lambda_j \binom{n+1}{i+j+1} v_{m-n+i+j},
 \end{aligned}
 \tag{2.8}$$

for any  $n \geq 0$ , where  $v_{-n} = 0$  for  $n > 0$ , because

$$L_n \Phi_{\Delta, \Lambda}^{[r], \lambda}(z) |\Delta\rangle = [L_n, \Phi_{\Delta, \Lambda}^{[r], \lambda}(z)] |\Delta\rangle + \delta_{n,0} \Delta \Phi_{\Delta, \Lambda}^{[r], \lambda}(z) |\Delta\rangle.$$

In this subsection, suppose that (2.8) holds, that is, we assume the existence of a rank  $r$  vertex operator satisfying (2.3) and (2.4). First, let us examine the case of  $n = 0$ . Set  $v_m = \tilde{v}_m |\Lambda\rangle$ .

*Proposition 2.13.*  $D_0(\beta_i) = i\beta_i$  ( $i = 1, \dots, r$ ) and

$$[L_0, \tilde{v}_m] |\Lambda\rangle = (m\tilde{v}_m + D_0(\tilde{v}_m)) |\Lambda\rangle. \tag{2.9}$$

*Proof.* From (2.8),

$$L_0 \tilde{v}_m |\Lambda\rangle = (\alpha + m + D_0 + \Delta - \rho\lambda_0) \tilde{v}_m |\Lambda\rangle + \sum_{i=1}^r (D_0(\beta_i) - i\beta_i) \tilde{v}_{m+i} |\Lambda\rangle + \frac{1}{2} \lambda_0^2 \tilde{v}_m |\Lambda\rangle. \tag{2.10}$$

Hence, substituting  $m = -r, \dots, -1$  yields  $D_0(\beta_i) = i\beta_i$  ( $i = 1, \dots, r$ ). Consequently,

$$L_0 \tilde{v}_m |\Lambda\rangle = \left( \alpha + m + D_0 + \Delta + \lambda_0 \left( -\rho + \frac{1}{2} \lambda_0 \right) \right) \tilde{v}_m |\Lambda\rangle. \tag{2.11}$$

In particular,

$$L_0 |\Lambda\rangle = \left( D_0 + \alpha + \Delta + \lambda_0 \left( -\rho + \frac{1}{2} \lambda_0 \right) \right) |\Lambda\rangle. \tag{2.12}$$

Then, by (2.12) and (2.11) is rewritten as (2.9). □

*Proposition 2.14.*

$$D_i(\beta_k) = (-1)^i (k + i) \beta_{k+i} \quad (k = 1, \dots, r), \tag{2.13}$$

for  $i = 0, \dots, r - 1$ , where  $\beta_k = 0$  if  $k > r$ , and for  $n = 1, \dots, r - 1$ ,

$$\begin{aligned}
 L_n \tilde{v}_m |\Lambda\rangle &= \sum_{k=1}^n \left( \binom{n+1}{k+1} \left( D_k - (k+1)\rho\lambda_k + \frac{1}{2} \sum_{i=0}^k \lambda_i \lambda_{k-i} \right) + \sum_{i=0}^k \binom{n+1}{i} (-1)^{i-1} k \beta_k \right) \tilde{v}_{m-n+k} |\Lambda\rangle \\
 &+ \left( \alpha + m - n + (n + 1)D_0 + (n + 1)\lambda_0 \left( -\rho + \frac{1}{2} \lambda_0 \right) \right) \tilde{v}_{m-n} |\Lambda\rangle.
 \end{aligned}
 \tag{2.14}$$

In particular,

$$L_n |\Lambda\rangle = \left( D_n - (n + 1)\rho\lambda_n + \frac{1}{2} \sum_{i=0}^n \lambda_i \lambda_{n-i} + (-1)^{n+1} n \beta_n \right) |\Lambda\rangle.$$

*Proof.* Substituting  $m = -r, \dots, n - 1 - r$  into (2.8) for  $n = 1, \dots, r - 1$  yields

$$\sum_{i=k-r}^{n+1} \binom{n+1}{i} D_{i-1}(\beta_{k-i}) = 0, \tag{2.15}$$

for  $k = r + 2, \dots, n + 1 + r$ . Thus, substituting  $m = n - r, \dots, -1$  into (2.8) again yields

$$\left( \sum_{i=1}^{n+1} \binom{n+1}{i} D_{i-1}(\beta_{k+1-i}) - k\beta_k \right) = 0, \tag{2.16}$$

for  $k = n + 1, \dots, r$ . From (2.15), inductively,

$$D_{i-1}(\beta_{r-j}) = 0 \quad (j = 0, 1, \dots, i - 2)$$

for  $i = 2, \dots, r$ . From (2.16), inductively,

$$D_i(\beta_k) = (-1)^i(k + i)\beta_{k+i} \quad (k = 1, \dots, r - i),$$

for  $i = 1, \dots, r - 1$ . Hence, we obtain (2.13), which yields (2.14). □

In a similar manner, we have the following proposition.

*Proposition 2.15.* for  $n = r, r + 1, \dots, 2r$ ,

$$L_n|\Lambda\rangle = \frac{1}{2} \sum_{i=0}^r \lambda_i \lambda_{n-i} |\Lambda\rangle + \delta_{n,r} ((-1)^{r+1} r \beta_r - (r + 1) \rho \lambda_r) |\Lambda\rangle, \tag{2.17}$$

and for  $n > 2r$ ,

$$L_n \tilde{v}_m |\Lambda\rangle = 0 \quad (0 \leq m < n - 2r). \tag{2.18}$$

Now, we consider the case when  $\Delta = 0$  and  $V_0^{[0]}$  is the irreducible highest weight representation.

*Proposition 2.16.* A rank  $r$  vertex operator  $\Phi_{0,\Lambda}^{[r],\lambda}(z): V_0^{[0]} \rightarrow V_\Lambda^{[r]}$  exists uniquely and acts on the highest weight vector  $|0\rangle$  as

$$\Phi_{0,\Lambda}^{[r],\lambda}(z)|0\rangle = \sum_{m=0}^{\infty} \frac{1}{m!} L_{-1}^m |\Lambda\rangle z^m,$$

where

$$\Lambda_n = \frac{1}{2} \sum_{i=0}^r \lambda_i \lambda_{n-i} - \delta_{n,r} (r + 1) \rho \lambda_r \quad (n = r, r + 1, \dots, 2r). \tag{2.19}$$

Consequently,

$$\lim_{z \rightarrow 0} \Phi_{0,\Lambda}^{[r],\lambda}(z)|0\rangle = |\Lambda\rangle.$$

*Proof.* From  $L_{-1}|0\rangle = 0$ ,

$$\begin{aligned} L_{-1} \Phi_{0,\Lambda}^{[r],\lambda}(z)|0\rangle &= \frac{\partial}{\partial z} \Phi_{0,\Lambda}^{[r],\lambda}(z)|0\rangle \\ &= \left( \frac{\alpha}{z} - \sum_{i=1}^r \frac{i\beta_i}{z^{i+1}} \right) \Phi_{0,\Lambda}^{[r],\lambda}(z)|0\rangle + z^\alpha \exp\left( \sum_{i=1}^r \frac{\beta_i}{z^i} \right) \sum_{m=0}^{\infty} m v_m z^{m-1}. \end{aligned}$$

This relation is true if and only if  $\alpha = \beta_1 = \dots = \beta_r = 0$  and  $v_m = L_{-1}^m / m! |\Lambda\rangle$ . Hence, if the rank  $r$  vertex operator  $\Phi_{0,\Lambda}^{[r],\lambda}(z)$  exists, then it is unique. In addition, from Proposition 2.15, relations (2.19) hold.

We need to show relations (2.8) for  $n \geq 0$  are true, when  $\alpha = \beta_1 = \dots = \beta_r = 0$  and  $v_m = L_{-1}^m / m! |\Lambda\rangle$ . Propositions 2.13 and 2.14 imply the actions of  $D_n$  ( $n = 0, 1, \dots, r - 1$ ) on  $|\Lambda\rangle$  is

$$D_n |\Lambda\rangle = \left( L_n + \frac{1}{2} \sum_{i=0}^n \lambda_i \lambda_{n-i} - (n + 1) \rho \lambda_n \right) |\Lambda\rangle. \tag{2.20}$$

This is achieved by realizing  $|\Lambda\rangle$  as

$$|\Lambda\rangle = \exp\left( \lambda_0 q + \sum_{n=1}^r \frac{\lambda_n}{n} a_{-n} \right).$$

By (2.20), relations (2.8) read as

$$L_n \tilde{v}_m |\Lambda\rangle = \sum_{i=1}^{r-1} \binom{n+1}{i+1} \tilde{v}_{m-n+i} L_i |\Lambda\rangle + \tilde{v}_{m-n} (m-n+(n+1)L_0) |\Lambda\rangle,$$

which can be verified by straightforward computations. □

Therefore, if we consider zero or infinity as an irregular singular point, then we do not need the general rank  $r$  vertex operator  $\Phi_{\Delta,\Lambda}^{[r],\lambda}(z): V_{\Delta}^{[0]} \rightarrow V_{\Lambda}^{[r]}$ .

**D. On a rank 0 vertex operator from a rank  $r$  Verma module to a rank  $r$  Verma module**

If the rank 0 vertex operator  $\Phi_{\Lambda,\Lambda'}^{\Delta}(z): V_{\Lambda}^{[r]} \rightarrow V_{\Lambda'}^{[r]}$  satisfies commutation relations (2.5) and condition (2.6), then for  $n \geq r$ ,

$$L_n v_m = \sum_{i=0}^r \delta_{n,i+r} \Lambda_{i+r} v_m - \sum_{i=1}^r i \beta_i v_{m+i-n} + (\alpha + (n+1)\Delta + m-n) v_{m-n}. \tag{2.21}$$

Setting  $m = 0$  in (2.21) yields

$$\Lambda'_n = \Lambda_n - \delta_{n,r} \beta_r,$$

for  $n = r, r+1, \dots, 2r$ .

Set

$$\tilde{L}_n = L_n - \Lambda'_n \quad (n = r, \dots, 2r), \quad \tilde{L}_n = L_n \quad (n > 2r).$$

Then, relations (2.21) are rewritten as

$$\tilde{L}_{n+r} v_m = \delta_{n,0r} \beta_r v_m - \sum_{i=1}^r i \beta_i v_{m+i-n-r} + (\alpha + (n+1)\Delta + m-n) v_{m-n-r} \quad (n \geq 0). \tag{2.22}$$

Because of commutation relations (2.5), we need to show only the uniqueness and existence of  $v_m$  ( $m \geq 1$ ) such that (2.22) holds. Below, we prepare lemmas and corollaries needed to prove Theorem 2.12.

*Lemma 2.17.* For any tuple  $(\mu_1, \dots, \mu_k)$  for  $\mu_i \in \mathbb{Z}_{\geq 1}$ ,

$$L_{-\mu_1+r} \cdots L_{-\mu_k+r} |\Lambda'\rangle \in U_m,$$

where  $m = \sum_{i=1}^k \mu_i$ .

*Proof.* Because  $[L_{-i+r}, L_{-j+r}] = (j-i)L_{-(i+j-r)+r}$ , moving  $L_{-\lambda_i+r}$  to the right side, at least, reduces the degree by  $\min(i+j, r)$ . □

*Lemma 2.18.* For a partition  $\lambda$ ,

$$\begin{aligned} \deg(L_{n+r} L_{\lambda} |\Lambda'\rangle) &= |\lambda| \quad (n = 0, 1, \dots, r), \\ \deg(L_{n+r} L_{\lambda} |\Lambda'\rangle) &< |\lambda| \quad (n > r). \end{aligned}$$

*Proof.* Because  $[L_{n+r}, L_{-i+r}] = (n+i)L_{-(i-n-r)+r}$ , the degree of  $[L_{n+r}, L_{\lambda}] |\Lambda'\rangle$  is less than  $|\lambda|$ . □

*Lemma 2.19.* For a partition  $\lambda = (m^{k_m}, (m-1)^{k_{m-1}}, \dots, 2^{k_2}, 1^{k_1})$  ( $k_i \in \mathbb{Z}_{\geq 0}$ ) and  $n \geq 1$ ,

$$\tilde{L}_{r+n} L_{\lambda} |\Lambda'\rangle = 2n k_n \Lambda_{2r} L_{\bar{\lambda}} |\Lambda'\rangle + v, \tag{2.23}$$

where  $\bar{\lambda} = (m^{k_m}, \dots, (n+1)^{k_{n+1}}, n^{k_n-1}, (n-1)^{k_{n-1}}, \dots, 1^{k_1})$  with  $\deg(\bar{\lambda}) = |\lambda| - n$  and  $\deg(v) \leq |\lambda| - n - 1$ .

*Proof.* Since the left hand side  $\widetilde{L}_{r+n}L_\lambda|\Lambda'\rangle$  of (2.23) is computed as

$$\widetilde{L}_{r+n}L_\lambda|\Lambda'\rangle = \sum_{i=1}^m L_{-m+r}^{k_m} \cdots L_{-(i+1)+r}^{k_{i+1}} [L_{r+n}, L_{-i+r}^{k_i}] L_{-(i-1)+r}^{k_{i-1}} \cdots L_{-1+r}^{k_1} |\Lambda'\rangle,$$

we examine  $[L_{r+n}, L_{-i+r}] = (n+i)L_{-(i-r-n)+r}$ .

Case 1:  $1 \leq i - r - n$ . From Lemma 2.17, the degree of

$$L_{-m+r}^{k_m} \cdots L_{-(i+1)+r}^{k_{i+1}} [L_{r+n}, L_{-i+r}^{k_i}] L_{-(i-1)+r}^{k_{i-1}} \cdots L_{-1+r}^{k_1} |\Lambda'\rangle$$

is  $|\lambda| - r - n$ .

Case 2:  $-r \leq i - r - n \leq 0$ . From  $L_{-(i-r-n)+r}|\Lambda'\rangle = \Lambda'_{2r+n-i}|\Lambda'\rangle$  and Lemma 2.18, the degree of

$$L_{-m+r}^{k_m} \cdots L_{-(i+1)+r}^{k_{i+1}} [L_{r+n}, L_{-i+r}^{k_i}] L_{-(i-1)+r}^{k_{i-1}} \cdots L_{-1+r}^{k_1} |\Lambda'\rangle$$

is  $|\lambda| - i$ .

Case 3:  $i - r - n \leq -r - 1$ . From  $L_{-(i-r-n)+r}|\Lambda'\rangle = L_{2r+n-i}|\Lambda'\rangle = 0$  and Lemma 2.18, the degree of

$$L_{-m+r}^{k_m} \cdots L_{-(i+1)+r}^{k_{i+1}} [L_{r+n}, L_{-i+r}^{k_i}] L_{-(i-1)+r}^{k_{i-1}} \cdots L_{-1+r}^{k_1} |\Lambda'\rangle$$

is less than  $|\lambda| - n - 1$ .

Thus, the degree is the highest, when  $i = n$  in Case 2. Therefore, (2.23) holds. □

The key lemma is the following.

*Lemma 2.20.* For any  $u \in U_m$  ( $u = \sum_\lambda a_\lambda L_\lambda|\Lambda'\rangle$ ) and  $n = 1, \dots, m$ ,

$$\widetilde{L}_{n+r}u = \sum_\lambda \Lambda_{2r} a_\lambda b_\lambda L_\lambda|\Lambda'\rangle + v,$$

where the sum is over all partitions  $\lambda$  of  $m$  that have a component  $\lambda_i = n$  of  $\lambda = (\lambda_1, \dots, \lambda_k)$  and  $\bar{\lambda} = (\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_k)$ , and  $\deg(v) \leq m - n - 1$ ,  $b_\lambda \in \mathbb{Z}_{\geq 1}$ .

*Proof.* This follows immediately from Lemma 2.19. □

*Corollary 2.21.* For any positive integer  $n$ , if  $\widetilde{L}_{n+r}u = 0$  for  $u \in V_{\Lambda'}^{[r]}$ , then  $u \in U_0$ .

Let a bilinear pairing  $\langle \cdot, \cdot \rangle: V_{\Lambda'}^{*[0]} \times V_{\Lambda'}^{[r]} \rightarrow \mathbb{C}$  be defined as in Subsection II A by the Heisenberg algebra and its Fock space. For a partition  $\lambda = (\lambda_1, \dots, \lambda_k)$  ( $\lambda_i \geq \lambda_{i+1}$ ), set  $\widetilde{L}_\lambda = \widetilde{L}_{\lambda_1+r} \cdots \widetilde{L}_{\lambda_k+r}$ .

*Lemma 2.22.* For partitions  $\lambda$  and  $\mu$  such that  $|\lambda| \geq |\mu|$ ,

$$\langle \Delta^* | \widetilde{L}_\lambda L_\mu | \Lambda' \rangle = \begin{cases} 0 & (\lambda \neq \mu), \\ (2\Lambda_{2r})^{|\lambda|} \prod_{i=1}^m i^{k_i} k_i! & (\lambda = \mu), \end{cases}$$

where  $\lambda = (m^{k_m}, (m-1)^{k_{m-1}}, \dots, 2^{k_2}, 1^{k_1})$  ( $k_i \in \mathbb{Z}_{\geq 0}$ ).

*Proof.* From Lemma 2.20,  $\widetilde{L}_{\lambda_1+r} \cdots \widetilde{L}_{\lambda_k+r}$  reduces the degree of  $L_\mu$  to  $|\mu| - |\lambda|$ . Hence, if  $|\lambda| > |\mu|$ , then  $\langle \Delta^* | \widetilde{L}_\lambda L_\mu | \Lambda' \rangle = 0$ . Suppose  $|\lambda| = |\mu|$ . Then, from Lemma 2.19  $\widetilde{L}_{\lambda_i+r}$  reduces the degree of  $L_{-j_1+r} \cdots L_{-j_k+r}|\Lambda'\rangle$  to  $j_1 + \cdots + j_k - \lambda_i$  if there exists some  $j_l$  such that  $j_l = \lambda_i$ , and to less than  $j_1 + \cdots + j_k - \lambda_i$  if  $j_l \neq \lambda_i$  for any  $l$ . Hence,  $\langle \Delta^* | \widetilde{L}_\lambda L_\mu | \Lambda' \rangle \neq 0$  if and only if  $\lambda = \mu$ . The commutation relation  $[L_{\lambda_i+r}, L_{-\lambda_i+r}] = 2\lambda_i L_{2r}$  yields the value of  $\langle \Delta^* | \widetilde{L}_\lambda L_\lambda | \Lambda' \rangle$ . □

Let a square matrix  $(\langle \Delta^* | \widetilde{L}_\lambda L_\mu | \Lambda' \rangle)_{n \leq |\lambda|, |\mu| \leq m}$  for  $n, m \in \mathbb{Z}_{\geq 0}$  ( $n < m$ ) be defined in such a way that  $\langle \Delta^* | \widetilde{L}_\lambda, L_\mu | \Lambda' \rangle$  are arranged in order of increasing with respect to  $|\lambda|, |\mu|$ . For example,

$$(\langle \Delta^* | \widetilde{L}_\lambda L_\mu | \Lambda' \rangle)_{1 \leq |\lambda|, |\mu| \leq 2} = \begin{pmatrix} \langle \Delta^* | \widetilde{L}_{1+r} L_{-1+r} | \Lambda' \rangle & \langle \Delta^* | \widetilde{L}_{1+r} L_{-2+r} | \Lambda' \rangle & \langle \Delta^* | \widetilde{L}_{1+r} L_{-1+r}^2 | \Lambda' \rangle \\ \langle \Delta^* | \widetilde{L}_{2+r} L_{-1+r} | \Lambda' \rangle & \langle \Delta^* | \widetilde{L}_{2+r} L_{-2+r} | \Lambda' \rangle & \langle \Delta^* | \widetilde{L}_{2+r} L_{-1+r}^2 | \Lambda' \rangle \\ \langle \Delta^* | \widetilde{L}_{1+r}^2 L_{-1+r} | \Lambda' \rangle & \langle \Delta^* | \widetilde{L}_{1+r}^2 L_{-2+r} | \Lambda' \rangle & \langle \Delta^* | \widetilde{L}_{1+r}^2 L_{-1+r}^2 | \Lambda' \rangle \end{pmatrix}.$$

Corollary 2.23. For any  $n, m \in \mathbb{Z}_{\geq 0}$ ,

$$\det \left( \left( \langle \Delta^* | \tilde{L}_\lambda L_\mu | \Lambda' \rangle \right)_{n \leq |\lambda|, |\mu| \leq m} \right) = \Lambda_{2r}^{\sum_{i=n}^m i p(i)},$$

where  $p(i)$  is the partition number of  $i$ .

*Proof.* Lemma 2.22 implies that the matrix  $(\langle \Delta^* | \tilde{L}_\lambda L_\mu | \Lambda' \rangle)_{n \leq |\lambda|, |\mu| \leq m}$  is an upper triangular matrix. Hence, its determinant is the product of all diagonal entries that are also computed in Lemma 2.22.  $\square$

Lemma 2.24. For any tuple  $(\lambda_1, \dots, \lambda_k)$  such that  $\lambda_i \in \mathbb{Z}_{\geq 1}$  and  $\sum_{i=1}^k \lambda_i \leq m$  ( $m \in \mathbb{Z}_{\geq 1}$ ), and  $u \in U_m$ ,

$$\tilde{L}_{\lambda_1+r} \cdots \tilde{L}_{\lambda_k+r} u = \sum_{\substack{\mu, \\ |\lambda| \leq |\mu| \leq m}} a_\mu \tilde{L}_\mu u.$$

*Proof.* From the commutation relation  $[L_{i+r}, L_{j+r}] = (i - j)L_{(i+j+r)+r}$  for  $i, j > 0$ ,

$$\tilde{L}_{\lambda_1+r} \cdots \tilde{L}_{\lambda_k+r} = \sum_{\substack{\mu, \\ |\lambda| \leq |\mu|}} a_\mu \tilde{L}_\mu.$$

Since  $u \in U_m$ , Lemma 2.20 implies  $\tilde{L}_\mu u = 0$  for  $|\mu| > m$ .  $\square$

### 1. Proof of the existence of $v_m$

Let us construct  $v_m$  in  $U_m$ . When  $m = 1$ ,  $v_1 = c_{(1)}^{(1)} L_{-1+r} | \Lambda' \rangle + c_\phi^{(1)} | \Lambda' \rangle$ . From (2.22),

$$\begin{aligned} \tilde{L}_r v_1 &= -(r - 1) \beta_{r-1} | \Lambda' \rangle, \\ \tilde{L}_{1+r} v_1 &= -r \beta_r | \Lambda' \rangle, \\ \tilde{L}_{n+r} v_1 &= 0 \quad (n > 1). \end{aligned}$$

Hence,  $c_{(1)}^{(1)}$  and  $\beta_{r-1}$  are solved as

$$c_{(1)}^{(1)} = -\frac{r \beta_r}{2 \Lambda_{2r}}, \quad \beta_{r-1} = \frac{r \beta_r \Lambda_{2r-1}}{2(r - 1) \Lambda_{2r}}.$$

Note that  $v_1$  satisfies relation (2.22), even though  $c_\phi^{(1)}$  is not determined.

Suppose that  $v_i \in U_i$  ( $1 \leq i \leq k \leq r - 1$ ) satisfy relation (2.22), and the coefficients  $c_\lambda^{(i)}$  in  $v_i = \sum_\lambda c_\lambda^{(i)} L_\lambda | \Lambda' \rangle$  and  $\beta_{r-i}$  for  $i = 1, \dots, k$  are determined as polynomials in  $\Delta, \beta_r, \Lambda_r, \dots, \Lambda_{2r}, c_\phi^{(1)}, \dots, c_\phi^{(k-1)}$ . Then,  $v_{k+1} = \sum_\lambda c_\lambda^{(k+1)} L_\lambda | \Lambda' \rangle$  is constructed as follows.

**Step 1.** We compute  $\langle \Delta^* | \tilde{L}_\lambda v_{k+1} \rangle$  by

$$\left( \langle \Delta^* | \tilde{L}_\lambda v_{k+1} \rangle \right)_{2 \leq |\lambda| \leq k+1} = \left( \langle \Delta^* | \tilde{L}_\lambda L_\mu | \Lambda' \rangle \right)_{2 \leq |\lambda|, |\mu| \leq k+1} \cdot \left( c_\mu^{(k+1)} \right)_{2 \leq |\mu| \leq k+1}.$$

The left-hand side is expressed by  $\Lambda_r, \dots, \Lambda_{2r}, \Lambda_{2r}^{-1}, \beta_r, c_\phi^{(1)}, \dots, c_\phi^{(k-1)}$  from (2.22). Because of Corollary 2.23, the coefficients  $c_\lambda^{(k+1)}$  ( $2 \leq |\lambda| \leq k + 1$ ) are uniquely solved as polynomials in  $\Lambda_r, \dots, \Lambda_{2r}, \Lambda_{2r}^{-1}, \beta_r, c_\phi^{(1)}, \dots, c_\phi^{(k-1)}$ . Note that the coefficients  $c_\lambda^{(k+1)}$  ( $2 \leq |\lambda| \leq k + 1$ ) do not depend on  $\Delta^*$ .

**Step 2.** We show that relation (2.22) for  $n = 1$  is true except for the constant term and it is true for  $n \geq 2$ . Denote the right-hand side of (2.22) by  $X_{n,m}$ . From Step 1 and Lemma 2.24,

$$\langle \Delta^* | \tilde{L}_\lambda \left( \tilde{L}_{n+r} v_{k+1} - X_{n,k+1} \right) = 0, \tag{2.24}$$

for  $1 \leq |\lambda| \leq k$  when  $n = 1$  and  $0 \leq |\lambda| \leq k + 1 - n$  when  $n > 1$ . Since  $\tilde{L}_{n+r} v_{k+1} - X_{n,k+1}$  is an element of  $U_{k+1-n}$  as a result of Lemma 2.20, let  $\tilde{L}_{n+r} v_{k+1} - X_{n,k+1} = \sum_\mu a_\mu^{(n)} L_\mu | \Lambda' \rangle$  where the sum

is over all  $\mu$  such that  $0 \leq |\mu| \leq k + 1 - n$ . Then, from Lemma 2.20, we obtain

$$\left( \langle \Delta^* | \tilde{L}_\lambda (\tilde{L}_{n+r} v_{k+1} - X_{n,k+1}) \rangle_{\delta_{n,1} \leq |\lambda| \leq k+1-n} \right) = \left( \langle \Delta^* | \tilde{L}_\lambda L_\mu | \Lambda' \rangle \right)_{\delta_{n,1} \leq |\lambda|, |\mu| \leq k+1-n} \cdot \left( a_\mu^{(n)} \right)_{\delta_{n,0} \leq |\mu| \leq k+1-n}.$$

Hence, by (2.24) and Corollary 2.23, for  $1 \leq |\mu| \leq k$  when  $n = 1$  and  $0 \leq |\mu| \leq k + 1 - n$  when  $n > 1$ ,  $a_\mu^{(n)}$  are equal to zero. Therefore, relation (2.22) for  $n = 1$  is true except for the constant term and is also true for  $n \geq 2$ . It is easy to see that  $\tilde{L}_{n+r} v_{k+1} = 0$  for  $n > k + 1$  from Lemma 2.20, because  $v_{k+1} \in U_{k+1}$ .

**Step 3.** We determine  $c_{(1)}^{(k+1)}$  and  $\beta_{r-k-1}$  by looking at the constant terms of relation (2.22) for  $n = 0$  and  $n = 1$ . They are equivalent to

$$\begin{aligned} \Lambda_{2r-1} c_{(1)}^{(k+1)} + (r - k + 1) \beta_{r-k-1} &= Y_{0,k+1}, \\ 2\Lambda_{2r} c_{(1)}^{(k+1)} &= Y_{1,k+1}, \end{aligned}$$

where  $Y_{0,k+1}$  and  $Y_{1,k+1}$  are polynomials in  $\Lambda_r, \dots, \Lambda_{2r}, \Lambda_{2r}^{-1}, \beta_r, \Delta, c_\phi^{(1)}, \dots, c_\phi^{(k)}$ . Thus, since  $\Lambda_{2r} \neq 0$ ,  $c_{(1)}^{(k+1)}$  and  $\beta_{r-k-1}$  are solved as polynomials in  $\Lambda_r, \dots, \Lambda_{2r}, \Lambda_{2r}^{-1}, \beta_r, \Delta, c_\phi^{(1)}, \dots, c_\phi^{(k)}$ . Thus, we have proved that the relation (2.22) is true for  $n = 1$ .

**Step 4.** Finally, we show that relation (2.22) is true for  $n = 0$ . Set  $\tilde{L}_r v_{k+1} - X_{0,k+1} = u \in U_k$ . From the relation (2.22) for  $n \geq 1$  which was proved in Steps 2 and 3, and  $[\tilde{L}_{n+r}, \tilde{L}_r] = n\tilde{L}_{n+2r}$  for  $n \geq 1$ , we obtain  $\tilde{L}_{n+r} u = 0$  for any  $n \geq 1$ . Hence, from Corollary 2.21,  $u$  is an element in  $U_0$ . In Step 3, we determined  $c_{(1)}^{(k+1)}$  and  $\beta_{r-k-1}$ , hence the constant term of  $\tilde{L}_{n+r} v_{k+1} - X_{n,k+1}$  is equal to zero. Therefore,  $u = 0$ .

In the same way,  $v_m$  ( $m \geq 1$ ) is uniquely constructed. When  $m = r$ , instead of  $\beta_i$ , the parameter  $\alpha$  is uniquely determined and when  $m > r$ ,  $c_\phi^{(m-r)}$  is uniquely determined. Therefore, we have proved the existence of rank 0 vertex operator  $\Phi_{\Lambda, \Lambda'}^\Delta(z): V_\Lambda^{[r]} \rightarrow V_{\Lambda'}^{[r]}$  such that

$$\Lambda'_n = \Lambda_n - \delta_{n,r} \beta_r \quad (n = r, \dots, 2r),$$

$v_m \in U_m$ , and  $\alpha, \beta_n$  for  $n = 1, \dots, r - 1$  and the coefficients  $c_\lambda^{(m)}$  of the vectors  $L_\lambda | \Lambda' \rangle$  in  $v_m$  are polynomials in  $\Delta, \beta_r, \Lambda_r, \dots, \Lambda_{2r}, \Lambda_{2r}^{-1}$ . □

**2. Proof of the uniqueness of  $v_m$**

In the proof of the existence of  $v_m$ , we proved that if we suppose  $v_m \in U_m$ , then  $v_m$  is uniquely determined by the parameters  $\Delta, \beta_r, \Lambda_r, \dots, \Lambda_{2r}$ . We need to show that if an element  $v_m \in V_{\Lambda'}^{[r]}$  satisfies relation (2.22) and elements  $v_k \in U_k$  for  $k = 1, \dots, m - 1$  satisfy relation (2.22), then  $v_m$  is also an element in  $U_m$ .

Suppose  $v_m \in U_k$  ( $k > m$ ) and  $v_i \in U_i$  for  $i = 1, \dots, m - 1$ . From Lemma 2.20, for any positive integer  $n$ ,

$$\tilde{L}_{n+r} v_m = \sum_\lambda \Lambda_{2r} a_\lambda b_\lambda L_{\bar{\lambda}} | \Lambda' \rangle + v,$$

where the sum is over all partitions  $\lambda$  of  $k$  that have a component  $\lambda_i = n$  of  $\lambda = (\lambda_1, \dots, \lambda_k)$  and  $\bar{\lambda} = (\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_k)$ , and  $\deg(v) \leq k - n - 1$ ,  $b_\lambda \in \mathbb{Z}_{\geq 1}$ . On the other hand, from  $\tilde{L}_{n+r} v_m = X_{n,m}$  and the assumption,  $\tilde{L}_{n+r} v_m \in U_{m-n}$ . Hence, since  $k > m$ , we obtain  $a_\lambda = 0$  for  $\lambda = (\lambda_1, \dots, \lambda_k)$  such that  $\lambda_i = n$  for some  $i$ . Therefore,  $v_m \in U_{k-1}$ . □

**III. IRREGULAR CONFORMAL BLOCKS**

**A. Definition**

At the present stage, we can define irregular conformal blocks with at most two irregular singular points. It is convenient to consider zero and infinity as irregular singular points. Then, an

irregular conformal block is defined as

$$\Phi(\Delta, \Delta', \Lambda, \Lambda', z, w) = \left( \langle \Lambda'_\infty | \Phi_{\Lambda'_\infty, \Lambda'_1}^{*, \Delta'_1}(w_1) \circ \dots \circ \Phi_{\Lambda'_{N-1}, \Lambda'_N}^{*, \Delta'_N}(w_N) \rangle \right) \cdot \left( \Phi_{\Lambda_{M-1}, \Lambda_M}^{\Delta_M}(z_M) \circ \dots \circ \Phi_{\Lambda_0, \Lambda_1}^{\Delta_1}(z_1) | \Lambda_0 \right), \tag{3.1}$$

by the bilinear pairing  $\langle \cdot \rangle: V_{\Lambda'_N}^{*, [s]} \times V_{\Lambda_M}^{[r]} \rightarrow \mathbb{C}$ . Here,  $\Phi_{\Lambda'_{k-1}, \Lambda'_k}^{\Delta'_k}(w_k)$  and  $\Phi_{\Lambda_{k-1}, \Lambda_k}^{\Delta_k}(z_k)$  are rank 0 vertex operators,

$$\Phi_{\Lambda'_{k-1}, \Lambda'_k}^{\Delta'_k}(w_k): V_{\Lambda'_{k-1}}^{*, [s]} \rightarrow V_{\Lambda'_k}^{*, [s]} \quad (k = 1, \dots, N), \quad \Phi_{\Lambda_{k-1}, \Lambda_k}^{\Delta_k}(z_k): V_{\Lambda_{k-1}}^{[r]} \rightarrow V_{\Lambda_k}^{[r]} \quad (k = 1, \dots, M),$$

with  $\Lambda'_0 = \Lambda'_\infty$ . From Theorem 2.12 and the definition of the bilinear pairing by the Fock spaces, the number of free parameters of the irregular conformal block  $\Phi(\Delta, \Delta', \Lambda, \Lambda', z, w)$  is actually  $2M + 2N + r + s + 1$ .

The irregular conformal block  $\Phi(\Delta, \Delta', \Lambda, \Lambda', z, w)$  is expected to be a divergent series with respect to both the  $z$  and  $w$  variables if  $s, r > 0$ . If  $s = 0$  and  $r > 0$ , then  $\Phi(\Delta, \Delta', \Lambda, \Lambda', z, w)$  is expected to be a convergent series with respect to the  $w$  variables and a divergent series with respect to the  $z$  variables. In addition, if  $s = r = 0$ , then a regular conformal block  $\Phi(\Delta, \Delta', \Lambda, \Lambda', z, w)$  is believed to be absolutely convergent in the domain  $|z_1| < \dots < |z_M| < |w_1| < \dots < |w_N|$ .

**B. Null vector condition**

In the case of regular singularities, if one of the vertex operators satisfies the null vector condition

$$\frac{\partial^2}{\partial z^2} \Phi_{\Delta_1, \Delta_3}^{\Delta_{2,1}}(z) + b^2 : T(z) \Phi_{\Delta_1, \Delta_3}^{\Delta_{2,1}}(z) := 0,$$

where  $c = 1 + 6(b + 1/b)^2$  and  $\Delta_{2,1} = -(3b^2 + 2)/4$ , then the conformal block is a solution to the BPZ equation mentioned in Section I. From the null vector condition, we know that if we set  $\Delta_1 = Q^2/4 - P^2$  with  $Q = b + 1/b$ , then  $\Delta_3$  must be  $Q^2/4 - (P \pm b/2)^2$ .

In the case of irregular singularities, we have the following proposition.

*Proposition 3.1.* Let a rank 0 vertex operator  $\Phi_{\Lambda, \Lambda'}^\Delta(z): V_\Lambda^{[r]} \rightarrow V_{\Lambda'}^{[r]}$  such that

$$\Phi_{\Lambda, \Lambda'}^\Delta(z) | \Lambda \rangle = z^\alpha \exp \left( \sum_{i=1}^r \beta_i z^{-i} \right) (| \Lambda' \rangle + O(z)),$$

satisfy the null vector condition

$$\frac{\partial^2}{\partial z^2} \Phi_{\Lambda, \Lambda'}^\Delta(z) + b^2 : T(z) \Phi_{\Lambda, \Lambda'}^\Delta(z) := 0. \tag{3.2}$$

Then,

$$-b^2 \Lambda_n = \sum_{i=n-r}^r i(n-i) \beta_i \beta_{n-i} \quad (r+1 \leq n \leq 2r), \tag{3.3}$$

$$-b^2 \Lambda_r = r \beta_r ((r+1)(b^2+1) - 2\alpha) + \sum_{i=1}^{r-1} i(r-i) \beta_i \beta_{r-i}. \tag{3.4}$$

*Proof.* A straightforward computation of

$$\frac{\partial^2}{\partial z^2} \Phi_{\Lambda, \Lambda'}^\Delta(z) | \Lambda \rangle + b^2 : T(z) \Phi_{\Lambda, \Lambda'}^\Delta(z) : | \Lambda \rangle$$

yields relations (3.3) and (3.4). □

Recall that due to Theorem 2.12,  $\alpha$  and  $\beta_i$  ( $i = 1, \dots, r-1$ ) are solved by  $\Lambda_n$  ( $n = r, \dots, 2r$ ),  $\beta_r$ , and  $\Delta$ . From the example, one can observe that relation (3.4) implies that the conformal dimension  $\Delta$  is equal to  $\Delta_{2,1}$ . Therefore, together with the relation  $-b^2 \Lambda_{2r} = r^2 \beta_r^2$ , the null vector condition of a rank 0 vertex operator provides the condition to the two parameters  $\Delta$  and  $\beta_r$ .



Based on the null vector condition, let us give an example of irregular conformal blocks satisfying BPZ-type differential equations. In Subsection III B 1, we explain the irregular conformal block for the case of the confluent hypergeometric equation, that is, Kummer’s equation. The differential equations satisfied by irregular conformal blocks have been already presented in Refs. 3 and 11. Also refer to Ref. 19, where quantization of the Lax equations of the Painlevé equations was derived as partial differential systems satisfied by irregular conformal blocks.

**1. Kummer**

The first example is the irregular conformal blocks having one irregular singular point  $z_1$  and two regular singular points  $z_2, z_3$  with one null vector condition. Keeping in mind that  $z_1, z_3$  will be set to  $0, \infty$ , respectively, we express them as

$$\left( \langle 0 | \Phi_{0, \theta_\infty^2}^{*, \theta_\infty^2}(z_3) \Phi_{\theta_\infty^2, (\theta_\infty \pm 1/2)^2}^{*, 1/4}(z_2) \right) \cdot \left( \Phi_{0, \Lambda}^{\lambda, [1]}(z_1) | 0 \rangle \right) \tag{3.5}$$

or

$$\left( \langle 0 | \Phi_{0, \theta_\infty^2}^{*, \theta_\infty^2}(z_3) \right) \cdot \left( \Phi_{\Lambda, \Lambda^\pm}^{1/4}(z_2) \Phi_{0, \Lambda}^{\lambda, [1]}(z_1) | 0 \rangle \right), \tag{3.6}$$

where we set the central charge  $c = 1$ ,  $\lambda = (\lambda_0, \lambda_1)$ ,  $\Lambda = (\Lambda_1, \Lambda_2)$ , and  $\Lambda^\pm = (\Lambda_1 \pm \Lambda_2^{1/2}, \Lambda_2)$ .

From null vector condition (3.2), and commutation relations (2.3) and (2.5), these two irregular conformal blocks satisfy the confluent BPZ equation

$$\left( \frac{\partial^2}{\partial z_2^2} - \frac{1}{2} \left( \sum_{i=0}^1 \frac{\lambda_i}{(z_2 - z_1)^{i+1}} \right)^2 - \frac{1}{z_2 - z_1} \frac{\partial}{\partial z_1} - \frac{\lambda_1}{(z_2 - z_1)^2} \frac{\partial}{\partial \lambda_1} - \frac{1}{z_2 - z_3} \frac{\partial}{\partial z_3} - \frac{\theta_\infty^2}{(z_2 - z_3)^2} \right) \Psi(z, \lambda) = 0,$$

and from the  $\mathfrak{sl}_2$  invariance  $L_n | 0 \rangle = \langle 0 | L_n = 0$  for  $n = 0, \pm 1$ , they also satisfy

$$\begin{aligned} \sum_{i=1}^N \frac{\partial}{\partial z_i} \Psi(z, \lambda) &= 0, \\ \sum_{i=1}^N \left( z_i \frac{\partial}{\partial z_i} + \sum_{p=1}^{r_i} p \lambda_p^{(i)} \frac{\partial}{\partial \lambda_p^{(i)}} + \frac{\lambda_0^{(i)}}{2} (\lambda_0^{(i)} - 2\rho) \right) \Psi(z, \lambda) &= 0, \\ \sum_{i=1}^N \left( z_i^2 \frac{\partial}{\partial z_i} + 2z_i \sum_{p=1}^{r_i} p \lambda_p^{(i)} \frac{\partial}{\partial \lambda_p^{(i)}} + \sum_{p=1}^{r_i-1} p \lambda_{p+1}^{(i)} \frac{\partial}{\partial \lambda_p^{(i)}} + (\lambda_0^{(i)} z_i + \lambda_1^{(i)}) (\lambda_0^{(i)} - 2\rho) \right) \Psi(z, \lambda) &= 0, \end{aligned}$$

where  $N = 3$ ,  $r_1 = 1$ ,  $r_2 = r_3 = 0$ ,  $\lambda_i^{(1)} = \lambda_i$  ( $i = 0, 1$ ),  $\lambda_i^{(2)} = 0$  ( $i = 0, 1$ ),  $\lambda_0^{(3)} = \sqrt{2}\theta_\infty$ ,  $\lambda_1^{(3)} = 0$ ,  $\rho = 0$ .

Together with these four equations, the two irregular conformal blocks become solutions to the ordinary differential equation with respect to  $z_2$ . We set  $z_1 = 0, z_2 = 1/x, z_3 = \infty$  and by scaling the variables  $x$ , we can set  $\Lambda_2 = 1/4$  so that  $\lambda_1 = 1/\sqrt{2}$ . Furthermore, set

$$\lambda_0 = \frac{2\alpha - \gamma}{\sqrt{2}}, \quad \theta_\infty = \frac{\gamma - 1}{2}.$$

Then, the ordinary differential equation is transformed to Kummer’s confluent hypergeometric equation

$$\left( \frac{d^2}{dx^2} + (\gamma - x) \frac{d}{dx} - \alpha \right) F(x, \alpha, \gamma) = 0. \tag{3.7}$$

Here,  $F(x, \alpha, \gamma) = g(x) \Psi(1/x, \alpha, \gamma)$  with  $g(x) = x^{-\gamma/2} \exp(x/2)$ . Therefore, since

$$\lim_{z_3 \rightarrow \infty} z_3^{2\theta_\infty^2} \langle 0 | \Phi_{0, \theta_\infty^2}^{*, \theta_\infty^2}(z_3) = \langle \theta_\infty^2 |, \quad \lim_{z_1 \rightarrow 0} \Phi_{0, \Lambda}^{\lambda, [1]}(z_1) | 0 \rangle = |(\Lambda_1, \Lambda_2)\rangle,$$

the irregular conformal blocks multiplied by the gauge factor  $g(x)$ ,

$$g(x) \left( \langle \theta_\infty^2 | \Phi_{\theta_\infty^2, (\theta_\infty \pm 1/2)^2}^{*, 1/4}(1/x) \right) \cdot |(\lambda_0/\sqrt{2}, 1/4)\rangle, \quad g(x) \langle \theta_\infty^2 | \cdot \left( \Phi_{(\lambda_0/\sqrt{2}, 1/4), (\lambda_0/\sqrt{2} \pm 1/2, 1/4)}^{1/4} | (\lambda_0/\sqrt{2}, 1/4) \right) \rangle,$$

are solutions to Kummer’s confluent hypergeometric equation.

Let us examine these irregular conformal blocks. From the definition,

$$\begin{aligned} \langle \theta_{\infty}^2 | \Phi_{\theta_{\infty}^2, (\theta_{\infty} \pm 1/2)^2}^{*, 1/4}(1/x) \rangle \cdot |(\lambda_0/\sqrt{2}, 1/4)\rangle &= x^{\alpha'} \sum_{i=0}^{\infty} A_i x^i, \\ \langle \theta_{\infty}^2 | \cdot \left( \Phi_{(\lambda_0/\sqrt{2}, 1/4), (\lambda_0/\sqrt{2} \pm 1/2, 1/4)}^{1/4} |(\lambda_0/\sqrt{2}, 1/4)\rangle \right) &= x^{\alpha} e^{\beta x} \sum_{i=0}^{\infty} B_i x^{-i}, \end{aligned}$$

with  $A_0 = B_0 = 1$ . Computing  $\alpha'$ ,  $\alpha$ ,  $\beta$  and a few terms of  $A_i$ ,  $B_i$  (see the [Appendix](#)) yields

$$\begin{aligned} \langle \theta_{\infty}^2 | \Phi_{\theta_{\infty}^2, (\theta_{\infty} \pm 1/2)^2}^{*, 1/4}(1/x) \rangle \cdot |(\lambda_0/\sqrt{2}, 1/4)\rangle &= x^{\pm\theta_{\infty} + 1/2} e^{-x/2} F \left( \begin{matrix} \Lambda_1 \pm \theta_{\infty} + \frac{1}{2}; x \\ \pm 2\theta_{\infty} + 1 \end{matrix} \right), \\ \langle \theta_{\infty}^2 | \cdot \left( \Phi_{(\lambda_0/\sqrt{2}, 1/4), (\lambda_0/\sqrt{2} \pm 1/2, 1/4)}^{1/4} |(\lambda_0/\sqrt{2}, 1/4)\rangle \right) &= x^{\pm\Lambda_1} e^{\pm x/2} F \left( \begin{matrix} \theta_{\infty} + \frac{1}{2} \mp \Lambda_1, -\theta_{\infty} + \frac{1}{2} \mp \Lambda_1; \pm \frac{1}{x} \end{matrix} \right), \end{aligned}$$

where

$$F \left( \begin{matrix} \alpha \\ \gamma \end{matrix}; x \right) = \sum_{i=0}^{\infty} \frac{(\alpha)_i}{i!(\gamma)_i} x^i, \quad F(\alpha, \gamma; x) = \sum_{i=0}^{\infty} \frac{(\alpha)_i (\gamma)_i}{i!} x^i.$$

#### IV. PAINLEVÉ TAU FUNCTIONS

In this section, as an application of the theory of irregular conformal blocks developed in Sections II and III, we propose series expansions of the tau functions of the fourth and fifth Painlevé equations in terms of irregular conformal blocks. Before proceeding to the detail, let us briefly review the Painlevé equations.

The relation between the Painlevé functions and their tau functions is similar to that between the elliptic functions and theta functions. Let us illustrate this by taking the first Painlevé equation as an example. The first Painlevé equation is

$$\frac{d^2 y}{dt^2} = 6y^2 + t.$$

The differential equation obtained by replacing the last term  $t$  with the constant term is

$$\frac{d^2 y}{dt^2} = 6y^2 - \frac{1}{2}g_2, \tag{4.1}$$

which is derived by differentiating the differential equation

$$\left( \frac{dy}{dt} \right)^2 = 4y^3 - g_2 y - g_3.$$

Hence, the Weierstrass  $\wp$  function is a solution to (4.1). The Weierstrass  $\sigma$  function is defined by

$$\wp = -\frac{d^2}{dt^2} \log \sigma.$$

The Weierstrass  $\sigma$  function is one of the theta functions. Conversely, any elliptic function is expressed by a ratio of theta functions, which have explicit series expansions yielding various formulas involving the theta functions.

On the other hand, for any solution  $y(t)$  of the first Painlevé equation, define the tau function  $\tau_1(t)$  by

$$y(t) = -\frac{d^2}{dt^2} \log \tau_1(t).$$

The Weierstrass  $\wp$  function is an entire function as well as  $\tau_1(t)$ . The tau functions of the other Painlevé functions are defined in a similar manner, and they play an important role in the study of the Painlevé equations. The interested reader is referred to Refs. 24 and 9 for details and further references.

Although the theta functions have explicit series expansions, explicit series expansions of the Painlevé tau functions were not known until recently. In 2012, a remarkable discovery was reported by Gamayun, Iorgov, and Lisovyy.<sup>12</sup> They found that the sixth Painlevé tau function has a series expansion in terms of the four point conformal block,

$$\tau_{\text{VI}}(t) = \sum_{n \in \mathbb{Z}} s^n C \left( \begin{matrix} \theta_1, \theta_t \\ \theta_\infty, \sigma + n, \theta_0 \end{matrix} \right) \mathcal{F} \left( \begin{matrix} \theta_1, \theta_t \\ \theta_\infty, \sigma + n, \theta_0 \end{matrix} ; t \right),$$

where  $s, \sigma \in \mathbb{C}$  are constants of integration,  $\theta_i$  are complex parameters in the sixth Painlevé equation,  $\mathcal{F}(\theta, \sigma; t)$  is a 4-point conformal block with the central charge  $c = 1$ :

$$\mathcal{F} \left( \begin{matrix} \theta_1, \theta_t \\ \theta_\infty, \sigma, \theta_0 \end{matrix} ; t \right) = \langle \theta_\infty^2 | \Phi_{\theta_\infty^2, \sigma^2}^{*, \theta_1^2}(1) \cdot \Phi_{\sigma^2, \theta_0^2}^{\theta_t^2}(t) | \theta_0^2 \rangle$$

and

$$C \left( \begin{matrix} \theta_1, \theta_t \\ \theta_\infty, \sigma, \theta_0 \end{matrix} \right) = \frac{\prod_{\epsilon, \epsilon' = \pm} G(1 + \theta_t + \epsilon \theta_0 + \epsilon' \sigma) G(1 + \theta_1 + \epsilon \theta_\infty + \epsilon' \sigma)}{\prod_{\epsilon = \pm} G(1 + 2\epsilon \sigma)},$$

where  $G(z)$  is the Barnes G-function such that  $G(z + 1) = \Gamma(z)G(z)$ . By the AGT correspondence,

$$\mathcal{F} \left( \begin{matrix} \theta_1, \theta_t \\ \theta_\infty, \sigma, \theta_0 \end{matrix} ; t \right) = t^{\sigma^2 - \theta_0^2 - \theta_t^2} (1 - t)^{2\theta_1 \theta_t} \sum_{\lambda, \mu \in \mathbb{Y}} \mathcal{F}_{\lambda, \mu} \left( \begin{matrix} \theta_1, \theta_t \\ \theta_\infty, \sigma, \theta_0 \end{matrix} \right) t^{|\lambda| + |\mu|},$$

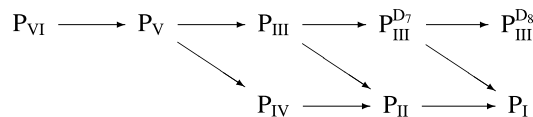
where  $\mathbb{Y}$  stands for the set of all Young diagrams,

$$\begin{aligned} \mathcal{F}_{\lambda, \mu} \left( \begin{matrix} \theta_1, \theta_t \\ \theta_\infty, \sigma, \theta_0 \end{matrix} \right) &= \prod_{(i, j) \in \lambda} \frac{((\theta_t + \sigma + i - j)^2 - \theta_0^2)((\theta_1 + \sigma + i - j)^2 - \theta_\infty^2)}{h_\lambda^2(i, j)(\lambda'_j + \mu_i - i - j + 1 + 2\sigma)^2} \\ &\times \prod_{(i, j) \in \mu} \frac{((\theta_t - \sigma + i - j)^2 - \theta_0^2)((\theta_1 - \sigma + i - j)^2 - \theta_\infty^2)}{h_\mu^2(i, j)(\mu'_j + \lambda_i - i - j + 1 - 2\sigma)^2}. \end{aligned}$$

Here,  $\lambda = (\lambda_1, \dots, \lambda_n)$  ( $\lambda_i \geq \lambda_{i+1}$ ),  $\lambda'$  denotes the transpose of  $\lambda$ , and  $h_\lambda(i, j)$  is the hook length defined by  $h_\lambda(i, j) = \lambda_i + \lambda'_j - i - j + 1$ . Therefore, we have an explicit series expansion of  $\tau_{\text{VI}}(t)$ .

A proof of the expansion of  $\tau_{\text{VI}}(t)$  was given in Ref. 14 by constructing a fundamental solution to the linear problem of  $\text{P}_{\text{VI}}$ , using Virasoro conformal field theory and another proof was given in Ref. 5 by proving that some conformal block satisfies the bilinear equations for  $\text{P}_{\text{VI}}$ , using embedding of the direct sum of two Virasoro algebras in the sum of fermion and super-Virasoro algebra.

Later, series expansions of the tau functions in  $t$ , in other words, expansions of the tau functions at a *regular singular point*, of the first line of the degeneration scheme



were obtained in Ref. 13 by taking scaling limits. It was conjectured in Ref. 15 that a long-distance expansion of the tau function for  $\text{P}_{\text{III}}^{\text{D}_8}$ , namely, an expansion in  $t^{-1}$ , can be represented as  $\sum_{n \in \mathbb{Z}} s^n \mathcal{G}(v + n; t^{-1})$ . The first few terms of  $\mathcal{G}(v; t^{-1})$  were explicitly obtained.

Therefore, it is natural to expect that the tau functions of the Painlevé equations have series expansions in terms of conformal blocks. In the following, we present conjectural formulas of series expansions in  $t^{-1}$  of the tau functions of the fifth and fourth Painlevé equations.

**A. Expansions of the P<sub>V</sub> and P<sub>IV</sub> tau functions**

The fourth and fifth Painlevé equations are the following second order nonlinear differential equations:

$$\begin{aligned}
 P_{IV} \quad & \frac{d^2q}{dt^2} = \frac{1}{2q} \left( \frac{dq}{dt} \right)^2 + \frac{3}{2}q^3 + 4tq^2 + 2(t^2 - \alpha)q + \frac{\beta}{q}, \\
 P_V \quad & \frac{d^2q}{dt^2} = \left( \frac{1}{2q} + \frac{1}{q-1} \right) \left( \frac{dq}{dt} \right)^2 - \frac{1}{t} \frac{dq}{dt} + \frac{(q-1)^2}{t^2} \left( \alpha q + \frac{\beta}{q} \right) + \frac{\gamma q}{t} + \frac{\delta q(q+1)}{q-1},
 \end{aligned}$$

$\alpha, \beta, \gamma, \delta$  being complex constants. They are equivalent to the Hamiltonian system,

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q},$$

with the Hamiltonians

$$H_{IV} = 2qp^2 - (q^2 + 2tq - 2(\theta + \theta_t))p - 2\theta_t q,$$

where  $\alpha = 1 + \theta - 3\theta_t$ ,  $\beta = -2(\theta + \theta_t)^2$ , and

$$tH_V = (q-1)(qp - 2\theta_t)(qp - p + 2\theta) - tqp + ((\theta + \theta_t)^2 - \theta_0^2)q + \left( \theta_t - \frac{\theta}{2} \right) t - 2 \left( \theta_t + \frac{\theta}{2} \right)^2,$$

where  $\alpha = 2\theta_0^2$ ,  $\beta = -2\theta_t^2$ ,  $\gamma = 2\theta - 1$ ,  $\delta = -1/2$ .

For a solution  $(q(t), p(t))$  to the Hamiltonian system, we define the Hamiltonian function by

$$H_J(t) = H(t; q(t), p(t)) \quad (J = IV, V).$$

The  $\tau$ -functions  $\tau_J = \tau_J(t)$  defined by

$$H_{IV}(t) = \frac{d}{dt} \log \tau_{IV}(t), \quad H_V(t) = t \frac{d}{dt} \log \tau_V(t)$$

play a central role in the study of the Painlevé functions, such as the construction of Bäcklund transformations,<sup>22</sup> relations to Soliton equations.

A key to construct birational canonical transformations on the Painlevé functions is the nonlinear differential equations satisfied by the Hamiltonian functions.<sup>22</sup> In fact,

$$(H''_{IV})^2 - 4(tH'_{IV} - H_{IV})^2 + 4H'_{IV}(H'_{IV} - 2(\theta + \theta_t))(H'_{IV} - 4\theta_t) = 0, \tag{4.2}$$

$$(th''_V)^2 - (h_V - th'_V + 2(h'_V)^2)^2 + \frac{1}{4}((2h'_V - \theta)^2 - 4\theta_0^2)((2h'_V + \theta)^2 - 4\theta_t^2) = 0, \tag{4.3}$$

where  $f' = df/dt$  and  $h_V = tH_V$ . As mentioned above, the Hamiltonian functions are defined by a solution to the Painlevé equations. Inversely a function  $q(t)$  defined by

$$\begin{aligned}
 P_{IV} \quad & q(t) = \frac{H''_{IV} - 2tH'_{IV} + 2H_{IV}}{2(H'_{IV} - 4\theta_t)}, \\
 P_V \quad & q(t) = \frac{2(th''_V + h_V - th'_V + 2(h'_V)^2)}{(2h'_V - \theta)^2 - 4\theta_0^2}
 \end{aligned}$$

provides solutions to the fourth and fifth Painlevé equations, respectively.

Based on the previous results on the series expansions of the tau functions of the Painlevé equations P<sub>VI</sub>, P<sub>V</sub>, P<sub>III</sub>, P<sub>II</sub><sup>D7</sup>, and P<sub>III</sub><sup>D8</sup>,<sup>12,13,15</sup> we expect that the tau functions of the other cases also admit series expansions in terms of irregular conformal blocks. Let us recall that the building block of  $\tau_{VI}(t)$  is the 4-point regular conformal block with  $c = 1$ ,

$$\langle \theta_\infty^2 | \Phi_{\theta_\infty^2, \sigma^2}^{*, \theta_1^2}(1) \cdot \Phi_{\sigma^2, \theta_0^2}^{\theta_t^2}(t) | \theta_0^2 \rangle.$$

Thus, it is natural to expect that a building block of  $\tau_V(t)$  is the irregular conformal block having one irregular singular point of rank 1 and two regular singular points with  $c = 1$ ,

$$\langle \theta_\infty^2 | \cdot \left( \Phi_{(\Lambda_1 - \beta, \Lambda_2), (\Lambda_1, \Lambda_2)}^{\theta_t^2}(t) | (\Lambda_1, \Lambda_2) \right) \rangle, \quad \left( \langle \theta_\infty^2 | \Phi_{\theta_\infty^2, \sigma^2}^{*, \theta_t^2}(t) \right) \cdot | (\Lambda_1, \Lambda_2) \rangle \rangle.$$

The latter is equal to the building block of the series expansion of the tau function of  $P_V$  obtained by a degeneration limit from the series expansion of  $\tau_{VI}(t)$ .<sup>13</sup> Using the former irregular conformal block, we present a conjectural formula for the tau function of  $P_V$ . In addition, it is natural to expect that a building block of  $\tau_{IV}(t)$  is the irregular conformal block having one irregular singular point of rank 2 and one regular singular point with  $c = 1$ ,

$$\langle 0 | \cdot \left( \Phi_{(\Lambda_2-2\beta_2, \Lambda_3, \Lambda_4), (\Lambda_2, \Lambda_3, \Lambda_4)}^{\theta_t^2}(t) | (\Lambda_2, \Lambda_3, \Lambda_4) \right),$$

where  $\langle 0 | \in \bar{V}_0^{*,[0]}$ .

After some computation, we arrive at the following two conjectures.

*Conjecture 4.1 ( $P_V$  case). Let*

$$\begin{aligned} \tau(t) = & \sum_{n \in \mathbb{Z}} (-1)^{n(n+1)/2} s^n \prod_{\epsilon = \pm 1} G(1 + \epsilon\theta_0 + \theta - \beta - n) G(1 + \theta_t + \epsilon(\beta + n)) \\ & \times \langle \theta_0^2 | \cdot \left( \Phi_{(\theta, 1/4), (\theta - \beta - n, 1/4)}^{\theta_t^2}(1/t) | (\theta, 1/4) \right) \end{aligned}$$

and  $H = (\log(t^{-2\theta_t^2 - \theta^2/2} e^{-\theta t/2} \tau(t)))'$ . Then,  $H$  satisfies differential equation (4.3).

*Conjecture 4.2 ( $P_{IV}$  case). Let*

$$\begin{aligned} \tau(t) = & t^{-2\theta_t^2} e^{\theta_t t^2} \sum_{n \in \mathbb{Z}} s^n G(1 + \theta - \beta - n) \prod_{\epsilon = \pm 1} G(1 + \theta_t + \epsilon(\beta + n)) \\ & \times \langle 0 | \cdot \left( \Phi_{(\theta, 0, 1/4), (\theta - \beta - n, 0, 1/4)}^{\theta_t^2}(1/\sqrt{2}t) | (\theta, 0, 1/4) \right) \end{aligned}$$

and  $H = (\log \tau(t))'$ . Then,  $H$  satisfies the differential equation (4.2).

Here,

$$\begin{aligned} & \langle \theta_0^2 | \cdot \left( \Phi_{(\theta, 1/4), (\theta - \beta, 1/4)}^{\theta_t^2}(1/t) | (\theta, 1/4) \right) \\ & = t^{2\theta_t^2 + 2\beta(\theta - \beta)} e^{\beta t} \left( 1 + 2(2\beta^3 - 3\beta^2\theta + \beta\theta^2 - \beta\theta_0^2 - \beta\theta_t^2 + \theta\theta_t^2) t^{-1} \right. \\ & + 2(4\beta^6 - 12\beta^5\theta + 13\beta^4\theta^2 - 4\beta^4\theta_0^2 - 4\beta^4\theta_t^2 + 5\beta^4 - 6\beta^3\theta^3 + 6\beta^3\theta\theta_0^2 + 10\beta^3\theta\theta_t^2 - 10\beta^3\theta + \beta^2\theta^4 \\ & - 2\beta^2\theta^2\theta_0^2 - 8\beta^2\theta^2\theta_t^2 + 6\beta^2\theta^2 + \beta^2\theta_0^4 + 2\beta^2\theta_0^2\theta_t^2 - 3\beta^2\theta_0^2 + \beta^2\theta_t^4 - 3\beta^2\theta_t^2 + 2\beta\theta^3\theta_t^2 - \beta\theta^3 \\ & \left. - 2\beta\theta\theta_0^2\theta_t^2 + \beta\theta\theta_0^2 - 2\beta\theta\theta_t^4 + 5\beta\theta\theta_t^2 + \theta^2\theta_t^4 - 2\theta^2\theta_t^2 + \theta_0^2\theta_t^2) t^{-2} + \dots \right) \end{aligned}$$

and

$$\begin{aligned} & \langle 0 | \cdot \left( \Phi_{(\theta, 0, 1/4), (\theta - \beta, 0, 1/4)}^{\theta_t^2}(1/t) | (\theta, 0, 1/4) \right) \\ & = t^{3\theta_t^2 + \beta(2\theta - 3\beta)} e^{\beta t^2/2} \left( 1 + (\theta^2\beta + 2\theta\theta_t^2 - 6\theta\beta^2 - 3\theta_t^2\beta + 6\beta^3) t^{-2} \right. \\ & + \frac{1}{4}(2\theta^4\beta^2 + 8\theta^3\theta_t^2\beta - 24\theta^3\beta^3 - 4\theta^3\beta + 8\theta^2\theta_t^4 - 60\theta^2\theta_t^2\beta^2 - 16\theta^2\theta_t^2 + 96\theta^2\beta^4 \\ & + 48\theta^2\beta^2 - 24\theta\theta_t^4\beta + 120\theta\theta_t^2\beta^3 + 72\theta\theta_t^2\beta - 144\theta\beta^5 - 140\theta\beta^3 - 2\theta\beta + 18\theta_t^4\beta^2 \\ & \left. + \theta_t^4 - 72\theta_t^2\beta^4 - 66\theta_t^2\beta^2 - \theta_t^2 + 72\beta^6 + 105\beta^4 + 3\beta^2) t^{-4} + \dots \right). \end{aligned}$$

If we substitute  $H$  into (4.3) or (4.2), then the coefficient of  $s^i$  for  $i \in \mathbb{Z}$  is of the form

$$t^A e^B (a_0 + a_1 t^{-1} + a_2 t^{-2} + \dots).$$

Because  $a_i$  ( $i = 0, 1, \dots$ ) are finite sums of the coefficients of  $t^{-k}$  in the corresponding irregular conformal blocks, we can check that the first several  $a_i$ 's are zero.

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**APPENDIX: DATA OF IRREGULAR VERTEX OPERATORS**

**1. Vertex operators from a Verma module to an irregular Verma module**

Let  $\lambda = (\lambda_0, \dots, \lambda_r)$ ,  $(\lambda_r \neq 0)$ ,  $\Lambda = (\Lambda_r, \dots, \Lambda_{2r})$   $(\Lambda_{2r} \neq 0)$ . We conjecture that the rank  $r$  vertex operator  $\Phi_{\Delta, \Lambda}^{[r], \lambda}(z): V_{\Delta}^{[0]} \rightarrow V_{\Lambda}^{[r]}$  such that

$$\Phi_{\Delta, \Lambda}^{[r], \lambda}(z)|\Delta\rangle = z^{\alpha} \exp\left(\sum_{n=1}^r \frac{\beta_n}{z^n}\right) \sum_{m=0}^{\infty} v_m|\Lambda\rangle z^m$$

exists uniquely, where  $v_0 = 1$ ,  $v_m|\Lambda\rangle \in V_{\Lambda}^{[r]}$   $(m \geq 1)$ ,  $\beta_r = a\lambda_r$   $(a \in \mathbb{C})$ ,  $\alpha = \alpha(\lambda_0, a, \Delta)$ ,  $\beta_i(\lambda_0, \dots, \lambda_r, a, \Delta)$   $(i = 1, \dots, r - 1)$ , and

$$\Lambda_n = \frac{1}{2} \sum_{i=0}^r \lambda_i \lambda_{n-i} + \delta_{n,r}(-1)^{r+1} r \beta_r \lambda_r \quad (n = r, \dots, 2r),$$

$$D_i(\beta_k) = (-1)^i (k + i) \beta_{k+i} \quad (k = 0, 1, \dots, r),$$

with  $D_i = \sum_{j=1}^{r-i} j \lambda_{j+i} \partial / \partial \lambda_j$ . Below, we present  $\alpha$ ,  $\beta_i$   $(i = 1, \dots, r - 1)$ , a few terms of  $v_m$ .

**a. Rank zero case**

Set  $\Delta_1 = \Delta$ ,  $\Delta_2 = \lambda_0^2/2 - \lambda_0\rho$ , and  $\Delta_3 = \Lambda_0$ .

$$\alpha = \Delta_3 - \Delta_2 - \Delta_1,$$

$$v_1 = \frac{(-\Delta_1 + \Delta_2 + \Delta_3)}{2\Delta_3} L_{-1},$$

$$v_2 = \frac{c(\Delta_1 - \Delta_2)^2 - \Delta_3(c\Delta_3 + 8\Delta_3^2 + 12\Delta_2) - (\Delta_1 - \Delta_2 - \Delta_3)(2c\Delta_3 + c + 16\Delta_3^2 - 4\Delta_3) + 8\Delta_3(\Delta_1 - \Delta_2)^2}{4\Delta_3(2c\Delta_3 + c + 16\Delta_3^2 - 10\Delta_3)} L_{-1}^2$$

$$- \frac{3(\Delta_1 - \Delta_2)^2 - 3\Delta_3^2 - (\Delta_1 + \Delta_2 - \Delta_3)(2\Delta_3 + 1)}{2c\Delta_3 + c + 16\Delta_3^2 - 10\Delta_3} L_{-2}.$$

**b. Rank one case**

$$\alpha = a^2 - 2\Delta + 2a\rho + a\lambda_0,$$

$$v_1 = \frac{(a^2 + \alpha)(\alpha + 2\Delta)}{2a\lambda_1} - \frac{a}{\lambda_1} L_0 + L_{-1},$$

$$v_2 = \frac{a^2}{2\lambda_1^2} L_0^2 - \frac{a}{\lambda_1} L_{-1} L_0 + \frac{b_{21}}{\lambda_1^2} + \frac{1}{2} L_{-1}^2 + \frac{a^2 + (a^2 + \alpha)(\alpha + 2\Delta)}{2a\lambda_1} L_{-1} - \frac{a^2 + (a^2 + \alpha)(1 + \alpha + 2\Delta)}{2\lambda_1^2} L_0,$$

$$v_3 = \frac{1}{6} L_{-1}^3 - \frac{a^3}{5\lambda_1^3} L_0^3 + \frac{a^2}{2\lambda_1^2} L_{-1} L_0^2 - \frac{a}{3\lambda_1} L_{-2} - \frac{a}{2\lambda_1} L_{-1}^2 L_0 + \frac{a(3(a^2 + \alpha)(\alpha + 2\Delta + 2) + 2(3a^2 + 1))}{12\lambda_1^3} L_0^2$$

$$+ \frac{b_{32}}{\lambda_1^3} L_0 - \frac{(2a^2 + (a^2 + \alpha)(1 + \alpha + 2\Delta))}{2\lambda_1^2} L_{-1} L_0 + \frac{b_{34}}{\lambda_1^2} L_{-1} + \frac{(2a^2 + (a^2 + \alpha)(\alpha + 2\Delta))}{4a\lambda_1} L_{-1}^2 + \frac{b_{36}}{\lambda_1^3},$$

where  $b_{ij}$  are polynomials in  $a$ ,  $\lambda_0$ ,  $\Delta$ ,  $\rho$ .

**c. Rank two case**

$$\begin{aligned} \alpha &= 6a^2 - 3\Delta - 6a\rho - 2a\lambda_0, \quad \beta_1 = -2a\lambda_1, \\ v_1 &= L_{-1} + \frac{2a}{\lambda_2}L_1 + \frac{(\alpha + 2\Delta)\lambda_1}{\lambda_2}, \\ v_2 &= \frac{1}{2}L_{-1}^2 + \frac{2a^2}{\lambda_2^2}L_1^2 + \frac{a\lambda_1(1 + 2\alpha + 4\Delta)}{\lambda_2^2}L_1 + \frac{\lambda_1^2(\alpha + 2\Delta)(1 + \alpha + 2\Delta)}{2\lambda_2^2} \\ &\quad + \frac{-12a^4 + a^2(12\alpha - c + 36\Delta + 1) + \alpha^2 + 2\alpha\Delta - 3\Delta^2}{8a\lambda_2} + \frac{a}{\lambda_2}L_0 \\ &\quad + \frac{2a}{\lambda_2}L_{-1}L_1 + \frac{\lambda_1(\alpha + 2\Delta)}{\lambda_2}L_{-1}, \\ v_3 &= \frac{1}{6}L_{-1}^3 + \frac{4a^3}{3\lambda_2^3}L_1^3 + \frac{2a^2\lambda_1(1 + \alpha + 2\Delta)}{\lambda_2^2}L_1^2 - \frac{2a^2}{\lambda_2^2}L_0L_1 + \frac{2a^2}{\lambda_2^2}L_{-1}L_1^2 - \frac{a}{\lambda_2}L_{-1}L_0 + \frac{a}{\lambda_2}L_{-1}^2L_1 \\ &\quad + \left( b_{31}\frac{\lambda_1^2}{\lambda_2^3} + b_{32}\frac{1}{\lambda_2^2} \right) L_1 - \frac{a\lambda_1(2 + 3\alpha + 6\Delta)}{3\lambda_2^2}L_0 + \frac{a\lambda_1(1 + 2\alpha + 4\Delta)}{\lambda_2^2}L_{-1}L_1 \\ &\quad + b_{33}\frac{\lambda_1^3}{\lambda_2^3} + b_{34}\frac{\lambda_1}{\lambda_2^2} + \left( b_{35}\frac{\lambda_1^2}{\lambda_2^2} + b_{36}\frac{1}{\lambda_2} \right) L_{-1} + \frac{\lambda_1(\alpha + 2\Delta)}{2\lambda_2}L_{-1}^2, \end{aligned}$$

where  $b_{ij}$  are polynomials in  $a, \lambda_0, \Delta, \rho$ .

**d. Rank three case**

$$\begin{aligned} \alpha &= 18a^2 - 4\Delta + 12a\rho + 3a\lambda_0, \quad \beta_1 = 3a\lambda_1, \quad \beta_2 = -\frac{3}{2}a\lambda_2, \\ v_1 &= L_{-1} - \frac{3a}{\lambda_3}L_2 + \frac{3a\lambda_1^2 + (2\alpha - 9a^2 + 6\Delta)\lambda_2}{2\lambda_3}, \\ v_2 &= \frac{1}{2}L_{-1}^2 - \frac{3a}{\lambda_3}L_{-1}L_2 + \frac{3a}{2\lambda_3}L_1 + \frac{9a^2}{2\lambda_3^2}L_2^2 + \frac{12a\lambda_1^2 - 4(9a^2 - 2\alpha - 6\Delta)\lambda_2}{8\lambda_3}L_{-1} \\ &\quad - \frac{36a^2\lambda_1^2 + 12a(1 - 9a^2 + 2\alpha + 6\Delta)\lambda_2}{8\lambda_3^2}L_2 + \frac{b_1\lambda_1^4 + b_2\lambda_1^2\lambda_2 + b_3\lambda_2^2}{\lambda_3^2} + \frac{b_4\lambda_1}{\lambda_3}, \\ v_3 &= -\frac{9a^3}{2\lambda_3^3}L_2^3 - \frac{9a^2}{2\lambda_3^2}L_1L_2 + \frac{9a^2}{2\lambda_3^2}L_{-1}L_2^2 + \frac{3a}{2\lambda_3}L_{-1}L_1 - \frac{3a}{2\lambda_3}L_{-1}^2L_2 + \frac{1}{6}L_{-1}^3 - \frac{a}{\lambda_3}L_0 \\ &\quad + \frac{9a^2(3a\lambda_1^2 + (2 - 9a^2 + 2\alpha + 6\Delta)\lambda_2)}{4\lambda_3^3}L_2^2 + \left( \frac{b_5\lambda_1^4 + b_6\lambda_1^2\lambda_2 + b_7\lambda_2^2}{\lambda_3^3} + \frac{b_8\lambda_1}{\lambda_3^2} \right) L_2 \\ &\quad + \frac{3a(-3a\lambda_1^2 + (-1 + 9a^2 - 2\alpha - 6\Delta)\lambda_2)}{2\lambda_3^2}L_{-1}L_2 + \frac{a(9a\lambda_1^2 + (4 + 27a^2 + 6\alpha + 18\Delta)\lambda_2)}{4\lambda_3^2}L_1 \\ &\quad + \left( \frac{b_9\lambda_1^4 + b_{10}\lambda_1^2\lambda_2 + b_{11}\lambda_2^2}{\lambda_3^2} + \frac{b_{12}\lambda_1}{\lambda_3} \right) L_{-1} + \frac{3a\lambda_1^2 + (-9a^2 + 2\alpha + 6\Delta)\lambda_2}{4\lambda_3}L_{-1}^2 \\ &\quad + \frac{b_{13}\lambda_1^6 + b_{14}\lambda_1^4\lambda_2 + b_{15}\lambda_1^2\lambda_2^2 + b_{16}\lambda_2^3}{\lambda_3^3} + \frac{b_{17}\lambda_1^3 + b_{18}\lambda_1\lambda_2}{\lambda_3^2}. \end{aligned}$$

**2. Vertex operators from an irregular Verma module to an irregular Verma module**

Let  $\Lambda = (\Lambda_r, \dots, \Lambda_{2r})$  ( $\Lambda_{2r} \neq 0$ ). The rank zero vertex operator  $\Phi_{\Lambda, \Lambda}^\Delta(z): V_\Lambda^{[r]} \rightarrow V_{\Lambda'}^{[r]}$  is defined by the commutation relations (2.5) and

$$\Phi_{\Lambda, \Lambda'}^\Lambda(z|\Lambda) = z^\alpha \exp\left(\sum_{n=1}^r \frac{\beta_n}{z^n}\right) \sum_{m=0}^\infty v_m |\Lambda'\rangle z^m,$$

where  $v_0 = 1, v_m |\Lambda'\rangle \in V_{\Lambda'}^{[r]}$  ( $m \geq 1$ ). Below, we present  $\alpha, \beta_n$  ( $n = 1, \dots, r - 1$ ), a few terms of  $v_m$ .

**a. Rank one case**

$$\begin{aligned} \alpha &= -\frac{\beta_1(\Lambda_1 - \beta_1)}{2\Lambda_2} - 2\Delta, \\ v_1 &= -\frac{\beta_1}{2\Lambda_2} L_0 + \frac{4\Delta\Lambda_1\Lambda_2 - 4\Delta\beta_1\Lambda_2 - 3\beta_1^2\Lambda_1 + 2\beta_1^3}{8\Lambda_2^2}, \\ v_2 &= \frac{\beta_1^2}{8\Lambda_2^2} L_0^2 - \frac{\beta_1}{4\Lambda_2} L_{-1} - \frac{-8\Delta\Lambda_2^2 - 2(1 - 2\Delta)\beta_1\Lambda_1\Lambda_2 + 2(3 - 2\Delta)\beta_1^2\Lambda_2 + \beta_1^2\Lambda_1^2 - 3\beta_1^3\Lambda_1 + 2\beta_1^4}{16\Lambda_2^3} L_0 \\ &\quad + (48\Delta(\Delta - 2)\Lambda_1^2\Lambda_2^2 + 8(c - 1 + 30\Delta - 12\Delta^2)\beta_1\Lambda_1\Lambda_2^2 + 12(2\Delta - 1)\beta_1\Lambda_1^3\Lambda_2 \\ &\quad - 8(c - 1 + 18\Delta - 6\Delta^2)\beta_1^2\Lambda_2^2 - 24(4\Delta - 3)\beta_1^2\Lambda_1^2\Lambda_2 + 3\beta_1^2\Lambda_1^4 + 120(\Delta - 1)\beta_1^3\Lambda_1\Lambda_2 \\ &\quad - 18\beta_1^3\Lambda_1^3 - 12(4\Delta - 5)\beta_1^4\Lambda_2 + 39\beta_1^4\Lambda_1^2 - 36\beta_1^5\Lambda_1 + 12\beta_1^6) \frac{1}{384\Lambda_2^4}, \\ v_3 &= -\frac{\beta_1}{6\Lambda_2} L_{-2} + \frac{\beta_1^2}{8\Lambda_2^2} L_{-1} L_0 - \frac{\beta_1^3}{48\Lambda_2^3} L_0^3 + \frac{1}{\Lambda_2^3} \sum_{i+j+2k=4} c_{ijk}^{(-1)} \beta_1^i \Lambda_1^j \Lambda_2^k L_{-1} + \frac{1}{\Lambda_2^4} \sum_{i+j+2k=5} c_{ijk}^{(0,0)} \beta_1^i \Lambda_1^j \Lambda_2^k L_0^2 \\ &\quad + \frac{1}{\Lambda_2^5} \sum_{i+j+2k=7} c_{ijk}^{(0)} \beta_1^i \Lambda_1^j \Lambda_2^k L_0 + \frac{1}{\Lambda_2^6} \sum_{i+j+2k=9} c_{ijk} \beta_1^i \Lambda_1^j \Lambda_2^k, \end{aligned}$$

where  $c_{ijk}^{(l)}$  are polynomials in  $c, \Delta$ .

**b. Rank two case**

$$\begin{aligned} \alpha &= \frac{\beta_2\Lambda_3^2}{4\Lambda_4^2} + \frac{3\beta_2^2}{\Lambda_4} - \frac{\beta_2\Lambda_2}{\Lambda_4} - 3\Delta, \quad \beta_1 = \frac{\beta_2\Lambda_3}{\Lambda_4}, \\ v_1 &= b_{10} - \frac{\beta_2}{\Lambda_4} L_1, \\ v_2 &= \frac{\beta_2^2}{2\Lambda_4^2} L_1^2 - \frac{\beta_2}{2\Lambda_4} L_0 + \frac{\beta_2}{\Lambda_4} \left(\frac{\Lambda_3}{4\Lambda_4} - b_{10}\right) L_1 + b_{20}, \\ v_3 &= \frac{\beta_2^2}{2\Lambda_4^2} L_0 L_1 - \frac{\beta_2}{3\Lambda_4} L_{-1} - \frac{\beta_2^3}{6\Lambda_4^3} L_1^3 - \frac{\beta_2^2}{4\Lambda_4^3} (\Lambda_3 - 2b_{10}\Lambda_4) L_1^2 + \frac{\beta_2}{6\Lambda_4^2} (\Lambda_3 - 3b_{10}\Lambda_4) L_0 \\ &\quad + \left(\frac{b_{10}\beta_2\Lambda_3}{4\Lambda_4^2} - \frac{\beta_2(2\Lambda_2 - \beta_2)}{6\Lambda_4^2} + \frac{b_{10}}{\Lambda_3} - \frac{b_{20}\beta_2}{\Lambda_4}\right) L_1 + b_{30}. \end{aligned}$$

Here,

$$\begin{aligned} b_{10} &= \frac{\Lambda_3}{\Lambda_4} \left(\frac{\Delta}{2} - \frac{\beta_2\Lambda_3^2}{8\Lambda_4^2} - \frac{3\beta_2^2}{2\Lambda_4} + \frac{\beta_2\Lambda_2}{2\Lambda_4}\right), \\ b_{20} &= \frac{1}{\Lambda_4^6} \sum_{2i+2j+3k+4l=22} b_{ijkl}^{(20)} \beta_2^i \Lambda_2^j \Lambda_3^k \Lambda_4^l, \quad b_{30} = \frac{1}{\Lambda_4^9} \sum_{2i+2j+3k+4l=33} b_{ijkl}^{(30)} \beta_2^i \Lambda_2^j \Lambda_3^k \Lambda_4^l, \end{aligned}$$

where  $b_{ijkl}^{(20)}, b_{ijkl}^{(30)}$  are polynomials in  $c, \Delta$ .



### c. Rank three case

$$\alpha = \frac{3\beta_3\Lambda_4\Lambda_5}{\Lambda_6^2} - \frac{3\beta_3\Lambda_5^3}{16\Lambda_6^3} - \frac{3\beta_3\Lambda_3}{2\Lambda_6} - 4\Delta, \quad \beta_1 = \frac{3\beta_3\Lambda_4}{2\Lambda_6} - \frac{3\beta_3\Lambda_5^2}{8\Lambda_6^2}, \quad \beta_2 = \frac{3\beta_3\Lambda_5}{4\Lambda_6},$$

$$v_1 = b_{10} - \frac{3\beta_3}{2\Lambda_6}L_2,$$

$$v_2 = b_{20} - \frac{3\beta_3}{4\Lambda_6}L_1 + \frac{9\beta_3^2}{8\Lambda_6^2}L_2^2 - \frac{3\beta_3}{8\Lambda_6^2}(4b_{10}\Lambda_6 - \Lambda_5)L_2,$$

$$v_3 = b_{30} - \frac{\beta_3}{2\Lambda_6}L_0 + \frac{9\beta_3^2}{8\Lambda_6^2}L_1L_2 + \frac{9\beta_3^2}{16\Lambda_6^3}(2b_{10}\Lambda_6 - \Lambda_5)L_2^2 - \frac{9\beta_3^3}{16\Lambda_6^3}L_2^3 - \frac{\beta_3}{4\Lambda_6^2}(3b_{10}\Lambda_6 - \Lambda_5)L_1$$

$$- \frac{\beta_3}{16\Lambda_6^3}(3\Lambda_5(\Lambda_5 - 2b_{10}\Lambda_6) + 4\Lambda_6(6b_{20}\Lambda_6 - \Lambda_4))L_2.$$

Here,

$$b_{10} = \frac{15\beta_3\Lambda_5^4}{128\Lambda_6^4} - \frac{9\beta_3\Lambda_4\Lambda_5^2}{16\Lambda_6^3} - \frac{27\beta_3^2\Lambda_5}{8\Lambda_6^2} + \frac{3\beta_3\Lambda_4^2}{8\Lambda_6^2} + \frac{\Delta\Lambda_5}{2\Lambda_6} + \frac{3\beta_3\Lambda_3\Lambda_5}{4\Lambda_6^2},$$

$$b_{20} = \frac{1}{\Lambda_6^8} \sum_{3i+3j+4k+5l+6m=46} b_{ijklm}^{(20)} \beta_3^i \Lambda_3^j \Lambda_4^k \Lambda_5^l \Lambda_6^m, \quad b_{30} = \frac{1}{\Lambda_6^{12}} \sum_{3i+3j+4k+5l+6m=69} b_{ijklm}^{(30)} \beta_3^i \Lambda_3^j \Lambda_4^k \Lambda_5^l \Lambda_6^m,$$

where  $b_{ijklm}^{(20)}$ ,  $b_{ijklm}^{(30)}$  are polynomials in  $c$ ,  $\Delta$ .

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