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# Path Integral Representation for Schrödinger Operators with Bernstein Functions of the Laplacian

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**Abstract**

Path integral representations for generalized Schrödinger operators obtained under a class of Bernstein functions of the Laplacian are established. The one-to-one correspondence of Bernstein functions with Lévy subordinators is used, thereby the role of Brownian motion entering the standard Feynman-Kac formula is taken here by subordinate Brownian motion. As specific examples, fractional and relativistic Schrödinger operators with magnetic field and spin are covered. Results on self-adjointness of these operators are obtained under conditions allowing for singular magnetic fields and singular external potentials as well as arbitrary integer and half-integer spin values. This approach also allows to propose a notion of generalized Kato class for which an $L^p$-$L^q$ bound of the associated generalized Schrödinger semigroup is shown. As a consequence, diamagnetic and energy comparison inequalities are also derived.

# 1 Introduction

## 1.1 Context and motivation

Feynman-Kac-type formulae proved to be a useful device in the analysis of spectral properties of a wide class of self-adjoint operators. Besides their prolific uses in the physics literature, functional integration poses remarkable new mathematical problems which can be addressed in terms of modern stochastic analysis.

The Feynman-Kac formula is a functional integral representation of the kernel of the semigroup generated by the Schrödinger operator

$$H = \frac{1}{2}\mathrm{p}^2 + V, \tag{1.1}$$

for which it was originally derived. Here $\mathrm{p} = -i\nabla$ is the momentum operator and $V$ is a real-valued potential. The Laplacian gives rise to an integral representation of the kernel of $e^{-tH}$ in terms of the Wiener measure, while $V$ introduces a density with respect to it. This implies that the ground state and various other properties of $H$ can be analyzed by running a Brownian motion under the potential $V$. Standard references on applications to the spectral analysis of Schrödinger operators include [Lie73, Lie80, Shi87, Sim82], with updated bibliography in [Sim04]. We also refer to [DC00] for an approach with the Feynman-Kac formula. While functional integration can be extended to include several other operators also covering quantum field models (see [LHB09] and references therein), the analysis based on random processes having almost surely continuous paths remained a basic feature.

In the mathematical physics literature there appear to be relatively few systematic attempts in going beyond continuous paths to replace them with *càdlàg* paths (right-continuous with left limits), also allowing jump discontinuities. On the other hand, such more general Lévy processes than Brownian motion prove to be useful in describing important features such as spin in terms of path measures. Another source of problems leading to paths with jump discontinuities are models featuring fractional Laplacians.

The aim of the present paper is to construct path integral representations for generalized Schrödinger operators including both non-relativistic and relativistic Schrödinger operators with vector potentials and spin. We propose a thorough study of this problem, extending the methods developed in [HL08] to the case of Lévy processes with *càdlàg* paths.

By a generalized Schrödinger operator here we mean a Schrödinger operator in which the Laplacian is replaced by a suitable pseudo-differential operator. Namely, instead of the operator

$$\frac{1}{2}(\sigma \cdot (\mathrm{p} - a))^2 + V \tag{1.2}$$

studied in [HL08], where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices and $a$ is a vector potential, we consider a class of general self-adjoint operators of the form

$$\Psi\left(\frac{1}{2}(\sigma \cdot (\mathrm{p} - a))^2\right) + V, \tag{1.3}$$

where $\Psi$ is a Bernstein function on the positive semi-axis (see below). In particular, this class includes not only relativistic Schrödinger operators

$$\sqrt{(\sigma \cdot (\mathrm{p} - a))^2 + m^2} - m + V \tag{1.4}$$

but also more general fractional Schrödinger operators

$$\left(\frac{1}{2}(\sigma \cdot (\mathrm{p} - a))^2\right)^{\alpha/2} + V, \tag{1.5}$$

with $\alpha \in (0, 2)$. The vector potential plays the role of magnetic field in appropriate contexts, however, we will use this terminology for all cases we consider, even when they may have other interpretations.

The application of functional integral techniques to relativistic Schrödinger operators, without magnetic field or spin, has been earlier on addressed in [CMS90]. The process involved is closely related to 1/2-stable processes, which can be understood in terms of a first hitting time process of Brownian motion. In the interesting papers [ALS83, ARS91] a path integral for relativistic Schrödinger operators with vector potential and spin 1/2 is presented, however, in a non-rigorous language. A functional integral representation also has been established for the Schrödinger semigroup with vector potential in [ITa86], applied in [Ich87] and completed in [Ich94], where, however, the operator concerned was a pseudo-differential operator associated with the symbol of the classical relativistic Hamiltonian defined through Weyl quantization. It should be noted that the terms in (1.3)–(1.5) involving a vector potential cannot be defined as pseudo-differential operators associated with simple and plain symbols. A further step has been made by addressing various problems of potential theory and heat kernel estimates of more general $\alpha$-stable processes [BB99, BJ07, BKM06, CS97, Ryz02, KS06, GR07, B09, KL10]; see also the influential early work [Bak87] involving the Cauchy process. Such processes relate with fractional Schrödinger operators

$$\left(\frac{1}{2}\mathrm{p}^2\right)^{\alpha/2} + V, \quad 0 < \alpha < 2, \tag{1.6}$$

and are motivated by further models of physics, chemistry, biology and, more recently, financial mathematics [BG90, BBACT02, EK95, MK04].

Fractional Schrödinger operators and stable processes provide just one special case of a sensible class of extensions. In the present paper we consider generalized Schrödinger operators obtained as Bernstein functions of the Laplacian to which we add an external potential $V$ and, in various versions, a vector potential and a contribution from a spin operator. In a sense, this is the greatest desirable generality as Bernstein functions with vanishing right limits at the origin stand in a one-to-one correspondence with Lévy subordinators. So our contribution may be more to mathematics than to physics. However, as will be seen in this paper, it is more natural to consider the path integral for generalized Schrödinger operators than the relativistic Schrödinger operators with spin themselves. Subordinators are random processes with jump discontinuities and can be uniquely described by specifying two parameters, the Lévy measure accounting for the jumps, and the drift function accounting for the continuous component of the paths. Given a Bernstein function $\Psi$ and a generalized Schrödinger operator $H^\Psi$ thereby obtained, the properties of the semigroup $e^{-tH^\Psi}$ can now be analyzed in terms of subordinate Brownian motion $B_{T_t^\Psi}$. Here $T_t^\Psi$ is the subordinator uniquely associated with $\Psi$. Roughly speaking, $B_{T_t^\Psi}$ is a *càdlàg* process which samples Brownian paths at random times distributed by the law of $T_t^\Psi$.

## 1.2 Main results

Throughout this paper we will use the following conditions on the vector potential.

**Assumption 1.1** *The vector potential $a = (a_1, ..., a_d)$ is a vector-valued function whose components $a_\mu$, $\mu = 1, ..., d$, are real-valued functions. Furthermore, we consider the following regularity conditions:*

**(A1)** $a \in (L^2_{\rm loc}(\mathbb{R}^d))^d$.

**(A2)** $a \in (L^2_{\rm loc}(\mathbb{R}^d))^d$ *and* $\nabla \cdot a \in L^1_{\rm loc}(\mathbb{R}^d)$.

**(A3)** $a \in (L^4_{\rm loc}(\mathbb{R}^d))^d$ *and* $\nabla \cdot a \in L^2_{\rm loc}(\mathbb{R}^d)$.

**(A4)** $d = 3$, $a \in (L^4_{\rm loc}(\mathbb{R}^3))^3$, $\nabla \cdot a \in L^2_{\rm loc}(\mathbb{R}^3)$ *and* $\nabla \times a \in (L^2_{\rm loc}(\mathbb{R}^3))^3$.

Since we discuss several variants of Schrödinger operators, different by whether they do or do not include spin, it is appropriate to explain here the notation. We define the spinless operator through a quadratic form for $a$ satisfying (A1) and denote it by

$$h = \frac{1}{2}(\mathrm{p} - a)^2 \qquad \text{(with no spin).} \tag{1.7}$$

We also define a Schrödinger operator with spin 1/2 through a quadratic form and will be denote it by

$$h_{1/2} = \frac{1}{2}(\sigma \cdot (\mathrm{p} - a))^2 \qquad \text{(with spin)}. \tag{1.8}$$

Using a suitable unitary map, we transform $h_{1/2}$ on the space $L^2(\mathbb{R}^3; \mathbb{C}^2) = L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$ to a self-adjoint operator $h_{\mathbb{Z}_2}$ on $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$. Here $\mathbb{Z}_2 = \{-1, 1\}$ describes the state space of a two-valued spin variable. Furthermore, we generalize spin from $\mathbb{Z}_2$ to $\mathbb{Z}_p$ and denote the so obtained Schrödinger operator by

$$h_{\mathbb{Z}_p} \qquad \text{(with generalized spin)} \tag{1.9}$$

acting on $L^2(\mathbb{R}^d \times \mathbb{Z}_p)$, for $d \geq 1$ and $p \geq 2$. The relativistic versions of (1.7) and (1.8) will be denoted by

$$\begin{aligned} h^{\mathrm{rel}} &= \sqrt{(\mathrm{p} - a)^2 + m^2} - m, \quad m \geq 0, \\ h^{\mathrm{rel}}_{1/2} &= \sqrt{(\sigma \cdot (\mathrm{p} - a))^2 + m^2} - m, \quad m \geq 0. \end{aligned} \tag{1.10}$$

In this paper we will consider generalized versions of (1.10). Let $\Psi$ be a Bernstein function. Our main objects are

$$\begin{aligned} H^{\Psi} &= \Psi(h) + V \qquad \text{(with no spin)}, \\ H^{\Psi}_{\mathbb{Z}_p} &= \Psi(h_{\mathbb{Z}_p}) + V \quad \text{(with generalized spin)}. \end{aligned} \tag{1.11}$$

In particular,

$$\Psi(u) = \sqrt{2u + m^2} - m$$

corresponds to (1.10). Under Assumptions (A2) (resp. (A3)), we will show that $C_0^\infty(\mathbb{R}^d)$ is a form core (resp. operator core) of both $\Psi(h)$ and $\Psi(h_{\mathbb{Z}_p})$. This is the content of Theorem 3.3 below.

The key results of this paper are the functional integral representations of $e^{-tH^{\Psi}}$ and $e^{-tH^{\Psi}_{\mathbb{Z}_p}}$ derived under Assumption (A2) for bounded potentials $V$. They are presented in Theorems 3.8 and 5.10, respectively. These are then further generalized to more singular potentials in Theorems 3.14 and 5.14. Recall that the standard Feynman-Kac-Itô formula says that

$$(f, e^{-t(h+V)}g) = \int_{\mathbb{R}^d} dx\, \mathbb{E}^x_P \left[ \overline{f(B_0)} g(B_t) e^{-i\int_0^t a(B_s)\circ dB_s} e^{-\int_0^t V(B_s)ds} \right], \tag{1.12}$$

with $d$-dimensional Brownian motion $(B_t)_{t\geq 0}$ on Wiener space $(\Omega_P, \mathscr{F}_P, P)$, where the stochastic integral in the exponent is to be interpreted as a Stratonovich integral. For $H^{\Psi} = \Psi(h) + V$ this formula modifies to (see Theorem 3.14 below)

$$(f, e^{-t(\Psi(h)+V)}g) = \int_{\mathbb{R}^d} dx\, \mathbb{E}^{x,0}_{P\times\nu} \left[ \overline{f(B_0)} g(B_{T^{\Psi}_t}) e^{-i\int_0^{T^{\Psi}_t} a(B_s)\circ dB_s} e^{-\int_0^t V(B_{T^{\Psi}_s})ds} \right], \tag{1.13}$$

where $T_t^\Psi$ is the Lévy subordinator on a probability space $(\Omega_\nu, \mathscr{F}_\nu, \nu)$ associated with $\Psi$. In particular, it should be noted that the integrands change as

$$\exp\left(-i\int_0^t a(B_s)\circ dB_s\right) \to \exp\left(-i\int_0^{T_t^\Psi} a(B_s)\circ dB_s\right)$$

and

$$\exp\left(-\int_0^t V(B_s)ds\right) \to \exp\left(-\int_0^t V(B_{T_s^\Psi})ds\right).$$

A similar situation occurs in the case including a generalized spin, see Theorem 5.14 below. By means of these formulae we are able to extend the definition of generalized Schrödinger operators $H^\Psi$ and $H^\Psi_{\mathbb{Z}_p}$ to the case of external potentials having singularities.

Having the functional integral representations at hand allows us to construct a strongly continuous symmetric Feynman-Kac semigroup for a large class of potentials $V$ which we call $\Psi$-Kato class. This will be dealt with in Theorem 4.8. Extension of the standard Kato-class is also derived in e.g., [CMS90, Zha91] The generator of this semigroup can be identified as a self-adjoint operator, which we denote by

$$K^\Psi \quad \text{(with } \Psi\text{-Kato class potential).} \tag{1.14}$$

This offers then a notion of generalized Schrödinger operator with vector potential for $\Psi$-Kato potentials. As a further result, in Theorem 4.11 we show that the semigroup $e^{-tK^\Psi}$ is $L^p$-$L^q$ bounded for $1 \le p \le q \le \infty$.

Our results improve and generalize those of [BHL00, CMS90, ITa86, ALS83, ARS91, Sim82, GV81]. Further applications to relativistic quantum field theory are discussed in [HS09, Lor09a, Lor09b].

The paper is organized as follows. In Section 2 we discuss the details of the relationship between Bernstein functions $\Psi$ and Lévy subordinators $(T_t^\Psi)_{t\ge0}$. In Section 3 we consider the spinless case. We establish the functional integral representation for their semigroup and obtain diamagnetic inequalities. Furthermore, we show essential self-adjointness of $\Psi(h)$ on $C_0^\infty(\mathbb{R}^d)$. In Section 4, we define the space of $\Psi$-Kato class potentials and discuss their relationship with the Lévy measure of the associated subordinators. In addition we prove that the generalized Schrödinger semigroups obtained for the $\Psi$-Kato class is $L^p$-$L^q$ bounded for $1 \le p \le q \le \infty$. In Section 5 we consider generalized Schrödinger operators with spin. We extend $\pm1$ spins to spins of $p$ possible orientations by describing them in terms of the cyclic group of the $p$th roots of unity. This gives rise to a random process driven by a weighted sum of $p$ independent Poisson variables of intensity 1. As a corollary, we derive diamagnetic inequalities.

## 2 Bernstein functions and Lévy subordinators

We start by considering some basic facts on Bernstein functions and their connection with subordinators. For standard definitions and results on Bernstein functions we refer to [Boc55, BF73, SSV10], for Lévy processes to [Sat99], to [Ber99] for a detailed study on subordinators, and to [Huf69] for details on subordinate Brownian motion.

Bernstein functions appear in the analysis of convolution semigroups, in particular they are a key concept in Bochner's theory of subordination.

**Definition 2.1 (Bernstein function)** Let

$$\mathscr{B} = \left\{ f \in C^{\infty}((0,\infty)) \,\middle|\, f(x) \geq 0 \;\text{ and }\; (-1)^n \left(\frac{d^n f}{dx^n}\right)(x) \leq 0 \;\text{ for all }\; n = 1, 2, ..., \right\}.$$

An element of $\mathscr{B}$ is called a *Bernstein function*. We also define the subclass

$$\mathscr{B}_0 = \left\{ f \in \mathscr{B} \,\middle|\, \lim_{u \to 0+} f(u) = 0 \right\}.$$

Bernstein functions are positive, increasing and concave. $\mathscr{B}$ is a convex cone containing the nonnegative constants. Examples of functions in $\mathscr{B}_0$ include $\Psi(u) = cu^{\alpha}$, $c \geq 0$, $0 < \alpha \leq 1$, and $\Psi(u) = 1 - e^{-au}$, $a \geq 0$.

A real-valued function $f$ on $(0, \infty)$ is a Bernstein function if and only if $g_t = e^{-tf}$ is a completely monotone function for all $t > 0$, i.e., exactly when $(-1)^n \dfrac{d^n g_t}{dx^n} \geq 0$, for all integers $n \geq 0$. On the other hand, a result by Bernstein says that a function is completely monotone if and only if it is the Laplace transform of a positive measure, which for each such function is unique. This leads to the following integral representation of Bernstein functions.

**Definition 2.2 (Class $\mathscr{L}$)** Let $\mathscr{L}$ be the set of Borel measures $\lambda$ on $\mathbb{R} \setminus \{0\}$ such that

(1) $\lambda((-\infty, 0)) = 0$;

(2) $\displaystyle\int_{\mathbb{R}\setminus\{0\}} (y \wedge 1)\lambda(dy) < \infty$.

Note that each $\lambda \in \mathscr{L}$ satisfies that $\int_{\mathbb{R}\setminus\{0\}}(y^2 \wedge 1)\lambda(dy) < \infty$ so that $\lambda$ is a Lévy measure.

Denote $\mathbb{R}_+ = [0, \infty)$. In the following proposition we give the integral representation of Bernstein functions with vanishing right limits at the origin.

**Proposition 2.3** *For every Bernstein function* $\Psi \in \mathscr{B}_0$ *there exists* $(b, \lambda) \in \mathbb{R}_+ \times \mathscr{L}$ *such that*

$$\Psi(u) = bu + \int_0^\infty (1 - e^{-uy})\lambda(dy). \tag{2.1}$$

*Conversely, the right hand side of (2.1) is in* $\mathscr{B}_0$ *for each pair* $(b, \lambda) \in \mathbb{R}_+ \times \mathscr{L}$.

For a given $\Psi \in \mathscr{B}_0$, the constant $b$ is uniquely determined by $b = \lim_{u\to\infty} \Psi(u)/u$. Moreover, since $\frac{d\Psi}{du} = b + \int_0^\infty ye^{-yu}\lambda(dy)$ and $\frac{d\Psi}{du}$ is a completely monotone function, the measure $\lambda$ is also uniquely determined; for details, see [BF73, Theorem 9.8]. Thus the map $\mathscr{B}_0 \to \mathbb{R}_+ \times \mathscr{L}$, $\Psi \mapsto (b, \lambda)$ with $\Psi$ and $(b, \lambda)$ as in (2.1) is a one-to-one correspondence.

Next we consider a probability space $(\Omega_\nu, \mathscr{F}_\nu, \nu)$ given and the following special class of Lévy processes.

**Definition 2.4 (Lévy subordinator)** A random process $(T_t)_{t\geq 0}$ on $(\Omega_\nu, \mathscr{F}_\nu, \nu)$ is called a *(Lévy) subordinator* whenever

(1) $(T_t)_{t\geq 0}$ is a Lévy process starting at 0, i.e., $\nu(T_0 = 0) = 1$;

(2) $T_t$ is almost surely non-decreasing in $t$.

Subordinators have thus independent and stationary increments, almost surely no negative jumps, and are of bounded variation. These properties also imply that they are Markov processes.

Let $\mathscr{S}$ denote the set of subordinators on $(\Omega_\nu, \mathscr{F}_\nu, \nu)$. In what follows we denote expectation by $\mathbb{E}^x_m[\cdots] = \int \cdots dm^x$ with respect to the path measure $m^x$ of a process starting at $x$.

**Proposition 2.5** *Let* $\Psi \in \mathscr{B}_0$ *or, equivalently, a pair* $(b, \lambda) \in \mathbb{R}_+ \times \mathscr{L}$ *be given. Then there exists a unique* $(T_t)_{t\geq 0} \in \mathscr{S}$ *such that*

$$\mathbb{E}^0_\nu[e^{-uT_t}] = e^{-t\Psi(u)}. \tag{2.2}$$

*Conversely, let* $(T_t)_{t\geq 0} \in \mathscr{S}$. *Then there exists* $\Psi \in \mathscr{B}_0$, *i.e., a pair* $(b, \lambda) \in \mathbb{R}_+ \times \mathscr{L}$ *such that (2.2) is satisfied.*

In particular, (2.1) coincides with the Lévy-Khintchine formula for Laplace exponents of subordinators.

By the above there is a one-to-one correspondence between $\mathscr{B}_0$ and $\mathscr{S}$, or equivalently, between $\mathscr{B}_0$ and $\mathbb{R}_+ \times \mathscr{L}$. For clarity, we will use the notation $T^\Psi_t$ for the Lévy subordinator associated with $\Psi \in \mathscr{B}_0$.

**Example 2.6 (Stable subordinator)** Let $b = 0$, $0 < \alpha < 2$ and $\lambda \in \mathscr{L}$ be defined by

$$\lambda(dy) = \frac{\alpha/2}{\Gamma(1-\alpha/2)} \frac{1_{(0,\infty)}(y)}{y^{1+\alpha/2}} dy,$$

where $\Gamma$ denotes the Gamma function. Then $\Psi(u) = u^{\alpha/2} \in \mathscr{B}_0$ and the corresponding subordinator $T_t^\Psi$ is given by

$$\mathbb{E}_\nu^0[e^{-uT_t^\Psi}] = e^{-tu^{\alpha/2}}.$$

**Example 2.7 (First hitting time)** Since $\Psi(u) = \sqrt{2u+m^2} - m \in \mathscr{B}_0$ for $m \geq 0$, there exists $T_t^\Psi \in \mathscr{S}$ such that

$$\mathbb{E}_\nu^0[e^{-uT_t^\Psi}] = \exp\left(-t(\sqrt{2u+m^2} - m)\right).$$

This case is thus related to the one-dimensional 1/2-stable process and it is known that the corresponding subordinator $T_t^\Psi$ can be represented as the first hitting time process

$$T_t^\Psi = \inf\{s > 0 \,|\, B_s + ms = t\} \tag{2.3}$$

for one-dimensional Brownian motion $(B_t)_{t\geq 0}$. In this case, moreover, the distribution $\rho(\cdot, t)$ on $\mathbb{R}$ also is known exactly to be [App09]

$$\rho(r,t) = \frac{t}{\sqrt{2\pi r^3}} e^{mt} \exp\left(-\frac{1}{2}\left(\frac{t^2}{r} + m^2 r\right)\right) 1_{[0,\infty)}(r), \quad m \geq 0. \tag{2.4}$$

# 3 Spinless case

## 3.1 Generalized Schrödinger operators with no spin

Now we define the class of generalized Schrödinger operators on $L^2(\mathbb{R}^d)$, which we consider in this paper. In order to cover interactions with a magnetic field we add a vector potential to the momentum operator. Let $\partial_{x_\mu} : \mathscr{D}'(\mathbb{R}^d) \to \mathscr{D}'(\mathbb{R}^d)$, $\mu = 1, ..., d$, denote the derivative on the Schwartz distribution space $\mathscr{D}'(\mathbb{R}^d)$ relative to the $\mu$th coordinate. With the notation $\mathrm{p} = -i\nabla$ and $\nabla = (\partial_{x_1}, \ldots, \partial_{x_d})$, the Schrödinger operator with vector potential $a$ is formally given by $\frac{1}{2}(\mathrm{p} - a)^2$. We will define it as a self-adjoint operator rigorously through a quadratic form.

Let $\mathrm{D}_\mu = \mathrm{p}_\mu - a_\mu$, $\mu = 1, ..., d$, where $\mathrm{p}_\mu = -i\partial_{x_\mu}$. Define the quadratic form

$$q(f,g) = \sum_{\mu=1}^d (\mathrm{D}_\mu f, \mathrm{D}_\mu g) \tag{3.1}$$

with domain

$$Q(q) = \left\{ f \in L^2(\mathbb{R}^d) \,|\, \mathrm{D}_\mu f \in L^2(\mathbb{R}^d),\, \mu = 1, ..., d \right\}. \tag{3.2}$$

It can be shown that, under Assumption (A1), the subspace $Q(q)$ is complete with respect to the norm $\|f\|_q = \sqrt{q(f,f)+\|f\|^2}$, $f \in Q(q)$. Thus $q$ is a non-negative closed form and there exists a unique self-adjoint operator $h$ satisfying

$$(hf,g) = q(f,g), \quad f \in D(h), \quad g \in Q(q), \tag{3.3}$$

with domain

$$D(h) = \left\{ f \in Q(q) \,|\, q(f,\cdot) \in L^2(\mathbb{R}^d)' \right\}. \tag{3.4}$$

The self-adjoint operator $h$ is our main object in this section. We summarize some facts about the form core and operator core of $h$ [LS81].

**Proposition 3.1** *(1) Let Assumption (A1) be satisfied. Then $C_0^\infty(\mathbb{R}^d)$ is a form core of $h$. (2) Let Assumption (A3) be satisfied. Then $C_0^\infty(\mathbb{R}^d)$ is an operator core for $h$.*

Note that in case (2) of Proposition 3.1,

$$hf = \frac{1}{2}\mathrm{p}^2 f - a\cdot \mathrm{p} f + \left(\frac{1}{2}a\cdot a - (\mathrm{p}\cdot a)\right) f.$$

**Definition 3.2** (**Generalized Schrödinger operator with vector potential and bounded** $V$) Let $\Psi \in \mathscr{B}_0$ and take Assumption (A1). Whenever $V$ is a real-valued bounded multiplication operator we call

$$H^\Psi = \Psi(h) + V \tag{3.5}$$

*generalized Schrödinger operator with vector potential $a$.*

Note that $\Psi \geq 0$ and $\Psi(h)$ is defined through the spectral projection of the self-adjoint operator $h$. Furthermore, $H^\Psi$ is self-adjoint on the domain $D(\Psi(h))$ as $V$ is bounded.

## 3.2 Essential self-adjointness

**Theorem 3.3** *Take $\Psi \in \mathscr{B}_0$.*

*(1) Let Assumption (A3) be satisfied. Then $C_0^\infty(\mathbb{R}^d)$ is an operator core of $\Psi(h)$.*

*(2) Let Assumption (A1) be satisfied. Then $C_0^\infty(\mathbb{R}^d)$ is a form core of $\Psi(h)$.*

Proof. (1) Recall the representation (2.1). Since we have $\int_0^1 y\lambda(dy) < \infty$ and $\int_1^\infty \lambda(dy) < \infty$ by Definition 2.2, there exist non-negative constants $c_1$ and $c_2$ such that $\Psi(u) \leq c_1 u + c_2$ for all $u \geq 0$. This gives the bound

$$\|\Psi(h)f\| \leq c_1\|hf\| + c_2\|f\| \tag{3.6}$$

for all $f \in D(h)$. Hence it can be proven that $C_0^\infty(\mathbb{R}^d)$ is contained in $D(\Psi(h))$ and $(\Psi(h)+1)C_0^\infty(\mathbb{R}^d)$ is dense. Then (1) follows.

(2) Note that $\|\Psi(h)^{1/2}f\|^2 \le c_1\|h^{1/2}f\|^2 + c_2\|f\|^2$ for $f \in Q(h) = D(h^{1/2})$, and $C_0^\infty(\mathbb{R}^d)$ is contained in $Q(\Psi(h)) = D(\Psi(h)^{1/2})$. Since $\Psi(h)^{1/2}+1$ has also bounded inverse, it is seen by the same argument as above that $C_0^\infty(\mathbb{R}^d)$ is a core of $\Psi(h)^{1/2}$ or a form core of $\Psi(h)$. **qed**

## 3.3 Singular magnetic fields

Before constructing a functional integral representation of $e^{-th}$, we extend stochastic integration to a class including $L^2_{\rm loc}(\mathbb{R}^d)$ functions since the vector potentials we consider may be more singular than $f$ satisfying (3.7) below.

Let $(B_t)_{t\ge 0}$ denote $d$-dimensional Brownian motion starting at $x \in \mathbb{R}^d$ on standard Wiener space $(\Omega_P, \mathscr{F}_P, P^x)$. Let $f$ be a $\mathbb{C}^d$-valued Borel measurable function on $\mathbb{R}^d$ such that

$$\mathbb{E}_P^x\left[\int_0^t |f(B_s)|^2 ds\right] < \infty. \tag{3.7}$$

Then the stochastic integral $\int_0^t f(B_s)\cdot dB_s$ is defined as a martingale and the Itô isometry

$$\mathbb{E}_P^x\left[\left|\int_0^t f(B_s)\cdot dB_s\right|^2\right] = \mathbb{E}_P^x\left[\int_0^t |f(B_s)|^2 ds\right]$$

holds. However, vector potentials $a$ under (A.1) of Assumption 1.1 do not necessarily satisfy (3.7). As we show next, a stochastic integral can indeed be defined for a wider class of functions than (3.7), and then $\int_0^t f(B_s)\cdot dB_s$ will be defined as a local martingale instead of a martingale. This extension will allow us to derive a functional integral representation of $e^{-th}$ with $a \in \left(L^2_{\rm loc}(\mathbb{R}^d)\right)^d$.

Consider the following class of vector valued functions on $\mathbb{R}^d$.

**Definition 3.4** We say that $f = (f_1, ..., f_d) \in \mathscr{E}_{\rm loc}$ if and only if for almost every $x \in \mathbb{R}^d$, the equality

$$P^x\left(\int_0^t |f(B_s)|^2 ds < \infty\right) = 1 \tag{3.8}$$

holds for all $t \ge 0$.

Let $R_n(\omega) = n \wedge \inf\left\{t \ge 0 \,\middle|\, \int_0^t |f(B_s(\omega))|^2 ds \ge n\right\}$ be a sequence of stopping times with respect to the natural filtration $\mathscr{F}_t^P = \sigma(B_s, 0 \le s \le t)$. Let $1_X$ denote the indicator function on $X$. Define

$$f_n(s,\omega) = f(B_s(\omega))1_{\{R_n(\omega)>s\}}(\omega). \tag{3.9}$$

Each of these functions satisfies $\int_0^\infty |f_n(s,\omega)|^2 ds = \int_0^{R_n} |f_n(s,\omega)|^2 ds \le n$. In particular, we have $\mathbb{E}_P^x\left[\int_0^t |f_n|^2 ds\right] < \infty$ and thus $\int_0^t f_n \cdot dB_s$ is well defined. Moreover, it can be seen that

$$\int_0^{t\wedge R_m} f_n(s,\omega)\cdot dB_s = \int_0^t f_m(s,\omega)\cdot dB_s \tag{3.10}$$

for $m < n$.

**Definition 3.5** For $f \in \mathscr{E}_{\rm loc}$ we define the integral

$$\int_0^t f(B_s)\cdot dB_s = \int_0^t f_n(s,\omega)\cdot dB_s, \quad 0 \le t \le R_n. \tag{3.11}$$

This definition is consistent with (3.10).

**Lemma 3.6** *The space $\mathscr{E}_{\rm loc}$ has the properties below:*

*(1) Let $f \in \mathscr{E}_{\rm loc}$. Suppose that a sequence of step functions $f_n$, $n = 1, 2, \dots$, satisfies $\int_0^t |f_n(B_s) - f(B_s)|^2 ds \to 0$ in probability as $n \to \infty$. Then*

$$\lim_{n\to\infty} \int_0^t f_n(B_s)\cdot dB_s = \int_0^t f(B_s)\cdot dB_s \quad \textit{in probability.}$$

*(2) The following inclusion holds: $(L^2_{\rm loc}(\mathbb{R}^d))^d \subset \mathscr{E}_{\rm loc}$.*

*(3) Let $a \in \left(L^2_{\rm loc}(\mathbb{R}^d)\right)^d$ and $\nabla\cdot a \in L^1_{\rm loc}(\mathbb{R}^d)$. Then*

$$\left|\int_0^t a(B_s)\cdot dB_s + \frac{1}{2}\int_0^t \nabla\cdot a(B_s) ds\right| < \infty \quad \textit{almost surely.}$$

Proof. Property (1) is standard: see e.g., [KS91, Proposition 2.26, p.147]. To see (2) take $f \in \left(L^2_{\rm loc}(\mathbb{R}^d)\right)^d$, then

$$\int dx \mathbb{E}_P^x\left[\int_0^t \chi_\xi(B_s)|f(B_s)|^2 ds\right] \le t\|\chi_\xi f\|^2 < \infty$$

and hence

$$\mathbb{E}_P^x\left[\int_0^t \chi_\xi(B_s)|f(B_s)|^2 ds\right] < \infty, \quad \xi > 0, \quad \text{a.e.} x \in \mathbb{R}^d,$$

for any indicator function $\chi_\xi$ of the set $\prod_{\mu=1}^d [-\xi,\xi]$. Hence $\int_0^t \chi_\xi(B_s)|f(B_s)|^2 ds < \infty$ for almost all $\omega$. For each $\omega$ there exists $b(\omega)$ such that $\sup_{0\le s\le t}|B_s(\omega)| < b(\omega)$. Take $\xi = \xi(\omega)$ such that $\xi > b(\omega)$. Then $\int_0^t |f(B_s(\omega))|^2 ds = \int_0^t \chi_\xi(B_s(\omega))|f(B_s(\omega))|^2 ds < \infty$, implying $P^x\left(\int_0^t |f(B_s)|^2 ds < \infty\right) = 1$, thus (2) follows. To see (3), note that

$$\mathbb{E}_P^x\left[\left|\int_0^t \chi_\xi(B_s)\nabla\cdot a(B_s) ds\right|\right] \le \int_0^t ds \int_{\mathbb{R}^d} dy \chi_\xi(sy)|(\nabla\cdot a)(sy)|\frac{te^{-|y|^2/2}}{(2\pi)^{d/2}} < \infty$$

for any indicator function $\chi_\xi$, whence it follows that $\left|\int_0^t \nabla\cdot a(B_s)ds\right| < \infty$ for almost every $\omega$. Thus (3) is obtained. **qed**

For $a \in (L^2_{\rm loc}(\mathbb{R}^d))^d$ such that $\nabla\cdot a \in L^1_{\rm loc}(\mathbb{R}^d)$, we denote

$$\int_0^t a(B_s)\circ dB_s = \int_0^t a(B_s)\cdot dB_s + \frac{1}{2}\int_0^t \nabla\cdot a(B_s)ds.$$

**Proposition 3.7** *Under Assumption (A2) we have*

$$(f, e^{-th}g) = \int_{\mathbb{R}^d} dx \mathbb{E}_P^x\left[\overline{f(B_0)}g(B_t)e^{-i\int_0^t a(B_s)\circ dB_s}\right]. \tag{3.12}$$

Proof. Equality (3.12) is well known as the Feynman-Kac-Itô formula, which in [Sim04, Theorem 15.5] was shown for $a \in L^2_{\rm loc}(\mathbb{R}^d)$, however, with $\nabla\cdot a = 0$. We provide a proof of (3.12) under Assumption (A2) for a self-contained presentation.

By using a mollifier we can take a sequence $a_n \in (C_0^\infty(\mathbb{R}^d))^d$, $n = 1, 2, ...$, such that $a_n \to a$ in $(L^2_{\rm loc})^d$ and $\nabla\cdot a_n \to \nabla\cdot a$ in $L^1_{\rm loc}$ as $n\to\infty$. Let $\chi_R = \chi(x^1/R)\cdots\chi(x^d/R)$, $R\in\mathbb{N}$, where $\chi\in C_0^\infty(\mathbb{R})$ such that $0\le\chi\le 1$, $\chi(x) = 1$ for $|x| < 1$ and $\chi(x) = 0$ for $|x|\ge 2$. Denote $h = h(a)$. Since $\chi_R a_n \to \chi_R a$ as $n\to\infty$ in $(L^2_{\rm loc})^d$ and $\chi_R a\to a$ as $R\to\infty$ in $(L^2_{\rm loc})^d$, it follows [LS81, Lemma 5 (3.17)] that $e^{-th(\chi_R a_n)} \to e^{-th(\chi_R a)}$ as $n\to\infty$ and $e^{-th(\chi_R a)} \to e^{-th(a)}$ as $R\to\infty$ in strong sense. Furthermore, (3.12) remains true for $a$ replaced by $\chi_R a_n \in (C_0^\infty(\mathbb{R}^d))^d$.

Since $\chi_R a_n \in (C_0^\infty(\mathbb{R}^d))^d$ and $\chi_R a_n \to \chi_R a$ in $(L^2)^d$ as $n\to\infty$, it follows that

$$\int_0^t \chi_R(B_s)a_n(B_s)\cdot dB_s \to \int_0^t \chi_R(B_s)a(B_s)\cdot dB_s \tag{3.13}$$

almost surely and since $\nabla\cdot(\chi_R a_n) = (\nabla\chi_R)\cdot a_n + \chi_R(\nabla\cdot a_n) \to (\nabla\chi_R)\cdot a + \chi_R(\nabla\cdot a)$ in $L^1(\mathbb{R}^d)$, it furthermore follows that

$$\int_0^t \nabla\cdot(\chi_R(B_s)a_n(B_s))ds \to \int_0^t (\nabla\chi_R(B_s))\cdot a(B_s)ds + \chi_R(B_s)(\nabla\cdot a(B_s))ds \tag{3.14}$$

strongly in $L^1(\Omega_P, dP^x)$. Thus there exists a subsequence $n'$ such that (3.13) and (3.14) with $n$ replaced by $n'$ hold almost surely. Hence (3.12) results by a limiting argument for $a$ replaced by $\chi_R a$. Let

$$\Omega_+(R) = \{\omega\in\Omega_P \,|\, \max_{0\le s\le t, 1\le\mu\le d} B_s^\mu(\omega) \le R\},$$
$$\Omega_-(R) = \{\omega\in\Omega_P \,|\, \min_{0\le s\le t, 1\le\mu\le d} B_s^\mu(\omega) \ge -R\}$$

and

$$I(R) = \left|\int_0^t \chi_R(B_s)a(B_s)\cdot dB_s - \int_0^t a(B_s)\cdot dB_s\right|.$$

We show that $I(R) \to 0$ in probability as $R \to \infty$. Note that the random variables $\max_{0\le s\le t} B_s^\mu(\omega)$ and $\min_{0\le s\le t} B_s^\mu(\omega)$ have the same distribution and

$$P^0(\Omega_-(R)) = P^0(\Omega_+(R)) = \prod_{\mu=1}^d P(|B_t^\mu| \le R) = \left(\frac{2}{\sqrt{2\pi t}}\int_0^R e^{-y^2/(2t)}dy\right)^d.$$

Since $\chi_R(B_s) = 1$ for all $0 \le s \le t$ on $\Omega_+(R)\cap\Omega_-(R)$, $I(R) = 0$ on $\Omega_+(R)\cap\Omega_-(R)$, we have

$$P^0(I(R) \ge \varepsilon) = P^0(I(R) \ge \varepsilon,\, \Omega_+(R)^c \cup \Omega_-(R)^c) \le 2\left(\frac{2}{\sqrt{2\pi t}}\int_R^\infty e^{-y^2/(2t)}dy\right)^d.$$

Hence $\lim_{R\to\infty} P^0(I(R) \ge \varepsilon) = 0$. Thus there exists a subsequence $R'$ such that $\int_0^t \chi_{R'}(B_s)a(B_s)\cdot dB_s \to \int_0^t a(B_s)\cdot dB_s$ almost surely as $R' \to \infty$. In a similar way it is seen that $\int_0^t \chi_{R''}(B_s)\nabla\cdot a(B_s)ds \to \int_0^t \nabla\cdot a(B_s)ds$ as $R'' \to \infty$ almost surely for some subsequence $R''$ of $R'$. Moreover,

$$\int_0^t \nabla\chi_R(B_s)\cdot a(B_s)ds = \frac{1}{R}\int_0^t \nabla\chi(B_s/R)\cdot a(B_s)ds \to 0 \tag{3.15}$$

in probability, and then for some subsequence $R'''$ of $R''$, (3.15) converges to zero almost surely. Thus $\int_0^t \chi_{R'''}(B_s)a(B_s)\circ dB_s \to \int_0^t a(B_s)\circ dB_s$ almost surely, and (3.12) holds for any $a$ satisfying Assumption (A2). **qed**

## 3.4 Functional integral representation

Now we turn to constructing a functional integral representation for generalized Schrödinger operators including a vector potential term defined by (3.5).

A key element in our construction of a Feynman-Kac-type formula for $e^{-tH^\Psi}$ is to make use of a Lévy subordinator.

**Theorem 3.8** *Let $\Psi \in \mathscr{B}_0$ and $V \in L^\infty(\mathbb{R}^d)$. Under Assumption (A2) we have*

$$(f, e^{-tH^\Psi}g) = \int_{\mathbb{R}^d} dx\, \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_0)}g(B_{T_t^\Psi})e^{-i\int_0^{T_t^\Psi} a(B_s)\circ dB_s}e^{-\int_0^t V(B_{T_s^\Psi})ds}\right]. \tag{3.16}$$

Proof. We divide the proof into four steps. To simplify the notation, in this proof we drop the superscript $\Psi$ of the subordinator.

Note that (3.12) holds, since (A2) is assumed.

*(Step 1)* Suppose $V = 0$. Then we claim that

$$(f, e^{-t\Psi(h)}g) = \int_{\mathbb{R}^d} dx\, \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_0)}g(B_{T_t})e^{-i\int_0^{T_t} a(B_s)\circ dB_s}\right]. \tag{3.17}$$

To prove (3.17) let $E^h$ denote the spectral projection of the self-adjoint operator $h$. Then

$$(f, e^{-t\Psi(h)}g) = \int_{\mathrm{Spec}(h)} e^{-t\Psi(u)} d(f, E^h_u g). \tag{3.18}$$

By inserting identity (2.2) in (3.18) we obtain

$$(f, e^{-t\Psi(h)}g) = \int_{\mathrm{Spec}(h)} \mathbb{E}^0_\nu[e^{-T_t u}] d(f, E^h_u g) = \mathbb{E}^0_\nu\left[(f, e^{-T_t h} g)\right].$$

Then by the Feynman-Kac-Itô formula for $e^{-th}$ we have

$$(f, e^{-t\Psi(h)}g) = \mathbb{E}^0_\nu\left[\int_{\mathbb{R}^d} dx\, \mathbb{E}^x_P\left[\overline{f(B_0)} g(B_{T_t}) e^{-i\int_0^{T_t} a(B_s)\circ dB_s}\right]\right],$$

thus (3.17) follows.

*(Step 2)* Let $0 = t_0 < t_1 < \cdots < t_n$, $f_0, f_n \in L^2(\mathbb{R}^d)$ and assume that $f_j \in L^\infty(\mathbb{R}^d)$ for $j = 1, ..., n-1$. We claim that

$$\left(f_0, \prod_{j=1}^n e^{-(t_j - t_{j-1})\Psi(h)} f_j\right) = \int_{\mathbb{R}^d} dx\, \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f_0(B_0)} \left(\prod_{j=1}^n f_j(B_{T_{t_j}})\right) e^{-i\int_0^{T_t} a(B_s)\circ dB_s}\right]. \tag{3.19}$$

For a concise notation we write $G_j(\cdot) = f_j(\cdot)\left(\prod_{i=j+1}^n e^{-(t_i - t_{i-1})\Psi(h)} f_i\right)(\cdot)$. By (Step 1) the left hand side of (3.19) can be represented as

$$\int_{\mathbb{R}^d} dx\, \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f_0(B_0)} e^{-i\int_0^{T_{t_1 - t_0}} a(B_s)\circ dB_s} G_1(B_{T_{t_1 - t_0}})\right].$$

Let $\mathscr{F}^{\mathrm{P}}_t = \sigma(B_s, 0 \le s \le t)$ and $\mathscr{F}^\nu_t = \sigma(T_s, 0 \le s \le t)$ be the natural filtrations. An application of the Markov property of $B_t$ yields

$$\begin{aligned}
&\left(f_0, \prod_{j=1}^n e^{-(t_j - t_{j-1})\Psi(h)} f_j\right) \\
&= \int_{\mathbb{R}^d} dx\, \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f_0(B_0)} e^{-i\int_0^{T_{t_1}} a(B_s)\circ dB_s} \mathbb{E}^0_\nu \mathbb{E}^{B_{T_{t_1}}}_P\left[f_1(B_0) e^{-i\int_0^{T_{t_2 - t_1}} a(B_s)\circ dB_s} G_2(B_{T_{t_2 - t_1}})\right]\right] \\
&= \int_{\mathbb{R}^d} dx\, \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f_0(B_0)} e^{-i\int_0^{T_{t_1}} a(B_s)\circ dB_s}\right. \\
&\qquad \left.\mathbb{E}^0_\nu\left[\mathbb{E}^0_P\left[f_1(B_{T_{t_1}}) e^{-i\int_{T_{t_1}}^{T_{t_2 - t_1} + T_{t_1}} a(B_s)\circ dB_s} G_2(B_{T_{t_1} + T_{t_2 - t_1}}) \,\middle|\, \mathscr{F}^P_{T_{t_1}}\right]\right]\right].
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\left(f_0, \prod_{j=1}^n e^{-(t_j - t_{j-1})\Psi(h)} f_j\right) \\
&= \int_{\mathbb{R}^d} dx\, \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f_0(B_0)} e^{-i\int_0^{T_{t_1}} a(B_s)\circ dB_s} \mathbb{E}^0_\nu\left[f_1(B_{T_{t_1}}) e^{-i\int_{T_{t_1}}^{T_{t_2 - t_1} + T_{t_1}} a(B_s)\circ dB_s} G_2(B_{T_{t_1} + T_{t_2 - t_1}})\right]\right].
\end{aligned}$$

The right hand side above can be rewritten as

$$\int_{\mathbb{R}^d} dx\, \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f_0(B_0)}e^{-i\int_0^{T_{t_1}} a(B_s)\circ dB_s} f_1(B_{T_{t_1}})\mathbb{E}^{T_{t_1}}_{\nu}\left[e^{-i\int_0^{T_{t_2-t_1}} a(B_s)\circ dB_s} G_2(B_{T_{t_2-t_1}})\right]\right].$$

Using now the Markov property of $T_t$ we see that

$$\begin{aligned}
&\left(f_0, \prod_{j=1}^n e^{-(t_j-t_{j-1})\Psi(h)} f_j\right)\\
&\quad= \int_{\mathbb{R}^d} dx\, \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f_0(B_0)}e^{-i\int_0^{T_{t_1}} a(B_s)\circ dB_s} f_1(B_{T_{t_1}})\mathbb{E}^{0}_{\nu}\left[e^{-i\int_{T_{t_1}}^{T_{t_2}} a(B_s)\circ dB_s} G_2(B_{T_{t_2}})\middle| \mathscr{F}^{\nu}_{t_1}\right]\right]\\
&\quad= \int_{\mathbb{R}^d} dx\, \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f_0(B_0)}e^{-i\int_0^{T_{t_1}} a(B_s)\circ dB_s} f_1(B_{T_{t_1}})e^{-i\int_{T_{t_1}}^{T_{t_2}} a(B_s)\circ dB_s} G_2(B_{T_{t_2}})\right].
\end{aligned}$$

By the above procedure we obtain (3.19).

*(Step 3)* Suppose now that $0 \neq V \in L^\infty$ and it is continuous; we prove (3.16) for such $V$. Since $H^\Psi$ is self-adjoint on $D(\Psi(h)) \cap D(V)$ the Trotter product formula holds:

$$(f, e^{-tH^\Psi} g) = \lim_{n\to\infty} (f, (e^{-(t/n)\Psi(h)} e^{-(t/n)V})^n g).$$

(Step 2) yields

$$\begin{aligned}
(f, e^{-tH^\Psi} g) &= \lim_{n\to\infty} \int_{\mathbb{R}^d} dx\, \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_0)}g(B_{T_t})e^{-i\int_0^{T_t} a(B_s)\circ dB_s} e^{-\sum_{j=1}^n (t/n)V(B_{T_{tj/n}})}\right]\\
&= \text{r.h.s. } (3.19)
\end{aligned}$$

Here we used that $V(B_{T_s(\tau)}(\omega))$ is continuous in $s \in [0,t]$ for each $(\omega, \tau)$ except for at most finite points, since $s \mapsto B_{T_s(\tau)}(\omega)$ is continuous except for finite points. Therefore $\sum_{j=1}^n \frac{t}{n} V(B_{T_{tj/n}}) \to \int_0^t V(B_{T_s}) ds$ as $n \to \infty$ for each path and exists as a Riemann integral.

*(Step 4)* An application of the method in [Sim04, Theorem 6.2] will complete the proof of Theorem 3.8. To do that, suppose that $V \in L^\infty$ and $V_n = \phi(x/n)(V * j_n)$, where $j_n = n^d \phi(xn)$ with $\phi \in C_0^\infty(\mathbb{R}^d)$ such that $0 \le \phi \le 1$, $\int \phi(x)dx = 1$ and $\phi(0) = 1$. Then $V_n(x) \to V(x)$ almost everywhere. The function $V_n$ is bounded and continuous, moreover $V_n(x) \to V(x)$ as $n \to \infty$ for $x \notin \mathscr{N}$, where the Lebesgue measure of $\mathscr{N}$ is zero. Notice that

$$\mathbb{E}^{x,0}_{P,\nu}\left[1_{\{B_{T_s}\in\mathscr{N}\}}\right] = \mathbb{E}^0_\nu\left[1_{\{T_t>0\}}\int_{\mathbb{R}^d} 1_{\mathscr{N}}(y)\hat{\mathrm{P}}_{T_t}(x-y)dy\right] + 1_{\mathscr{N}}(x)\mathbb{E}^0_\nu[1_{\{T_t=0\}}] = 0$$

for $x \notin \mathscr{N}$. Here $\hat{\mathrm{P}}_s(x)$ denotes the $d$-dimensional heat kernel:

$$\hat{\mathrm{P}}_s(x) = (2\pi s)^{-d/2}\exp(-|x|^2/(2s)). \tag{3.20}$$

Therefore $0 = \int_0^t ds \mathbb{E}^{x,0}_{P,\nu}\left[1_{\{B_{T_s}\in\mathscr{N}\}}\right] = \mathbb{E}^{x,0}_{P,\nu}\left[\int_0^t ds 1_{\{B_{T_s}\in\mathscr{N}\}}\right]$, for $x \notin \mathscr{N}$, by the Fubini theorem. Thus for every $x \notin \mathscr{N}$, and almost every $(\omega, \tau) \in \Omega_P \times \Omega_N$ the measure of $\{t \in [0,\infty) \,|\, B_{T_t(\tau)}(\omega) \in \mathscr{N}\}$ is zero. Hence $\int_0^t V_n(B_{T_s})ds \to \int_0^t V(B_{T_s})ds$ as $n \to \infty$ almost surely under $P^x \times \nu^0$, $x \notin \mathscr{N}$, and

$$\int_{\mathbb{R}^d} dx\, \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_0)} g(B_{T_t}) e^{-i\int_0^{T_t} a(B_s)\circ dB_s} e^{-\int_0^t V_n(B_{T_s})ds}\right]$$
$$\to \int_{\mathbb{R}^d} dx\, \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_0)} g(B_{T_t}) e^{-i\int_0^{T_t} a(B_s)\circ dB_s} e^{-\int_0^t V(B_{T_s})ds}\right]$$

as $n \to \infty$. On the other hand, $e^{-t(\Psi(h)+V_n)} \to e^{-t(\Psi(h)+V)}$ strongly as $n \to \infty$, since $\Psi(h) + V_n$ converges to $\Psi(h) + V$ on the common domain $D(\Psi(h))$. Thus the theorem follows. **qed**

Setting $a = 0$ and $\Psi(u) = u^{\alpha/2}$ for $0 < \alpha < 2$, we have the so-called *fractional Schrödinger operator* with exponent $\alpha/2$:

$$H_\alpha = \left(\frac{1}{2}\mathrm{p}^2\right)^{\alpha/2} + V\,, \tag{3.21}$$

for which Theorem 3.8 holds for example. For analytic results on fractional Schrödinger operators for some potentials, e.g., ground state and heat kernel estimates, intrinsic ultracontractivity, and related Gibbs measures see [KL10].

For a self-adjoint operator $T$ which is bounded from below, we use the notation $\mathscr{E}_T = \inf \operatorname{Spec} T$ here and in Sections 5 and 6 below.

**Corollary 3.9 (Diamagnetic inequality)** *Let $\Psi \in \mathscr{B}_0$, $V \in L^\infty(\mathbb{R}^d)$, and Assumption (A2) hold. Then*

$$|(f, e^{-tH^\Psi} g)| \le (|f|, e^{-t(\Psi(\mathrm{p}^2/2)+V)}|g|) \tag{3.22}$$

*and the energy comparison inequality $\mathscr{E}_{\Psi(\mathrm{p}^2/2)+V} \le \mathscr{E}_{H^\Psi}$ holds.*

PROOF. By Theorem 3.8 we have

$$|(f, e^{-tH^\Psi} g)| \le \int_{\mathbb{R}^d} dx\, \mathbb{E}^{x,0}_{P\times\nu}\left[|f(B_0)||g(B_{T^\Psi_t})| e^{-\int_0^t V(B_{T^\Psi_s})ds}\right].$$

The right hand side above coincides with that of (3.22), and $\mathscr{E}_{\Psi(\mathrm{p}^2/2)+V} \le \mathscr{E}_{H^\Psi}$ follows directly from (3.22). **qed**

**Remark 3.10** In Definitions 3.13 and 4.9 below, we shall define the generalized Schrödinger operators with singular potential $V$ and see them also satisfy diamagnetic inequalities.

## 3.5 Singular external potentials

By making use of the functional integral representation obtained in the previous subsection we can now also consider more singular external potentials.

**Theorem 3.11** *Let Assumption (A2) be satisfied.*

*(1) Suppose that $|V|$ is relatively form bounded with respect to $\Psi(\mathrm{p}^2/2)$ with relative bound $b$. Then $|V|$ is also relatively form bounded with respect to $\Psi(h)$ with a relative bound not larger than $b$.*

*(2) Suppose that $|V|$ is relatively bounded with respect to $\Psi(\mathrm{p}^2/2)$ with relative bound $b$. Then $|V|$ is also relatively bounded with respect to $\Psi(h)$ with a relative bound not larger than $b$.*

Proof. By virtue of Corollary 3.9 we have

$$|(f, e^{-t\Psi(h)}g)| \leq (|f|, e^{-t\Psi(\mathrm{p}^2/2)}|g|). \tag{3.23}$$

Then the proof is parallel with that of [Sim04, Theorem 15.10]. **qed**

**Corollary 3.12** *(1) Suppose that Assumption (A2) holds and let $V$ be relatively bounded with respect to $\Psi(\mathrm{p}^2/2)$ with relative bound strictly smaller than one. Then $\Psi(h)+V$ is self-adjoint on $D(\Psi(h))$ and bounded from below. Moreover, it is essentially self-adjoint on any core of $\Psi(h)$. (2) Suppose furthermore that (A3) holds. Then $C_0^\infty(\mathbb{R}^d)$ is an operator core of $\Psi(h)+V$.*

Proof. (1) By (2) of Theorem 3.11, $V$ is relatively bounded with respect to $\Psi(h)$ with a relative bound strictly smaller than one. Then the corollary follows by the Kato-Rellich theorem. (2) follows from Theorem 3.3. **qed**

Theorem 3.11 also allows $\Psi(h)+V$ to be defined in form sense. Let $V = V_+ - V_-$ where $V_+ = \max\{V, 0\}$ and $V_- = \min\{-V, 0\}$. Theorem 3.11 implies that whenever $V_-$ is form bounded with respect to $\Psi(\mathrm{p}^2/2)$ with a relative bound strictly smaller than one, it is also form bounded with respect to $\Psi(h)$ with a relative bound strictly smaller than one. Moreover, assume that $V_+ \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. We see that given Assumption (A1), $Q(\Psi(h)) \cap Q(V_+) \supset C_0^\infty(\mathbb{R}^d)$ by Corollary 3.12. In particular, $Q(\Psi(h)) \cap Q(V_+)$ is dense. Define the quadratic form

$$\mathrm{q}(f,f) = (\Psi(h)^{1/2}f, \Psi(h)^{1/2}f) + (V_+^{1/2}f, V_+^{1/2}f) - (V_-^{1/2}f, V_-^{1/2}f) \tag{3.24}$$

on $Q(\Psi(h)) \cap Q(V_+)$. By the KLMN Theorem [RS78] q is a semibounded closed form.

**Definition 3.13 (Generalized Schrödinger operator with singular $V$)** Let Assumption (A2) be satisfied and $V = V_+ - V_-$ be such that $V_+ \in L^1_{\rm loc}(\mathbb{R}^d)$ and $V_-$ is form bounded with respect to $\Psi(\frac{1}{2}\mathrm{p}^2)$ with a relative bound strictly less than 1. We denote the self-adjoint operator associated with (3.24) by $\Psi(h) \dot{+} V_+ \dot{-} V_-$ defined as a quadratic form sum.

Since we need (A2) to show the relative form boundedness of $V_-$ with respect to $\Psi(h)$, (A2) is assumed in Definition (3.13).

Now we are in the position to extend Theorem 3.8 to potentials expressed in terms of form sums.

**Theorem 3.14** *Let Assumption (A2) be satisfied. Let $V = V_+ - V_-$ be such that $V_+ \in L^1_{\rm loc}(\mathbb{R}^d)$ and $V_-$ is infinitesimally small with respect to $\Psi(\frac{1}{2}\mathrm{p}^2)$ in form sense, i.e., for every $\varepsilon > 0$ there exists a non-negative constant $b_\varepsilon$ such that*

$$\|V_-^{1/2} f\|^2 \le \varepsilon \|\Psi(\frac{1}{2}\mathrm{p}^2)^{1/2} f\|^2 + b_\varepsilon \|f\|^2$$

*for all $f \in D(\Psi(\frac{1}{2}\mathrm{p}^2)^{1/2})$. Then the functional integral representation given by Theorem 3.8 also holds for $\Psi(h) \dot{+} V_+ \dot{-} V_-$.*

Proof. Write

$$V_{+,n}(x) = \begin{cases} V_+(x), & V_+(x) < n, \\ n, & V_+(x) \ge n, \end{cases} \qquad V_{-,m}(x) = \begin{cases} V_-(x), & V_-(x) < m, \\ m, & V_-(x) \ge m. \end{cases}$$

The proof is a slight modification of that of [Sim04, Theorem 6.2]. For simplicity we write just $\Psi$ for $\Psi(h)$. We see that

$$e^{-t\left(\Psi \dot{+} V_{+,n} \dot{-} V_{-,m}\right)} \to e^{-t\left(\Psi \dot{+} V_{+,n} \dot{-} V_-\right)} \tag{3.25}$$

strongly as $m \to \infty$, for all $t \ge 0$, and we also obtain

$$e^{-t\left(\Psi \dot{+} V_{+,n} \dot{-} V_-\right)} \to e^{-t\left(\Psi \dot{+} V_+ \dot{-} V_-\right)}, \tag{3.26}$$

for all $t \ge 0$, in strong sense as $n \to \infty$. On the other hand, we look at the convergence of the expression

$$\int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{P\times\nu} \left[ e^{-\int_0^t (V_{+,n} - V_{-,m})(B_{T_s^\Psi}) ds} I \right]. \tag{3.27}$$

Here $I = \overline{f(B_0)} e^{-i\int_0^{T_t} a(B_s)\circ dB_s} g(B_{T_t})$. Decompose $I$ into its real and imaginary parts, and further into their positive and negative parts $\Re I = \Re I_+ - \Re I_-$ and $\Im I = \Im I_+ - \Im I_-$. Then by (3.25) and the monotone convergence theorem

$$\int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{P\times\nu} \left[ e^{-\int_0^t (V_{+,n} - V_{-,m})(B_{T_s^\Psi}) ds} \Re I_+ \right] \longrightarrow \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{P\times\nu} \left[ e^{-\int_0^t (V_{+,n} - V_-)(B_{T_s^\Psi}) ds} \Re I_+ \right]$$

as $m \to \infty$. Similarly, the remaining three terms $\Re I_-$, $\Im I_+$ and $\Im I_-$ also converge. Thus (3.27) converges to $\displaystyle\int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{P\times\nu}\left[e^{-\int_0^t (V_{+,n}-V_-)(B_{T_s^\Psi})ds} I\right]$ as $m \to \infty$. Moreover,

$$\int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{P\times\nu}\left[e^{-\int_0^t (V_{+,n}-V_-)(B_{T_s^\Psi})ds} I\right] \longrightarrow \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{P\times\nu}\left[e^{-\int_0^t (V_{+}-V_-)(B_{T_s^\Psi})ds} I\right]$$

as $n \to \infty$, by (3.26) and the dominated convergence theorem. Thus the proof is complete. **qed**

# 4 Ψ-Kato class potentials

## 4.1 Definition of Ψ-Kato class potentials

In this section we give a meaning to Kato class for potentials $V$ relative to $\Psi$ and extend generalized Schrödinger operators with vector potential to such $V$.

It is known that the composition of a Brownian motion and a subordinator yields a Lévy process. Recall that for given $\Psi \in \mathscr{B}_0$, the random process

$$X_t : \Omega_P \times \Omega_\nu \ni (\omega, \tau) \mapsto B_{T_t^\Psi(\tau)}(\omega) \tag{4.1}$$

is called $d$-dimensional subordinated Brownian motion with respect to the subordinator $(T_t^\Psi)_{t\geq 0}$. It is a Lévy process whose properties are determined by the pair $(b, \lambda)$ in (2.1). Its characteristic function is

$$\mathbb{E}^{0,0}_{P\times\nu}[e^{i\xi\cdot X_t}] = e^{-t\Psi(\xi\cdot\xi/2)}, \quad \xi \in \mathbb{R}^d. \tag{4.2}$$

**Assumption 4.1** *Let $\Psi \in \mathscr{B}_0$ be such that*

$$\int_{\mathbb{R}^d} e^{-t\Psi(\xi\cdot\xi/2)} d\xi < \infty \text{ for all } t > 0. \tag{4.3}$$

Let $\Psi \in \mathscr{B}_0$ and $(b, \lambda) \in \mathbb{R}_+ \times \mathscr{L}$ be its corresponding non-negative drift coefficient and Lévy measure, i.e., $\Psi(u) = bu + \int_0^\infty (1 - e^{-uy})\, \lambda(dy)$. It is clear that if $b > 0$, then (4.3) is satisfied. In the case of $b = 0$ but $\int_0^1 \lambda(dy) < \infty$, since $\sup_{u\geq 0} \Psi(u) < \infty$, (4.3) is not satisfied. Thus $\Psi$ obeying (4.3) at least satisfies $\int_0^1 \lambda(dy) = \infty$ when $b = 0$. In this case we have

$$\Psi(u^2/2) \geq \int_0^1 (1 - e^{-u^2y/2})\lambda(dy) \geq (1 - e^{-1}) \int_0^1 (\frac{u^2 y}{2} \wedge 1)\lambda(dy) \geq (1 - e^{-1}) \int_{2/u^2}^1 \lambda(dy).$$

Thus in case $b = 0$ and $\int_0^1 \lambda(dy) = \infty$, assuming that there exists $\rho(u)$ such that $\int_{2/u^2}^1 \lambda(dy) \geq \rho(u)$ and $\int_{\mathbb{R}^d} e^{-t\rho(|\xi|)} d\xi < \infty$, we can make sure Assumption 4.1 holds.

Under Assumption 4.1 we define

$$p_t(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-t\Psi(\xi\cdot\xi/2)} d\xi \tag{4.4}$$

and

$$\Pi_\lambda(x) = \int_0^\infty e^{-\lambda t} p_t(x) dt. \tag{4.5}$$

The function $p_t(x)$ denotes the distribution density of $X_t$ in (4.1) and $\Pi_\lambda(x-y)$ is the integral kernel of the resolvent $(\Psi(\mathrm{p}^2/2)+\lambda)^{-1}$ with $\lambda > 0$, i.e.,

$$\left(f, \left(\Psi(\mathrm{p}^2/2)+\lambda\right)^{-1} g\right) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \overline{f(x)} g(y) \Pi_\lambda(x-y) dx dy.$$

Clearly, $p_t(x)$ and $\Pi_\lambda(x)$ are spherically symmetric. For $f \in C_0^\infty(\mathbb{R}^d)$ it follows that

$$\mathbb{E}^{0,0}_{P\times\nu}[f(X_t)] = \int_{\mathbb{R}^d} f(x) p_t(x) dx. \tag{4.6}$$

Hence for non-negative $f \in C_0^\infty(\mathbb{R}^d)$, the right hand side of (4.6) is non-negative since so is the left hand side. Thus $p_t(x) \geq 0$ for almost every $x \in \mathbb{R}^d$. By a limiting argument with $f \to 1$, we also see that $p_t \in L^1(\mathbb{R}^d)$ and $\|p_t\|_{L^1(\mathbb{R}^d)} = 1$ by (4.6).

We moreover compute $\Pi_\lambda$ as

$$\Pi_\lambda(x) = (2\pi)^{-d/2} \frac{1}{|x|^{(d-1)/2}} \int_0^\infty \frac{r^{(d-1)/2}}{\lambda + \Psi(r^2/2)} \sqrt{r|x|} J_{(d-2)/2}(r|x|) dr,$$

with the Bessel function given by

$$J_\nu(s) = \left(\frac{s}{2}\right)^\nu \frac{1}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})} \int_0^\pi e^{is\cos\theta} (\sin\theta)^{2\nu} d\theta = \sum_{n=0}^\infty \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{s}{2}\right)^{2n+\nu}.$$

Note that $\sup_{u\geq 0} \sqrt{u} J_\nu(u) < \infty$.

Let

$$\|f\|_{l^1(L^\infty)} = \sum_{\alpha\in\mathbb{Z}^d} \sup_{x\in C_\alpha} |f(x)|,$$

where $C_\alpha$ denotes the unit cube centered at $\alpha \in \mathbb{Z}^d$. We introduce an additional assumption on the distribution density $p_t$.

**Assumption 4.2** *Let $p_t$ be such that for each $\delta > 0$, $\sup_{t>0} \|1_{\{|x|>\delta\}} p_t\|_{l^1(L^\infty)} < \infty$.*

Let $f$ be a real valued function on $\mathbb{R}^d$. When $r \mapsto f(rx)$ is non-increasing on $[0,\infty)$ for all $x \in \mathbb{R}^d$, we say that $f$ is radially non-increasing. In $d = 1$ for a radially non-increasing $L^1$-function $f$ it can be seen, by the definition of $l^1(L^\infty)$, that there exists a constant $C_\delta = C_\delta(f)$ such that for each $\delta > 0$,

$$\|1_{\{|x|>\delta\}} f\|_{l^1(L^\infty)} \leq C_\delta \|f\|_{L^1}. \tag{4.7}$$

In the general case $d \geq 2$ it can be also seen that (4.7) holds for all radially non-increasing $f$, see [CMS90, p. 131, Corollary]. In particular, Assumption 4.2 is satisfied whenever $p_t$ is radially non-increasing, since $\|p_t\|_{L^1} = 1$.

**Example 4.3 ($\alpha/2$-stable subordinator)** In the case of $\Psi(u) = u^{\alpha/2}$, $0 < \alpha < 2$, it is clear that Assumption 4.1 is satisfied. It is also known that the distribution density of $B_{T_t^\Psi}$ (which in this case is symmetric $\alpha$-stable process) is radially non-increasing. This is obtained by a unimodality argument of spherically symmetric distribution functions; see [Kan77, Theorem 4.1],[Wol78, Theorem 2], [CMS90, p.132], [Yam78, Theorem 1], and [Sat99] for details on unimodality. Then Assumption 4.2 is again satisfied.

**Example 4.4** Let $\Psi(u) = \sqrt{2u + m^2} - m$, $m \geq 0$. It is clear that Assumption 4.1 is satisfied. The distribution function $p_t$ of $B_{T_t^\Psi}$ is expressed as

$$p_t(x) = (2\pi)^{-d} \frac{t}{\sqrt{|x|^2 + t^2}} \int_{\mathbb{R}^d} e^{mt} e^{-\sqrt{(|x|^2+t^2)(p^2+m^2)}} dp,$$

see [HS78, (2.7)]. Then $p_t$ is indeed radially non-increasing.

The next proposition allows an extension of $\Psi(\mathrm{p}^2/2)$ to Kato class.

**Proposition 4.5** *Let $V \geq 0$. Under Assumptions 4.1 and 4.2 the following three properties are equivalent:*

*(1)* $\lim_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} \int_0^t \mathbb{E}^{x,0}_{P \times \nu}[V(X_s)] ds = 0,$

*(2)* $\lim_{\lambda \to \infty} \sup_{x \in \mathbb{R}^d} \left( (\Psi(\mathrm{p}^2/2) + \lambda)^{-1} V \right)(x) = 0,$

*(3)* $\lim_{\delta \downarrow 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y|<\delta} \Pi_1(x-y) V(y) dy = 0.$

Proof. The proof is similar to that of Theorem III.1 in [CMS90], and is therefore omitted. **qed**

**Definition 4.6 ($\Psi$-Kato class)** Let Assumptions 4.1 and 4.2 be satisfied. Write $V = V_+ - V_-$ in terms of its positive and negative parts. The $\Psi$-*Kato class* is defined as the set of potentials $V$ for which $V_-$ and $1_C V_+$ with every compact subset $C \subset \mathbb{R}^d$ satisfy any of the three equivalent conditions in Proposition 4.5. Here $1_C$ denotes the indicator function of $C$.

By (3) of Proposition 4.5 we can derive explicit conditions defining $\Psi$-Kato class using the relation of the Lévy measure of the subordinator with the associated Bernstein function.

## 4.2 Ψ-Kato class potential and $L^p$-$L^q$ bound

In this section we construct Schrödinger semigroups with Ψ-Kato class potentials and show their $L^p$-$L^q$ boundedness. References on the $L^p$-$L^q$ bound for semigroups with usual Schrödinger operators with magnetic field include [Sim82, BHL00].

**Lemma 4.7** *Let $V \geq 0$ and $\Psi \in \mathscr{B}_0$. Suppose that Assumptions 4.1 and 4.2 hold. Suppose moreover that $V$ satisfies (1) of Proposition 4.5. Then for $t \geq 0$,*

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}^{x,0}_{P\times\nu}\left[e^{\int_0^t V(X_s)ds}\right] < \infty. \tag{4.8}$$

Proof. There exists $s > 0$ such that $\sup_{x\in\mathbb{R}^d} \mathbb{E}^{x,0}_{P\times\nu}[\int_0^s V(X_s)ds] = \epsilon < 1$ by (1) of Proposition 4.5. Then by the Khas'minskii Lemma we conclude that

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}^{x,0}_{P\times\nu}\left[e^{\int_0^s V(X_s)ds}\right] \leq (1-\epsilon)^{-1}.$$

Consider the image measure $\rho$ of $(X_t)_{t\geq 0}$ on the space $D([0,\infty);\mathbb{R}^d)$ of *cádlág* paths. Then $\mathbb{E}^x_\rho\left[e^{\int_0^s V(X_s)ds}\right] = \mathbb{E}^{x,0}_{P\times\nu}\left[e^{\int_0^s V(X_s)ds}\right]$ and clearly $(X_t)_{t\geq 0}$ is a Markov process with respect to $\rho$. Furthermore,

$$\mathbb{E}^x_\rho\left[e^{\int_0^{2s} V(X_s)ds}\right] = \mathbb{E}^x_\rho\left[e^{\int_0^s V(X_s)ds}\mathbb{E}^{X_s}_\rho\left[e^{\int_0^s V(X_s)ds}\right]\right] \leq (1-\epsilon)^{-2}.$$

Repeating this procedure we obtain (4.8) for all $t \geq 0$. **qed**

The next result says that we can define a Feynman-Kac semigroup for Ψ-Kato class potentials.

**Theorem 4.8** *Let $\Psi \in \mathscr{B}_0$ and suppose that Assumptions 4.1 and 4.2 hold. Let $V$ belong to Ψ-Kato class and let Assumption (A2) hold. Consider*

$$\mathrm{U}_t f(x) = \mathbb{E}^{x,0}_{P\times\nu}\left[e^{-i\int_0^{T_t^\Psi} a(B_s)\circ dB_s} e^{-\int_0^t V(B_{T_s^\Psi})ds} f(B_{T_t^\Psi})\right].$$

*Then $\mathrm{U}_t$ is a strongly continuous self-adjoint semigroup. In particular, there exists a self-adjoint operator $K^\Psi$ bounded from below such that $\mathrm{U}_t = e^{-tK^\Psi}$.*

Proof. Let $V = V_+ - V_-$. Hence by Lemma 4.7 we have

$$\begin{aligned}\|\mathrm{U}_t f\|^2 &\leq \int_{\mathbb{R}^d} dx\, \mathbb{E}^{x,0}_{P\times\nu}\left[e^{-2\int_0^t V_+(X_s)ds}|f(X_t)|^2\right]\mathbb{E}^{x,0}_{P\times\nu}\left[e^{2\int_0^t V_-(X_s)ds}\right] \\ &\leq C_t \int_{\mathbb{R}^d} dx\, \mathbb{E}^{x,0}_{P\times\nu}\left|f(X_t)|^2\right] \leq C_t\|f\|^2,\end{aligned}$$

where $C_t = \sup_{x\in\mathbb{R}^d} \mathbb{E}^{x,0}_{P\times\nu}[e^{2\int_0^t V_-(X_s)ds}]$. Thus $\mathrm{U}_t$ is a bounded operator from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. In the same manner as in Step 2 of the proof of Theorem 3.8 we conclude that the semigroup property $\mathrm{U}_t\mathrm{U}_s = \mathrm{U}_{t+s}$ holds for $t, s \geq 0$. We check strong continuity of $\mathrm{U}_t$ in $t$; it suffices to show weak continuity. Let $f, g \in C_0^\infty(\mathbb{R}^d)$ and we write $T_t$ for $T_t^\Psi$ for simplicity. Then we have

$$(f, \mathrm{U}_t g) = \int_{\mathbb{R}^d} dx\, \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_0)}g(B_{T_t})e^{-i\int_0^{T_t} a(B_s)\circ dB_s}e^{-\int_0^t V(B_{T_s})ds}\right].$$

Since $T_t(\tau) \to 0$ as $t \to 0$ for each $\tau \in \Omega_\nu$, the dominated convergence theorem gives $(f, \mathrm{U}_t g) \to (f, g)$.

Finally we check the symmetry property $\mathrm{U}_t^* = \mathrm{U}_t$. By a limiting argument it is enough to show this for $a \in (C_{\mathrm{b}}^2(\mathbb{R}^d))^d$. Let

$$\widetilde{B}_s = \widetilde{B}_s(\omega, \tau) = B_{T_t(\tau)-s}(\omega) - B_{T_t(\tau)}(\omega), \quad 0 \leq s \leq T_t(\tau).$$

Then for each $\tau \in \Omega_\nu$, $\widetilde{B}_s \overset{\mathrm{d}}{=} B_s$ for $0 \leq s \leq T_t$ with respect to $dP^x$. Here $Z \overset{\mathrm{d}}{=} Y$ denotes that $Z$ and $Y$ are identically distributed. Let

$$I_j = \frac{1}{2}\left(a(x + \widetilde{B}_{T_t j/n}) + a(x + \widetilde{B}_{T_t(j-1)/n})\right)(\widetilde{B}_{T_t j/n} - \widetilde{B}_{T_t(j-1)/n}).$$

Then $\sum_{j=1}^n I_j \to \int_0^{T_t} a(x + \widetilde{B}_s) \circ d\widetilde{B}_s$ in $L^2(\Omega_P, dP)$ as $n \to \infty$. Thus there exists a subsequence $\{\sum_{j=1}^{n'} I_j\}_{n'}$ of $\{\sum_{j=1}^n I_j\}_n$ such that $\sum_{j=1}^{n'} I_j \to \int_0^{T_t} a(x+\widetilde{B}_s)\circ d\widetilde{B}_s$ almost surely and

$$(f, \mathrm{U}_t g) = \lim_{n'\to\infty} \mathbb{E}^{0,0}_{P\times\nu}\left[\int_{\mathbb{R}^d} dx\, \overline{f(x)}e^{-i\sum_{j=1}^{n'} I_j}e^{-\int_0^t V(x+\widetilde{B}_{T_s})}g(x + \widetilde{B}_{T_t})\right]$$

by the dominated convergence theorem. We reset $n'$ as $n$. Changing the variable $x$ to $y = x + \widetilde{B}_{T_t}$, we have

$$(f, \mathrm{U}_t g) = \lim_{n\to\infty} \mathbb{E}^{0,0}_{P\times\nu}\left[\int_{\mathbb{R}^d} dy\overline{f(y - \widetilde{B}_{T_t})}e^{-i\sum_{j=1}^n \widetilde{I}_j}e^{-\int_0^t V(y-\widetilde{B}_{T_t}+\widetilde{B}_{T_s})}g(y)\right],$$

where

$$\widetilde{I}_j = \frac{1}{2}\left(a(y - \widetilde{B}_{T_t} + \widetilde{B}_{T_t j/n}) + a(y - \widetilde{B}_{T_t} + \widetilde{B}_{T_t(j-1)/n})\right)(\widetilde{B}_{T_t j/n} - \widetilde{B}_{T_t(j-1)/n}).$$

Since $\widetilde{B}_{T_s} - \widetilde{B}_{T_t} \overset{\mathrm{d}}{=} B_{T_t - T_s}$, we can compute $\lim_{n\to\infty}\sum_{j=1}^n \widetilde{I}_j$ in $L^2(\Omega_P, dP^0)$ as

$$\lim_{n\to\infty}\sum_{j=1}^n \widetilde{I}_j = -\int_0^{T_t} a(B_s) \circ dB_s.$$

Then we have

$$(f, \mathrm{U}_t g) = \int_{\mathbb{R}^d} dx\, \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_{T_t})} e^{+i\int_0^{T_t} a(B_s)\circ dB_s} e^{-\int_0^t V(B_{T_t - T_s})ds} g(x)\right].$$

Moreover, as $T_t - T_s \stackrel{\mathrm{d}}{=} T_{t-s}$ for $0 \le s \le t$, we obtain

$$(f, \mathrm{U}_t g) = \int_{\mathbb{R}^d} dx\, \overline{\mathbb{E}^{x,0}_{P\times\nu}\left[f(B_{T_t}) e^{-i\int_0^{T_t} a(B_s)\circ dB_s} e^{-\int_0^t V(B_{T_s})ds}\right]} g(x) = (\mathrm{U}_t f, g).$$

Since $\mathrm{U}_t$ is a strongly continuous self-adjoint semigroup, its generator $K^\Psi$ is a symmetric closed operator and there exists $a \in \mathbb{R}$ such that $(-\infty, a)$ includes the resolvent of $K^\Psi$ by the Hille-Yohsida theorem. Then the spectrum of $K^\Psi$ is included $\mathbb{R}$, and hence $K^\Psi$ is self-adjoint. **qed**

**Definition 4.9 ($\Psi$-Kato class Schrödinger operator)** Let Assumptions 4.1 and 4.2 be satisfied. Let $V$ be in $\Psi$-Kato class and take Assumption (A2). We call $K^\Psi$ given in Theorem 4.8 *generalized Schrödinger operator for $\Psi$-Kato class potentials*. We refer to the one-parameter operator semigroup $\{e^{-tK^\Psi} : t \ge 0\}$, as the *$\Psi$-Kato class generalized Schrödinger semigroup*.

For $\Psi$-Kato class potentials $V$ condition (2) of Proposition 4.5 implies that $V_-$ is infinitesimally form bounded with respect to $\Psi(\mathrm{p}^2/2)$. In this case $\Psi(\mathrm{p}^2/2) + V$ can be defined in form sense.

**Theorem 4.10** *Suppose that Assumptions 4.1 and 4.2 hold. Let $V$ be in $\Psi$-Kato class and take Assumption (A2). Then*

$$K^\Psi = \Psi(h) \dotplus V_+ \dot{-} V_-. \tag{4.9}$$

Proof. The proof is similar to that of Theorem 3.14. In the same approximation of $V$ as in the proof of Theorem 3.14, we see that $e^{-t\left(\Psi(h) \dotplus V_{+,n} \dot{-} V_{-,m}\right)} \to e^{-t\left(\Psi(h) \dotplus V_{+,n} \dot{-} V_{-}\right)}$ as $n \to \infty$, and then $e^{-t\left(\Psi(h) \dotplus V_{+,n} \dot{-} V_{-}\right)} \to e^{-t\left(\Psi(h) \dotplus V_{+} \dot{-} V_{-}\right)}$ as $m \to \infty$ strongly. On the other hand Feynman-Kac formula of $I_{nm} = (f, e^{-t\left(\Psi(h) \dotplus V_{+,n} \dot{-} V_{-,m}\right)} g)$ satisfies that

$$\lim_{m\to\infty}\lim_{n\to\infty} I_{mn} = \int_{\mathbb{R}^d} dx\, \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(x)} e^{-i\int_0^{T_t} a(B_s)\circ dB_s} e^{-\int_0^t V(B_{T_s})} g(B_{T_t})\right].$$

Hence the theorem follows. **qed**

Put $K^\Psi_0$ for the operator defined by $K^\Psi$ with $a$ replaced by 0.

**Theorem 4.11 ($L^p$-$L^q$ bound)** *Let $V$ be a $\Psi$-Kato class potential and assume (A2) to hold. Suppose that Assumptions 4.1 and 4.2 hold. Then $e^{-tK^\Psi}$ is a bounded operator from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$, for all $1 \leq p \leq q \leq \infty$. Moreover, $\|e^{-tK^\Psi}\|_{p,q} \leq \|e^{-tK_0^\Psi}\|_{p,q}$ holds for all $t \geq 0$.*

PROOF. By the Riesz-Thorin theorem it suffices to show that $e^{-tK^\Psi}$ is bounded as an operator of (1) $L^\infty(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$, (2) $L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ and (3) $L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$. Since

$$|e^{-tK^\Psi} f(x)| \leq e^{-tK_0^\Psi} |f|(x), \tag{4.10}$$

we will prove (1)-(3) for $e^{-tK_0^\Psi}$ in a similar way to [Sim82]. **qed**

# 5 The case of operators with spin

## 5.1 Generalized Schrödinger operator with spin $\mathbb{Z}_p$, $p \geq 2$

Besides operators describing interactions with magnetic fields we now consider operators also including a spin variable. The Schrödinger operator with spin 1/2 is formally given by

$$h_{1/2} = \frac{1}{2}(\sigma \cdot (\mathrm{p} - a))^2 \tag{5.1}$$

on $L^2(\mathbb{R}^3; \mathbb{C}^2)$, where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Note that under Assumption (A4)

$$h_{1/2} f = \frac{1}{2}\mathrm{p}^2 f - a \cdot \mathrm{p} f + \left( \frac{1}{2} a \cdot a - \frac{1}{2}(\mathrm{p} \cdot a) - \frac{1}{2}\sigma \cdot (\nabla \times a) \right) f \tag{5.2}$$

holds for $f \in \mathbb{C}^2 \otimes C_0^\infty(\mathbb{R}^3)$. In order to construct a functional integral representation for $e^{-th_{1/2}}$ we make a unitary transform of the operator $h_{1/2}$ on $L^2(\mathbb{R}^3; \mathbb{C}^2)$ to an operator on the space $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$. This is a space of $L^2$-functions of $x \in \mathbb{R}^3$ and an additional two-valued spin variable $\theta \in \mathbb{Z}_2$, where

$$\mathbb{Z}_2 = \{-1, 1\} = \{\theta_1, \theta_2\}. \tag{5.3}$$

Also, we define on $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ the operator

$$(h_{\mathbb{Z}_2} f)(x, \theta) = (hf)(x, \theta) - \frac{1}{2}\theta b_3(x) f(x, \theta) - \frac{1}{2}\Big(b_1(x) - i\theta b_2(x)\Big) f(x, -\theta), \tag{5.4}$$

where $x \in \mathbb{R}^3$, $\theta \in \mathbb{Z}_2$ and $(b_1, b_2, b_3) = \nabla \times a$. The operators $h_{\mathbb{Z}_2}$ and $h_{1/2}$ are unitary equivalent, as is shown in [HL08].

Next we generalize the operator $h_{\mathbb{Z}_2}$ on $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ by considering a similar operator on $L^2(\mathbb{R}^d \times \mathbb{Z}_p)$ for $d \geq 1$ and $p \geq 2$. Define $\mathbb{Z}_p$ as the cyclic group of the $p$th roots of unity by

$$\mathbb{Z}_p = \{\theta_1^{(p)}, ..., \theta_p^{(p)}\}, \tag{5.5}$$

where

$$\theta_\alpha^{(p)} = \exp\left(2\pi i \frac{\alpha}{p}\right), \quad \alpha \in \mathbb{N}. \tag{5.6}$$

In what follows we fix $p \geq 2$ and abbreviate $\theta_\beta^{(p)}$ simply to $\theta_\beta$ for notational convenience. Consider the finite dimensional vector space $\ell^2(\mathbb{Z}_p) = \{f : \mathbb{Z}_p \to \mathbb{C}\}$ equipped with the scalar product $(f, g)_{\ell^2(\mathbb{Z}_p)} = \sum_{\beta=1}^p \overline{f(\theta_\beta)} g(\theta_\beta)$.

Now we consider the Schrödinger operator with spin $\mathbb{Z}_p$. We define a spin operator with its diagonal part $U$ and off-diagonal part $U_\beta$, $\beta = 1, ..., p-1$, separately.

**Definition 5.1 (Generalized spin operator)** We define two functions below:

(1) (Diagonal part) Let $U_p : \mathbb{R}^d \times \mathbb{Z}_p \to \mathbb{R}$ be such that $\max_{\theta \in \mathbb{Z}_p} |U_p(x, \theta)|$ is a multiplication operator, relatively bounded with respect to $\frac{1}{2}\mathrm{p}^2$.

(2) (Off-diagonal part) Let $W_\beta : \mathbb{R}^d \times \mathbb{Z}_p \to \mathbb{C}$, $1 \leq \beta \leq p-1$, be such that $\max_{\theta \in \mathbb{Z}_p} |W_\beta(x, \theta)|$ is a multiplication operator, relatively bounded with respect to $\frac{1}{2}\mathrm{p}^2$. Moreover, let $U_\beta : \mathbb{R}^d \times \mathbb{Z}_p \to \mathbb{C}$ be defined

$$U_\beta(x, \theta_\alpha) = \frac{1}{2}\left(W_\beta(x, \theta_{\alpha+\beta}) + \overline{W_{p-\beta}(x, \theta_\alpha)}\right), \quad \alpha = 1, ..., p, \ \beta = 1, ..., p-1. \tag{5.7}$$

Furthermore, we call $M_{\mathbb{Z}_p} : L^2(\mathbb{R}^d \times \mathbb{Z}_p) \to L^2(\mathbb{R}^d \times \mathbb{Z}_p)$,

$$M_{\mathbb{Z}_p} : f(x, \theta_\alpha) \mapsto \sum_{\beta=1}^p U_\beta(x, \theta_\alpha) f(x, \theta_{\alpha+\beta}) \tag{5.8}$$

the *generalized spin operator* on $L^2(\mathbb{R}^d \times \mathbb{Z}_p)$.

Below we will use the notation

$$u_\beta(x) = \max_{\theta \in \mathbb{Z}_p} |U_\beta(x, \theta)|, \quad 1 \leq \beta \leq p. \tag{5.9}$$

Clearly, $u_\beta(x)$ is a multiplication operator relatively bounded with respect to $\frac{1}{2}\mathrm{p}^2$, i.e., there exist $c_\beta > 0$ and $b_\beta \geq 0$ such that

$$\|u_\beta f\| \leq c_\beta \|\frac{1}{2}\mathrm{p}^2 f\| + b_\beta \|f\|, \quad \beta = 1, ..., p, \tag{5.10}$$

for all $f \in D((1/2)\mathrm{p}^2)$. These definitions of $U_\beta$ cover, in particular, the $\mathbb{Z}_2$ case of the Schrödinger operator associated with spin 1/2.

**Example 5.2 (Spin 1/2)** Let $d = 3$ and $p = 2$. Define

$$W_1(x,\theta) = -\frac{1}{2}(b_1(x) + i\theta b_2(x)), \quad \theta \in \mathbb{Z}_2.$$

Then $\theta_1 = -1$, $\theta_2 = 1$ and by (5.7) we see that

$$U_1(x,\theta) = \frac{1}{2}(W_1(x,\theta\theta_1) + \overline{W_1(x,\theta)}), \quad \theta \in \mathbb{Z}_2.$$

It is straightforward to see that $W_1(x,\theta\theta_1) = -\frac{1}{2}(b_1(x) - i\theta b_2(x)) = \overline{W_1(x,\theta)}$, hence the off-diagonal part is $U_1(x,\theta) = -\frac{1}{2}(b_1(x) - i\theta b_2(x))$, while the diagonal part is given by $U_2(x,\theta) = -\frac{1}{2}\theta b_3(x)$, both of which coincide with the interaction in (5.4)

**Example 5.3** Let $p \geq 2$, and $W_\beta(\theta) = W(\theta) = -\frac{1}{2}(b_1 + i\theta b_2)$ for $1 \leq \beta \leq p-1$. Then

$$U_\beta(\theta_\alpha) = \frac{1}{2}\left(W_\beta(\theta_{\alpha+\beta}) + \overline{W_{p-\beta}(\theta_\alpha)}\right) = -\frac{1}{2}\left(b_1 + i\frac{\theta_{\alpha+\beta} - \theta_{p-\alpha}}{2}b_2\right). \tag{5.11}$$

This gives one possible generalization of the case of spin 1/2 of Example 5.2.

**Definition 5.4 (Schrödinger operator with generalized spin)** Let $h$ be the generalized Schrödinger operator defined in (3.3). Under Assumption (A1) we define the *Schrödinger operator with generalized spin* $M_{\mathbb{Z}_p}$ by

$$h_{\mathbb{Z}_p} = 1 \otimes h + M_{\mathbb{Z}_p}. \tag{5.12}$$

Above we made the identification $L^2(\mathbb{R}^d \times \mathbb{Z}_p) \cong \ell^2(\mathbb{Z}_p) \otimes L^2(\mathbb{R}^d)$. Formally, $h_{\mathbb{Z}_p}$ is written as

$$(h_{\mathbb{Z}_p} f)(x,\theta_\alpha) = \left(\frac{1}{2}(\mathrm{p} - a(x))^2 + U_p(x,\theta_\alpha)\right) f(x,\theta_\alpha) + \sum_{\beta=1}^{p-1} U_\beta(x,\theta_\alpha) f(x,\theta_{\alpha+\beta}). \tag{5.13}$$

We introduce assumptions on the generalized spin:

**Assumption 5.5** *Let $U_\beta$ be defined in Definition 5.1, $u_\beta$ by (5.10) and $c_\beta$ by (5.9). We consider the following conditions:*

**(U1)** $\sum_{\beta=1}^p c_\beta < 1$.

**(U2)** $u_\beta \in L^\infty(\mathbb{R}^d)$ *for* $\beta = 1, ..., p$

Assumption (U1) is a sufficient condition for self-adjointness of Schrödinger operator with generalized spin. (U2) is a stronger assumption than (U1). (U2) is used to construct a functional integral representation for the Schrödinger operator with generalized spin. Although (U2) can be relaxed, we do not consider weaker conditions here; see [HIL11], where we shall discuss relativistic Schrödinger operator with spin 1/2 under weaker conditions than (U2).

**Theorem 5.6** *Let Assumption (A2) and (U1) be satisfied. Then $h_{\mathbb{Z}_p}$ is self-adjoint on $\ell^2(\mathbb{Z}_p)\otimes D(h)$ and bounded from below. Moreover, it is essentially self-adjoint on any core of $1\otimes h$. In particular, $\ell^2(\mathbb{Z}_p)\otimes C_0^\infty(\mathbb{R}^d)$ is an operator core of $h_{\mathbb{Z}_p}$.*

PROOF. It can be seen that

$$\sum_{\alpha=1}^p \overline{g(x,\theta_\alpha)}\left(\sum_{\beta=1}^{p-1} W_\beta(x,\theta_{\alpha+\beta})f(x,\theta_{\alpha+\beta})\right) = \sum_{\gamma=1}^p \left(\sum_{\beta=1}^{p-1} \overline{\overline{W_{p-\beta}(x,\theta_\gamma)}g(x,\theta_{\gamma+\beta})}\right) f(x,\theta_\gamma)$$

for every $x\in\mathbb{R}^d$. Then it follows that

$$(g(x,\cdot), M_{\mathbb{Z}_p}f(x,\cdot))_{\ell^2(\mathbb{Z}_p)} = (M_{\mathbb{Z}_p}g(x,\cdot), f(x,\cdot))_{\ell^2(\mathbb{Z}_p)}$$

and $M_{\mathbb{Z}_p}$ is symmetric. Its norm can be estimated as $\|M_{\mathbb{Z}_p}f\|\le\sum_{\beta=1}^p\|(1\otimes u_\beta)f\|$ by Definition 5.1. Then with $h_0=\frac{1}{2}\mathrm{p}^2$ and $E>0$, we have by the proof of Theorem 3.11, $\|u_\beta(h+E)^{-1}g\|\le\|u_\beta(h_0+E)^{-1}|g|\|$, and hence

$$\|M_{\mathbb{Z}_p}f\|\le\sum_{\beta=1}^p\|u_\beta(h_0+E)^{-1}\|\|1\otimes(h+E)f\|\le\sum_{\beta=1}^p c_\beta\|(1\otimes h)f\|+b\|f\|$$

with a suitable constant $b$. Thus the claim follows from the Kato-Rellich theorem. **qed**

**Definition 5.7 (Generalized Schrödinger operator with spin)** Suppose that Assumption (A2) and (U1) hold. Recall that $\mathscr{E}_{h_{\mathbb{Z}_p}}$ denotes $\inf\mathrm{Spec}(h_{\mathbb{Z}_p})$. Let $\Psi\in\mathscr{B}_0$ and put

$$\overline{h_{\mathbb{Z}_p}} = \begin{cases} h_{\mathbb{Z}_p} & \text{if } \mathscr{E}_{h_{\mathbb{Z}_p}}\ge 0,\\ h_{\mathbb{Z}_p}-\mathscr{E}_{h_{\mathbb{Z}_p}} & \text{if } \mathscr{E}_{h_{\mathbb{Z}_p}}<0.\end{cases} \tag{5.14}$$

We call the operator

$$H_{\mathbb{Z}_p}^\Psi = \Psi\left(\overline{h_{\mathbb{Z}_p}}\right)+V \tag{5.15}$$

*generalized Schrödinger operator with vector potential $a$ and spin $\mathbb{Z}_p$.*

**Corollary 5.8** *Suppose that Assumption (A2) and (U1) hold. If $\Psi\in\mathscr{B}_0$, then $\ell^2(\mathbb{Z}_p)\otimes C_0^\infty(\mathbb{R}^d)$ is an operator core of $\Psi(h_{\mathbb{Z}_p})$.*

PROOF. Since $h_{\mathbb{Z}_p}$ is essentially self-adjoint on $\ell^2(\mathbb{Z}_p)\otimes C_0^\infty(\mathbb{R}^d)$, the corollary can be proven in the same way as Theorem 3.3. **qed**

## 5.2 Functional integral representation

In this subsection we give a functional integral representation of $e^{-tH_{\mathbb{Z}_p}^{\Psi}}$ by means of Brownian motion, a jump process and a subordinator.

Let $(N_t^\beta)_{t\geq 0}$, $\beta = 1, ..., p-1$, be $p-1$ independent Poisson processes with unit intensity on a probability space $(\Omega_N, \mathscr{F}_N, \mu)$, i.e., $\mu(N_t^\beta = n) = e^{-t}t^n/n!$. Define the random process $(N_t)_{t\geq 0}$ by

$$N_t = \sum_{\beta=1}^{p-1} \beta N_t^\beta. \tag{5.16}$$

Let $\mathscr{F}_t^N = \sigma(N_t, t \leq s)$ be the natural filtration. Then since $N_t$ is a Lévy process, it is a Markov process with respect to $\mathscr{F}_t^N$. We write $\mathbb{E}_\mu[f(N_t + \alpha)]$ as $\mathbb{E}_\mu^\alpha[f(N_t)]$. Also, $\mathbb{E}_\mu^\alpha[1_{N_0=\alpha}] = 1$. Define

$$\int_v^{w+} g(N_{s-})dN_s^\beta = \sum_{\substack{v\leq r\leq w \\ N_{r+}^\beta \neq N_{r-}^\beta}} g(N_{r-}). \tag{5.17}$$

It can be seen that

$$\mathbb{E}_\mu\left[\int_v^{w+} g(N_{s-})dN_s^\beta\right] = \mathbb{E}_\mu\left[\int_v^w g(N_s)ds\right]. \tag{5.18}$$

The next lemma is an extension of a result obtained in [ALS83, HL08].

**Lemma 5.9** *Suppose that Assumptions (A2) and (U2) hold, and*

$$\int_0^t ds \int_{\mathbb{R}^d} dy \hat{\mathrm{P}}_s(x-y)|\log u_\beta(y)| < \infty, \quad \beta = 1, ..., p-1, \tag{5.19}$$

*where* $\hat{\mathrm{P}}_s(x-y)$ *is the heat kernel given by (3.20). Then*

$$(f, e^{-th_{\mathbb{Z}_p}}g) = e^{(p-1)t}\sum_{\alpha=1}^{p}\int_{\mathbb{R}^d} dx\, \mathbb{E}_{P\times\mu}^{x,\alpha}\left[\overline{f(B_0, \theta_{N_0})}g(B_t, \theta_{N_t})e^{\mathrm{S}}\right], \tag{5.20}$$

*where* $\mathrm{S} = \mathrm{S}_a + \mathrm{S}_{\mathrm{spin}}$ *and*

$$\mathrm{S}_a = -i\int_0^t a(B_s) \circ dB_s,$$

$$\mathrm{S}_{\mathrm{spin}} = -\int_0^t U_p(B_s, \theta_{N_s})ds + \sum_{\beta=1}^{p-1}\int_0^{t+} \log(-U_\beta(B_s, \theta_{N_{s-}}))dN_s^\beta.$$

*Here we take* $\log z$ *with the principal branch for* $z \in \mathbb{C}$.

PROOF. First assume that the diagonal part $U_p(x,\theta_\alpha)$ and the off-diagonal part $U_\beta(x,\theta_\alpha)$ are continuous in $x$ and $a \in (C_0^\infty(\mathbb{R}^d))^d$. Since from (5.19) and (5.18) it follows that

$$\mathbb{E}_{P\times\mu}^{x,\alpha}\left[\int_0^{t+} |\log(-U_\beta(B_s,\theta_{N_{s-}}))|dN_s^\beta\right] \le \int_0^t ds \int_{\mathbb{R}^d} \hat{\mathrm{P}}_s(x-y)|\log u_\beta(y)|dy < \infty,$$

we note that

$$\int_0^{t+} |\log(-U_\beta(B_s,\theta_{N_{s-}}))|dN_s^\beta < \infty \tag{5.21}$$

almost surely. By the estimate $|c\mathrm{S}_{\mathrm{spin}}| \le c\|u_p\|_\infty t + \sum_{\beta=1}^{p-1} |\log \|u_\beta\|_\infty^c| N_t^\beta$ and the equality

$$\mathbb{E}_\mu^0\left[\exp\left(\sum_{\beta=1}^{p-1} r_\beta N_t^\beta\right)\right] = \exp\left(t\sum_{\beta=1}^{p-1}(e^{r_\beta}-1)\right)$$

for $r_\beta \in \mathbb{R}$, we have for $c>0$,

$$\left|\mathbb{E}_{P\times\mu}^{x,\alpha}[e^{c\mathrm{S}_{\mathrm{spin}}}]\right| \le \exp\left(t\left(c\|u_p\|_\infty + \sum_{\beta=1}^{p-1}(\|u_\beta\|_\infty^c - 1)\right)\right), \tag{5.22}$$

where $u_\beta$ is given in (5.9). Denote

$$Z_{[v,w]} = -i\int_v^w a(B_s)\circ dB_s - \int_v^w U_p(B_s,\theta_{N_s})ds + \sum_{\beta=1}^{p-1}\int_v^{w+} \log(-U_\beta(B_s,\theta_{N_{s-}}))dN_s^\beta$$

and let

$$\mathrm{P}_t g(x,\theta_\alpha) = \mathbb{E}_{P\times\mu}^{x,\alpha}\left[e^{Z_{[0,t]}} g(B_t,\theta_{N_t})\right].$$

Let $g \in \ell^2(\mathbb{Z}_p)\otimes C_0^\infty(\mathbb{R}^d)$. By using the Schwarz inequality and setting $c=2$ in (5.22) we have the estimate

$$\|\mathrm{P}_t g\|^2 \le \exp\left(t\left(2\|u_p\|_\infty + \sum_{\beta=1}^{p-1}(\|u_\beta\|_\infty^2 - 1)\right)\right)\|g\|^2.$$

Thus $\mathrm{P}_t$ is bounded. We show now that $\{\mathrm{P}_t : t\ge 0\}$ is a $C_0$-semigroup with generator $-(h_{\mathbb{Z}_p}+p-1)$, i.e., (1) $\mathrm{P}_0 = I$, (2) $\mathrm{P}_s\mathrm{P}_t = \mathrm{P}_{s+t}$, (3) $\mathrm{P}_t g$ is continuous in $t$ and (4) $\lim_{t\to 0}\frac{1}{t}(\mathrm{P}_t g - g) = -(h_{\mathbb{Z}_p}+(p-1))g$ in strong sense. First, (1) is trivial. To check (2) notice that

$$\mathrm{P}_t\mathrm{P}_s g(x,\theta_\alpha) = \mathbb{E}_{P\times\mu}^{x,\alpha}\left[e^{Z_{[0,t]}}\mathbb{E}_{P\times\mu}^{B_t,N_t}\left[e^{Z_{[0,s]}}g(B_s,\theta_{N_s})\right]\right]. \tag{5.23}$$

By the Markov property of $B_t$ we have

$$\begin{aligned}(5.23) = \mathbb{E}_{P\times\mu}^{x,\alpha}\Bigg[&e^{Z_{[0,t]}}\exp\left(-i\int_t^{t+s} a(B_r)\circ dB_r\right)\\ \mathbb{E}_\mu^{N_t}\Bigg[&\exp\left(-\int_0^s U_p(B_{t+r},\theta_{N_r})dr + \sum_{\beta=1}^{p-1}\int_0^{s+}\log(-U_\beta(B_{t+r-},\theta_{N_{r-}}))dN_r^\beta\right)g(B_{t+s},\theta_{N_s})\Bigg]\Bigg].\end{aligned} \tag{5.24}$$

Furthermore the Markov property of $N_t$ yields that

$$(5.24) = \mathbb{E}^{x,\alpha}_{P\times\mu}\left[e^{Z_{[0,t]}}e^{Z_{[t,t+s]}}g(B_{t+s},\theta_{N_{t+s}})\right] = \mathrm{P}_{s+t}g(x,\theta_\alpha).$$

This proves the semigroup property (2). Next we obtain the generator of $\mathrm{P}_t$. An application of the Itô formula (see Appendix A) yields that

$$dN_t = \sum_{\beta=1}^{p-1}\int_0^{t+}\beta dN_s^\beta, \quad d\theta_{N_t} = \sum_{\beta=1}^{p-1}(\theta_{N_t+\beta}-\theta_{N_t})$$

and

$$\begin{aligned} dg(B_t,\theta_{N_t}) &= \int_0^t \nabla g(B_s,\theta_{N_s})\cdot dB_s + \frac{1}{2}\int_0^t \Delta g(B_s,\theta_{N_s})ds \\ &\qquad + \sum_{\beta=1}^{p-1}\int_0^{t+}(g(B_s,\theta_{N_s+\beta})-g(B_s,\theta_{N_s}))dN_s^\beta \\ de^{Z_{[0,t]}} &= \int_0^t e^{Z_{[0,s]}}(-ia(B_s))\cdot dB_s + \frac{1}{2}\int_0^t e^{Z_{[0,s]}}(-i\nabla\cdot a(B_s)-a(B_s)^2)ds \\ &\quad - \int_0^t e^{Z_{[0,s]}}U_p(B_s,\theta_{N_s})ds + \sum_{\beta=1}^{p-1}\int_0^{t+}e^{Z_{[0,s-]}}\left(e^{\log(-U_\beta(B_s,\theta_{N_{s-}}))}-1\right)dN_s^\beta. \end{aligned}$$

The product formula (see Appendix A) $d\left(e^{Z_{[0,t]}}g\right) = de^{Z_{[0,t]}}\cdot g + e^{Z_{[0,t]}}\cdot dg + de^{Z_{[0,t]}}\cdot dg$ furthermore gives

$$\begin{aligned} d\left(e^{Z_{[0,t]}}g\right)(B_t,\theta_{N_t}) &= \int_0^t e^{Z_{[0,s]}}\left\{\frac{1}{2}\Delta g(B_s,\theta_{N_s}) - ia(B_s)\cdot\nabla g(B_s,\theta_{N_s})\right. \\ &\quad \left. + \left(\frac{1}{2}(-i\nabla\cdot a)(B_s) - \frac{1}{2}a(B_s)^2 - U_p(B_s,\theta_{N_s})\right)g(B_s,\theta_{N_s})\right\}ds \\ &\quad + \int_0^t e^{Z_{[0,s]}}\Big(\nabla g(B_s,\theta_{N_s}) - ia(B_s)g(B_s,\theta_{N_s})\Big)\cdot dB_s \\ &\quad + \sum_{\beta=1}^{p-1}\int_0^{t+}e^{Z_{[0,s-]}}\Big(g(B_s,\theta_{N_{s-}+\beta})e^{\log(-U_\beta(B_s,\theta_{N_{s-}}))} - g(B_s,\theta_{N_{s-}})\Big)dN_s^\beta. \end{aligned}$$

Taking expectation values on both sides above yields

$$\frac{1}{t}(f,(\mathrm{P}_t-1)g) = \frac{1}{t}\int_0^t ds\int_{\mathbb{R}^d}dx\overline{f(x)}\mathbb{E}^{x,\alpha}_{P\times\mu}\left[G(s)\right],$$

where

$$G(s) = e^{Z_{[0,s]}} \left( \frac{1}{2}\Delta - ia(B_s)\cdot\nabla + \frac{1}{2}(-i\nabla\cdot a)(B_s) - \frac{1}{2}a(B_s)^2 - U_p(B_s,\theta_{N_s}) \right) g(B_s,\theta_{N_s})$$
$$+ \sum_{\beta=1}^{p-1} e^{Z_{[0,s]}} \left( g(B_s,\theta_{N_s+\beta}) e^{\log(-U_\beta(B_s,\theta_{N_s}))} - g(B_s,\theta_{N_s}) \right),$$
$$G(0) = \left( \frac{1}{2}\Delta - ia(B_0)\cdot\nabla + \frac{1}{2}(-i\nabla\cdot a)(B_0) - \frac{1}{2}a(B_0)^2 - U_p(B_0,\theta_{N_0}) \right) g(B_0,\theta_{N_0})$$
$$+ \sum_{\beta=1}^{p-1} (-U_\beta(B_0,\theta_{N_0}) g(B_0,\theta_{N_0+\beta}) - g(B_0,\theta_{N_0}))$$
$$= -(h_{\mathbb{Z}_p} + (p-1)) g(x,\theta_\alpha).$$

Note that $U_\beta(x,\theta)$ and $a_\mu(x)$ are continuous in $x$. Therefore $G(s)$ is continuous at $s=0$ for each $(\omega,\tau)\in\Omega_P\times\Omega_N$, and $\mathbb{E}^{x,\alpha}_{P\times\mu}[G(s)]$ is continuous at $s=0$ by the dominated convergence theorem. Thus

$$\lim_{t\to 0}\frac{1}{t}(f,(\mathrm{P}_t-1)g) = (f,-(h_{\mathbb{Z}_p}+(p-1))g)$$

follows. Finally, the strong continuity (3) follows from (2) and (4), and hence

$$e^{t(p-1)}\mathrm{P}_t g = e^{-th_{\mathbb{Z}_p}} g. \tag{5.25}$$

By a similar approximation argument as in the proof of Proposition 3.7, (5.25) can be extended to $a$ obeying Assumption (A2). Finally, we extend (5.25) for $U_\beta$ given in Definition 5.1. By using a mollifier it is seen that there exists a sequence $U_\beta^{(n)}(x,\theta_\alpha)$, $n=1,2,3,...$, such that they are continuous in $x$ and converge to $U_\beta(x,\theta_\alpha)$ for each $x$ as $n\to\infty$, and $\|U_\beta^{(n)}(\cdot,\theta_\alpha)\|_\infty \le \|U_\beta(\cdot,\theta_\alpha)\|_\infty$. For each fixed $\tau\in\Omega_N$ there exists $r_1=r_1(\tau),...,r_M=r_M(\tau)$, where $M=M(\tau)$, such that

$$e^{\sum_{\beta=1}^{p-1}\int_0^{t+}\log(-U_\beta(B_s,\theta_{N_{s-}}))dN_s^\beta} = \prod_{\beta=1}^{p-1}\prod_{i=1}^{M}(-U_\beta(B_{r_i},\theta_{N_{r_i}})). \tag{5.26}$$

Then for each $\tau\in\Omega_N$,

$$\lim_{n\to\infty} e^{\sum_{\beta=1}^{p-1}\int_0^{t+}\log(-U_\beta^{(n)}(B_s,\theta_{N_{s-}}))dN_s^\beta} = e^{\sum_{\beta=1}^{p-1}\int_0^{t+}\log(-U_\beta(B_s,\theta_{N_{s-}}))dN_s^\beta}. \tag{5.27}$$

In the same way as above we can also see that $e^{-\int_0^t U_p^{(n)}(B_s,\theta_{N_s})ds} \to e^{-\int_0^t U_p(B_s,\theta_{N_s})ds}$ as $n\to\infty$ almost surely. Therefore by the dominated convergence theorem (5.25) holds for such $U_\beta$ and $U$. **qed**

Now we can state and prove the functional integral representation of $e^{-tH^\Psi_{\mathbb{Z}_p}}$.

**Theorem 5.10** *Let $\Psi \in \mathscr{B}_0$ and $V \in L^\infty(\mathbb{R}^d)$. Let Assumptions (A2) and (U2) be satisfied, and suppose*

$$\int_{\mathbb{R}} \rho(r,t)dr \int_0^r ds \int_{\mathbb{R}^d} dy \hat{\mathrm{P}}_s(x-y)|\log u_\beta(y)| < \infty, \quad \beta = 1,...,p-1, \tag{5.28}$$

*where $\rho(r,t)$ is the distribution density of $T_t^\Psi$ on $\mathbb{R}$ and $u_\beta$ is given in (5.9). Then*

$$(f, e^{-tH^\Psi_{\mathbb{Z}_p}} g) = \sum_{\alpha=1}^p \int_{\mathbb{R}^d} dx\, \mathbb{E}^{x,\alpha,0}_{P\times\mu\times\nu}\left[e^{(p-1)T_t^\Psi}\overline{f(B_0,\theta_{N_0})}g(B_{T_t^\Psi},\theta_{N_{T_t^\Psi}})e^{\mathrm{S}^\Psi}\right], \tag{5.29}$$

*where $\mathrm{S}^\Psi = \mathrm{S}^\Psi_V + \mathrm{S}^\Psi_a + \mathrm{S}^\Psi_{\mathrm{spin}}$ and*

$$\mathrm{S}^\Psi_V = -\int_0^t V(B_{T_s^\Psi})ds, \quad \mathrm{S}^\Psi_a = -i\int_0^{T_t^\Psi} a(B_s)\circ dB_s,$$

$$\mathrm{S}^\Psi_{\mathrm{spin}} = \begin{cases} -\int_0^{T_t^\Psi}\left(U_p(B_s,\theta_{N_s}) - \mathscr{E}_{h_{\mathbb{Z}_p}}\right)ds + \sum_{\beta=1}^{p-1}\int_0^{T_t^\Psi+}\log(-U_\beta(B_s,\theta_{N_{s-}}))dN_s^\beta \\ \qquad\qquad \text{if } \mathscr{E}_{h_{\mathbb{Z}_p}} < 0, \\ -\int_0^{T_t^\Psi}\left(U_p(B_s,\theta_{N_s})\right)ds + \sum_{\beta=1}^{p-1}\int_0^{T_t^\Psi+}\log(-U_\beta(B_s,\theta_{N_{s-}}))dN_s^\beta \\ \qquad\qquad \text{if } \mathscr{E}_{h_{\mathbb{Z}_p}} \geq 0. \end{cases}$$

Proof. Since from (5.28) it follows that

$$\begin{aligned}\mathbb{E}^{x,\alpha,0}_{P\times\mu\times\nu}&\left[\int_0^{T_t^\Psi+}|\log(-U_\beta(B_s,\theta_{N_{s-}}))|dN_s^\beta\right] \\ &\leq \int_{\mathbb{R}}\rho(r,t)dr\int_0^r ds\int_{\mathbb{R}^d}\hat{\mathrm{P}}_s(x-y)|\log u_\beta(y)|dy < \infty,\end{aligned}$$

we notice that $\int_0^{T_t^\Psi+}|\log(-U_\beta(B_s,\theta_{N_{s-}}))|dN_s^\beta < \infty$ almost surely. Using Lemma 5.9 we obtain similarly like in the proof of (3.16) that

$$\left(f, e^{-t\Psi(\overline{h_{\mathbb{Z}_p}})}g\right) = \sum_{\alpha=1}^p \int_{\mathbb{R}^d} dx\, \mathbb{E}^{x,\alpha,0}_{P\times\mu\times\nu}\left[e^{(p-1)T_t^\Psi}\overline{f(B_0,\theta_{N_0})}g(B_{T_t^\Psi},\theta_{N_{T_t^\Psi}})e^{\mathrm{S}^\Psi_a+\mathrm{S}^\Psi_{\mathrm{spin}}}\right]. \tag{5.30}$$

Let $0 = t_0 < t_1 < \cdots < t_n = t$. We show that

$$\begin{aligned}&\left(f_0, \prod_{j=1}^n e^{-(t_j-t_{j-1})\Psi(\overline{h_{\mathbb{Z}_p}})}f_j\right) \\ &\quad = \sum_{\alpha=1}^p \int_{\mathbb{R}^d} dx\, \mathbb{E}^{x,\alpha,0}_{P\times\mu\times\nu}\left[e^{(p-1)T_t^\Psi}\overline{f_0(B_0,\theta_{N_0})}\left(\prod_{j=1}^n f_j(B_{T^\Psi_{t_j}},\theta_{N_{T^\Psi_{t_j}}})\right)e^{\mathrm{S}^\Psi_a+\mathrm{S}^\Psi_{\mathrm{spin}}}\right]. \end{aligned} \tag{5.31}$$

This can be proven in the same way as in Step 2 of the proof of Theorem 3.8 with the $d$-dimensional Brownian motion $B_t$ on $(\Omega_P, \mathscr{F}_P, P^x)$ replaced by the $d+1$ dimensional Markov process $(B_t, N_t)$ on $(\Omega_P \times \Omega_N, \mathscr{F}_P \times \mathscr{F}_N, P^x \times \mu)$ under the natural filtration. Note that when $V$ is continuous, by the Trotter product formula and (5.31) it follows that

$$\begin{aligned}\left(f, e^{-tH^{\Psi}_{\mathbb{Z}_p}} g\right) &= \lim_{n\to\infty} \left(f, \left(e^{-(t/n)\Psi\left(h_{\mathbb{Z}_p}\right)} e^{-(t/n)V}\right)^n g\right) \\ &= \lim_{n\to\infty} \sum_{\alpha=1}^{p} \int_{\mathbb{R}^d} dx\, \mathbb{E}^{x,\alpha,0}_{P\times\mu\times\nu} \\ &\quad \left[ e^{(p-1)T^{\Psi}_t} \overline{f\left(B_0, \theta_{N_0}\right)} e^{-\sum_{j=1}^{n} \frac{t}{n} V\left(B_{T^{\Psi}_{jt/n}}\right)} g\left(B_{T_t}, \theta_{N_{T^{\Psi}_t}}\right) e^{\mathrm{S}^{\Psi}_a + \mathrm{S}^{\Psi}_{\mathrm{spin}}} \right] \\ &= \sum_{\alpha=1}^{p} \int_{\mathbb{R}^d} dx\, \mathbb{E}^{x,\alpha,0}_{P\times\mu\times\nu} \left[ e^{(p-1)T^{\Psi}_t} \overline{f\left(B_0, \theta_{N_0}\right)} g\left(B_{T^{\Psi}_t}, \theta_{N_{T^{\Psi}_t}}\right) e^{\mathrm{S}^{\Psi}} \right].\end{aligned}$$

Hence the theorem holds for continuous $V$. This can be extended for $V \in L^\infty(\mathbb{R}^d)$ in the same way as in Step 4 of the proof of Theorem 3.8. **qed**

**Remark 5.11** *In the case of* $\Psi(u) = \sqrt{2u + m^2} - m$*, the distribution of* $T^{\Psi}_t$ *is explicitly given by (2.4).*

Now let $h^0_{\mathbb{Z}_p}$ be defined by $h_{\mathbb{Z}_p}$ in (5.12) with $a$ and $U_\beta$, $\beta = 1, ..., p-1$, replaced by 0 and $|U_\beta|$, respectively, i.e.,

$$\left(h^0_{\mathbb{Z}_p} f\right)(x, \theta_\alpha) = \frac{1}{2}\mathrm{p}^2 f(x, \theta_\alpha) + U_p(x, \theta_\alpha) f(x, \theta_\alpha) - \sum_{\beta=1}^{p-1} |U_\beta(x, \sigma)| f(x, \theta_{\alpha+\beta}). \quad (5.32)$$

Let

$$\overline{h^0_{\mathbb{Z}_p}} = \begin{cases} h^0_{\mathbb{Z}_p} & \text{if } \mathscr{E}_{h^0_{\mathbb{Z}_p}} \geq 0, \\ h^0_{\mathbb{Z}_p} - \mathscr{E}_{h^0_{\mathbb{Z}_p}} & \text{if } \mathscr{E}_{h^0_{\mathbb{Z}_p}} < 0. \end{cases} \quad (5.33)$$

An immediate corollary of Theorem 5.10 is

**Corollary 5.12** (**Diamagnetic inequality**) *Under the assumptions of Theorem 5.10 we have* $h_{\mathbb{Z}_p} - \mathscr{E}_{h^0_{\mathbb{Z}_p}} \geq 0$. *Moreover,*

*(1) if* $\mathscr{E}_{h^0_{\mathbb{Z}_p}} \geq 0$, *then*

$$\left|\left(f, e^{-t\left(\Psi\left(h_{\mathbb{Z}_p}\right)+V\right)} g\right)\right| \leq \left(|f|, e^{-t\left(\Psi\left(\overline{h^0_{\mathbb{Z}_p}}\right)+V\right)} |g|\right) \quad (5.34)$$

*and* $\mathscr{E}_{\Psi\left(\overline{h^0_{\mathbb{Z}_p}}\right)+V} \leq \mathscr{E}_{\Psi\left(h_{\mathbb{Z}_p}\right)+V}$*;*

*(2) if* $\mathscr{E}_{h^0_{\mathbb{Z}_p}} < 0$, *then*

$$\left|\left(f, e^{-t\left(\Psi\left(h_{\mathbb{Z}_p}-\mathscr{E}_{h^0_{\mathbb{Z}_p}}\right)+V\right)}g\right)\right| \leq \left(|f|, e^{-t\left(\Psi\left(\overline{h^0_{\mathbb{Z}_p}}\right)+V\right)}|g|\right) \tag{5.35}$$

*and* $\mathscr{E}_{\Psi\left(\overline{h^0_{\mathbb{Z}_p}}\right)+V} \leq \mathscr{E}_{\Psi\left(h_{\mathbb{Z}_p}-\mathscr{E}_{h^0_{\mathbb{Z}_p}}\right)+V}$.

Proof. Note the estimate

$$\left|\exp\left(\sum_{\beta=1}^{p-1}\int_0^{T_t^\Psi+}\log\left(-U_\beta\left(\theta_{N^\beta_{s-}}\right)\right)dN_s^\beta\right)\right| \leq \exp\left(\sum_{\beta=1}^{p-1}\int_0^{T_t^\Psi+}\log|U_\beta\left(\theta_{N^\beta_{s-}}\right)|dN_s^\beta\right). \tag{5.36}$$

Let $\Psi(u)=u$ and then $T_t^\Psi = t$. Theorem 5.10 and (5.36) imply that

$$|\left(f, e^{-th_{\mathbb{Z}_p}}g\right)| \leq \left(|f|, e^{-th^0_{\mathbb{Z}_p}}|g|\right). \tag{5.37}$$

This further implies $\mathscr{E}_{h^0_{\mathbb{Z}_p}} \leq \mathscr{E}_{h_{\mathbb{Z}_p}}$, thus $h_{\mathbb{Z}_p} - \mathscr{E}_{h^0_{\mathbb{Z}_p}} \geq 0$ holds. (5.34) and (5.35) follow similarly by Theorem 5.10 and the estimate (5.36). $\mathscr{E}_{\Psi\left(\overline{h^0_{\mathbb{Z}_p}}\right)+V} \leq \mathscr{E}_{\Psi\left(h_{\mathbb{Z}_p}\right)+V}$ and $\mathscr{E}_{\Psi\left(\overline{h^0_{\mathbb{Z}_p}}\right)+V} \leq \mathscr{E}_{\Psi\left(h_{\mathbb{Z}_p}-\mathscr{E}_{h^0_{\mathbb{Z}_p}}\right)+V}$ are an immediate consequence of (5.34) and (5.35), respectively. **qed**

**Theorem 5.13** *Let Assumptions (A2), (U2) and (5.19) be satisfied. If* $|V|$ *is relatively bounded with respect to* $\Psi\left(\overline{h^0_{\mathbb{Z}_p}}\right)$ *with a relative bound* $b$*, then* $|V|$ *is relatively bounded with respect to* $\Psi\left(\overline{h_{\mathbb{Z}_p}}\right)$ *with a relative bound not larger than* $b$.

Proof. We prove the theorem in the case of $\mathscr{E}_{h^0_{\mathbb{Z}_p}} < 0$; the case $\mathscr{E}_{h^0_{\mathbb{Z}_p}} \geq 0$ is simpler. By assumption we have for every $\varepsilon > 0$,

$$\|Vf\| \leq (b+\varepsilon)\|\Psi\left(\overline{h^0_{\mathbb{Z}_p}}\right)f\| + c\|f\|. \tag{5.38}$$

In virtue of Corollary 5.12 we have

$$\frac{\||V|\left(\Psi\left(h_{\mathbb{Z}_p}-\mathscr{E}_{h^0_{\mathbb{Z}_p}}\right)+E\right)^{-1}f\|}{\|f\|} \leq \frac{\||V|\left(\Psi\left(\overline{h^0_{\mathbb{Z}_p}}\right)+E\right)^{-1}|f|\|}{\|f\|} \tag{5.39}$$

By (5.38) the right hand side of (5.39) converges to a number smaller than $b+\varepsilon$ as $E \to \infty$. Thus

$$\|Vf\| \leq (b+\varepsilon)\|\Psi\left(h_{\mathbb{Z}_p}-\mathscr{E}_{h^0_{\mathbb{Z}_p}}\right)f\| + c_b\|f\| \tag{5.40}$$

follows with some constant $c_b$. Let $X < Y$ and $X < 0$. From (2.1) we see that

$$\Psi(u-X) - \Psi(u-Y) = b(Y-X) + \int_0^\infty e^{-(u-Y)y}(1-e^{-(Y-X)y})\lambda(dy), \quad u \geq Y.$$

Hence $\sup_{u\geq Y}|\Psi(u-X)-\Psi(u-Y)|\leq\Psi(Y-X)$. From this and $\mathscr{E}_{h^0_{\mathbb{Z}_p}}\leq\mathscr{E}_{h_{\mathbb{Z}_p}}$ we obtain that

$$\sup_{u\geq\mathscr{E}_{h_{\mathbb{Z}_p}}}|\Psi(u-\mathscr{E}_{h^0_{\mathbb{Z}_p}})-\Psi(u-\mathscr{E}_{h_{\mathbb{Z}_p}})|\leq\Psi(\mathscr{E}_{h_{\mathbb{Z}_p}}-\mathscr{E}_{h^0_{\mathbb{Z}_p}}).$$

Thus the spectral decomposition yields that

$$\|\Psi(h_{\mathbb{Z}_p}-\mathscr{E}_{h^0_{\mathbb{Z}_p}})f\|\leq\|\Psi(h_{\mathbb{Z}_p}-\mathscr{E}_{h_{\mathbb{Z}_p}})f\|+\Psi(\mathscr{E}_{h_{\mathbb{Z}_p}}-\mathscr{E}_{h^0_{\mathbb{Z}_p}})\|f\|.$$

Then the theorem follows together with (5.40), since $\epsilon$ is arbitrary. **qed**

We have the immediate consequences below.

**Theorem 5.14** *Let Assumptions (U2) be satisfied, and suppose that $V$ is relatively bounded with respect to $\Psi(\overline{h^0_{\mathbb{Z}_p}})$ with a relative bound strictly less than 1. Moreover, assume (5.19).*

*(1) Let Assumption (A2) be satisfied. Then $H^\Psi_{\mathbb{Z}_p}$ is self-adjoint on $D\left(\Psi\left(\overline{h_{\mathbb{Z}_p}}\right)\right)$ and essentially self-adjoint on any core of $\Psi\left(\overline{h_{\mathbb{Z}_p}}\right)$. In particular, under Assumption (A3) the operator $H^\Psi_{\mathbb{Z}_p}$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$.*

*(2) Let Assumption (A3) be satisfied. Then the functional integral representation of $e^{-tH^\Psi_{\mathbb{Z}_p}}$ is given by (5.10).*

Proof. (1) is trivial. (2) is proven in a similar way to the approxiamtion argument in Step 4 of Theorem 3.8. **qed**

# A Appendix

Let $F\in C^2(\mathbb{R})$. The differential of the transformed process $dF(L_t)$ can be computed by the following Itô formula.

**Proposition A.1 (Itô formula)** *Let $\mathscr{F}_t=\sigma((B_s,N^\beta_s),0\leq s\leq t,\beta=1,...,p)$ be the natural filtration. Consider*

$$L^i_t=\int_0^t f^i(s,\omega)ds+\int_0^t g^i(s,\omega)\cdot dB_s+\sum_{\beta=1}^{p-1}\int_0^{t+}h^i_\beta(s,\omega)dN^\beta_s,\quad i=1,...,n$$

*where $f^i(\cdot,\omega)\in L^1_{\mathrm{loc}}(\mathbb{R})$ a.s, $g^i\in\mathscr{E}_{\mathrm{loc}}$ and $h^i_\beta(s,\omega)$ is adapted with respect to $\mathscr{F}_t$, left continuous in $s$ and $\int_0^{t+}|h^i_\beta(s,\omega)|dN^\beta_s<\infty$ a.s. Take $F\in C^2(\mathbb{R}^n)$. Then for the* *random process* $F(L_t)$ *the expression*

$$\begin{aligned} F(L_t) - F(L_0) &= \sum_{i=1}^{n} \int_0^t F_i(L_s) f^i(s) ds + \sum_{i,j=1}^{n} \int_0^t \frac{1}{2} F_{ij}(L_s) g^i(s) \cdot g^j(s) ds \\ &\quad + \sum_{i=1}^{n} \int_0^t F_i(L_s) g^i(s) \cdot dB_s + \sum_{\beta=1}^{p-1} \int_0^{t+} (F(L_{s-} + h_\beta(s)) - F(L_{s-})) dN_s^\beta \end{aligned}$$

*holds. Here* $F_i = \partial_i F$ *and* $F_{ij} = \partial_i \partial_j F$.

Furthermore, the following form of the product rule holds.

**Proposition A.2** (**Product rule**) *Let* $(L_t)_{t\geq 0}$ *and* $(M_t)_{t\geq 0}$ *be two random processes. Then* $L_t M_t - L_0 M_0 = \int_0^t dL_s \cdot M_s + \int_0^t L_s \cdot dM_s + \int_0^t dL_s \cdot dM_s$, *computed by the rules* $dtdt = 0$, $dB_t^\mu dt = 0$, $dB_t^\mu dB_t^\nu = \delta_{\mu\nu} dt$, $dN_t^\alpha dN_t^\beta = 0$, $dN_t^\alpha dt = 0$, *and* $dN_t^\alpha dB_t = 0$.

For proofs see, for instance, [IW81, LHB09].

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