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著者	Ito Tatsuro, Terwilliger Paul M.
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q-Inverting pairs of linear transformations and the q-tetrahedron algebra

Tatsuro Ito* and Paul Terwilliger†

Abstract

As part of our study of the q-tetrahedron algebra $\boxtimes_q$ we introduce the notion of a *q-inverting pair*. Roughly speaking, this is a pair of invertible semisimple linear transformations on a finite-dimensional vector space, each of which acts on the eigenspaces of the other according to a certain rule. Our main result is a bijection between the following two sets: (i) the isomorphism classes of finite-dimensional irreducible $\boxtimes_q$-modules of type 1; (ii) the isomorphism classes of q-inverting pairs.

Keywords. Tetrahedron algebra, q-tetrahedron algebra, Leonard pair, tridiagonal pair, q-tridiagonal pair.
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1 Introduction

Throughout this paper $\mathbb{F}$ denotes an algebraically closed field. We fix a nonzero $q \in \mathbb{F}$ that is not a root of 1.

The q-tetrahedron algebra $\boxtimes_q$ was introduced in [10] as part of the ongoing investigation of the Leonard pairs [4], [14], [15], [17], [18], [19], [20], [21], [22], [23], [24], [26] and tridiagonal pairs [1], [2], [5], [6], [7], [8], [11], [12], [13], [16]. The algebra $\boxtimes_q$ is a unital associative $\mathbb{F}$-algebra, and infinite-dimensional as a vector space over $\mathbb{F}$. We defined $\boxtimes_q$ by generators and relations. As explained in [10], $\boxtimes_q$ can be viewed as a q-analog of the three-point loop algebra $\mathfrak{sl}_2 \otimes_{\mathbb{F}} \mathbb{F}[t, t^{-1}, (t-1)^{-1}]$ (t indeterminate). The algebra $\boxtimes_q$ is related to the quantum group $U_q(\mathfrak{sl}_2)$ [10, Proposition 7.4], the $U_q(\mathfrak{sl}_2)$ loop algebra [10, Proposition 8.3], and the positive part of $U_q(\widehat{\mathfrak{sl}}_2)$ [10, Proposition 9.3]. In [10] we described the finite-dimensional irreducible $\boxtimes_q$-modules. From this description and from [8, Section 2] there emerges a characterization of the finite-dimensional irreducible $\boxtimes_q$-modules in terms of a certain kind of tridiagonal pair said to be *q-geometric*. For notational convenience, in the present paper we will refer to this as a *q-tridiagonal pair*. As we will review in Section 3, the following two sets are in bijection:

*Department of Computational Science, Faculty of Science, Kanazawa University, Kakuma-machi, Kanazawa 920-1192, Japan
†Department of Mathematics, University of Wisconsin, 480 Lincoln Drive, Madison WI 53706-1388 USA

(i) the isomorphism classes of finite-dimensional irreducible $\boxtimes_q$-modules of type 1; (ii) the isomorphism classes of q-tridiagonal pairs.

In the present paper we give a second characterization of the finite-dimensional irreducible $\boxtimes_q$-modules, this time using a linear algebraic object called a *q-inverting pair.* Roughly speaking, this is a pair of invertible semisimple linear transformations on a finite-dimensional vector space, each of which acts on the eigenspaces of the other according to a certain rule that we find attractive. Our main result is a bijection between the following two sets: (i) the isomorphism classes of finite-dimensional irreducible $\boxtimes_q$-modules of type 1; (ii) the isomorphism classes of q-inverting pairs.

The plan for the paper is as follows. In Section 2 we recall the algebra $\boxtimes_q$ and discuss its finite-dimensional irreducible modules. In Section 3 we review the notion of a q-tridiagonal pair, and show how these objects are related to the finite-dimensional irreducible $\boxtimes_q$-modules. In Section 4 we introduce the notion of a q-inverting pair, and discuss how these objects are related to the finite-dimensional irreducible $\boxtimes_q$-modules. Theorem 4.5 and Theorem 4.6 are the main results of the paper; Sections 5–8 are devoted to their proofs. In Section 9 we give some suggestions for further research.

2 The algebra $\boxtimes_q$

In this section we recall the q-tetrahedron algebra and discuss its finite-dimensional irreducible modules. We will use the following notation. Let $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$ denote the cyclic group of order 4. Define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad\qquad n = 0, 1, 2 \ldots$$

Definition 2.1 [10, Definition 6.1] Let $\boxtimes_q$ denote the unital associative $\mathbb{F}$-algebra that has generators

$$\{x_{rs} \mid r, s \in \mathbb{Z}_4,\ s - r = 1 \text{ or } s - r = 2\}$$

and the following relations:

(i) For $r, s \in \mathbb{Z}_4$ such that $s - r = 2$,

$$x_{rs} x_{sr} = 1. \tag{1}$$

(ii) For $r, s, t \in \mathbb{Z}_4$ such that the pair $(s - r, t - s)$ is one of $(1, 1), (1, 2), (2, 1)$,

$$\frac{q x_{rs} x_{st} - q^{-1} x_{st} x_{rs}}{q - q^{-1}} = 1. \tag{2}$$

(iii) For $r, s, t, u \in \mathbb{Z}_4$ such that $s - r = t - s = u - t = 1$,

$$x_{rs}^3 x_{tu} - [3]_q x_{rs}^2 x_{tu} x_{rs} + [3]_q x_{rs} x_{tu} x_{rs}^2 - x_{tu} x_{rs}^3 = 0. \tag{3}$$

We call $\boxtimes_q$ the *q-tetrahedron algebra.*

Note 2.2 The equations (3) are the cubic q-Serre relations [3].

We make some observations.

Lemma 2.3 *There exists an $\mathbb{F}$-algebra automorphism ρ of $\boxtimes_q$ that sends each generator x_{rs} to $x_{r+1,s+1}$. Moreover $\rho^4 = 1$.*

Lemma 2.4 *There exists an $\mathbb{F}$-algebra automorphism of $\boxtimes_q$ that sends each generator x_{rs} to $-x_{rs}$.*

We comment on the $\boxtimes_q$-modules. Let V denote a finite-dimensional irreducible $\boxtimes_q$-module. By [10, Theorem 12.3] there exist an integer $d \geq 0$ and a scalar $\varepsilon \in \{1, -1\}$ such that for each generator x_{rs} the action on V is semisimple with eigenvalues $\{\varepsilon q^{d-2i} \mid 0 \leq i \leq d\}$. We call d the *diameter* of V. We call ε the *type* of V. Replacing each generator x_{rs} by εx_{rs} the type becomes 1.

3 q-Tridiagonal pairs

In this section we recall the notion of a *q-tridiagonal pair* and discuss how these objects are related to the finite-dimensional irreducible $\boxtimes_q$-modules.

We will use the following notation. Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. Let $\{s_i\}_{i=0}^d$ denote a finite sequence consisting of positive integers whose sum is the dimension of V. By a *decomposition of V of shape* $\{s_i\}_{i=0}^d$ we mean a sequence $\{V_i\}_{i=0}^d$ of subspaces of V such that V_i has dimension s_i for $0 \leq i \leq d$ and

$$V = \sum_{i=0}^{d} V_i \qquad \text{(direct sum)}.$$

We call d the *diameter* of the decomposition. For $0 \leq i \leq d$ we call V_i the i^{th} *component* of the decomposition. For notational convenience we define $V_{-1} = 0$ and $V_{d+1} = 0$. By the *inversion* of the decomposition $\{V_i\}_{i=0}^d$ we mean the decomposition $\{V_{d-i}\}_{i=0}^d$.

Definition 3.1 ([5, Definition 1.1], [8, Definition 2.6]) Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. By a *q-tridiagonal pair on V* we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy (i)–(iii) below.

(i) There exists a decomposition $\{V_i\}_{i=0}^d$ of V such that

$$(A - q^{d-2i} I)V_i = 0 \qquad (0 \leq i \leq d),$$
$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \leq i \leq d).$$

(ii) There exists a decomposition $\{V_i^*\}_{i=0}^{\delta}$ of V such that

$$
\begin{aligned}
(A^* - q^{\delta-2i} I)V_i^* &= 0 \qquad\qquad (0 \le i \le \delta), \\
AV_i^* &\subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad\qquad (0 \le i \le \delta).
\end{aligned}
$$

(iii) There does not exist a subspace $W \subseteq V$ such that $AW \subseteq W$ and $A^*W \subseteq W$, other than $W = 0$ and $W = V$.

We say the pair A, A^* is *over* $\mathbb{F}$. We call V the *underlying vector space*.

Note 3.2 According to a common notational convention A^* denotes the conjugate transpose of A. We are not using this convention. In a q-tridiagonal pair A, A^* the linear transformations A and A^* are arbitrary subject to (i)–(iii) above.

Note 3.3 The integers d and δ from Definition 3.1 are equal [5, Lemma 4.5]; we call this common value the *diameter* of the pair.

We now recall the notion of *isomorphism* for q-tridiagonal pairs.

Definition 3.4 Let A, A^* and $A', A^{*\prime}$ denote q-tridiagonal pairs over $\mathbb{F}$. By an *isomorphism of q-tridiagonal pairs* from A, A^* to $A', A^{*\prime}$ we mean a vector space isomorphism σ from the vector space underlying A, A^* to the vector space underlying $A', A^{*\prime}$ such that $\sigma A = A'\sigma$ and $\sigma A^* = A^{*\prime}\sigma$. We say that A, A^* and $A', A^{*\prime}$ are *isomorphic* whenever there exists an isomorphism of q-tridiagonal pairs from A, A^* to $A', A^{*\prime}$.

Our results concerning q-tridiagonal pairs and $\boxtimes_q$-modules are contained in the following two theorems and subsequent remark.

Theorem 3.5 *([8, Theorem 2.7], [10, Theorem 10.3]) Let V denote a finite-dimensional irreducible $\boxtimes_q$-module of type 1. Then the generators x_{01}, x_{23} act on V as a q-tridiagonal pair.*

Theorem 3.6 *([8, Theorem 2.7]), [10, Theorem 10.4]) Let V denote a vector space over $\mathbb{F}$ with finite positive dimension and let A, A^* denote a q-tridiagonal pair on V. Then there exists a unique $\boxtimes_q$-module structure on V such that x_{01}, x_{23} act on V as A, A^* respectively. This module structure is irreducible and type 1.*

Remark 3.7 Combining Theorem 3.5 and Theorem 3.6 we get a bijection between the following two sets:

(i) the isomorphism classes of finite-dimensional irreducible $\boxtimes_q$-modules of type 1;

(ii) the isomorphism classes of q-tridiagonal pairs.

4 q-Inverting pairs

In this section we introduce the notion of a *q-inverting pair* and discuss how these objects are related to the finite-dimensional irreducible $\boxtimes_q$-modules. This section contains our main results.

Definition 4.1 Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. By a *q-inverting pair on V* we mean an ordered pair of invertible linear transformations $K : V \to V$ and $K^* : V \to V$ that satisfy (i)–(iii) below.

(i) There exists a decomposition $\{V_i\}_{i=0}^d$ of V such that

$$(K - q^{d-2i}I)V_i = 0 \qquad (0 \le i \le d), \tag{4}$$
$$K^*V_i \subseteq V_0 + V_1 + \cdots + V_{i+1} \qquad (0 \le i \le d), \tag{5}$$
$$(K^*)^{-1}V_i \subseteq V_{i-1} + V_i + \cdots + V_d \qquad (0 \le i \le d). \tag{6}$$

(ii) There exists a decomposition $\{V_i^*\}_{i=0}^\delta$ of V such that

$$(K^* - q^{\delta-2i}I)V_i^* = 0 \qquad (0 \le i \le \delta), \tag{7}$$
$$KV_i^* \subseteq V_{i-1}^* + V_i^* + \cdots + V_\delta^* \qquad (0 \le i \le \delta), \tag{8}$$
$$K^{-1}V_i^* \subseteq V_0^* + V_1^* + \cdots + V_{i+1}^* \qquad (0 \le i \le \delta). \tag{9}$$

(iii) There does not exist a subspace $W \subseteq V$ such that $KW \subseteq W$ and $K^*W \subseteq W$, other than $W = 0$ and $W = V$.

We say the pair K, K^* is *over* $\mathbb{F}$. We call V the *underlying vector space.*

Note 4.2 According to a common notational convention K^* denotes the conjugate transpose of K. We are not using this convention. In a q-inverting pair K, K^* the linear transformations K and K^* are arbitrary subject to (i)–(iii) above.

Note 4.3 The integers d and δ from Definition 4.1 turn out to be equal; we will show this in Lemma 8.4.

We now define the notion of *isomorphism* for q-inverting pairs.

Definition 4.4 Let K, K^* and $K', K^{*\prime}$ denote q-inverting pairs over $\mathbb{F}$. By an *isomorphism of q-inverting pairs* from K, K^* to $K', K^{*\prime}$ we mean a vector space isomorphism σ from the vector space underlying K, K^* to the vector space underlying $K', K^{*\prime}$ such that $\sigma K = K'\sigma$ and $\sigma K^* = K^{*\prime}\sigma$. We say K, K^* and $K', K^{*\prime}$ are *isomorphic* whenever there exists an isomorphism of q-inverting pairs from K, K^* to $K', K^{*\prime}$.

The main results of this paper are contained in the following two theorems and subsequent remark.

Theorem 4.5 *Let V denote a finite-dimensional irreducible $\boxtimes_q$-module of type 1. Then the generators x_{02}, x_{13} act on V as a q-inverting pair.*

Theorem 4.6 *Let V denote a vector space over $\mathbb{F}$ with finite positive dimension and let K, K^* denote a q-inverting pair on V. Then there exists a unique $\boxtimes_q$-module structure on V such that x_{02}, x_{13} act on V as K, K^* respectively. This module structure is irreducible and type 1.*

Remark 4.7 Combining Theorem 4.5 and Theorem 4.6 we get a bijection between the following two sets:

(i) the isomorphism classes of finite-dimensional irreducible $\boxtimes_q$-modules of type 1;

(ii) the isomorphism classes of q-inverting pairs.

The proof of Theorem 4.5 and Theorem 4.6 will take up Sections 5–8.

5 The $\mathbb{Z}_4$ action

In this section we display an action of the group $\mathbb{Z}_4$ on the set of q-inverting pairs.

Referring to the q-inverting pair K, K^* on V from Definition 4.1, if we replace

$$K; K^*; \{V_i\}_{i=0}^{d}; \{V_i^*\}_{i=0}^{\delta}$$

by

$$K^*; K^{-1}; \{V_i^*\}_{i=0}^{\delta}; \{V_{d-i}\}_{i=0}^{d}$$

then the axioms in Definition 4.1(i)–(iii) still hold; therefore the pair K^*, K^{-1} is a q-inverting pair on V. Consider the map ϱ which takes each q-inverting pair K, K^* to the q-inverting pair K^*, K^{-1}. The map ϱ is a permutation on the set of q-inverting pairs, and $\varrho^4 = 1$. Therefore ϱ induces an action of $\mathbb{Z}_4$ on the set of q-inverting pairs. We record a result for later use.

Corollary 5.1 *Let K, K^* denote a q-inverting pair. Then each of the following is a q-inverting pair:*

$$K, K^*; \qquad K^*, K^{-1}; \qquad K^{-1}, K^{*-1}; \qquad K^{*-1}, K. \tag{10}$$

Proof: Repeatedly apply ϱ to the q-inverting pair K, K^*. $\square$

Remark 5.2 The q-inverting pairs (10) might not be mutually nonisomorphic.

6 Some linear algebra

In this section we obtain some linear algebraic results that we will need to prove Theorem 4.5 and Theorem 4.6. We will use the following concepts. Let V denote a vector space over $\mathbb{F}$ with finite positive dimension and let $A : V \to V$ denote a linear transformation. For $\theta \in \mathbb{F}$ we define

$$V_A(\theta) = \{v \in V \mid Av = \theta v\}.$$

Observe that θ is an eigenvalue of A if and only if $V_A(\theta) \neq 0$, and in this case $V_A(\theta)$ is the corresponding eigenspace. The sum $\sum_{\theta \in \mathbb{F}} V_A(\theta)$ is direct. Moreover this sum is equal to V if and only if A is semisimple.

Lemma 6.1 *Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. Let $A : V \to V$ and $B : V \to V$ denote linear transformations. Then for all nonzero $\theta \in \mathbb{F}$ the following are equivalent:*

(i) The expression $A^3B - [3]_q A^2BA + [3]_q ABA^2 - BA^3$ vanishes on $V_A(\theta)$.

(ii) $BV_A(\theta) \subseteq V_A(q^2\theta) + V_A(\theta) + V_A(q^{-2}\theta)$.

Proof: For $v \in V_A(\theta)$ we have

$$\begin{aligned}
&(A^3B - [3]_q A^2BA + [3]_q ABA^2 - BA^3)v \\
&\quad = (A^3 - \theta[3]_q A^2 + \theta^2[3]_q A - \theta^3 I)Bv \qquad\qquad \text{since } Av = \theta v \\
&\quad = (A - q^2\theta I)(A - \theta I)(A - q^{-2}\theta I)Bv,
\end{aligned}$$

where $I : V \to V$ is the identity map. The scalars $q^2\theta, \theta, q^{-2}\theta$ are mutually distinct since $\theta \neq 0$ and since q is not a root of 1. The result follows. $\square$

Lemma 6.2 *Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. Let $A : V \to V$ and $B : V \to V$ denote linear transformations. Then for all nonzero $\theta \in \mathbb{F}$ the following are equivalent:*

(i) The expression $qAB - q^{-1}BA - (q - q^{-1})I$ vanishes on $V_A(\theta)$.

(ii) $(B - \theta^{-1}I)V_A(\theta) \subseteq V_A(q^{-2}\theta)$.

Proof: For $v \in V_A(\theta)$ we have

$$(qAB - q^{-1}BA - (q - q^{-1})I)v = q(A - q^{-2}\theta I)(B - \theta^{-1}I)v$$

and the result follows. $\square$

Lemma 6.3 *Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. Let $A : V \to V$ and $B : V \to V$ denote linear transformations. Then for all nonzero $\theta \in \mathbb{F}$ the following are equivalent:*

(i) The expression $qAB - q^{-1}BA - (q - q^{-1})I$ vanishes on $V_B(\theta)$.

(ii) $(A - \theta^{-1}I)V_B(\theta) \subseteq V_B(q^2\theta)$.

Proof: In Lemma 6.2 replace (A, B, q) by (B, A, q^{-1}). $\square$

7 From $\boxtimes_q$-modules to q-inverting pairs

Our goal in this section is to prove Theorem 4.5. We start with some comments on $\boxtimes_q$-modules.

Definition 7.1 Let V denote a finite-dimensional irreducible $\boxtimes_q$-module of type 1 and diameter d. For each generator x_{rs} of $\boxtimes_q$ we define a decomposition of V which we call $[r, s]$. The decomposition $[r, s]$ has diameter d. For $0 \le i \le d$ the i^{th} component of $[r, s]$ is the eigenspace of x_{rs} on V associated with the eigenvalue q^{d-2i}.

Note 7.2 With reference to Definition 7.1, for $r \in \mathbb{Z}_4$ the decomposition $[r, r+2]$ is the inversion of the decomposition $[r+2, r]$.

Proposition 7.3 *[10, Proposition 13.3] Let V denote a finite-dimensional irreducible $\boxtimes_q$-module of type 1 and diameter d. Choose a generator x_{rs} of $\boxtimes_q$ and consider the corresponding decomposition $[r, s]$ of V from Definition 7.1. Then the shape of this decomposition is independent of the choice of generator. Denoting the shape by $\{\rho_i\}_{i=0}^d$ we have $\rho_i = \rho_{d-i}$ for $0 \le i \le d$.*

Definition 7.4 Let V denote a finite-dimensional irreducible $\boxtimes_q$-module of type 1 and diameter d. By the *shape* of V we mean the sequence $\{\rho_i\}_{i=0}^d$ from Proposition 7.3.

Theorem 7.5 *[10, Theorem 14.1] Let V denote a finite-dimensional irreducible $\boxtimes_q$-module of type 1 and diameter d. Let $\{U_i\}_{i=0}^d$ denote a decomposition of V from Definition 7.1. Then for $r \in \mathbb{Z}_4$ and for $0 \le i \le d$ the action of $x_{r,r+1}$ on U_i is given as follows.*

decomposition	action of $x_{r,r+1}$ on U_i
$[r, r+1]$	$(x_{r,r+1} - q^{d-2i}I)U_i = 0$
$[r+1, r+2]$	$(x_{r,r+1} - q^{2i-d}I)U_i \subseteq U_{i-1}$
$[r+2, r+3]$	$x_{r,r+1}U_i \subseteq U_{i-1} + U_i + U_{i+1}$
$[r+3, r]$	$(x_{r,r+1} - q^{2i-d}I)U_i \subseteq U_{i+1}$
$[r, r+2]$	$(x_{r,r+1} - q^{d-2i}I)U_i \subseteq U_{i-1}$
$[r+1, r+3]$	$(x_{r,r+1} - q^{2i-d}I)U_i \subseteq U_{i-1}$

Theorem 7.6 *[10, Theorem 14.2] Let V denote a finite-dimensional irreducible $\boxtimes_q$-module of type 1 and diameter d. Let $\{U_i\}_{i=0}^d$ denote a decomposition of V from Definition 7.1. Then for $r \in \mathbb{Z}_4$ and for $0 \le i \le d$ the action of $x_{r,r+2}$ on U_i is given as follows.*

decomposition	action of $x_{r,r+2}$ on U_i
$[r, r+1]$	$(x_{r,r+2} - q^{d-2i}I)U_i \subseteq U_0 + \cdots + U_{i-1}$
$[r+1, r+2]$	$(x_{r,r+2} - q^{d-2i}I)U_i \subseteq U_{i+1} + \cdots + U_d$
$[r+2, r+3]$	$(x_{r,r+2} - q^{2i-d}I)U_i \subseteq U_{i-1}$
$[r+3, r]$	$(x_{r,r+2} - q^{2i-d}I)U_i \subseteq U_{i+1}$
$[r, r+2]$	$(x_{r,r+2} - q^{d-2i}I)U_i = 0$
$[r+1, r+3]$	$x_{r,r+2}U_i \subseteq U_{i-1} + \cdots + U_d$

We recall the notion of a *flag*. Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. Let $\{s_i\}_{i=0}^d$ denote a sequence of positive integers whose sum is the dimension of V. By a *flag on V of shape* $\{s_i\}_{i=0}^d$ we mean a nested sequence $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_d$ of subspaces of V such that the dimension of F_i is $s_0 + \cdots + s_i$ for $0 \leq i \leq d$. We call F_i the i^{th} *component* of the flag. We call d the *diameter* of the flag. We observe $F_d = V$.

The following construction yields a flag on V. Let $\{U_i\}_{i=0}^d$ denote a decomposition of V of shape $\{s_i\}_{i=0}^d$. Define

$$F_i = U_0 + U_1 + \cdots + U_i \qquad\qquad (0 \leq i \leq d).$$

Then the sequence $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_d$ is a flag on V of shape $\{s_i\}_{i=0}^d$. We say this flag is *induced* by the decomposition $\{U_i\}_{i=0}^d$.

We now recall what it means for two flags to be *opposite*. Suppose we are given two flags on V with the same diameter: $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_d$ and $F'_0 \subseteq F'_1 \subseteq \cdots \subseteq F'_d$. We say that these flags are *opposite* whenever there exists a decomposition $\{U_i\}_{i=0}^d$ of V such that

$$F_i = U_0 + U_1 + \cdots + U_i, \qquad\qquad F'_i = U_d + U_{d-1} + \cdots + U_{d-i}$$

for $0 \leq i \leq d$. In this case

$$F_i \cap F'_j = 0 \qquad \text{if} \quad i + j < d \qquad\qquad (0 \leq i, j \leq d) \tag{11}$$

and

$$U_i = F_i \cap F'_{d-i} \qquad\qquad (0 \leq i \leq d). \tag{12}$$

In particular the decomposition $\{U_i\}_{i=0}^d$ is uniquely determined by the given flags.

We now return our attention to $\boxtimes_q$.

Theorem 7.7 *[10, Theorem 16.1] Let V denote a finite-dimensional irreducible $\boxtimes_q$-module of type 1 and diameter d. Then there exists a collection of flags on V, denoted $[n], n \in \mathbb{Z}_4$, that have the following property: for each generator x_{rs} of $\boxtimes_q$ the decomposition $[r, s]$ of V induces $[r]$ and the inversion of $[r, s]$ induces $[s]$.*

Lemma 7.8 *[10, Lemma 16.2] Let V denote a finite-dimensional irreducible $\boxtimes_q$-module of type 1. Then for $n \in \mathbb{Z}_4$ the shape of the flag $[n]$ coincides with the shape of V.*

Theorem 7.9 *[10, Theorem 16.3] Let V denote a finite-dimensional irreducible $\boxtimes_q$-module of type 1. Then the flags $[n]$, $n \in \mathbb{Z}_4$ on V from Theorem 7.7 are mutually opposite.*

Theorem 7.10 *[10, Theorem 16.4] Let V denote a finite-dimensional irreducible $\boxtimes_q$-module of type 1 and diameter d. Pick a generator x_{rs} of $\boxtimes_q$ and consider the corresponding decomposition $[r, s]$ of V from Definition 7.1. For $0 \leq i \leq d$ the i^{th} component of $[r, s]$ is the intersection of the following two sets:*

(i) *component i of the flag $[r]$;*

(ii) *component $d-i$ of the flag $[s]$.*

Proposition 7.11 *Let V denote a finite-dimensional irreducible $\boxtimes_q$-module of type 1. Let W denote a nonzero subspace of V such that $x_{02}W \subseteq W$ and $x_{13}W \subseteq W$. Then $W = V$.*

Proof: Without loss we may assume that W is irreducible as a module for x_{02}, x_{13}. Since $x_{02}x_{20} = 1$ and $x_{13}x_{31} = 1$ we find W is invariant under each of x_{20}, x_{31}. Therefore W is invariant under $x_{r,r+2}$ for $r \in \mathbb{Z}_4$. For the moment fix $r \in \mathbb{Z}_4$ and let $\{U_i\}_{i=0}^d$ denote the decomposition $[r, r+2]$. Recall that $x_{r,r+2}$ is semisimple on V with eigenspaces $U_0, U_1, \ldots, U_d$. By this and since W is invariant under $x_{r,r+2}$ we find

$$W = \sum_{i=0}^{d} W \cap U_i.$$

Since $W \neq 0$ there exists an integer i $(0 \leq i \leq d)$ such that $W \cap U_i \neq 0$. Define

$$m_r = \min\{i \mid 0 \leq i \leq d, \quad W \cap U_i \neq 0\}.$$

We claim that m_r is independent of r for $r \in \mathbb{Z}_4$. Suppose the claim is false. Then there exists $r \in \mathbb{Z}_4$ such that $m_r > m_{r+1}$. By construction the space W is contained in component $d - m_r$ of the flag $[r+2]$. By construction W has nonzero intersection with component m_{r+1} of the flag $[r+1]$. Since $m_r > m_{r+1}$ the component $d - m_r$ of $[r+2]$ has zero intersection with component m_{r+1} of $[r+1]$, for a contradiction. We have proved the claim. For the rest of this proof let m denote the common value of m_r for $r \in \mathbb{Z}_4$. The claim implies that for all $r \in \mathbb{Z}_4$ the component $d - m$ of the flag $[r]$ contains W, and component m of $[r]$ has nonzero intersection with W. We can now easily show $W = V$. Since the $\boxtimes_q$-module V is irreducible and $W \neq 0$ it suffices to show that W is invariant under $\boxtimes_q$. We mentioned earlier that W is invariant under $x_{r,r+2}$ for $r \in \mathbb{Z}_4$. We now show that W is invariant under $x_{r,r+1}$ for $r \in \mathbb{Z}_4$. Let r be given and let W' denote the span of the set of vectors in W that are eigenvectors for $x_{r,r+1}$. By the construction $W' \subseteq W$ and $x_{r,r+1}W' \subseteq W'$. We show $W' = W$. To this end we show that W' is nonzero and invariant under each of x_{02}, x_{13}. We now show $W' \neq 0$. By the comment after the preliminary claim, W has nonzero intersection with component m of the flag $[r]$ and W is contained in component $d - m$ of the flag $[r+1]$. By Theorem 7.10 the intersection of component m of $[r]$ and component $d - m$ of $[r+1]$ is equal to component m of the decomposition $[r, r+1]$, which is an eigenspace for $x_{r,r+1}$. The intersection of W with this eigenspace is nonzero and contained in W', so $W' \neq 0$. We now show that W' is invariant under each of x_{02}, x_{13}. Since $x_{02}x_{20} = 1$ and $x_{13}x_{31} = 1$ it suffices to show that W' is invariant under $x_{r+1,r+3}$ and $x_{r+2,r}$. We now show that W' is invariant under $x_{r+1,r+3}$. To this end we pick $v \in W'$ and show $x_{r+1,r+3}v \in W'$. Without loss we may assume that v is an eigenvector for $x_{r,r+1}$; let θ denote the corresponding eigenvalue. Then $\theta \neq 0$ by the comment at the end of Section 2. Since $v \in W'$ and $W' \subseteq W$ we have $v \in W$. The space W is invariant under $x_{r+1,r+3}$ so $x_{r+1,r+3}v \in W$. By these comments $(x_{r+1,r+3} - \theta^{-1}I)v \in W$. By Lemma 6.2 (with $A = x_{r,r+1}$ and $B = x_{r+1,r+3}$) the vector $(x_{r+1,r+3} - \theta^{-1}I)v$ is contained in an eigenspace of $x_{r,r+1}$ so $(x_{r+1,r+3} - \theta^{-1}I)v \in W'$. By this and since $v \in W'$ we have $x_{r+1,r+3}v \in W'$. We have now shown that W' is invariant under $x_{r+1,r+3}$ as desired. Next we show that W' is invariant under $x_{r+2,r}$. To this end we pick

$u \in W'$ and show $x_{r+2,r}u \in W'$. Without loss we may assume that u is an eigenvector for $x_{r,r+1}$; let η denote the corresponding eigenvalue. Then $\eta \neq 0$ by the comment at the end of Section 2. Recall $u \in W'$ and $W' \subseteq W$ so $u \in W$. The space W is invariant under $x_{r+2,r}$ so $x_{r+2,r}u \in W$. By these comments $(x_{r+2,r} - \eta^{-1}I)u \in W$. By Lemma 6.3 (with $A = x_{r+2,r}$, $B = x_{r,r+1}$, $\theta = \eta$) the vector $(x_{r+2,r} - \eta^{-1}I)u$ is contained in an eigenspace of $x_{r,r+1}$ so $(x_{r+2,r} - \eta^{-1}I)u \in W'$. By this and since $u \in W'$ we have $x_{r+2,r}u \in W'$. We have now shown that W' is invariant under $x_{r+2,r}$ as desired. From our above comments W' is nonzero and invariant under each of x_{02}, x_{13}. Now $W' = W$ by the irreducibility of W, so $x_{r,r+1}W \subseteq W$. We have now shown that W is invariant under $x_{r,r+1}$ and $x_{r,r+2}$ for $r \in \mathbb{Z}_4$. Therefore W is $\boxtimes_q$-invariant. The $\boxtimes_q$-module V is irreducible and $W \neq 0$ so $W = V$. $\square$

It is now a simple matter to prove Theorem 4.5.

Proof of Theorem 4.5: Define the linear transformation $K : V \to V$ (resp. $K^* : V \to V$) to be the action of x_{02} (resp. x_{13}) on V. We show that K, K^* is a q-inverting pair on V. To do this we show that K, K^* satisfy the conditions (i)–(iii) of Definition 4.1. Concerning Definition 4.1(i), we denote the decomposition $[0, 2]$ by $\{V_i\}_{i=0}^d$ and show that this decomposition satisfies (4)–(6). Line (4) is satisfied by the construction. To get (5), (6) we refer to the last row in the table of Theorem 7.6. Line (5) holds by that row (with $r = 1$) and since the decomposition $[2, 0]$ is the inversion of $[0, 2]$. Line (6) holds by that row (with $r = 3$) and since $x_{13}x_{31} = 1$. We have now shown that K, K^* satisfy Definition 4.1(i). Concerning Definition 4.1(ii), we denote the decomposition $[1, 3]$ by $\{V_i^*\}_{i=0}^d$ and show that this decomposition satisfies (7)–(9). Line (7) holds by the construction. To get (8), (9) we refer to the last row in the table of Theorem 7.6. Line (8) holds by that row (with $r = 0$). Line (9) holds by that row (with $r = 2$), since $x_{02}x_{20} = 1$ and since the decomposition $[3, 1]$ is the inversion of $[1, 3]$. We have now shown that K, K^* satisfy Definition 4.1(ii). The maps K, K^* satisify Definition 4.1(iii) by Proposition 7.11. We have now verified that K, K^* satisfy Definition 4.1(i)–(iii) so K, K^* is a q-inverting pair on V. The result follows. $\square$

8 From q-inverting pairs to $\boxtimes_q$-modules

Our goal in this section is to prove Theorem 4.6. On our way to this goal we will show that the integers d and δ from Definition 4.1 are equal.

Definition 8.1 With reference to Definition 4.1 we set

$$V_{ij} = (V_0 + \cdots + V_i) \cap (V_0^* + \cdots + V_j^*) \tag{13}$$

for all integers i, j. We interpret the sum on the left in (13) to be 0 (resp. V) if $i < 0$ (resp. $i > d$). We interpret the sum on the right in (13) to be 0 (resp. V) if $j < 0$ (resp. $j > \delta$).

Lemma 8.2 *With reference to Definition 4.1 and Definition 8.1, the following (i), (ii) hold.*

(i) $V_{i\delta} = V_0 + \cdots + V_i \qquad (0 \leq i \leq d)$.

(ii) $V_{dj} = V_0^* + \cdots + V_j^* \qquad (0 \leq j \leq \delta)$.

Proof: (i) Set $j = \delta$ in (13) and recall $V = V_0^* + \cdots + V_\delta^*$.
(ii) Set $i = d$ in (13) and use $V = V_0 + \cdots + V_d$. $\square$

Lemma 8.3 *With reference to Definition 4.1 and Definition 8.1, the following (i), (ii) hold for* $0 \leq i \leq d$ *and* $0 \leq j \leq \delta$.

(i) $(K^{-1} - q^{2i-d}I)V_{ij} \subseteq V_{i-1,j+1}$.

(ii) $(K^* - q^{\delta-2j}I)V_{ij} \subseteq V_{i+1,j-1}$.

Proof: (i) Using (4) we find

$$(K^{-1} - q^{2i-d}I)\sum_{h=0}^{i} V_h = \sum_{h=0}^{i-1} V_h. \tag{14}$$

Using (9) we find

$$(K^{-1} - q^{2i-d}I)\sum_{h=0}^{j} V_h^* \subseteq \sum_{h=0}^{j+1} V_h^*. \tag{15}$$

Evaluating $(K^{-1} - q^{2i-d}I)V_{ij}$ using (13)–(15) we find it is contained in $V_{i-1,j+1}$.
(ii) Using (5) we find

$$(K^* - q^{\delta-2j}I)\sum_{h=0}^{i} V_h \subseteq \sum_{h=0}^{i+1} V_h. \tag{16}$$

Using (7) we find

$$(K^* - q^{\delta-2j}I)\sum_{h=0}^{j} V_h^* = \sum_{h=0}^{j-1} V_h^*. \tag{17}$$

Evaluating $(K^* - q^{\delta-2j}I)V_{ij}$ using (13) and (16), (17) we find it is contained in $V_{i+1,j-1}$. $\square$

Lemma 8.4 *The scalars* d *and* δ *from Definition 4.1 are equal. Moreover, with reference to Definition 8.1,*

$$V_{ij} = 0 \quad \text{if} \quad i + j < d \qquad\qquad (0 \leq i, j \leq d). \tag{18}$$

Proof: For all nonnegative integers r such that $r \leq d$ and $r \leq \delta$ we define

$$W_r = V_{0r} + V_{1,r-1} + \cdots + V_{r0}. \tag{19}$$

We have $K^{-1}W_r \subseteq W_r$ by Lemma 8.3(i) so $KW_r \subseteq W_r$. We have $K^*W_r \subseteq W_r$ by Lemma 8.3(ii). Now $W_r = 0$ or $W_r = V$ in view of Definition 4.1(iii). Suppose for the moment that $r \leq d-1$. Each term on the right in (19) is contained in $V_0 + \cdots + V_r$ so $W_r \subseteq V_0 + \cdots + V_r$. Therefore $W_r \neq V$ so $W_r = 0$. Next suppose $r = d$. Then $V_{d0} \subseteq W_r$. Recall $V_{d0} = V_0^*$ by Lemma 8.2(ii) and $V_0^* \neq 0$ so $V_{d0} \neq 0$. Now $W_r \neq 0$ so $W_r = V$. We have now shown that $W_r = 0$ if $r \leq d-1$ and $W_r = V$ if $r = d$. Similarly $W_r = 0$ if $r \leq \delta - 1$ and $W_r = V$ if $r = \delta$. Now $d = \delta$; otherwise we take $r = \min(d, \delta)$ in our above comments and find W_r is both 0 and V, for a contradiction. The result follows. $\square$

Definition 8.5 *With reference to Definition 4.1, for $0 \leq i \leq d$ we define*

$$U_i = (V_0 + \cdots + V_i) \cap (V_0^* + \cdots + V_{d-i}^*).$$

We observe U_i is equal to the space $V_{i,d-i}$ from Definition 8.1.

Lemma 8.6 *With reference to Definition 4.1 and Definition 8.5, the sequence $\{U_i\}_{i=0}^d$ is a decomposition of V.*

Proof: We first show

$$V = \sum_{i=0}^{d} U_i. \tag{20}$$

Let W denote the sum on the right in (20). We have $K^{-1}W \subseteq W$ by Lemma 8.3(i) so $KW \subseteq W$. We have $K^*W \subseteq W$ by Lemma 8.3(ii). Now $W = 0$ or $W = V$ in view of Definition 4.1(iii). The space W contains U_0 and $U_0 = V_0$ is nonzero so $W \neq 0$. Therefore $W = V$ and (20) follows. Next we show that the sum (20) is direct. To do this, we show

$$(U_0 + U_1 + \cdots + U_{i-1}) \cap U_i = 0$$

for $1 \leq i \leq d$. Let i be given. From the construction

$$U_j \subseteq V_0 + V_1 + \cdots + V_{i-1}$$

for $0 \leq j \leq i-1$, and

$$U_i \subseteq V_0^* + V_1^* + \cdots + V_{d-i}^*.$$

Therefore

$$\begin{aligned} &(U_0 + U_1 + \cdots + U_{i-1}) \cap U_i \\ &\quad \subseteq (V_0 + V_1 + \cdots + V_{i-1}) \cap (V_0^* + V_1^* + \cdots + V_{d-i}^*) \\ &\quad = V_{i-1,d-i} \\ &\quad = 0 \end{aligned}$$

in view of (18). We have now shown that the sum (20) is direct. Next we show that each of $U_0, \ldots, U_d$ is nonzero. Suppose there exists an integer i $(0 \leq i \leq d)$ such that $U_i = 0$. Observe that $i \neq 0$ since $U_0 = V_0$ is nonzero, and $i \neq d$ since $U_d = V_0^*$ is nonzero. Set

$$U = U_0 + U_1 + \cdots + U_{i-1}$$

and observe $U \neq 0$ and $U \neq V$ by our above remarks. By Lemma 8.3(i) we find $K^{-1}U \subseteq U$ so $KU \subseteq U$. By Lemma 8.3(ii) and since $U_i = 0$ we find $K^*U \subseteq U$. Now $U = 0$ or $U = V$ by Definition 4.1(iii), for a contradiction. We conclude that each of $U_0, \ldots, U_d$ is nonzero. We have now shown that the sequence $\{U_i\}_{i=0}^d$ is a decomposition of V. $\square$

Definition 8.7 We call the decomposition $\{U_i\}_{i=0}^d$ from Lemma 8.6 the *split decomposition* of V associated with K, K^*.

Lemma 8.8 *With reference to Definition 4.1 and Definition 8.5, the following (i), (ii) hold for* $0 \leq i \leq d$.

(i) $(K^{-1} - q^{2i-d}I)U_i \subseteq U_{i-1}$.

(ii) $(K^* - q^{2i-d}I)U_i \subseteq U_{i+1}$.

Proof: Immediate from Lemma 8.3 and Definition 8.5. □

Lemma 8.9 *With reference to Definition 4.1 and Definition 8.5, the following (i), (ii) hold for* $0 \leq i \leq d$.

(i) $U_0 + \cdots + U_i = V_0 + \cdots + V_i$.

(ii) $U_{d-i} + \cdots + U_d = V_0^* + \cdots + V_i^*$.

Proof: (i) Let $X_i = \sum_{j=0}^i U_j$ and $X_i' = \sum_{j=0}^i V_j$. We show $X_i = X_i'$. Define $R_i = \prod_{j=0}^i (K^{-1} - q^{2j-d}I)$. Then $X_i' = \{v \in V \mid R_i v = 0\}$, and $R_i X_i = 0$ by Lemma 8.8(i), so $X_i \subseteq X_i'$. Now define $T_i = \prod_{j=i+1}^d (K^{-1} - q^{2j-d}I)$. Observe that $T_i V = X_i'$, and $T_i V \subseteq X_i$ by Lemma 8.8(i), so $X_i' \subseteq X_i$. By these comments $X_i = X_i'$.
(ii) Similar to the proof of (i) above. □

Definition 8.10 With reference to Definition 4.1, by the *split operator* for K, K^* we mean the linear transformation $S : V \to V$ such that for $0 \leq i \leq d$ the space U_i from Definition 8.5 is the eigenspace for S with eigenvalue q^{d-2i}.

We are now ready to prove Theorem 4.6.

Proof of Theorem 4.6: We let the generators x_{rs} of $\boxtimes_q$ act on V as follows. We let x_{02} (resp. x_{13}) (resp. x_{20}) (resp. x_{31}) act on V as K (resp. K^*) (resp. K^{-1}) (resp. K^{*-1}). With reference to Corollary 5.1 and Definition 8.10 we let x_{01} (resp. x_{12}) (resp. x_{23}) (resp. x_{30}) act on V as the split operator for K, K^* (resp. K^*, K^{-1}) (resp. K^{-1}, K^{*-1}) (resp. K^{*-1}, K). We now show that the above actions induce a $\boxtimes_q$-module structure on V. To do this we show that they satisfy the defining relations for $\boxtimes_q$ given in Definition 2.1. The relations in Definition 2.1(i) hold by the construction, so consider the relations in Definition 2.1(ii). The three kinds of relations involved are treated in the following three claims.

Line (2) holds for all $r, s, t \in \mathbb{Z}_4$ *such that* $(s - r, t - s) = (1, 2)$.
To prove this claim, by $\mathbb{Z}_4$ symmetry we may assume that $r = 0$, $s = 1$, $t = 3$. Let Δ denote the left hand side of (2) minus the right hand side of (2), for $r = 0$, $s = 1$, $t = 3$. We show $\Delta = 0$. For $0 \leq i \leq d$ we combine Lemma 6.2 (with $A = x_{01}$, $B = x_{13}$, $\theta = q^{d-2i}$) and Lemma 8.8(ii) to find $\Delta U_i = 0$. Now $\Delta = 0$ in view of Lemma 8.6 and the claim is proved.

Line (2) holds for all $r, s, t \in \mathbb{Z}_4$ such that $(s-r, t-s) = (2, 1)$.
To prove this claim, by $\mathbb{Z}_4$ symmetry we may assume that $r = 2$, $s = 0$, $t = 1$. Let Δ denote the left hand side of (2) minus the right hand side of (2), for $r = 2$, $s = 0$, $t = 1$. We show $\Delta = 0$. For $0 \leq i \leq d$ we combine Lemma 6.3 (with $A = x_{20}$, $B = x_{01}$, $\theta = q^{d-2i}$) and Lemma 8.8(i) to find $\Delta U_i = 0$. Now $\Delta = 0$ in view of Lemma 8.6 and the claim is proved.

Line (2) holds for all $r, s, t \in \mathbb{Z}_4$ such that $(s-r, t-s) = (1, 1)$.
To prove the claim, by $\mathbb{Z}_4$ symmetry it suffices to show that (2) holds for $r = 0$, $s = 1$, $t = 2$. Combining (2) (with $r = 3$, $s = 1$, $t = 2$) and Lemma 6.2 we find

$$(x_{12} - q^{d-2i}I)V_i^* \subseteq V_{i-1}^* \qquad (0 \leq i \leq d).$$

By this and Lemma 8.9(ii) we have

$$(x_{12} - q^{2i-d}I)U_i \subseteq U_{i+1} + \cdots + U_d \qquad (0 \leq i \leq d). \tag{21}$$

Combining (2) (with $r = 1$, $s = 2$, $t = 0$) and Lemma 6.3 we find

$$(x_{12} - q^{d-2i}I)V_i \subseteq V_{i+1} \qquad (0 \leq i \leq d).$$

By this and Lemma 8.9(i) we have

$$(x_{12} - q^{2i-d}I)U_i \subseteq U_0 + \cdots + U_{i+1} \qquad (0 \leq i \leq d). \tag{22}$$

Combining (21), (22) we find

$$(x_{12} - q^{2i-d}I)U_i \subseteq U_{i+1} \qquad (0 \leq i \leq d). \tag{23}$$

Let Δ denote the left hand side of (2) minus the right hand side of (2), for $r = 0$, $s = 1$, $t = 2$. For $0 \leq i \leq d$ we combine Lemma 6.2 (with $A = x_{01}$, $B = x_{12}$, $\theta = q^{d-2i}$) and (23) to find $\Delta U_i = 0$. Now $\Delta = 0$ in view of Lemma 8.6. We have now shown that (2) holds for $r = 0$, $s = 1$, $t = 2$, and the claim follows.

We have now shown that the relations in Definition 2.1(ii) are satisfied. We now consider the relations in Definition 2.1(iii). Let $r, s, t, u \in \mathbb{Z}_4$ be given such that $s - r = t - s = u - t = 1$. We show that (3) holds. By $\mathbb{Z}_4$ symmetry we may assume that $r = 0$, $s = 1$, $t = 2$, $u = 3$. Combining (2) (with $r = 0$, $s = 2$, $t = 3$) and Lemma 6.2 we find

$$(x_{23} - q^{2i-d}I)V_i \subseteq V_{i+1} \qquad (0 \leq i \leq d).$$

By this and Lemma 8.9(i) we have

$$x_{23}U_i \subseteq U_0 + \cdots + U_{i+1} \qquad (0 \leq i \leq d). \tag{24}$$

By (2) (with $r = 2$, $s = 3$, $t = 1$) and Lemma 6.3 we find

$$(x_{23} - q^{d-2i}I)V_i^* \subseteq V_{i+1}^* \qquad (0 \leq i \leq d).$$

By this and Lemma 8.9(ii) we have

$$x_{23}U_i \subseteq U_{i-1} + \cdots + U_d \qquad (0 \le i \le d). \tag{25}$$

Combining (24), (25) we find

$$x_{23}U_i \subseteq U_{i-1} + U_i + U_{i+1} \qquad (0 \le i \le d). \tag{26}$$

Let Δ denote the left hand side of (3) for $r = 0$, $s = 1$, $t = 2$, $u = 3$. For $0 \le i \le d$ we combine Lemma 6.1 (with $A = x_{01}$, $B = x_{23}$, $\theta = q^{d-2i}$) and (26) to find $\Delta U_i = 0$. Now $\Delta = 0$ in view of Lemma 8.6. We have now shown that (3) holds for $r = 0$, $s = 1$, $t = 2$, $u = 3$ as desired. We have now verified all the relations in Definition 2.1(iii).

We have shown that the actions on V of the generators x_{rs} satisfy the defining relations for $\boxtimes_q$ given in Definition 2.1. Therefore these actions induce a $\boxtimes_q$-module structure on V. By the construction x_{02}, x_{13} act on V as K, K^* respectively. Apparently there exists a $\boxtimes_q$-module structure on V such that x_{02}, x_{13} act on V as K, K^* respectively. We now show that this $\boxtimes_q$-module structure is unique. Suppose we are given any $\boxtimes_q$-module structure on V such that x_{02}, x_{13} act as K, K^* respectively. This $\boxtimes_q$-module structure is irreducible by construction and Definition 4.1(iii). This $\boxtimes_q$-module structure is type 1 and diameter d, since the action of x_{01} on V has eigenvalues $\{q^{d-2i} \mid 0 \le i \le d\}$. For each generator x_{rs} of $\boxtimes_q$ the action on V is determined by the decomposition $[r, s]$. By Theorem 7.10 the decomposition $[r, s]$ is determined by the flags $[r]$ and $[s]$. Therefore our $\boxtimes_q$-module structure on V is determined by the flags $[n], n \in \mathbb{Z}_4$. By construction the flags $[0]$ and $[2]$ are determined by the decomposition $[0, 2]$ and hence by the action of x_{02} on V. Similarly the flags $[1]$ and $[3]$ are determined by the decomposition $[1, 3]$ and hence by the action of x_{13} on V. Therefore the given $\boxtimes_q$-module structure on V is determined by the action of K and K^* on V, so this $\boxtimes_q$-module structure is unique. We have now shown that there exists a unique $\boxtimes_q$-module structure on V such that x_{02}, x_{13} act as K, K^* respectively. We mentioned earlier that this $\boxtimes_q$-module structure is irreducible and has type 1. $\square$

9 Directions for further research

In this section we give some directions for further research. We start with a definition.

Definition 9.1 Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. Let $\{V_i\}_{i=0}^d$ denote a decomposition of V. For $0 \le i \le d$ let $F_i : V \to V$ denote the linear transformation that satisfies

$$\begin{aligned} (F_i - I)V_i &= 0, \\ F_i V_j &= 0 \quad \text{if} \quad i \ne j \qquad (0 \le j \le d). \end{aligned}$$

We observe that F_i is the projection from V onto V_i. We note that $I = \sum_{i=0}^d F_i$ and $F_i F_j = \delta_{ij} F_i$ for $0 \le i, j \le d$. Therefore the sequence $F_0, \ldots, F_d$ is a basis for a commutative subalgebra $\mathcal{D}$ of $\mathrm{End}(V)$.

Problem 9.2 Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. Let $\{V_i\}_{i=0}^d$ and $\{V_i^*\}_{i=0}^\delta$ denote decompositions of V. Let $\mathcal{D}$ and $\mathcal{D}^*$ denote the corresponding commutative algebras from Definition 9.1. Investigate the case in which (i)–(v) hold below.

(i) $\mathcal{D}$ has a generator A_+ such that

$$A_+ V_i^* \subseteq V_0^* + \cdots + V_{i+1}^* \qquad (0 \le i \le \delta).$$

(ii) $\mathcal{D}$ has a generator A_- such that

$$A_- V_i^* \subseteq V_{i-1}^* + \cdots + V_\delta^* \qquad (0 \le i \le \delta).$$

(iii) $\mathcal{D}^*$ has a generator A_+^* such that

$$A_+^* V_i \subseteq V_0 + \cdots + V_{i+1} \qquad (0 \le i \le d).$$

(iv) $\mathcal{D}^*$ has a generator A_-^* such that

$$A_-^* V_i \subseteq V_{i-1} + \cdots + V_d \qquad (0 \le i \le d).$$

(v) There does not exist a subspace $W \subseteq V$ such that $\mathcal{D}W \subseteq W$ and $\mathcal{D}^*W \subseteq W$, other than $W = 0$ and $W = V$.

Note 9.3 Let A, A^* denote a tridiagonal pair on V, as in [5, Definition 1.1]. Then the conditions (i)–(v) of Problem 9.2 are satisfied with

$$A_+ = A, \qquad A_- = A, \qquad A_+^* = A^*, \qquad A_-^* = A^*.$$

Note 9.4 Let K, K^* denote a q-inverting pair on V. Then the conditions (i)–(v) of Problem 9.2 are satisfied with

$$A_+ = K^{-1}, \qquad A_- = K, \qquad A_+^* = K^*, \qquad A_-^* = K^{*-1}.$$

Problem 9.5 Referring to Problem 9.2, assume conditions (i)–(v) hold. Show that the decompositions $\{V_i\}_{i=0}^d$ and $\{V_i^*\}_{i=0}^\delta$ have the same shape and in particular that $d = \delta$. Denoting this common shape by $\{\rho_i\}_{i=0}^d$, show that $\rho_i = \rho_{d-i}$ for $0 \le i \le d$.

Problem 9.6 Referring to Problem 9.2, assume conditions (i)–(v) hold. Consider the four decompositions of V consisting of $\{V_i\}_{i=0}^d$ and $\{V_i^*\}_{i=0}^\delta$, together with their inversions. Show that the flags on V induced by these four decompositions are mutually opposite.

We motivate our last problem with a comment. By Theorem 3.6 and (3) every q-tridiagonal pair satisfies the cubic q-Serre relations. More generally, every tridiagonal pair [5, Definition 1.1] satisfies a pair of equations called the *tridiagonal relations* [19, Theorem 3.7].

Problem 9.7 Find some polynomial relations satisfied by every q-inverting pair.

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Tatsuro Ito
Department of Computational Science
Faculty of Science
Kanazawa University
Kakuma-machi
Kanazawa 920-1192, Japan
email: `ito@kappa.s.kanazawa-u.ac.jp`

Paul Terwilliger
Department of Mathematics
University of Wisconsin
Van Vleck Hall
480 Lincoln Drive
Madison, WI 53706-1388 USA
email: `terwilli@math.wisc.edu`