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Dynamics of an Open System for Repeated Harmonic Perturbation

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ABSTRACT

We use the Kossakowski-Lindblad-Davies formalism to study an open dynamical system defined as Markovian extension of the one-mode quantum resonator $\mathcal{S}$, perturbed by repeated harmonic interaction with a chain of multi-level harmonic atoms $\mathcal{C}$. The long-time asymptotic behaviour and correlations of various subsystems of the system $\mathcal{S} + \mathcal{C}$ are treated in the framework of the W^*-dynamical system approach.

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Contents

1 Introduction

A quantum Hamiltonian system with time-dependent repeated harmonic interaction was proposed and investigated in [TZ]. The corresponding open system can be defined through the Kossakowski-Lindblad-Davies *dissipative* extension of the Hamiltonian dynamics. In our paper [TZ1] the existence and uniqueness of the evolution map for density matrices of the open system are established and its dual W^*-dynamics on the CCR C^*-algebra was described explicitly.

Behind our model [TZ], there is a physical phenomenon known as "one-atom maser" [MWM], when the pumping of a resonator (or a quantum cavity) is caused by a chain of atoms which proceed one-by-one through the cavity. The mathematical study of this repeated interaction system first appeared in [AP].

The quality factor of the leaky cavity measures the effect of losses and indicates that the system is open. For mathematical description of the leaky cavity we use in this paper a well-known Kossakowski-Lindblad-Davies formalism for Markovian approach to dissipative dynamics of open systems [AJP2].

Note that a subtle point of analysis is the nature of the atom-cavity interaction. A standard motivated by the quantum optics choice is the Jaynes-Cummings inelastic interaction of *two-level* atoms with a one-mode resonator [BJM]. Instead of this interaction, a purely elastic one, which does not change the "hard" atom internal state, was considered in [NVZ] both for the isolated and for the leaky cavity. It was found there that the properties of these two models for repeated perturbation are drastically different.

This motivated us to study repeated inelastic interaction for a very "soft" *multi-level* atoms. To this aim we proposed in [TZ] an exactly soluble model of an isolated system with Hamiltonian dynamics generated by repeated interaction of a one-mode resonator (cavity) with atoms, which have infinitely many *harmonic* levels of internal states, when the interaction is *linear*. We call it the *harmonic perturbation* of the cavity.

In the present paper we consider the *open* version of the model [TZ] with dynamics à la Kossakowski-Lindblad-Davies [TZ1]. Our aim is to analyse the long-time asymptotic behaviour and the quantum correlations of subsystems for this open system.

Let a and a^* be the annihilation and the creation operators defined in the Fock space $\mathscr{F}$ generated by a cyclic vector Ω (*vacuum*). That is, the Hilbert space $\mathscr{F}$ is the completion of the algebraic span $\mathscr{F}_{\rm fin}$ of vectors $\{(a^*)^m\Omega\}_{m\geqslant 0}$ and a, a^* satisfy the Canonical Commutation Relations (CCR)

$$[a, a^*] = \mathbb{1}, \quad [a, a] = 0, \quad [a^*, a^*] = 0 \quad \text{on} \quad \mathscr{F}_{\rm fin}. \tag{1.1}$$

We denote by $\{\mathscr{H}_k\}_{k=0}^N$ the copies of $\mathscr{F}$ for an arbitrary but finite $N \in \mathbb{N}$ and by $\mathscr{H}^{(N)}$ the Hilbert space tensor product of these copies:

$$\mathscr{H}^{(N)} := \bigotimes_{k=0}^{N} \mathscr{H}_k = \mathscr{F}^{\otimes(N+1)} \ . \tag{1.2}$$

In this space we define for $k = 0, 1, 2, \ldots, N$ the operators

$$b_k := \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes a \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} \ , \quad b_k^* := \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes a^* \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} \ , \tag{1.3}$$

where operator a (respectively a^*) is the $(k+1)$th factor in (1.3). They satisfy the CCR:

$$[b_k, b_{k'}^*] = \delta_{k,k'}\mathbb{1}, \quad [b_k, b_{k'}] = [b_k^*, b_{k'}^*] = 0 \qquad (k, k' = 0, 1, 2, \ldots, N) \tag{1.4}$$

on the algebraic tensor product $(\mathscr{F}_{\rm fin})^{\otimes(N+1)}$.

Recall that non-autonomous system with Hamiltonian for time-dependent *repeated* harmonic perturbation proposed in [TZ] has the form

$$H_N(t) := E b_0^* b_0 + \epsilon \sum_{k=1}^{N} b_k^* b_k + \eta \sum_{k=1}^{N} \chi_{[(k-1)\tau, k\tau)}(t) \, (b_0^* b_k + \ b_k^* b_0) \ . \tag{1.5}$$

Here $t \in [0, N\tau)$, the parameters: τ, E, ϵ, η are *positive*, and $\chi_{[x,y)}(\cdot)$ is the characteristic function of the semi-open interval $[x, y) \subset \mathbb{R}$. Operator $H_N(t)$ is self-adjoint on the time-independent domain

$$\mathcal{D}_0 = \bigcap_{k=0}^{N} \mathrm{dom}\,(b_k^* b_k) \subset \mathscr{H}^{(N)} \, . \tag{1.6}$$

The model (1.5) presents the system $\mathcal{S} + \mathcal{C}_N$, E where $\mathcal{S}$ is the quantum one-mode *cavity*, which is repeatedly perturbed by a time-equidistant *chain* of subsystem: $\mathcal{C}_N = \mathcal{S}_1 + \mathcal{S}_2 + \ldots + \mathcal{S}_N$. Here $\{\mathcal{S}_k\}_{k\geq 1}$ can be considered as *atoms* with harmonic internal degrees of freedom. This interpretation is motivated by certain physical models known as the "one-atom maser" [BJM], [NVZ]. The Hilbert space $\mathscr{H}_{\mathcal{S}} := \mathscr{H}_0$ corresponds to subsystem $\mathcal{S}$ and the Hilbert space $\mathscr{H}_k$ to subsystems $\mathcal{S}_k$ $(k = 1, \ldots, N)$, respectively. Then (1.2) is

$$\mathscr{H}^{(N)} = \mathscr{H}_{\mathcal{S}} \otimes \mathscr{H}_{\mathcal{C}_N} , \quad \mathscr{H}_{\mathcal{C}_N} := \bigotimes_{k=1}^{N} \mathscr{H}_k \ . \tag{1.7}$$

By (1.5) exactly one subsystem $\mathcal{S}_n$ (atom) interacts with $\mathcal{S}$ for $t \in [(n-1)\tau, n\tau)$. In this sense, the interaction is *tuned* [TZ]. The system $\mathcal{S} + \mathcal{C}_N$ is *autonomous* on each interval $[(n-1)\tau, n\tau)$ governed by the self-adjoint Hamiltonian

$$H_n := E\, b_0^* b_0 + \epsilon \sum_{k=1}^{N} b_k^* b_k + \eta\, (b_0^* b_n + b_n^* b_0) \ , \ n = 1, 2, \ldots, N \ , \tag{1.8}$$

on domain $\mathcal{D}_0$. Note that if

$$\eta^2 \leqslant E\, \epsilon \ , \tag{1.9}$$

Hamiltonians (1.5) and (1.8) are semi-bounded from below.

We note that Hamiltonian (1.8) is *gauge-invariant*: $e^{-i\phi \mathcal{N}_N} H_n e^{i\phi \mathcal{N}_N} = H_n$, $\phi \in \mathbb{R}$ (conserve the total number of bosons), but it is not *locally* gauge-invariant since $e^{-i(\phi_0 b_0^* b_0 + \phi_n b_n^* b_n)} H_n e^{i(\phi_0 b_0^* b_0 + \phi_n b_n^* b_n)} \neq H_n$ for nontrivial $\phi_0 \neq \phi_n$. Here $\mathcal{N}_N = \sum_{k=0}^{N} b_k^* b_k$ denotes the total number operator for *bosons* in the system $\mathcal{S} + \mathcal{C}_N$.

We denote by $\mathfrak{C}_1(\mathscr{H}^{(N)})$ the Banach space of the *trace-class* operators on $\mathscr{H}^{(N)}$. Its dual space is isometrically isomorph to the Banach space of bounded operators on $\mathscr{H}^{(N)}$: $\mathfrak{C}_1^*(\mathscr{H}^{(N)}) \simeq \mathcal{L}(\mathscr{H}^{(N)})$. The corresponding dual pair is defined by the bilinear functional

$$\langle \phi\, | A \rangle_{\mathscr{H}^{(N)}} = \mathrm{Tr}_{\mathscr{H}^{(N)}}(\phi\, A) \quad \text{for} \ \ (\phi, A) \in \mathfrak{C}_1(\mathscr{H}^{(N)}) \times \mathcal{L}(\mathscr{H}^{(N)}) \ . \tag{1.10}$$

The positive operators $\rho \in \mathfrak{C}_1(\mathscr{H}^{(N)})$ with unit trace is the set of *density matrices*. Recall that the state ω_ρ over $\mathcal{L}(\mathscr{H}^{(N)})$ is *normal* if there is a density matrix ρ such that

$$\omega_\rho(\,\cdot\,) = \langle \rho\, |\, \cdot\, \rangle_{\mathscr{H}^{(N)}} \ . \tag{1.11}$$

1.1 Master equation

To make the system $\mathcal{S} + \mathcal{C}_N$ open, we couple it to the *boson* reservoir $\mathcal{R}$, [AJP3]. More precisely, we follow the scheme $(\mathcal{S} + \mathcal{R}) + \mathcal{C}_N$, i.e. we study repeated perturbation of the open system $\mathcal{S} + \mathcal{R}$ [NVZ].

Evolution of normal states of the open system $(\mathcal{S} + \mathcal{R}) + \mathcal{C}_N$ can be described by the Kossakowski-Lindblad-Davies dissipative extension of the Hamiltonian dynamics to the Markovian dynamics with the time-dependent generator [AL], [AJP2]

$$\begin{aligned} L_\sigma(t)(\rho) := -i\, [H_N(t), \rho] + \\ + \mathcal{Q}(\rho) - \frac{1}{2}(\mathcal{Q}^*(\mathbb{1})\rho + \rho\, \mathcal{Q}^*(\mathbb{1}))\,, \end{aligned} \tag{1.12}$$

for $t \geqslant 0$ and $\rho \in \mathrm{dom}L_\sigma(t) \subset \mathfrak{C}_1(\mathscr{H}^{(N)})$. Here the first operator $\mathcal{Q} : \rho \mapsto \mathcal{Q}(\rho) \in \mathfrak{C}_1(\mathscr{H}^{(N)})$ in the dissipative part of (1.12) has the form:

$$\mathcal{Q}(\cdot) = \sigma_- \, b_0 \, (\cdot) \, b_0^* + \sigma_+ \, b_0^* \, (\cdot) \, b_0 \ , \quad \sigma_\mp \geqslant 0 \ , \tag{1.13}$$

and the operator $\mathcal{Q}^*$ is its dual via relation $\langle \mathcal{Q}(\rho) \, | A \rangle_{\mathscr{H}^{(N)}} = \langle \rho \, | \mathcal{Q}^*(A) \rangle_{\mathscr{H}^{(N)}}$:

$$\mathcal{Q}^*(\cdot) = \sigma_- \, b_0^* \, (\cdot) \, b_0 + \sigma_+ \, b_0 \, (\cdot) \, b_0^* \, . \tag{1.14}$$

By virtue of (1.5), for $t \in [(n-1)\tau, n\tau)$, the generator (1.12) takes the form

$$L_{\sigma,n}(\rho) := -i[H_n, \rho] + \mathcal{Q}(\rho) - \frac{1}{2}(\mathcal{Q}^*(\mathbb{1})\rho + \rho \mathcal{Q}^*(\mathbb{1})) \, . \tag{1.15}$$

The mathematical problem concerning the open quantum system is to solve the Cauchy problem for the non-autonomous quantum Master Equation [AJP2]

$$\partial_t \rho(t) = L_\sigma(t)(\rho(t)) \ , \quad \rho(0) = \rho \, . \tag{1.16}$$

For the tuned repeated perturbation, this solution is a strongly continuous family $\{T^\sigma_{t,0}\}_{t\geq 0}$, which is defined by composition of the one-step evolution semigroups:

$$T^\sigma_{t,0} = T^\sigma_{t,(n-1)\tau} \, T^\sigma_{n-1} \ldots T^\sigma_2 \ T^\sigma_1 \ ,$$

where $t = (n-1)\tau + \nu(t)$, $n \leqslant N, \nu(t) < \tau$. Here we put

$$T^\sigma_k := T^\sigma_k(\tau), \qquad T^\sigma_k(s) := e^{sL_{\sigma,k}} \quad (s \geqslant 0), \tag{1.17}$$

and then $T^\sigma_{t,(n-1)\tau} = T^\sigma_n(\nu(t))$ holds. The evolution map is connected to solution of the Cauchy problem (1.16) by

$$T^\sigma_{t,0} : \rho \mapsto \rho(t) = T^\sigma_{t,0}(\rho). \tag{1.18}$$

The construction of unique positivity- and trace-preserving dynamical semigroup on $\mathfrak{C}_1(\mathscr{H}^{(N)})$ for *unbounded* generator (1.15) is a nontrivial problem. It is done in [TZ1] under the conditions (1.9) and

$$0 \leqslant \sigma_+ < \sigma_- \ . \tag{1.19}$$

for the coefficients in (1.13, 1.14). Then, $\{T^\sigma_k(s)\}_{s\geqslant 0}$ for each k (1.17) is the Markov dynamical semigroup, and (1.18) is automorphism on the set of density matrices.

1.2 Evolution in the dual space

In order to control the evolution of normal states, it is usual to consider the W^*-dynamical system $(\mathcal{L}(\mathscr{H}^{(N)}), \{T^{\sigma\,*}_{t,0}\}_{t\geqslant 0})$, where $\{T^{\sigma\,*}_{t,0}\}_{t\geqslant 0}$ are weak*-continuous evolution maps on the von Neumann algebra $\mathcal{L}(\mathscr{H}^{(N)}) \simeq \mathfrak{C}_1^*(\mathscr{H}^{(N)})$ [AJP1]. They are dual to the evolution (1.18) on $\mathfrak{C}_1(\mathscr{H}^{(N)})$ by the relation (1.10):

$$\langle T^\sigma_{t,0}(\rho) \mid A \rangle_{\mathscr{H}^{(N)}} = \langle \rho \mid T^{\sigma\,*}_{t,0}(A) \rangle_{\mathscr{H}^{(N)}} \quad \text{for} \ \ (\rho, A) \in \mathfrak{C}_1(\mathscr{H}^{(N)}) \times \mathcal{L}(\mathscr{H}^{(N)}) \, , \tag{1.20}$$

which uniquely defines the map $A \mapsto T^{\sigma}_{t,0}{}^{*}(A)$ for $A \in \mathcal{L}(\mathscr{H}^{(N)})$. The corresponding dual time-dependent generator is formally given by

$$L^{*}_{\sigma}(t)(\cdot) = i\,[H_N(t), \cdot\,] + \\ + \mathcal{Q}^{*}(\cdot) - \frac{1}{2}(\mathcal{Q}^{*}(\mathbb{1})(\cdot) + (\cdot)\mathcal{Q}^{*}(\mathbb{1})) \quad \text{for} \quad t \geqslant 0\,. \tag{1.21}$$

When $t \in [(k-1)\tau, k\tau)$, the above generator has the form

$$L^{*}_{\sigma,k}(\cdot) = i[H_k, \cdot] + \mathcal{Q}^{*}(\cdot) - \frac{1}{2}(\mathcal{Q}^{*}(\mathbb{1})(\cdot) + (\cdot)\mathcal{Q}^{*}(\mathbb{1}))\,. \tag{1.22}$$

We adopt the notations

$$T^{\sigma}_{k}{}^{*} = T^{\sigma}_{k}(\tau)^{*}\ ,\ T^{\sigma}_{t,\,(n-1)\tau}{}^{*} = T^{\sigma}_{n}(\nu(t))^{*}\ ,\ \text{and}\ \ T^{\sigma}_{k}(s)^{*} := e^{sL^{*}_{\sigma,k}} \quad (s \geqslant 0)\ , \tag{1.23}$$

dual to (1.17) for $t = (n-1)\tau + \nu(t)$, $n \leqslant N$, $\nu(t) < \tau$. Then, we obtain

$$T^{\sigma}_{t,0}{}^{*}(A) = T^{\sigma}_{1}{}^{*}T^{\sigma}_{2}{}^{*} \ldots T^{\sigma}_{n-1}{}^{*}T^{\sigma}_{t,(n-1)\tau}{}^{*}(A) \quad \text{for} \ \ A \in \mathcal{L}(\mathscr{H}^{(N)})\,. \tag{1.24}$$

Let $\mathscr{A}(\mathscr{F})$ (or CCR($\mathbb{C}$)) denote the Weyl CCR-algebra on $\mathscr{F}$. This unital C^*-algebra is generated as operator-norm completion of the linear span $\mathscr{A}_w$ of the set of Weyl operators

$$\widehat{w}(\alpha) = e^{i\Phi(\alpha)} \qquad (\,\alpha \in \mathbb{C}\,), \tag{1.25}$$

where $\Phi(\alpha) = (\overline{\alpha}a + \alpha a^{*})/\sqrt{2}$ is the self-adjoint Segal operator in $\mathscr{F}$. [The closure of the sum is understood.] Then CCR (1.1) take the Weyl form

$$\widehat{w}(\alpha_1)\widehat{w}(\alpha_2) = e^{-i\,\mathrm{Im}(\overline{\alpha}_1\alpha_2)/2}\ \widehat{w}(\alpha_1+\alpha_2) \qquad \text{for} \qquad \alpha_1, \alpha_2 \in \mathbb{C}\,. \tag{1.26}$$

We note that $\mathscr{A}(\mathscr{F})$ is contained in the C^*-algebra $\mathcal{L}(\mathscr{F})$ of all *bounded* operators on $\mathscr{F}$.

Similarly we define the Weyl CCR-algebra $\mathscr{A}(\mathscr{H}^{(N)}) \subset \mathcal{L}(\mathscr{H}^{(N)})$ over $\mathscr{H}^{(N)}$. This algebra is generated by operators

$$W(\zeta) = \bigotimes_{j=0}^{N} \widehat{w}(\zeta_j) \quad \text{for} \ \ \zeta = \begin{pmatrix} \zeta_0 \\ \zeta_1 \\ \cdot \\ \cdot \\ \cdot \\ \zeta_N \end{pmatrix} \in \mathbb{C}^{N+1}\,. \tag{1.27}$$

By (1.3), the Weyl operators (1.27) can be rewritten as

$$W(\zeta) = \exp[i\big(\langle\zeta, b\rangle + \langle b, \zeta\rangle\big)/\sqrt{2}]\ , \tag{1.28}$$

where the sesquilinear form notations

$$\langle\zeta, b\rangle := \sum_{j=0}^{N} \bar{\zeta}_j b_j, \qquad \langle b, \zeta\rangle := \sum_{j=0}^{N} \zeta_j b^{*}_j \tag{1.29}$$

are used. Let us recall that $\mathscr{A}(\mathscr{H}^{(N)})$ is weakly dense in $\mathcal{L}(\mathscr{H}^{(N)})$[AJP1].

Explicit formulae for evolution operators (1.23) acting on the Weyl operators has been established in [TZ1]. For $n = 1, 2, \ldots .N$, let J_n and X_n be $(N+1) \times (N+1)$ Hermitian matrices:

$$(J_n)_{jk} = \begin{cases} 1 & (j = k = 0 \;\text{ or }\; j = k = n) \\ 0 & \text{otherwise} \end{cases}, \tag{1.30}$$

$$(X_n)_{jk} = \begin{cases} (E-\epsilon)/2 & (j,k) = (0,0) \\ -(E-\epsilon)/2 & (j,k) = (n,n) \\ \eta & (j,k) = (0,n) \\ \eta & (j,k) = (n,0) \\ 0 & \text{otherwise} \end{cases}. \tag{1.31}$$

We define the matrices

$$Y_n := \epsilon I + \frac{E-\epsilon}{2} J_n + X_n \quad (n = 1, \ldots, N), \tag{1.32}$$

where I is the $(N+1) \times (N+1)$ identity matrix. Then Hamiltonian (1.8) takes the form

$$H_n = \sum_{j,k=0}^{N} (Y_n)_{jk} b_j^* b_k. \tag{1.33}$$

We also need the $(N+1) \times (N+1)$ matrix P_0 defined by $(P_0)_{jk} = \delta_{j0}\delta_{k0}$ $(j, k = 0, 1, 2, \ldots, N)$. Then one obtains the following proposition which is proved in [TZ1]:

Proposition 1.1 *Let $n = 1, 2, \ldots, N$ and $\zeta \in \mathbb{C}^{N+1}$. Then for $s \geqslant 0$, the dual Markov dynamical semigroup (1.23) on the Weyl C^*-algebra has the form*

$$T_n^{\sigma *}(s)(W(\zeta)) = \Omega_{n,s}^{\sigma}(\zeta) W(U_n^{\sigma}(s)\zeta) , \tag{1.34}$$

where

$$\Omega_{n,s}^{\sigma}(\zeta) := \exp\Big[-\frac{1}{4}\,\frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+}\big(\langle\zeta,\zeta\rangle - \langle U_n^{\sigma}(s)\zeta, U_n^{\sigma}(s)\zeta\rangle\big)\Big] \tag{1.35}$$

and

$$U_n^{\sigma}(s) = \exp\Big[i\, s\Big(Y_n + i\,\frac{\sigma_- - \sigma_+}{2} P_0\Big)\Big] \tag{1.36}$$

under the conditions (1.9) and (1.19). Therefore, the k-step evolution ($t = k\tau, k \leqslant N$ in (1.24)) of the Weyl operator is given by

$$T_{k\tau,0}^{\sigma\;*}(W(\zeta)) = \exp\Big[-\frac{\sigma_- + \sigma_+}{4(\sigma_- - \sigma_+)}\big(\langle\zeta,\zeta\rangle - \langle U_1^{\sigma}\ldots U_k^{\sigma}\,\zeta,\, U_1^{\sigma}\ldots U_k^{\sigma}\,\zeta\rangle\big)\Big]$$
$$\times\, W(U_1^{\sigma}\ldots U_k^{\sigma}\,\zeta) , \tag{1.37}$$

where $T_{k\tau,0}^{\sigma\;*} = T_1^{\sigma\;*} T_2^{\sigma\;*}\, \ldots\, T_k^{\sigma\;*}$ and $U_n^{\sigma} := U_n^{\sigma}(\tau)$.

Remark 1.2 *The explicit expression of the matrix* $U_n^\sigma(t)$ *in (1.36) is given by* $U_n^\sigma(t) = e^{it\epsilon}V_n^\sigma(t)$*, where*

$$(V_n^\sigma(t))_{jk} = \begin{cases} g^\sigma(t)z^\sigma(t)\,\delta_{k0} + g^\sigma(t)w^\sigma(t)\,\delta_{kn} & (j=0) \\ g^\sigma(t)w^\sigma(t)\,\delta_{k0} + g^\sigma(t)z^\sigma(-t)\,\delta_{kn} & (j=n) \\ \quad\delta_{jk} & (otherwise) \end{cases} . \tag{1.38}$$

Here $E_\sigma := E + i\ (\sigma_- - \sigma_+)/2$ *and*

$$g^\sigma(t) := e^{it(E_\sigma-\epsilon)/2}, \qquad w^\sigma(t) := \frac{2i\eta}{\sqrt{(E_\sigma-\epsilon)^2+4\eta^2}}\sin t\sqrt{\frac{(E_\sigma-\epsilon)^2}{4}+\eta^2}\ , \tag{1.39}$$

$$z^\sigma(t) := \cos t\sqrt{\frac{(E_\sigma-\epsilon)^2}{4}+\eta^2} + \frac{i(E_\sigma-\epsilon)}{\sqrt{(E_\sigma-\epsilon)^2+4\eta^2}}\sin t\sqrt{\frac{(E_\sigma-\epsilon)^2}{4}+\eta^2}\ . \tag{1.40}$$

Note that the relation $z^\sigma(t)z^\sigma(-t) - w^\sigma(t)^2 = 1$ *holds for any* $\sigma_\pm \geqslant 0$*, whereas one has* $|g^\sigma(t)|^2(|z^\sigma(t)|^2 + |w^\sigma(t)|^2) < 1$ *and* $z^\sigma(-t) \neq \overline{z^\sigma(t)}$ *for* $0 \leqslant \sigma_+ < \sigma_-$*.*

Hereafter, together with (1.37) we also use the following short-hand notations:

$$g^\sigma = g^\sigma(\tau),\ w^\sigma := w^\sigma(\tau),\ z^\sigma = z^\sigma(\tau) \ \text{ and } \ V_n^\sigma := V_n^\sigma(\tau)\,. \tag{1.41}$$

Remark 1.3 *Dual dynamical semigroups (1.34) and the evolution operator (1.37) are examples of the quasi-free maps on the Weyl* C^**-algebra. Using the arguments of [DVV], we have shown in [TZ1] that they can be extended to the unity-preserving completely positive linear maps on* $\mathcal{L}(\mathscr{H}^{(N)})$ *under the conditions (1.9) and (1.19).*

The aim of the rest of the paper is to study evolution of the reduced density matrices for subsystems of the total system $(\mathcal{S}+\mathcal{R})+\mathcal{C}_N$.

In Section 2, we consider the subsystem $\mathcal{S}$. This includes analysis of convergence to stationary states in the infinite-time limit $N \to \infty$. We also perform a similar analysis for the subsystems $\mathcal{S}+\mathcal{S}_m$ and $\mathcal{S}_m+\mathcal{S}_n$. Section 3 is devoted to a more complicated problem of evolution of reduced density matrices for finite subsystems, which include $\mathcal{S}$ and a part of $\mathcal{C}_N$. This allows us to detect an asymptotic behaviour of the quantum correlations between $\mathcal{S}$ and a part of $\mathcal{C}_N$ caused by repeated perturbation and dissipation for large N in terms of those for small N with the stable initial state.

For the brevity, we hereafter suppress the dependence on N of the Hilbert space $\mathscr{H}^{(N)}$ as well as of the Hamiltonian $H_N(t)$ and the subsystem $\mathcal{C}_N$, when it will not cause any confusion.

2 Time Evolution of Subsystems I

2.1 Subsystem S

We start by analysis of the simplest subsystem $\mathcal{S}$. Let the initial state of the total system $\mathcal{S}+\mathcal{C}$ be defined by a density matrix $\rho \in \mathfrak{C}_1(\mathscr{H}_\mathcal{S} \otimes \mathscr{H}_\mathcal{C})$. Then for any $t \geqslant 0$, the evolved state $\omega^t_{\mathcal{S}}(\cdot)$ on the Weyl C^*-algebra $\mathscr{A}(\mathscr{H}_{\mathcal{S}})$ of subsystem $\mathcal{S}$ is given by the partial trace:

$$\omega^t_{\mathcal{S}}(A) = \omega_{\rho(t)}(A \otimes \mathbb{1}) = \mathrm{Tr}_{\mathscr{H}_{\mathcal{S}}\otimes\mathscr{H}_{\mathcal{C}_N}}(T^{\sigma}_{t,0}(\rho_{\mathcal{S}} \otimes \rho_{\mathcal{C}})\, A \otimes \mathbb{1}) \quad \text{for} \quad A \in \mathscr{A}(\mathscr{H}_{\mathcal{S}}) \ , \tag{2.1}$$

where $\rho(t) = T^{\sigma}_{t,0}\rho$ and $\mathbb{1} \in \mathscr{A}(\mathscr{H}_{\mathcal{C}})$. Recall that for a density matrix $\varrho \in \mathfrak{C}_1(\mathscr{H}_{\mathcal{S}} \otimes \mathscr{H}_{\mathcal{C}})$, the *partial trace* of ϱ with respect to the Hilbert space $\mathscr{H}_{\mathcal{C}}$ is a bounded linear map $\mathrm{Tr}_{\mathscr{H}_{\mathcal{C}}} : \varrho \mapsto \widehat{\varrho} \in \mathfrak{C}_1(\mathscr{H}_{\mathcal{S}})$ characterised by the identity

$$\mathrm{Tr}_{\mathscr{H}_{\mathcal{S}}\otimes\mathscr{H}_{\mathcal{C}}}(\varrho\,(A \otimes \mathbb{1})) = \mathrm{Tr}_{\mathscr{H}_{\mathcal{S}}}(\widehat{\varrho}\, A) \quad \text{for} \quad A \in \mathcal{L}(\mathscr{H}_{\mathcal{S}}) \ . \tag{2.2}$$

If one puts

$$\rho_{\mathcal{S}}(t) := \mathrm{Tr}_{\mathscr{H}_{\mathcal{C}}}(T^{\sigma}_{t,0}(\rho))\,, \tag{2.3}$$

then one gets the identity

$$\omega^t_{\mathcal{S}}(A) = \mathrm{Tr}_{\mathscr{H}_{\mathcal{S}}}(\rho_{\mathcal{S}}(t)\, A) =: \omega_{\rho_{\mathcal{S}}(t)}(A) \ , \tag{2.4}$$

by (2.1), i.e., $\rho_{\mathcal{S}}(t)$ is the density matrix defining the normal state $\omega^t_{\mathcal{S}}$.

As initial states $\omega^t_{\mathcal{S}+\mathcal{C}}\mid_{t=0}$ of the total system we consider the normal product states $\omega^t_{\mathcal{S}} \otimes \omega^t_{\mathcal{C}}\mid_{t=0} = \omega_{\rho_{\mathcal{S}}\otimes\rho_{\mathcal{C}}}$ for density matrices, which are *stationary* for the subsystem $\mathcal{C}$:

$$\rho = \rho_{\mathcal{S}} \otimes \rho_{\mathcal{C}} \quad \text{for} \quad \rho_{\mathcal{S}} = \rho_0 \ , \quad \rho_{\mathcal{C}} = \bigotimes_{k=1}^{N} \rho_k \quad \text{with} \quad \rho_1 = \rho_2 = \ldots = \rho_N \,. \tag{2.5}$$

Note that the *characteristic* function $E_{\omega_{\mathcal{S}}}\colon \mathbb{C} \to \mathbb{C}$ of the state $\omega_{\mathcal{S}}$ on the algebra $\mathscr{A}(\mathscr{H}_{\mathcal{S}})$ is

$$E_{\omega_{\mathcal{S}}}(\theta) = \omega_{\mathcal{S}}(\widehat{w}(\theta)) \tag{2.6}$$

and that (2.6) can uniquely determine the state $\omega_{\mathcal{S}}$ by the Araki-Segal theorem [AJP1].

Lemma 2.1 *Let $A = \widehat{w}(\theta)$. Then evolution of (2.1) on the interval* $[0,\tau)$ *yields*

$$E_{\omega^t_{\mathcal{S}}}(\theta) = \exp\Big[-\frac{|\theta|^2}{4}\frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+}\big(1 - |g^{\sigma}(t) z^{\sigma}(t)|^2 - |g^{\sigma}(t) w^{\sigma}(t)|^2\big)\Big]$$

$$\times \omega_{\rho_0}\big(\widehat{w}(e^{i\tau\epsilon} g^{\sigma}(t) z^{\sigma}(t)\theta)\big)\omega_{\rho_1}\big(\widehat{w}(e^{i\tau\epsilon} g^{\sigma}(t) w^{\sigma}(t)\theta)\big) \ , \ t \in [0,\tau) \ . \tag{2.7}$$

Proof : By (1.27), we obtain that $W(\theta e) = \widehat{w}(\theta) \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1}$ for the vector $e = {}^t(1, 0, \ldots, 0) \in \mathbb{C}^{N+1}$, where ${}^t(\ldots)$ means the *vector-transposition*, cf (1.27). Then (2.1)-(2.4) yield

$$\omega^t_{\mathcal{S}}(\widehat{w}(\theta)) = \omega_{\rho(t)}(\widehat{w}(\theta) \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1}) = \omega_{\rho_{\mathcal{S}}(t)}(\widehat{w}(\theta))\,. \tag{2.8}$$

By virtue of duality (1.20) and (1.37) for $k = 1$, we obtain

$$\omega_{\rho_{\mathcal{S}}(t)}(\widehat{w}(\theta)) = \omega_{\rho_{\mathcal{S}}\otimes\rho_{\mathcal{C}}}((T^{\sigma\,*}_{t,0} W)(\theta e)) = \omega_{\bigotimes_{j=0}^{N}\rho_j}((T^{\sigma\,*}_{t,0} W)(\theta e))$$

$$= \exp\Big[-\frac{|\theta|^2}{4}\frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+}\big(1 - \langle U^{\sigma}_1(t)e, U^{\sigma}_1(t)e\rangle\big)\Big]\omega_{\bigotimes_{j=0}^{N}\rho_j}\big(W(\theta\, U^{\sigma}_1(t)e)\big)\,.$$

Taking into account (1.38) and (2.6), one obtains for (2.8) the expression which coincides with assertion (2.7). $\square$

Similarly, for $t = m\tau$ we obtain the characteristic function

$$E_{\omega_{\mathcal{S}}^{m\tau}}(\theta) = \omega_{\rho_{\mathcal{S}}\otimes\rho_{\mathcal{C}}}(T^{\sigma\,*}_{m\tau,0}(W(\theta e)) = \exp\Big[-\frac{|\theta|^2}{4}\frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+}\big(1 - \langle U_1^\sigma \ldots U_m^\sigma e, U_1^\sigma \ldots U_m^\sigma e\rangle\big)\Big]$$

$$\times\omega_{\bigotimes_{j=0}^N \rho_j}\big(W(\theta\, U_1^\sigma \ldots U_m^\sigma e)\big) = \tag{2.9}$$

$$= \exp\Big[-\frac{|\theta|^2}{4}\frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+}\big(1 - \langle U_1^\sigma \ldots U_m^\sigma e, U_1^\sigma \ldots U_m^\sigma e\rangle\big)\Big]\prod_{j=0}^{N}\omega_{\rho_j}\big(\widehat{w}(\theta\,(U_1^\sigma \ldots U_m^\sigma e)_j)\big)\,,$$

where we have used (1.27) and (1.37). By (1.38) we obtain

$$(U_1^\sigma \ldots U_m^\sigma\; e)_k = \begin{cases} e^{im\tau\epsilon}(g^\sigma z^\sigma)^m & (k=0) \\ e^{im\tau\epsilon} g^\sigma w^\sigma (g^\sigma z^\sigma)^{m-k} & (1 \leqslant k \leqslant m) \\ 0 & (m < k \leqslant N)\,. \end{cases} \tag{2.10}$$

Then taking into account $|g^\sigma z^\sigma| < 1$ (Remark 1.2), we find

$$\begin{aligned}&\langle e, e\rangle - \langle U_1^\sigma \ldots U_m^\sigma\, e\,, U_1^\sigma \ldots U_m^\sigma\, e\rangle \\ &= (1 - |g^\sigma z^\sigma|^{2m})\Big[1 - \frac{|g^\sigma w^\sigma|^2}{1 - |g^\sigma z^\sigma|^2}\Big]\,.\end{aligned} \tag{2.11}$$

By setting $m = N$, (2.6), (2.9)-(2.11) yield the following result.

Lemma 2.2 *The state of the subsystem $\mathcal{S}$ after N-step evolution has the characteristic function*

$$\begin{aligned}&E_{\omega_{\mathcal{S}}^{N\tau}}(\theta) = \omega_{\rho_{\mathcal{S}}(N\tau)}(\widehat{w}(\theta)) \\ &= \exp\Big[-\frac{|\theta|^2}{4}\frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+}(1 - |g^\sigma z^\sigma|^{2N})\Big(1 - \frac{|g^\sigma w^\sigma|^2}{1 - |g^\sigma z^\sigma|^2}\Big)\Big] \\ &\times\omega_{\rho_0}\big(\widehat{w}(e^{iN\tau\epsilon}(g^\sigma)^N (z^\sigma)^N\theta)\big)\prod_{k=1}^{N}\omega_{\rho_k}\big(\widehat{w}(e^{iN\tau\epsilon}(g^\sigma)^{N-k+1}(z^\sigma)^{N-k} w^\sigma\theta)\big)\,.\end{aligned} \tag{2.12}$$

To study the asymptotic behaviour of the state $\omega_{\mathcal{S}}^{N\tau}$ for large time $t = N\tau$, we *assume* that the states $\{\omega_{\rho_k}\}_{k\geq 1}$ are gauge-invariant, i.e., one has

$$e^{-i\,\phi_k\, b_k^* b_k}\rho_k\, e^{i\,\phi_k\, b_k^* b_k} = \rho_k \qquad (\phi_k \in \mathbb{R})\ ,\ k \in \mathbb{N}\ , \tag{2.13}$$

for each component of the initial density matrix $\rho_{\mathcal{C}}$ (2.5) for atoms $\mathcal{C}$ but not for the cavity $\mathcal{S}$. We note that under this condition there exists an example of the cavity-atom interaction [NVZ], such that the limit state: $\lim_{N\to\infty}\omega_{\mathcal{S}}^{N\tau}$, is not gauge-invariant even for a normal gauge-invariant initial state ω_{ρ_0} of the cavity $\mathcal{S}$. We stick to condition (2.13) to check a possibility of the gauge-invariance breaking for the "soft" interaction (1.8), see discussion in Section 1.

Theorem 2.3 *Let ω_{ρ_k} be gauge-invariant for $k = 1, 2, \ldots, N$ and suppose that the product*

$$D(\theta) := \prod_{s=0}^{\infty} \omega_{\rho_1}(\widehat{w}((g^\sigma z^\sigma)^s \theta)) , \tag{2.14}$$

converges for any $\theta \in \mathbb{C}$ and let the map $\mathbb{R} \ni r \mapsto D(r\,\theta) \in \mathbb{C}$ be continuous. Then for any initial normal state $\omega_{\mathcal{S}}^0(\cdot) = \omega_{\rho_0}(\cdot)$ of the subsystem $\mathcal{S}$, the following properties hold.
(a) The pointwise limit of the characteristic functions (2.12) exists

$$E_*(\theta) = \lim_{N\to\infty} \omega_{\rho_{\mathcal{S}}(N\tau)}(\widehat{w}(\theta)) , \quad \theta \in \mathbb{C} . \tag{2.15}$$

(b) There exists a unique density matrix $\rho_*^{\mathcal{S}}$ such that the limit (2.15) is a characteristic function of the gauge-invariant normal state: $E_*(\theta) = \omega_{\rho_*^{\mathcal{S}}}(\widehat{w}(\theta))$.
(c) The states $\{\omega_{\mathcal{S}}^{m\tau}\}_{m\geqslant 1}$ converge to $\omega_{\rho_*^{\mathcal{S}}}$ for $m \to \infty$ in the weak-topology.*

Proof: (a) By (1.25) and by the gauge-invariance (2.13), one gets $\omega_{\rho_k}(\widehat{w}(e^{i\phi_k}\theta)) = \omega_{\rho_k}(\widehat{w}(\theta))$ for every $\phi_k \in \mathbb{R}$. Hence, for $1 \leqslant k \leqslant N$ the characteristic functions $E_{\omega_{\rho_k}}(\theta)$ depend only on $|\theta|$, and we can skip the factor $e^{iN\tau\epsilon}$ in the arguments of the factors in the right-hand side of (2.12). Note that for $N \to \infty$ the factor ω_{ρ_0} converges to one, since the normal states are regular and $|g^\sigma z^\sigma| < 1$ (see Remark 1.2). Hence, the pointwise limit (2.15) follows from (2.12) and the hypothesis (2.14). It does not depend on the initial state ω_{ρ_0} of the subsystem $\mathcal{S}$ and the explicit expression of (2.15) is given by

$$E_*(\theta) = \exp\Big[-\frac{|\theta|^2}{4}\frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+}\Big(1 - \frac{|g^\sigma w^\sigma|^2}{1 - |g^\sigma z^\sigma|^2}\Big)\Big] D(g^\sigma w^\sigma \theta) . \tag{2.16}$$

(b) The limit (2.16) inherits the properties of characteristic functions $E_{\omega_{\mathcal{S}}^{m\tau}}(\theta) = \omega_{\mathcal{S}}^{m\tau}(\widehat{w}(\theta))$:
(*i*) normalisation: $E_*(0) = 1$,
(*ii*) unitary : $\overline{E_*(\theta)} = E_*(-\theta)$,
(*iii*) positive definiteness: $\sum_{k,k'=1}^{K} \overline{z_k} z_{k'} e^{-i\,Im(\overline{\theta_k}\theta_{k'})/2}\, E_*(\theta_k - \theta_{k'}) \geqslant 0$ for any $K \geqslant 1$ and $z_k \in \mathbb{C}$ $(k = 1, 2, \ldots, K)$,
(*iv*) regularity: the continuity of the map $r \mapsto D(r\theta)$ implies that the function $r \mapsto E_*(r\,\theta)$ is also continuous.

Note that by the Araki-Segal theorem, the properties (i)-(iv) guarantee the existence of the unique normal state $\omega_{\rho_*^{\mathcal{S}}}$ over the CCR algebra $\mathscr{A}(\mathscr{H}_{\mathcal{S}})$ such that $E_*(\theta) = \omega_{\rho_*^{\mathcal{S}}}(\widehat{w}(\theta))$. Taking into account (a) and (2.16) we conclude that in contrast to the initial state $\omega_{\mathcal{S}}^0$ the limit state $\omega_{\rho_*^{\mathcal{S}}}$ is gauge-invariant.
(c) The convergence (2.14) can be extended by linearity to the algebraic span of the set of Weyl operators $\{\widehat{w}(\alpha)\}_{\alpha\in\mathbb{C}}$. Since it is norm-dense in C^*-algebra $\mathscr{A}(\mathscr{H}_{\mathcal{S}})$, the weak*-convergence of the states $\omega_{\mathcal{S}}^{m\tau}$ to the limit state $\omega_{\rho_*^{\mathcal{S}}}$ follows (see [BR1], [AJP1]). □

Remark 2.4 *(a) By Theorem 2.3 (a)-(b), one has $\rho_*^{\mathcal{S}} = \rho_*^{\mathcal{S}}(\tau)$, i.e. the limit state $\omega_{\rho_*^{\mathcal{S}}}$ is invariant under the one-step evolution $T_{\tau,0}^\sigma$. Comparing (2.7) and (2.16) one finds that $\rho_*^{\mathcal{S}} \neq T_{t,0}^\sigma(\rho_*^{\mathcal{S}})$ for $0 < t < \tau$. Instead, the evolution for repeated perturbation yields the asymptotic periodicity (cyclicity):*

$$\lim_{n\to\infty} (\omega_{\rho_{\mathcal{S}}(t)}(\widehat{w}(\theta)) - \omega_{\rho_*^{\mathcal{S}}(\nu(t))}(\widehat{w}(\theta)) = 0 \quad \text{for} \quad t = (n-1)\tau + \nu(t) . \tag{2.17}$$

(b) Consider the simplest case when density matrix ρ_1 in (2.5) corresponds to the gauge-invariant quasi-free Gibbs state for the inverse temperature $\beta > 0$:

$$\rho_1 = Z^{-1}\ e^{-\beta\,\epsilon\, b_1^* b_1}\ ,\ Z = \mathrm{Tr}_{\mathscr{H}_{\mathcal{S}_1}} e^{-\beta\,\epsilon\, b_1^* b_1}\ , \tag{2.18}$$

and let $\omega_{\rho_0}(\cdot)$ be any initial normal state of subsystem $\mathcal{S}$. Since

$$\omega_{\rho_1}(\widehat{w}(\theta)) = \exp\Big[-\frac{1}{4}\ |\theta|^2 \coth\frac{\beta\,\epsilon}{2}\Big]\ , \tag{2.19}$$

holds, we obtain for (2.14):

$$D(\theta) = \exp\Big[-\frac{1}{4}\ \frac{|\theta|^2}{1-|g^\sigma z^\sigma|^2}\ \ \coth\frac{\beta\,\epsilon}{2}\Big]\ . \tag{2.20}$$

Put $\lambda^\sigma(\tau) := |g^\sigma w^\sigma|^2(1-|g^\sigma z^\sigma|^2)^{-1} \in [0,1)$ (Remark 1.2). Then for the characteristic function of the limit state in Theorem 2.3, we get

$$\omega_{\rho_*}(\widehat{w}(\theta)) = \exp\left[-\frac{|\theta|^2}{4}\left((1-\lambda^\sigma(\tau))\frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+} + \lambda^\sigma(\tau)\ \coth\frac{\beta\,\epsilon}{2}\right)\right]. \tag{2.21}$$

If there is no such cavity-atom repeated interaction (i.e., $w^\sigma = 0$ and $\lambda^\sigma(\tau) = 0$), then the open subsystem $\mathcal{S}$ is only in contact with reservoir $\mathcal{R}$, and it evolves to a steady state with characteristic function

$$E_{*0}(\theta) = \exp\left[-\frac{|\theta|^2}{4}\frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+}\right]\ , \qquad 0 \le \sigma_+ < \sigma_-\ , \tag{2.22}$$

which corresponds to the gauge-invariant quasi-free Gibbs state for the inverse temperature $\beta_{*0} := E^{-1}\ln(\sigma_-/\sigma_+)$. It describes a thermal equilibrium between $\mathcal{S}$ and $\mathcal{R}$ for the effective temperature $1/\beta_{*0}$ of reservoir $\mathcal{R}$ measured in the harmonic cavity $\mathcal{S}$.

If $w^\sigma \neq 0$, the steady state (2.21) of subsystem $\mathcal{S}$ has the characteristic function

$$E_*(\theta) = \exp\left[-\frac{|\theta|^2}{4}\ \ \coth\frac{\beta_*^\sigma(\tau)E}{2}\right], \tag{2.23}$$

where the inverse temperature $\beta_*^\sigma(\tau)$ is defined by equation

$$\coth\frac{\beta_*^\sigma(\tau)E}{2} = (1-\lambda^\sigma(\tau))\ \coth\frac{\beta_{*0}E}{2} + \lambda^\sigma(\tau)\ \coth\frac{\beta\,\epsilon}{2}\ .$$

Note that now $\beta_*^\sigma(\tau)$ has an intermediate value between β_{*0} and $\beta\,\epsilon/E$ and satisfies either (i) $\beta_{*0} \leqslant \beta_*^\sigma(\tau) \leqslant \beta\,\epsilon/E$, or (ii) $\beta_{*0} \geqslant \beta_*^\sigma(\tau) \geqslant \beta\,\epsilon/E$.

Taking into account Remark 2.4(b), the physical interpretation of (a) is the following. Since at the moment $t = (n-1)\tau$ a new atom in the state (2.18) comes into cavity $\mathcal{S}$, which is different to that of the state of the outgoing atom, for the tuned interaction (see Section 1) the cavity starts to evolve on the interval $[0,\tau)$ as in Lemma 2.1 for the Gibbs state with temperature $1/\beta_*^\sigma(\tau)$ as the initial. Since in the limit $n \to \infty$ this Gibbs state is invariant under the one-step evolution $T^\sigma_{\tau,0}$, the cavity return back at the moment $t = n\tau$ to the initial Gibbs state with temperature $1/\beta_*^\sigma(\tau)$ and the atom at this moment leaves the resonator in the same state as the previous atom. This steady *cyclic* evolution of $\mathcal{S}$ is forced by repeated perturbation due to atoms and it is expressed by the limit (2.17).

As it follows from Remark 2.4(b) the cavity can be either heated (i) or cooled (ii) by the atomic beam as a function of the value of its temperature $1/\beta$. Note that we control the temperature $\beta_*^\sigma(\tau)$ only at the moments $t = n\tau$. Out of these moments the cavity $\mathcal{S}$ performs a cyclic evolution from the Gibbs state with temperature $1/\beta_*^\sigma(\tau)$ to itself with the period of repeated perturbation τ.

Note that if the atomic beam temperature is given by $1/\beta = \epsilon/(E\beta_{*0})$, then by Remark 2.4(b), (i)-(ii) the cavity temperature $1/\beta_*^\sigma(\tau)$ at $t = n\tau$ coincides with equilibrium temperature β_{*0} for the non-interacting case. Although this temperature varies on the interval $[0, \tau)$.

2.2 Correlations: subsystems $\mathcal{S} + \mathcal{S}_n$ and $\mathcal{S}_m + \mathcal{S}_n$

To study quantum correlations induced by repeated perturbation, we cast the first glance on the *bipartite* subsystems $\mathcal{S} + \mathcal{S}_n$ and $\mathcal{S}_m + \mathcal{S}_n$. We consider the initial density matrix (2.5) satisfying

$$\omega_{\rho_0}(\widehat{w}(\theta)) = \exp\Big[-\frac{|\theta|^2}{4}\coth\frac{\beta_0 E}{2}\Big]\ ,\ \omega_{\rho_j}(\widehat{w}(\theta)) = \exp\Big[-\frac{|\theta|^2}{4}\coth\frac{\beta\,\epsilon}{2}\Big]\,. \qquad (2.24)$$

From (1.20) and (1.37), we have:

Proposition 2.5 *For evolved density matrix* $\rho(N\tau) = T^\sigma_{N\tau,0}\,\rho$ *the characteristic function of the state* $\omega_{\rho(N\tau)}(\cdot)$ *is*

$$\omega_{\rho(N\tau)}(W(\zeta)) = \langle \rho\,|\,T^{\sigma\,*}_{N\tau,0}(W(\zeta))\rangle_{\mathscr{H}} = \exp\Big[-\frac{1}{4}\langle\zeta, X^\sigma(N\tau)\zeta\rangle\Big], \qquad (2.25)$$

where $X^\sigma(N\tau)$ *is the* $(N+1)\times(N+1)$ *matrix given by*

$$\begin{aligned} X^\sigma(N\tau) \;&=\; U_N^{\sigma\,*}\ldots U_1^{\sigma\,*}\Big[\Big(-\frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+} + \frac{1+e^{-\beta\epsilon}}{1-e^{-\beta\epsilon}}\Big)I + \Big(\frac{1+e^{-\beta_0 E}}{1-e^{-\beta_0 E}} - \frac{1+e^{-\beta\epsilon}}{1-e^{-\beta\epsilon}}\Big)P_0\Big] \\ &\quad \times U_1^\sigma \ldots U_N^\sigma \;+\; \frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+}\, I. \end{aligned} \qquad (2.26)$$

Remark 2.6 *In the theory of quantum correlation and entanglement for quasi-free states the matrix* $X^\sigma(t)$ *is known as the covariant matrix for Gaussian states, see [AdIl], [Ke]. Indeed, differentiating (2.25) with respect to components of* ζ *and* $\overline{\zeta}$ *at* $\zeta = 0$, *one can identify the entries of* $X^\sigma(t)$ *with expectations of monomials generated by the creation and the annihilation operators involved in (1.28), (1.29).*

Subsystem $\mathcal{S}+\mathcal{S}_n$. For $1 < n \leqslant N$ the initial state $\omega^0_{\mathcal{S}+\mathcal{S}_n}(\cdot)$ on the Weyl C^*-algebra $\mathscr{A}(\mathscr{H}_0 \otimes \mathscr{H}_n) \simeq \mathscr{A}(\mathscr{H}_0) \otimes \mathscr{A}(\mathscr{H}_n)$ of this *composite* subsystem is given by the partial trace

$$\omega^0_{\mathcal{S}+\mathcal{S}_n}(\widehat{w}(\alpha_0) \otimes \widehat{w}(\alpha_1)) = \omega_\rho(\widehat{w}(\alpha_0) \otimes \bigotimes_{k=1}^{n-1} \mathbb{1} \otimes \widehat{w}(\alpha_1) \otimes \bigotimes_{k=n+1}^{N} \mathbb{1})$$

$$= \exp\Big[-\frac{|\alpha_0|^2}{4}\coth\frac{\beta_0 E}{2}\Big]\exp\Big[-\frac{|\alpha_1|^2}{4}\coth\frac{\beta\,\epsilon}{2}\Big]\ . \tag{2.27}$$

This is the characteristic function of the product state corresponding to two isolated systems with different temperatures. Put $\zeta^{(0,n)} := {}^t(\alpha_0, 0, \ldots, 0, \alpha_1, 0, \ldots, 0) \in \mathbb{C}^{N+1}$ (cf. (1.27)), where α_1 occupies the $(n+1)$th position. Then we get

$$\omega^{N\tau}_{\mathcal{S}+\mathcal{S}_n}(\widehat{w}(\alpha_0) \otimes \widehat{w}(\alpha_1)) = \omega_{\rho(N\tau)}(W(\zeta^{(0,n)}))\ . \tag{2.28}$$

For the components of the vector $U^\sigma_1 \ldots U^\sigma_N \zeta^{(0,n)}$, we get from Remark 1.2 that

$$(U^\sigma_1 \ldots U^\sigma_N\ \zeta^{(0,n)})_k = \tag{2.29}$$

$$\begin{cases} e^{iN\tau\epsilon}\ [(g^\sigma z^\sigma)^N\ \alpha_0 + (g^\sigma z^\sigma)^{n-1} g^\sigma w^\sigma\, \alpha_1], & (k=0)\\ e^{iN\tau\epsilon}[(g^\sigma z^\sigma)^{N-k} g^\sigma w^\sigma \alpha_0 + (g^\sigma z^\sigma)^{n-k-1}(g^\sigma w^\sigma)^2\, \alpha_1], & (1 \leqslant k < n)\\ e^{iN\tau\epsilon}\ [(g^\sigma z^\sigma)^{N-n} g^\sigma w^\sigma\, \alpha_0 + g^\sigma z^\sigma(-\tau)\, \alpha_1], & (k=n)\\ e^{iN\tau\epsilon}\ (g^\sigma z^\sigma)^{N-k} g^\sigma w^\sigma \alpha_0 & (n < k \leqslant N). \end{cases}$$

Substitution of these expressions into (2.25) and (2.26) allows to calculate off-diagonal entries of the matrix $X^\sigma(N\tau)$ for $\zeta = \zeta^{(0,n)}$, which correspond to the cross-terms involving α_0 and α_1.

Because of $|g^\sigma z^\sigma| < 1$ (Remark 1.2), these non-zero off-diagonal entries will disappear when $N \to \infty$ for a fixed n. Hence, in the long-time limit the composite subsystem $\mathcal{S}+\mathcal{S}_n$ evolves from the product of two initial equilibrium states (2.27) to another product-state:

$$\omega^\infty_{\mathcal{S}+\mathcal{S}_n}(\widehat{w}(\alpha_0) \otimes \widehat{w}(\alpha_1)) = \exp\left[-\frac{|\alpha_0|^2}{4}\left((1-\lambda^\sigma(\tau))\frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+} + \lambda^\sigma(\tau)\ \coth\frac{\beta\,\epsilon}{2}\right)\right]$$
$$\times \exp\left[-\frac{|\alpha_1|^2}{4}\left((1-\mu^\sigma(\tau)-\nu^\sigma(\tau))\frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+} + \mu^\sigma(\tau)\ \coth\frac{\beta\,\epsilon}{2} + \nu^\sigma(\tau)\ \coth\frac{\beta_0 E}{2}\right)\right]\ ,$$

where $\lambda^\sigma(\tau)$ is the same as in (2.21),

$$\mu^\sigma(\tau) := |g^\sigma w^\sigma|^4 \frac{1-|g^\sigma z^\sigma|^{2(n-1)}}{1-|g^\sigma z^\sigma|^2} + |g^\sigma z^\sigma(-\tau)|^2\ ,$$

and $\nu^\sigma(\tau) := |g^\sigma w^\sigma|^2 |g^\sigma z^\sigma|^{(n-1)}$.

On the other hand, the cross-terms will not disappear in the limit $N, n \to \infty$, when $N-n$ is fixed [TZ]. It is interesting that in this case the steady state of the subsystem $\mathcal{S}$

keeps a correlation with subsystem $\mathcal{S}_n$ in the long-time limit and the limit reduced density of the combined subsystem $\mathcal{S}+\mathcal{S}_n$ is expressed in terms of ρ_*. In fact, for $n=N$, we obtain

$$\lim_{N\to\infty}\omega^{N\tau}_{\mathcal{S}+\mathcal{S}_N}(\widehat{w}(\alpha_0)\otimes\widehat{w}(\alpha_1))=\lim_{N\to\infty}\omega_{T^\sigma_{\tau,0}(\rho_{\mathcal{S}}((N-1)\tau)\otimes\rho_1)}(\widehat{w}(\alpha_0)\otimes\widehat{w}(\alpha_1))$$
$$=\omega_{T^\sigma_{\tau,0}(\rho_*\otimes\rho_1)}(\widehat{w}(\alpha_0)\otimes\widehat{w}(\alpha_1))\,.$$

This observation and the implication of the following example for subsystem $\mathcal{S}_m+\mathcal{S}_n$ will be generalised in the next Section 3.

Subsystem $\mathcal{S}_m+\mathcal{S}_n$. We suppose that $1\leqslant m<n\leqslant N$. Then the initial state $\omega^0_{\mathcal{S}_m+\mathcal{S}_n}(\cdot)$ on $\mathscr{A}(\mathscr{H}_m\otimes\mathscr{H}_n)\simeq\mathscr{A}(\mathscr{H}_m)\otimes\mathscr{A}(\mathscr{H}_n)$ of this composed subsystem is given by the partial trace

$$\omega^0_{\mathcal{S}_m+\mathcal{S}_n}(\widehat{w}(\alpha_1)\otimes\widehat{w}(\alpha_2))=\omega_\rho\Big(\bigotimes_{k=0}^{m-1}\mathbb{1}\otimes\widehat{w}(\alpha_1)\otimes\bigotimes_{k=m+1}^{n-1}\mathbb{1}\otimes\widehat{w}(\alpha_2)\otimes\bigotimes_{k=n+1}^{N}\mathbb{1}\Big)$$
$$=\exp\Big[-\frac{|\alpha_1|^2}{4}\coth\frac{\beta}{2}\Big]\exp\Big[-\frac{|\alpha_2|^2}{4}\coth\frac{\beta}{2}\Big]\,. \tag{2.30}$$

This is the characteristic function of the product-state corresponding to two isolated systems with the same temperature.

We define the vector $\zeta^{(m,n)}:={}^t(0,0,\ldots,0,\alpha_1,0,\ldots,0,\alpha_2,0,\ldots,0)\in\mathbb{C}^{N+1}$, where α_1 occupies the $(m+1)$th position and α_2 occupies the $(n+1)$th position, then

$$\omega^{N\tau}_{\mathcal{S}_m+\mathcal{S}_n}(\widehat{w}(\alpha_1)\otimes\widehat{w}(\alpha_2))=\omega_{\rho(N\tau)}(W(\zeta^{(m,n)}))\,. \tag{2.31}$$

Again with help of Remark 1.2, we can calculate the components of $U^\sigma_1\ldots U^\sigma_N\,\zeta^{(m,n)}$ as

$$(U^\sigma_1\ldots U^\sigma_N\,\zeta^{(m,n)})_k= \tag{2.32}$$

$$\begin{cases} e^{iN\tau\epsilon}\,(g^\sigma z^\sigma)^{m-1}\,g^\sigma w^\sigma[\alpha_1+(g^\sigma z^\sigma)^{n-m}\alpha_2] & (k=0)\\ e^{iN\tau\epsilon}\,(g^\sigma z^\sigma)^{m-k-1}(g^\sigma w^\sigma)^2\,[\alpha_1+(g^\sigma z^\sigma)^{n-m}\,\alpha_2] & (1\leqslant k<m)\\ e^{iN\tau\epsilon}\,[g^\sigma z^\sigma(-\tau)\,\alpha_1+(g^\sigma w^\sigma)^2\,(g^\sigma z^\sigma)^{n-m-1}\,\alpha_2] & (k=m)\\ e^{iN\tau\epsilon}\,(g^\sigma z^\sigma)^{n-k-1}\,(g^\sigma w^\sigma)^2\,\alpha_2 & (m<k<n)\\ e^{iN\tau\epsilon}\,g^\sigma z^\sigma(-\tau)\,\alpha_2 & (k=n)\\ 0 & (n<k\leqslant N)\end{cases}.$$

The correlation between $\mathcal{S}_m$ and $\mathcal{S}_n$, i.e. the corresponding off-diagonal elements of $X^\sigma(N\tau)$ are non-zero when $w\neq0$, and large for small $n-m$ and they decrease to zero as $n-m$ increase. Note that in contrast to the case $\mathcal{S}+\mathcal{S}_n$ (2.29) the last components $n<k\leqslant N$ in (2.32) as well as the state (2.31) do not depend on N. This reflects the fact that correlation involving $\mathcal{S}_m$ and $\mathcal{S}_n$ via subsystem $\mathcal{S}$ is switched off after the moment $t=n\tau$. If $w=0$, then (2.32) implies that $X^\sigma(N\tau)$ is always diagonal and that dynamics (2.31) keeps $\mathcal{S}_m+\mathcal{S}_n$ uncorrelated.

3 Time Evolution of Subsystems II

The results of Section 2.2 indicate that the *two*-component subsystems $\mathcal{S}+\mathcal{S}_n$ and $\mathcal{S}_m+\mathcal{S}_n$ of $\mathcal{S}+\mathcal{S}_N+\ldots+\mathcal{S}_{N-n}$ have important correlations for small n at the moment $t=N\tau$, even when N is large. Moreover, these correlations are asymptotically stable as $N\to\infty$.

In this section, we consider the corresponding *many*-component correlations for the initially uncorrelated product states. To this aim, for any fixed moment $t=k\tau$ we split the total system into two subsystems $\mathcal{S}_{n,k}$ and $\mathcal{C}_{n,k}$, where

$$\mathcal{S}_{n,k}=\mathcal{S}+\mathcal{S}_k+\mathcal{S}_{k-1}+\ldots+\mathcal{S}_{k-n+1}, \tag{3.1}$$

and

$$\mathcal{C}_{n,k}=\mathcal{S}_N+\ldots+\mathcal{S}_{k+1}\quad+\mathcal{S}_{k-n}+\ldots+\mathcal{S}_1. \tag{3.2}$$

Here, $n\in\mathbb{N}$ is supposed to be fixed and small with respect to large $N\in\mathbb{N}$. Then cavity $\mathcal{S}$ and atomic beam $\mathcal{S}_1,\ldots,\mathcal{S}_N$ at the moment $t=k\tau$ can be visualised as the line:

$$\mathcal{S}_N,\ldots,\mathcal{S}_{k+1},\ \mathcal{S},\ \mathcal{S}_k,\ldots,\mathcal{S}_{k-n+1},\ \mathcal{S}_{k-n},\ldots,\mathcal{S}_1\ . \tag{3.3}$$

Note that since the interaction between $\mathcal{S}$ and each of $\mathcal{S}_1,\ldots,\mathcal{S}_k$ is already ended, and they may be correlated. Whereas atoms $\mathcal{S}_{k+1},\ldots,\mathcal{S}_N$ have not yet interacted with $\mathcal{S}$ and hence they are still in uncorrelated initial product state.

From now on, we are going to treat $\mathcal{S}_{n,k}$ (3.1) as a *configuration* of the subsystem (or the object) denoted by $\mathcal{S}_{\sim n}$ at the moment $t=k\tau$. In other words, the subsystem $\mathcal{S}_{\sim n}$ possesses $\mathcal{S},\mathcal{S}_k,\ldots,\mathcal{S}_{k-n}$ as components, i.e. it contains those atoms passed up to the moment $t=k\tau$ the cavity $\mathcal{S}$ that are visible in the "window of observation" of the size n including the subsystem $\mathcal{S}$, see (3.3).

Note that the subsystem $\mathcal{S}_{\sim n}$ is an open system: when time passes from $t=k\tau$ to $t=(k+1)\tau$ the atom $\mathcal{S}_{k+1}$ enters into $\mathcal{S}_{\sim n}$ and the atom $\mathcal{S}_{k-n+1}$ leaves $\mathcal{S}_{\sim n}$.

We are interested in analysis of $\mathcal{S}_{\sim n}$ since it can be interpreted as a model of subsystem which is open for exchange of its constituent particles as well as of the energy with environment.

Below we concentrate on the large-time asymptotic behaviour of the state of $\mathcal{S}_{\sim n}$. To this aim we consider initial product states (2.5) with general density matrices $\rho_0,\rho_1\in\mathfrak{C}_1(\mathscr{F})$ for subsystems $\mathcal{S}_{n,k}$. Here we fix n and we treat k as a large varying parameter.

To express the state of $\mathcal{S}_{\sim n}$ at $t=k\tau$, we decompose the Hilbert space $\mathscr{H}$ into a tensor product of two Hilbert spaces

$$\mathscr{H}=\mathscr{H}_{\mathcal{S}_{n,k}}\bigotimes\mathscr{H}_{\mathcal{C}_{n,k}}\ .$$

Here $\mathscr{H}_{\mathcal{S}_{n,k}}$ is the Hilbert space for the subsystem (3.1) and $\mathscr{H}_{\mathcal{C}_{n,k}}$ for (3.2):

$$\mathscr{H}_{\mathcal{S}_{n,k}}=\mathscr{H}_0\bigotimes\Big(\bigotimes_{j=k-n+1}^{k}\mathscr{H}_j\Big),\qquad \mathscr{H}_{\mathcal{C}_{n,k}}=\Big(\bigotimes_{j=1}^{k-n}\mathscr{H}_j\Big)\bigotimes\Big(\bigotimes_{l=k+1}^{N}\mathscr{H}_l\Big). \tag{3.4}$$

If $\rho \in \mathfrak{C}_1(\mathscr{H})$ is the initial density matrix of the total system $\mathcal{S}_{n,k} + \mathcal{C}_{n,k}$, the reduced density matrix $\rho_{\mathcal{S}_{\sim n}}(k\tau)$ of $\mathcal{S}_{\sim n}$ at $t = k\tau$ is given by the partial trace

$$\rho_{\mathcal{S}_{\sim n}}(k\tau) = \mathrm{Tr}_{\mathscr{H}_{\mathcal{C}_{n,k}}}(T^{\sigma}_{k\tau,0}\,\rho) = \mathrm{Tr}_{\mathscr{H}_{c_1}}\big(\mathrm{Tr}_{\mathscr{H}_{c_2}}(T^{\sigma}_{k\tau,0}\,\rho)\big)\ , \tag{3.5}$$

for $k \geqslant n$ as in (2.2), where we decompose $\mathscr{H}_{\mathcal{C}_{n,k}}$ as

$$\mathscr{H}_{\mathcal{C}_{n,k}} = \mathscr{H}_{c_1} \bigotimes \mathscr{H}_{c_2}\,, \quad \mathscr{H}_{c_1} = \bigotimes_{j=1}^{k-n} \mathscr{H}_j\,, \quad \mathscr{H}_{c_2} = \bigotimes_{l=k+1}^{N} \mathscr{H}_l\,.$$

3.1 Preliminaries

Here we introduce notations and definitions to study evolution of subsystems in somewhat more general setting than in the previous sections.

In order to avoid the confusion caused by the fact that every $\mathscr{H}_j$ coincides with $\mathscr{F}$ in our case, we treat the Weyl algebra on the subsystem and the corresponding reduced density matrix of $\rho \in \mathfrak{C}_1(\mathscr{H})$ in the following way. On the Fock space $\mathcal{F}^{\otimes(m+1)}$ for $m = 0, 1, \ldots, N$, we define the Weyl operators

$$W_m(\zeta) := \exp\Big(i\ \frac{\langle\zeta, \tilde{b}\rangle_{m+1} + \langle\tilde{b}, \zeta\rangle_{m+1}}{\sqrt{2}}\Big)\ , \tag{3.6}$$

where $\zeta \in \mathbb{C}^{m+1}$, $\tilde{b}_0, \ldots, \tilde{b}_m$ and $\tilde{b}_0^*, \ldots, \tilde{b}_m^*$ are the annihilation and the creation operators in $\mathcal{F}^{\otimes(m+1)}$, which are constructed as in (1.3) satisfying the corresponding CCR and

$$\langle\zeta, \tilde{b}\rangle_{m+1} = \sum_{j=0}^{m} \bar{\zeta}_j \tilde{b}_j, \qquad \langle\tilde{b}, \zeta\rangle_{m+1} = \sum_{j=0}^{m} \zeta_j \tilde{b}_j^*.$$

By $\mathscr{A}(\mathscr{F}^{\otimes(m+1)})$, we denote the C^*-algebra generated by the Weyl operators (3.6).

Below, we adopt the abbreviations:

$$\mathscr{A}^{(m)} = \mathscr{A}(\mathscr{F}^{\otimes(m+1)}) \quad \text{and} \quad \mathscr{C}^{(m)} = \mathfrak{C}_1(\mathscr{F}^{\otimes(m+1)}) \tag{3.7}$$

for the Weyl C^* algebra on $\mathscr{F}^{\otimes(m+1)}$ and the algebra of all trace class operators on $\mathscr{F}^{\otimes(m+1)}$ for $m = 0, 1, 2, \ldots$, respectively. Note that the bilinear form

$$\langle\ \cdot \mid \cdot\ \rangle_m : \mathscr{C}^{(m)} \times \mathscr{A}^{(m)} \ni (\rho, A) \mapsto \mathrm{Tr}[\rho A] \in \mathbb{C} \tag{3.8}$$

yields the dual pair $(\mathscr{C}^{(m)}, \mathscr{A}^{(m)})$. Indeed, the following properties hold:

(i) $\langle\rho \mid A\rangle_m = 0$ for every $A \in \mathscr{A}^{(m)}$ implies $\rho = 0$;

(ii) $\langle\rho \mid A\rangle_m = 0$ for every $\rho \in \mathscr{C}^{(m)}$ implies $A = 0$;

(iii) $|\langle\rho \mid A\rangle_m| \leqslant \|\rho\|_{\mathfrak{C}_1} \|A\|_{\mathcal{L}}$.

These properties are a direct consequence of the fact that $\mathscr{A}^{(m)}$ is weakly dense in $\mathcal{L}(\mathscr{F}^{\otimes(m+1)})$ the dual space of $\mathscr{C}^{(m)}$. Below we shall use the topology $\sigma(\mathscr{C}^{(m)},\mathscr{A}^{(m)})$ induced by the dual pair $(\mathscr{C}^{(m)},\mathscr{A}^{(m)})$ on $\mathscr{C}^{(m)}$. We refer to it as the weak*-$\mathscr{A}^{(m)}$ topology, see e.g. [Ro], [BR1].

For $k \leqslant N$, we need the $(k+1)\times(k+1)$ matrix $U_\ell^{\sigma(k)}$ whose components are given by

$$(U_\ell^{\sigma(k)})_{ij} = \begin{cases} e^{i\tau\epsilon} g^\sigma(\tau)(\delta_{j0} z^\sigma(\tau) + \delta_{j\ell} w^\sigma(\tau)) & (i=0) \\ e^{i\tau\epsilon} g^\sigma(\tau)(\delta_{j0} w^\sigma(\tau) + \delta_{j\ell} z^\sigma(-\tau)) & (i=\ell) \\ e^{i\tau\epsilon}\delta_{ij} & (\text{otherwise}) \end{cases}, \tag{3.9}$$

for $\ell = 1,2,\ldots,k$ (c.f. Remark 1.2 Here N in the remark is replaced by k.). Then the one step evolution $T_\ell^{\sigma(k)}$ on $\mathscr{C}^{(k)}$ is given by

$$\langle T_\ell^{\sigma(k)}\rho \,|\, W_k(\zeta)\rangle_k = \langle \rho \,|\, T_\ell^{\sigma(k)*} W_k(\zeta)\rangle_k$$

where

$$T_\ell^{\sigma(k)*} W_k(\zeta) = \exp\Big[-\frac{\sigma_- + \sigma_+}{4(\sigma_- - \sigma_+)}\big(\langle\zeta,\zeta\rangle_{k+1} - \langle U_\ell^{\sigma(k)}\zeta, U_\ell^{\sigma(k)}\zeta\rangle_{k+1}\big)\Big] W_k(U_\ell^{\sigma(k)}\zeta)\,, \tag{3.10}$$

$\rho \in \mathscr{C}^{(k)}$ and $\zeta \in \mathbb{C}^{k+1}$ (see Proposition 1.1).

Now we introduce the "*free*" one-step evolution $\mathcal{T} : \mathscr{C}^{(0)} \longmapsto \mathscr{C}^{(0)}$ of density matrix corresponding to any of subsystems $\mathcal{S}_k$ by its dual

$$\mathcal{T}^* \widehat{w}(\theta) := \widehat{w}(e^{i\tau\epsilon}\theta)\,. \tag{3.11}$$

From (3.9) and (3.10), we have

$$\begin{aligned} & T_l^{\sigma(k+m)*}\big(W_k(\zeta)\otimes\widehat{w}(\zeta_{k+1})\cdots\otimes\widehat{w}(\zeta_{k+m})\big) \\ &= \big(T_l^{\sigma(k)*}\otimes(\mathcal{T}^*)^{\otimes m}\big)\big(W_k(\zeta)\otimes\widehat{w}(\zeta_{k+1})\cdots\otimes\widehat{w}(\zeta_{k+m})\big) \end{aligned} \tag{3.12}$$

for $l = 1,\cdots,k$ and $\zeta \in \mathbb{C}^{k+1}, \zeta_{k+1},\cdots,\zeta_{k+m} \in \mathbb{C}$. By composition, we also have

$$T_{k\tau,0}^{\sigma(k+m)*} = T_{k\tau,0}^{\sigma(k)*} \otimes (\mathcal{T}^{*k})^{\otimes m} \tag{3.13}$$

and its pre-dual

$$T_{k\tau,0}^{\sigma(k+m)} = T_{k\tau,0}^{\sigma(k)} \otimes (\mathcal{T}^{k})^{\otimes m}\,. \tag{3.14}$$

Now the calculation of the partial trace over $\mathscr{H}_{c_2}$ in (3.5) for the initial normal *product* state (2.5) is obvious:

$$\mathrm{Tr}_{\mathscr{H}_{c_2}}(T_{k\tau,0}^{\sigma\,(N)} \bigotimes_{j=0}^{N}\rho_j) = T_{k\tau,0}^{\sigma\,(k)} \bigotimes_{j=0}^{k}\rho_j \tag{3.15}$$

since $\mathcal{T}$ does not affect the trace:

$$\mathrm{Tr}[\mathcal{T}\rho_j] = \langle \mathcal{T}\rho_j \,|\, \widehat{w}(0)\rangle = \langle \rho_j \,|\, \mathcal{T}^*\widehat{w}(0)\rangle = \langle \rho_j \,|\, \widehat{w}(0)\rangle = \mathrm{Tr}[\rho_j]\,.$$

To calculate the partial trace with respect to $\mathscr{H}_{c_1}$ in (3.5), we introduce the imbedding:

$$r_{m+1,m} : \mathbb{C}^{m+1} \ni \zeta = \begin{pmatrix} \zeta_0 \\ \zeta_1 \\ \zeta_2 \\ \cdot \\ \cdot \\ \cdot \\ \zeta_m \end{pmatrix} \longmapsto \begin{pmatrix} \zeta_0 \\ 0 \\ \zeta_1 \\ \zeta_2 \\ \cdot \\ \cdot \\ \cdot \\ \zeta_m \end{pmatrix} = r_{m+1,m}\zeta \in \mathbb{C}^{m+2} \tag{3.16}$$

for $m = 0, 1, 2, \ldots, N$ and the partial trace over the *second* component $R_{m,m+1} : \mathscr{C}^{(m+1)} \to \mathscr{C}^{(m)}$ characterised by

$$\langle R_{m,m+1}\rho | \widehat{w}(\zeta_0) \otimes \widehat{w}(\zeta_1) \otimes \ldots \otimes \widehat{w}(\zeta_m) \rangle_m = \langle \rho | \widehat{w}(\zeta_0) \otimes \mathbb{1} \otimes \widehat{w}(\zeta_1) \otimes \ldots \otimes \widehat{w}(\zeta_m) \rangle_{m+1} \tag{3.17}$$

for $\rho \in \mathscr{C}^{(m+1)}$, where $\mathbb{1} = \widehat{w}(0)$ is the unit in $\mathscr{A}^{(0)}$. Therefore, its dual operator $R^*_{m,m+1}$ has the expression:

$$R^*_{m,m+1} W_m(\zeta) = W_{m+1}(r_{m+1,m}\zeta) \quad \text{for} \quad \zeta \in \mathbb{C}^{m+1} . \tag{3.18}$$

Lemma 3.1 *For $m \in \mathbb{N}$ and $\ell = 1, 2, \ldots, m$,*

$$U^{\sigma(m+1)}_{\ell+1} r_{m+1,m} = r_{m+1,m} U^{\sigma(m)}_{\ell} \ , \tag{3.19}$$

holds.

Proof : In fact, for the vector $\zeta = {}^t(\zeta_0, \zeta_1, \cdots, \zeta_m) \in \mathbb{C}^{m+1}$, one obtains

$$(U^{\sigma(m+1)}_{\ell+1} r_{m+1,m}\zeta)_j = (r_{m+1,m} U^{\sigma(m)}_{\ell} \zeta)_j$$

$$= \begin{cases} e^{i\tau\epsilon} g^\sigma(\tau)(z^\sigma(\tau)\zeta_0 + w^\sigma(\tau)\zeta_\ell) & (j = 0) \\ 0 & (j = 1) \\ e^{i\tau\epsilon}\zeta_{j-1} & (2 \leqslant j \leqslant \ell) \\ e^{i\tau\epsilon} g^\sigma(\tau)(w^\sigma(\tau)\zeta_0 + z^\sigma(-\tau)\zeta_\ell) & (j = \ell + 1) \\ e^{i\tau\epsilon}\zeta_{j-1} & (\ell + 2 \leqslant j \leqslant m + 1) \end{cases}$$

by explicit calculations. This proves the claim (3.19). □

For $k \in \mathbb{N}$ and $m = 0, 1, 2, \ldots, k-1$, let the maps $r_{k,m} : \mathbb{C}^{m+1} \to \mathbb{C}^{k+1}$ and $R_{m,k} : \mathscr{C}^{(k)} \to \mathscr{C}^{(m)}$ be defined by composition of the one-step maps (3.16), (3.17):

$$r_{k,m} = r_{k,k-1} \circ r_{k-1,k-2} \circ \ldots \circ r_{m+1,m} \ ,$$

and

$$R_{m,k} = R_{m,m+1} \circ R_{m+1,m+2} \circ \ldots \circ R_{k-1,k} \ ,$$

respectively. These definitions together with (3.17) and (3.18) imply that $R^*_{m,k} : \mathscr{A}^{(m)} \to \mathscr{A}^{(k)}$ and

$$R^*_{m,k}\, \widehat{w}(\zeta_0) \otimes \widehat{w}(\zeta_1) \otimes \ldots \otimes \widehat{w}(\zeta_m) = \widehat{w}(\zeta_0) \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes \widehat{w}(\zeta_1) \otimes \ldots \otimes \widehat{w}(\zeta_m)\ . \quad (3.20)$$

Hence, by (3.17) the map $R_{m,k}$, which is predual to (3.20), acts as the partial trace over the components with indices $j = 1, 2, \ldots, k-m$ of the tensor product $\bigotimes_{j=0}^{k} \rho_j \in \mathscr{C}^{(k)}$. Therefore, the map $R_{n,k}$ coincides with the partial trace Tr_{c_1} in (3.5). Then $R_{n,k}$ combined with (3.15) gives the expression

$$\rho_{\mathcal{S}_{\sim n}}(k\tau) = R_{n,k} T^{\sigma(k)}_{k\tau,0}(\bigotimes_{j=0}^{k} \rho_j) \quad \text{for} \quad k \geqslant n+1\,. \quad (3.21)$$

Obviously, it follows from (3.20) that

$$R^*_{m,m+k}\big(\widehat{w}(\zeta_0) \otimes \widehat{w}(\zeta_1) \otimes \cdots \otimes \widehat{w}(\zeta_m)\big) = \big(R^*_{0,k} \otimes I_d^{\otimes m}\big)\widehat{w}(\zeta_0) \otimes \big(\widehat{w}(\zeta_1) \cdots \otimes \widehat{w}(\zeta_m)\big) \quad (3.22)$$

and its predual identity

$$R_{m,m+k} = R_{0,k} \otimes I_d^{\otimes m} \quad (3.23)$$

for $m, k \in \mathbb{N}$, where I_d is the identity operator on $\mathcal{L}(\mathscr{F}) \supset \mathfrak{C}_1(\mathscr{F})$.

The formulae (3.14), (3.19) and (3.23) represent the general aspects of repeated perturbation systems in the words of our conclete model. We will use them in the following fashon in the remaining arguments.

Lemma 3.2 *For $m, k \in \mathbb{N}$, $\ell = 1, 2, \ldots, m$, the following properties hold:*

$$\text{(i)} \quad R_{m,m+k} T^{\sigma(m+k)}_{\ell+k} = T^{\sigma(m)}_{\ell} R_{m,m+k} \quad ; \quad (3.24)$$

$$\text{(ii)} \quad R_{m,m+k} T^{\sigma(m+k)}_{k\tau,0} = \big(R_{m,m+1} T^{\sigma(m+1)}_1\big) \cdots \big(R_{m+k-1,m+k} T^{\sigma(m+k)}_1\big) \quad ; \quad (3.25)$$

$$\text{(iii)} \quad R_{m,m+k} T^{\sigma(m+k)}_{k\tau,0} = \big(R_{0,k} T^{\sigma(k)}_{k\tau,0}\big) \otimes \big(\mathcal{T}^k\big)^{\otimes m} \quad . \quad (3.26)$$

Proof : (i) It is enoough to show that $T^{\sigma(m+k)*}_{\ell+k} R^*_{m,m+k} W_m(\zeta) = R^*_{m,m+k} T^{\sigma(m)*}_{\ell} W_m(\zeta)$. However, it is reduced to $U^{\sigma(m+k)}_{\ell+k} r_{m+k,m} = r_{m+k,m} U^{\sigma(m)}_{\ell}$, which is given by multiple use of (3.19).
(ii) is derived by multiple application of (i) to $T^{\sigma(m+k)}_{k\tau,0} = T^{\sigma(m+k)}_k \cdots T^{\sigma(m+k)}_1$.
(iii) is a composition of (3.23) and (3.14). □

Note that the *free* one-step evolution $\mathcal{T}$ is nothing but a gauge transformation. In this sense, it is applied not only to subsystems $\mathcal{S}_k$'s but also to $\mathcal{S}$. Since the present model is made as a gauge invariant theory, the following simple assertions on gauge transformations hold.

Lemma 3.3 *For any $m, k \in \mathbb{N}$ and $\ell = 1, \cdots, m$, the following properties hold:*

$$\text{(i)} \quad (\mathcal{T}^{\pm 1})^{\otimes m} R_{m-1,m-1+k} = R_{m-1,m-1+k} (\mathcal{T}^{\pm 1})^{\otimes (m+k)}\,, \quad (3.27)$$

$$\text{(ii)} \quad (\mathcal{T}^{\pm 1})^{\otimes (m+1)} T^{\sigma(m)}_{\ell} = T^{\sigma(m)}_{\ell} (\mathcal{T}^{\pm 1})^{\otimes (m+1)}\,, \quad (3.28)$$

$$\text{(iii)} \quad \mathcal{T}^{-k} R_{0,k} T^{\sigma(k)}_{k\tau,0} = \mathcal{T}^{-1} R_{0,1} T^{\sigma(1)}_1 (\mathcal{T}^{-1})^{\otimes 2} R_{1,2} T^{\sigma(2)}_1 \cdots (\mathcal{T}^{-1})^{\otimes k} R_{k-1,k} T^{\sigma(k)}_1. \quad (3.29)$$

Proof : It is easy to see that the dual identities of (i) and (ii) hold on the Weyl operators. For (iii), it is enough to apply the above (ii) and the formula

$$(\mathcal{T}^{-k+\ell-1})^{\otimes\ell} R_{\ell-1,\ell} = (\mathcal{T}^{-1})^{\otimes\ell} R_{\ell-1,\ell} (\mathcal{T}^{-k+\ell})^{\otimes(\ell+1)} ,$$

which follows from the above (i), to Lemma 3.2 (ii). □

3.2 Reduced density matrices of finite subsystems

In this subsection, we consider evolution of the subsystem $\mathcal{S}_{\sim n}$. Our aim is to study the large-time asymptotic behaviour of its states, when initial density matrix is given by (2.5).

For the density matrix ρ_1 in (2.5), we assume the condition:

[H] $$D(\theta) = \prod_{l=0}^{\infty} \langle \rho_1 \,|\, \widehat{w}((g^\sigma z^\sigma)^l \theta) \rangle_0 \ \text{ converge for any } \theta \in \mathbb{C}$$
$$\text{and the map } \mathbb{R} \ni t \mapsto D(t\theta) \in \mathbb{C} \ \text{ is continuous.}$$

Here, we do not assume gauge invariance of ρ_1. (c.f. Theorem 2.3)
Under the condition [H], one obtains the following theorem:

Theorem 3.4 *There exists a unique density matrix ρ_* on $\mathscr{F}$ such that $R_{0,1} T_1^{\sigma(1)}(\rho_* \otimes \rho_1) = \mathcal{T}\rho_*$ holds. And ρ_* also satisfies*

(1) $$\omega_{\rho_*}(\widehat{w}(\theta)) = \exp\Big[- \frac{|\theta|^2}{4} \frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+} \Big(1 - \frac{|g^\sigma w^\sigma|^2}{1 - |g^\sigma z^\sigma|^2} \Big) \Big] D(g^\sigma w^\sigma \theta);$$

(2) $R_{0,k} T_{k\tau,0}^{\sigma(k)} (\rho_* \otimes \rho_1^{\otimes k}) = \mathcal{T}^k \rho_*$ *for* $k > 1$;

(3) *For any density matrix ρ_0 on $\mathscr{F}$, the convergence* $\lim_{k\to\infty} \mathcal{T}^{-k} R_{0,k} T_{k\tau,0}^{\sigma(k)} (\rho_0 \otimes \rho_1^{\otimes k}) = \rho_*$ *holds in the weak*-$\mathscr{A}^{(0)}$ topology on $\mathscr{C}^{(0)}$.*

Remark 3.5 *(a) The weak*-$\mathscr{A}^{(0)}$ topology on $\mathscr{C}^{(0)}$ induced by the pair $(\mathscr{C}^{(0)}, \mathscr{A}^{(0)})$ (3.8) is coarser than the weak*-$\mathcal{L}(\mathscr{F})$ topology, which coincides with the weak and the norm topologies on the set of normal states [Ro, BR1].*
(b) When ρ_1 is gauge-invariant, the characteristic function in (1) coincides with (2.16) and the present theorem reduces to Theorem 2.3. Especially, the free evolution $R_{0,1} T_1^{\sigma(1)}(\rho_* \otimes \rho_1) = \mathcal{T}\rho_*$ reduces to the invariance $R_{0,1} T_1^{\sigma(1)}(\rho_* \otimes \rho_1) = \rho_*$

Proof : By the use of versions of (1.37), (2.10) and (2.11), we get

$$E_k(\theta) := \langle \mathcal{T}^{-k} R_{0,k} T_{k\tau,0}^{\sigma(k)} (\rho_0 \otimes \rho_1^{\otimes k} \,|\, \widehat{w}(\theta) \rangle_0 = \langle \rho_0 \otimes \rho_1^{\otimes k} | T_{k\tau,0}^{\sigma(k)*} W_k (e^{-ik\epsilon\tau} \theta e) \rangle_0$$

$$= \exp\Big[- \frac{|\theta|^2}{4} \frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+} \big(1 - \| U_1^{\sigma(k)} \cdots U_k^{\sigma(k)} \, e \|_{k+1} \big) \Big] \langle \rho_0 \otimes \rho_1^{\otimes k} \,|\, W_k (e^{-ik\epsilon\tau} U_1^{\sigma(k)} \cdots U_k^{\sigma(k)} \, e \rangle_{k+1}$$

$$= \exp\Big[- \frac{|\theta|^2}{4} \frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+} (1 - |g^\sigma z^\sigma|^{2k}) \Big(1 - \frac{|g^\sigma w^\sigma|^2}{1 - |g^\sigma z^\sigma|^2} \Big) \Big] \langle \rho_0 \,|\, \widehat{w}((g^\sigma z^\sigma)^k \theta) \rangle_0 \qquad (3.30)$$

$$\times \prod_{j=1}^{k} \langle \rho_1 \,|\widehat{w}((g^\sigma z^\sigma)^{k-j} g^\sigma w^\sigma \theta)\rangle_0 \,.$$

Thanks to the assumption [H], $\lim_{k\to\infty} E_k$ exists and equals to the right-hand side of (1) in the theorem. We note that $\lim_{k\to\infty}\langle\rho_0|w^\sigma((g^\sigma z^\sigma)^k\theta)\rangle_0 = 1$ because of $|g^\sigma z^\sigma| < 1$ and of the weak continuity of the normal state $\omega_{\rho_0} = \langle\rho_0|\cdot\rangle_0$.

The right-hand side of (1) satisfies: (i) *normalization*, (ii) *unitarity* and (iii) *positivity*, and (vi) *regularity*, since it is a limit of characteristic function $E_k(\theta)$. Hence from the Araki-Segal theorem as in Section 2.1, there exists a state ω_* on the CCR-algebra $\mathscr{A}(\mathscr{F})$ such that its characteristic function is given by the right-hand side of (1). Moreover, the continuity assumption about the function D yields that the state ω_* is normal by the Stone-von Neumann uniqueness theorem [BR2]. Hence, there exists a density matrix ρ_* such that $\omega_* = \omega_{\rho_*}$, which conclude (1). Now, (3) is obvious.

Put $\rho_0 = \rho_*$ in (3.30). Then we get

$$\langle \mathcal{T}^{-k} R_{0,k} T^{\sigma(k)}_{k\tau,0}(\rho_* \otimes \rho_1^{\otimes k}) \,|\, \widehat{w}(\theta)\rangle_0$$

$$= \exp\Big[-\frac{|\theta|^2}{4}\frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+}\Big(1 - \frac{|g^\sigma w^\sigma|^2}{1-|g^\sigma z^\sigma|^2}\Big)\Big] \prod_{j=1}^{\infty} \langle\rho_1 \,|\widehat{w}((g^\sigma z^\sigma)^j g^\sigma w^\sigma\theta)\rangle_0$$

$$= \langle \rho_* \,|\widehat{w}(\theta)\rangle_0$$

for $k \in \mathbb{N}$.

To prove the uniqueness of ρ_*, let $\rho_\spadesuit$ be another density matrix satisfying $R_{0,1}T_1^{\sigma(1)}(\rho_\spadesuit \otimes \rho_1] = \mathcal{T}\rho_\spadesuit$. Combining Lemma 3.3(iii) with Lemma 3.2(iii), we get

$$\mathcal{T}^{-k} R_{0,k} T^{\sigma(k)}_{k\tau,0} = \Big(\mathcal{T}^{-1} R_{0,1} T_1^{\sigma(1)}\Big)\Big(\big(\mathcal{T}^{-1}R_{0,1}T_1^{\sigma(1)}\big) \times \mathbb{1}\Big)\cdots\Big(\big(\mathcal{T}^{-1}R_{0,1}T_1^{\sigma(1)}\big) \otimes \mathbb{1}^{\otimes(k-1)}\Big),$$

which yields

$$\mathcal{T}^{-k} R_{0,k} T^{\sigma(k)}_{k\tau,0}(\rho_\spadesuit \otimes \rho_1^{\otimes k}) = \rho_\spadesuit \,.$$

Then, $\rho_\spadesuit = \rho_*$ follows from (3). □

Now we consider the large-time behaviour of the states (3.5) of subsystems $\mathcal{S}_{\sim n}$. Let ρ_1 be a density matrix on $\mathscr{F}$ satisfying the condition [H]. Then we have the following theorem.

Theorem 3.6 *For any density matrix ρ_0 on $\mathscr{F}$ and $n, m \in \mathbb{N}$, $m \geqslant n$, the limit:*

$$(\mathcal{T}^{-k})^{\otimes(m+1)} R_{m,m+k} T^{\sigma(m+k)}_{(n+k)\tau,0}\big(\rho_0 \otimes \rho_1^{\otimes(m+k)}\big) \longrightarrow T^{\sigma(m)}_{n\tau,0}\big(\rho_* \otimes \rho_1^{\otimes m}\big) \quad as \quad k \to \infty \;,$$

holds in the weak-$\mathscr{A}^{(m)}$ topology on $\mathscr{C}^{(m)}$. Here ρ_* is the density matrix on $\mathscr{F}$ given in Theorem 3.4.*

Proof : From Lemma 3.2 and Lemma 3.3, we obtain

$$(\mathcal{T}^{-k})^{\otimes(m+1)} R_{m,m+k} T^{\sigma(m+k)}_{(n+k)\tau,0}\big(\rho_0 \otimes \rho_1^{\otimes(m+k)}\big)$$

$$= T^{\sigma(m)}_{n\tau,0}(\mathcal{T}^{-k})^{\otimes(m+1)} R_{m,m+k} T^{\sigma(m+k)}_{k\tau,0}(\rho_0 \otimes \rho_1^{\otimes(m+k)}) = T^{\sigma(m)}_{n\tau,0}\big((\mathcal{T}^{-k} R_{0,k} T^{\sigma(k)}_{k\tau,0}) \otimes (\mathbb{1}^{\otimes m})\big)(\rho_0 \otimes \rho_1^{\otimes(m+k)})$$

$$= T^{\sigma(m)}_{n\tau,0}\big(\big(\mathcal{T}^{-k} R_{0,k} T^{\sigma(k)}_{k\tau,0}(\rho_0 \otimes \rho_1^{\otimes k})\big) \otimes (\rho_1^{\otimes m})\big)$$

By Theorem 3.4, one has

$$\lim_{k\to\infty} \mathcal{T}^{-k} R_{0,k} T^{\sigma(k)}_{k\tau,0}(\rho_0 \otimes \rho_1^{\otimes k}) = \rho_*$$

in the weak*-$\mathscr{A}^{(0)}$ topology. Then, we obtain also the weak*-$\mathscr{A}^{(m)}$ convergence

$$\big(\mathcal{T}^{-k} R_{0,k} T^{\sigma(k)}_{k\tau,0}(\rho_0 \otimes \rho_1^{\otimes k})\big) \otimes (\rho_1^{\otimes m}) \longrightarrow \rho_* \otimes \rho_1^{\otimes m} \quad \text{as } k \to \infty\,.$$

By the continuity of $T^{\sigma(m)}_{n\tau,0}$, one gets the weak*-$\mathscr{A}^{(m)}$ convergence

$$(\mathcal{T}^{-k})^{\otimes(m+1)} R_{m,m+k} T^{\sigma(m+k)}_{(n+k)\tau,0}\big(\rho_0 \otimes \rho_1^{\otimes(m+k)}\big) \longrightarrow T^{\sigma(m)}_{n\tau,0}\big(\rho_* \otimes \rho_1^{\otimes m}\big) \quad \text{as } k \to \infty\,,$$

claimed in the theorem. □

Let us put $m = n$ in the theorem. Then by (3.21), we obtain the limit of the reduced density matrix $\rho_{\mathcal{S}_{\sim n}}(\cdot)$ for the subsystem $\mathcal{S}_{\sim n}$:

Corollary 3.7 *The convergence*

$$\lim_{k\to\infty} (\mathcal{T}^{-k})^{\otimes(n+1)} \rho_{\mathcal{S}_{\sim n}}((n+k)\tau) = T^{\sigma(n)}_{n\tau,0}(\rho_* \otimes \rho_1^{\otimes n}) \tag{3.31}$$

holds in the weak-$\mathscr{A}^{(n)}$ *topology on* $\mathscr{C}^{(n)}$.

Since $\mathcal{T}$ is the free evolution (3.11), the limit (3.31) means that dynamics of subsystem $\mathcal{S}_{\sim n}$ is the asymptotically-free evolution of the state, which is given by the n-step evolution of the initial density matrix $\rho_* \otimes \rho_1^{\otimes n}$ of the system $\mathcal{S} + \mathcal{C}_n$.

From the continuous time point of view, the subsystem $\mathcal{S}_{\sim n}$ shows the asymptotic behaviour, which is a combination of the free and periodic evolutions, cf Remark 2.4(a).

Remark 3.8 *There are three energy parameters E, ϵ and η in the Hamiltonian (1.5) and two corresponding parameters $\sigma_\pm$ in the dissipative term. However, the subsystems described above indicate asymptotically free evolution governed by the energy ϵ alone. For an intuitive understanding of this phenomenon, let us consider an example of evolution of a coherent state.*

Let both density matrices ρ_0 and ρ_1 be the pure state corresponding to coherent vector-state $|\alpha\rangle$ satisfying $a|\alpha\rangle = \alpha|\alpha\rangle$, where $\alpha \in \mathbb{C} - \{0\}$. Then, we obtain

$$\langle \rho_0 \,|\, \widehat{w}(\theta)\rangle_0 = \langle \rho_1 \,|\, \widehat{w}(\theta)\rangle_0 = \langle \alpha \,|\, \widehat{w}(\theta) \,|\, \alpha\rangle = \exp\Big[-\frac{|\theta|^2}{4} + \frac{i(\overline{\alpha}\theta + \alpha\overline{\theta})}{\sqrt{2}}\Big]\,. \tag{3.32}$$

Put $\rho^{(k)} = R_{0,k}T^{\sigma(k)}_{k\tau,0}(\rho_0 \otimes \rho_1^{\otimes k})$*. Then the recursion formulra*

$$\rho^{(k+1)} = R_{0,1}T_1^{\sigma(1)}(\rho^{(k)} \otimes \mathcal{T}^k\rho_1), \qquad (k \in \mathbb{N}) \tag{3.33}$$

and $\rho^{(0)} = \rho_0$*. This relation follows from*

$$R_{0,k+1}T^{\sigma(k+1)}_{(k+1)\tau,0} = R_{0,1}T_1^{\sigma(1)}R_{1,k+1}T^{\sigma(k+1)}_{k\tau,0} = R_{0,1}T_1^{\sigma(1)}\big[\big(R_{0,k}T^{\sigma(k)}_{k\tau,0}\big) \otimes \mathcal{T}^k\big]/,,$$

where we have used Lemma 3.2(i), (iii).

As the expectation of the Weyl operator by $\rho^{(k)}$*,*

$$\langle \rho^{(k)} \mid \widehat{w}(\theta)\rangle_0 = \exp\Big[-\frac{|\theta|^2}{4}A_k + \frac{i(\overline{\alpha}B_k\theta + \alpha\overline{B_k\theta})}{\sqrt{2}}\Big]$$

is valid for the sequences $\{A_k\}_{k\geq 0}, \{B_k\}_{k\geq 0} \subset \mathbb{C}$ *given by recursions*

$$\begin{aligned} A_{k+1} &= |g^\sigma z^\sigma|^2 A_k + \frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+}(1 - |g^\sigma z^\sigma|^2 - |g^\sigma w^\sigma|^2) + |g^\sigma w^\sigma|^2 \ , \\ B_{k+1} &= e^{i\epsilon\tau}g^\sigma z^\sigma B_k + e^{i(k+1)\epsilon\tau}g^\sigma w^\sigma \ , \end{aligned}$$

and $A_0 = B_0 = 1$*. From the second recursion, it is easy to see that*

$$\lim_{k\to\infty} e^{-ik\epsilon\tau}B_k = \frac{g^\sigma w^\sigma}{1 - g^\sigma z^\sigma} \ .$$

This implies that asymptotic behaviour of B_k *is described by oscillating factor* $e^{ik\epsilon\tau}$ *and the constant which depend on the other parameters* E, η *and* $\sigma_\pm$ *as well as* ϵ *through* g^σ*,* z^σ *and* w^σ*.*

We comment that this recursion can be read as that $(k+1)$*-th state of* $\mathcal{S}$ *results from a mixed evolution of the* k*-th state of* $\mathcal{S}$ *and the* k*-th state of* $\mathcal{S}_k$*, where subsystem* $\mathcal{S}_k$ *has evolved* k *times freely with parameter* ϵ *corresponding to the atomic energy oscillator spectrum. The example demonstrates how the asymptotic behaviour is imposed by these oscillations. It also indicates that the asymptotic evolution is analogous to the forced oscillator, when a beam of atoms plays the roll of an external force.*

As an example to Corollary 3.7, we consider the asymptotic form, when the state ρ_1 is *coherent*.

Let ρ_* be the state given by Theorem 3.4 for ρ_1 with characteristic function (3.32). Then the density matrix in the right-hand side of (3.31) has the characteristic function:

$$\langle T^{\sigma(n)}_{n\tau,0}(\rho_* \otimes \rho_1^{\otimes n})|W_n(\zeta)\rangle_n \tag{3.34}$$

$$= \exp\Big[-\frac{1}{4}\langle\zeta, X_n^\sigma\zeta\rangle + \frac{i}{\sqrt{2}}\Big(\overline{\alpha}\big(C_n^\sigma\zeta_0 + D_n^\sigma\sum_{j=1}^n\zeta_j\big) + \alpha\,\overline{\big(C_n^\sigma\zeta_0 + D_n^\sigma\sum_{j=1}^n\zeta_j\big)}\Big)\Big],$$

where $\zeta \in \mathbb{C}^{n+1}$,

$$X_n^\sigma = \frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+} I + \Big(1 - \frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+}\Big) U_n^{\sigma(n)*} \dots U_1^{\sigma(n)*} \Big(I + \Big(\frac{|g^\sigma w^\sigma|^2}{1 - |g^\sigma z^\sigma|^2} - 1\Big) P_0\Big) U_1^{\sigma(n)} \dots U_n^{\sigma(n)} .$$

is a $(n+1) \times (n+1)$ matrix and

$$C_n^\sigma := e^{in\tau\epsilon} \frac{g^\sigma w^\sigma}{1 - g^\sigma z^\sigma} , \qquad D_n^\sigma := e^{in\tau\epsilon} g^\sigma \frac{z^\sigma(-\tau) - g^\sigma}{1 - g^\sigma z^\sigma} .$$

In order to obtain (3.34), we first note that the function of condition [H] has the form:

$$D(\theta) = \exp\Big[- \frac{|\theta|^2}{4} \frac{1}{1 - |g^\sigma z^\sigma|^2} + \frac{i}{\sqrt{2}} \Big(\overline{\alpha} \frac{\theta}{1 - g^\sigma z^\sigma} + \alpha \frac{\overline{\theta}}{1 - \overline{g^\sigma z^\sigma}}\Big)\Big] .$$

This yields

$$\langle \rho_* \,|\, \widehat{w}(\theta)\rangle_0 = \exp\Big[- \frac{|\theta|^2}{4} \Big(\frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+} \Big(1 - \frac{|g^\sigma w^\sigma|^2}{1 - |g^\sigma z^\sigma|^2}\Big) + \frac{|g^\sigma w^\sigma|^2}{1 - |g^\sigma z^\sigma|^2}\Big)\Big]$$

$$\times \exp\Big[\frac{i}{\sqrt{2}} \Big(\bar{\alpha} \frac{g^\sigma w^\sigma \theta}{1 - g^\sigma z^\sigma} + \alpha \frac{\overline{g^\sigma w^\sigma \theta}}{1 - \overline{g^\sigma z^\sigma}}\Big)\Big] . \tag{3.35}$$

Now taking into account (3.35), by duality (1.20), (1.24), by (3.10), we obtain the representation

$$\begin{aligned}
&\langle T_{n\tau,0}^{\sigma(n)} (\rho_* \otimes \rho_1^{\otimes n}) | W_n(\zeta)\rangle_n \\
&= \exp\Big[- \frac{1}{4} \frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+} \big(\|\zeta\|_n - \|U_1^{\sigma(n)} \dots U_n^{\sigma(n)} \zeta\|_n\big)\Big] \\
&\quad \times \langle \rho_* \otimes \rho_1^{\otimes n} | W_n(U_1^{\sigma(n)} \dots U_n^{\sigma(n)} \zeta)\rangle_n \\
&= \exp\Big[- \frac{1}{4} \frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+} \big(\|\zeta\|_n - \|U_1^{\sigma(n)} \dots U_n^{\sigma(n)} \zeta\|_n\big)\Big] \\
&\quad \times \langle \rho_* | \widehat{w}((U_1^{\sigma(n)} \dots U_n^{\sigma(n)} \zeta)_0)\rangle_0 \prod_{j=1}^n \langle \rho_1 | \widehat{w}((U_1^{\sigma(n)} \dots U_n^{\sigma(n)} \zeta)_j)\rangle_0 .
\end{aligned}$$

Then the assertion (3.34) follows if one notes that

$$\frac{g^\sigma w^\sigma}{1 - g^\sigma z^\sigma} (U_1^{\sigma(n)} \dots U_n^{\sigma(n)} \zeta)_0 + \sum_{j=1}^n (U_1^{\sigma(n)} \dots U_n^{\sigma(n)} \zeta)_j$$

$$= e^{in\tau\epsilon} \Big(\frac{g^\sigma w^\sigma}{1 - g^\sigma z^\sigma} \zeta_0 + g^\sigma \frac{z^\sigma(-\tau) - g^\sigma}{1 - g^\sigma z^\sigma} \sum_{j=1}^n \zeta_j\Big) ,$$

which is a consequence of a straightforward calculation using (3.9) and of identity $z^\sigma(\tau) z^\sigma(-\tau) - (w^\sigma(\tau))^2 = 1$, see Remark 1.2.

4 Conclusions

In this paper we addressed to the problem: how interaction with resonator of a beam of initially independent atoms (product state) might produce correlated/entengled states in the beam ?

We note in Remark 2.6 that the answer is given by the properties of the matrix $X^{\sigma}(t)$ (2.26). This matrix is initially *diagonal* since it corresponds to uncorrelated at $t = 0$ tensor product of states (2.24). For $k > 0$ the off-diagonal elements of $X^{\sigma}(kt)$ encode the quantum correlations between subsystems of ensemble $\mathcal{S} + \{\mathcal{S}_k\}_{k=1}^{N}$. Although there is still a room for approximating this state by a trace-norm convergent convex sum of product states (known as the *separability*), the next level of correlation leads to *entanglement* [AdIl], [Ke].

A transition between separable and entangled states is explicitly established for the model of two-mode quasi-free squeezed thermal state for large squeeze parameter [MMS]. This *bipartite* model is similar to our case (Proposition 2.5), when only *two* components of the vector $\{\zeta_j\}_{j=0}^{N}$ are non-zero.

In the present paper we do not aim to study a subtle problem of the separability-entanglement transition, but instead we concentrate our attention on correlations for the *multipartite* case, see Sections 2 and 3. To elucidate the setup of the problem we first analysed correlations for the two bipartite cases: $\mathcal{S} + \mathcal{S}_m$ and $\mathcal{S}_m + \mathcal{S}_n$, see Sections 2.

Then in Section 3, we treated a more general case of the subsystem $\mathcal{S}_{\sim n}$. For any moment it constitutes of the cavity $\mathcal{S}$ and the closest to it n atoms just after interaction with the cavity. The atomic constituents in $\mathcal{S}_{\sim n}$ are exchanging with environment with the running time $t = k\tau$, Section 3. The constituents evolve asymptotically freely with the energy parameter of the atoms.

In other words, the subsystem $\mathcal{S}_{\sim n}$ is similar to a "grand-canonical ensemble", which is open for exchange of particles. They are the atoms migrating through $\mathcal{S}_{\sim n}$. Again, similar to the grand-canonical ensemble the configurations $\mathcal{S}_{n,k}$ of $\mathcal{S}_{\sim n}$ are visible in a "window of observation" of the size n, which includes $\mathcal{S}$ and n atoms that passed $\mathcal{S}$ when $k \geq n$.

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