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journal or publication title	Mathematische Annalen
volume	364
number	3-4
page range	1043-1068
year	2016-04-01
URL	http://hdl.handle.net/2297/43342

doi: 10.1007/s00208-015-1243-7

On the effect of equivalent constraints on a maximizing problem associated with the Sobolev type embeddings in $\mathbb{R}^N$

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Abstract

In this paper, we consider the attainability of a maximizing problem

$$D := \sup_{\|u\|_{H^{1,N}_\gamma}=1} \left(\|u\|_N^N + \alpha \|u\|_p^p \right),$$

where $N \geq 2$, $N < p < \infty$, $\alpha > 0$, $0 < \gamma \leq N$ and $\|u\|_{H^{1,N}_\gamma} = (\|u\|_N^\gamma + \|\nabla u\|_N^\gamma)^{\frac{1}{\gamma}}$. The existence of a maximizer for D is closely related to the exponent γ. In fact, we show that the value

$$\alpha = \alpha_* := \inf_{\|u\|_{H^{1,N}_\gamma}=1} \left(\frac{1 - \|u\|_N^N}{\|u\|_p^p} \right)$$

is a threshold in terms of the attainability of D.

2010 Mathematics Subject Classification. 47J30; 46E35; 26D10.
Key words: attainability of maximizing problem, critical Sobolev space, Sobolev's embedding theorem, normalized vanishing sequence

1 Introduction and main results

The following standard Sobolev inequality is well-known:

$$\|u\|_q^p \leq S(\|\nabla u\|_p^p + \|u\|_p^p), \quad u \in H^{1,p}(\mathbb{R}^N), \tag{1.1}$$

where $N \geq 2$, $1 \leq p < N$, $p \leq q \leq p^* := \frac{Np}{N-p}$ and S is a constant which depends only on N and p. We can consider the associated variational problem, namely, the attainability of the value

$$S := \sup_{u \in H^{1,p}(\mathbb{R}^N),\ \|\nabla u\|_p^p + \|u\|_p^p = 1} \|u\|_q^p \tag{1.2}$$

and now the existence of a maximizer associated with S is a standard fact.

In the case where $p = N$, the situation is changed. For bounded domains, Trudinger introduced the so-called Trudinger inequality in [17] (see also [15, 19]) and later Moser found its best-constant in

[9]. For unbounded domain case, (1.1) holds for every $q \in [p, \infty)$ and the limiting inequality is the following Trudinger-Moser type one. Let $\beta_N := N\omega_{N-1}^{\frac{1}{N-1}}$, where ω_{N-1} denotes the surface area of the $(N-1)$-dimensional unit sphere. Then the following generalization of the Trudinger-Moser inequality to the whole space case is known:

$$d_{N,\beta} := \sup_{u \in H^{1,N}(\mathbb{R}^N),\ \|\nabla u\|_N^N + \|u\|_N^N = 1} \int_{\mathbb{R}^N} \Phi_{N,\beta}(u) < \infty, \quad \beta \in (0, \beta_N], \tag{1.3}$$

where $\Phi_{N,\beta}(t) := e^{\beta|t|^{\frac{N}{N-1}}} - \sum_{j=0}^{N-2} \frac{\beta^j}{j!}|t|^{\frac{N}{N-1}j}$, see Cao [2], Ruf [16]. For another generalization, see e.g. [1, 6, 10, 11, 12, 13, 14] and references therein.

For bounded domains, the attainability of the supremum is discussed e.g. in [3, 4, 8]. As for the attainability of the supremum $d_{N,\beta}$ in unbounded domains, the first author proved the following fact [5].

Proposition 1.1.
(a) *Let $N \geq 3$. Then $d_{N,\beta}$ is attained for any $\beta \in (0, \beta_N)$.*
(b) *Let $N = 2$. Then $d_{N,\beta}$ is attained for $\beta \in (\frac{2}{B_2}, 4\pi]$ and never attained for sufficiently small β, where $B_2 := \sup_{u \in H^1(\mathbb{R}^2)\setminus\{0\}} \frac{\|u\|_4^4}{\|\nabla u\|_2^2 \|u\|_2^2}$.*

Also for the case (a), the attainability of d_{N,β_N} is proved by Li-Ruf [7].

Proposition 1.1 is rather strange since, different from the usual Sobolev case (1.2), the situation for the attainability of $d_{N,\beta}$ heavily depends on the dimension. Moreover, in the two dimensional case, $d_{N,\beta}$ is attained if β is nearly critical and never attained if β is sufficiently subcritical, which is against, in a sense, with the natural expectation for the existence and nonexistence of maximizers. The method of the proof for Proposition 1.1 relies on the careful analysis using the concentration-compactness type argument together with the behavior of the functional

$$J_{N,\beta}(u) := \|u\|_N^N + \frac{\beta}{N}\|u\|_{\frac{N^2}{N-1}}^{\frac{N^2}{N-1}}, \tag{1.4}$$

which corresponds to the first two terms of the original functional $\int_{\mathbb{R}^N} \Phi_{N,\beta}(u)$.

Just after the publication of the paper [5], the second author pointed out that, even in the higher dimensional case, the attainability of $d_{N,\beta}$ heavily depends on the value β if one replaces the normalizing condition $\|\nabla u\|_N^N + \|u\|_N^N = 1$ by $\|\nabla u\|_N + \|u\|_N = 1$. This result suggests that the attainability of the supremum value depends delicately on the choice of normalizing conditions even if conditions are equivalent.

In this paper, we consider the following model problem to clarify the effect of the dimension, α and the equivalent normalizing condition on the solvability of the associated maximizing problem. Throughout this paper, we assume $N \geq 2$, $N < p < \infty$ and $0 < \gamma \leq N$. We consider the attainability of D defined by

$$D = D(N, p, \gamma, \alpha) := \sup_{\substack{u \in H^{1,N}(\mathbb{R}^N), \\ \|u\|_{H_\gamma^{1,N}} = 1}} \left(\|u\|_N^N + \alpha\|u\|_p^p\right),$$

where $\alpha > 0$ and $\|u\|_{H_\gamma^{1,N}} = (\|u\|_N^\gamma + \|\nabla u\|_N^\gamma)^{\frac{1}{\gamma}}$. When we are mainly concerned with the relationship between α and D, sometimes we denote D by D_α. We need

$$\alpha_* = \alpha_*(N, p, \gamma) := \inf_{\substack{u \in H^{1,N}(\mathbb{R}^N), \\ \|u\|_{H_\gamma^{1,N}} = 1}} \left(\frac{1 - \|u\|_N^N}{\|u\|_p^p}\right).$$

It is easy to see

$$\alpha > \alpha_* \text{ is equivalent to } D_\alpha > 1. \tag{1.5}$$

Also let

$$B(v) := \frac{\|v\|_p^p}{\|\nabla v\|_N^{p-N}\|v\|_N^N}$$

for any $v \in H^{1,N}(\mathbb{R}^N) \setminus \{0\}$ and

$$B := \sup_{H^{1,N}(\mathbb{R}^N)\setminus\{0\}} \frac{\|v\|_p^p}{\|\nabla v\|_N^{p-N}\|v\|_N^N}.$$

The critical Gagliardo-Nirenberg-Sobolev inequality implies that $B < \infty$. Moreover, it is somewhat well-known that for any $N \geq 2$, B is attained. This fact is proved by Weinstein [18] for $N = 2$ and we give a proof for the case $N \geq 3$ in the appendix A for the sake of the completeness. Here and henceforth, a maximizer associated with B which is normalized in $\|\cdot\|_{H^{1,N}_\gamma}$ is denoted by V (see Proposition A.1 for the existence).

Let us recall our assumption $N \geq 2$, $N < p < \infty$ and $0 < \gamma \leq N$. Our main results are the following:

Theorem 1.2.
Let $\gamma > p - N$. Then $\alpha_* = 0$ and D is attained for any $\alpha > 0$.

Theorem 1.3.
Let $\gamma = p - N$. Then $\alpha_* = \frac{N}{B(p-N)}$ and D is attained for any $\alpha > \frac{N}{B(p-N)}$ while never attained for $\alpha \leq \frac{N}{B(p-N)}$.

Theorem 1.4.
Let $\gamma < p - N$. Then D is attained for any $\alpha \geq \alpha_*$ and never attained for $\alpha < \alpha_*$. Moreover, $\alpha_* = \alpha_*(\gamma)$ satisfies the following:

$$\frac{N}{B\gamma}\left(\frac{p}{p-N-\gamma}\right)^{\frac{p-N}{\gamma}-1} < \alpha_*(\gamma) < \left[\left(\frac{p}{N}\right)^{\frac{N}{\gamma}} - 1\right]\frac{1}{B}\left(\frac{p}{p-N-\gamma}\right)^{\frac{p-N}{\gamma}}, \tag{1.6}$$

$$\lim_{\gamma\downarrow 0} \alpha_*(\gamma) = \infty, \tag{1.7}$$

$$\lim_{\gamma\uparrow p-N} \alpha_*(\gamma) = \frac{N}{B(p-N)}. \tag{1.8}$$

By putting $\gamma = N$, we have an immediate corollary of Theorem 1.2 to Theorem 1.4.

Corollary 1.5.
(a) *Let $2N > p$. Then D is attained for all $\alpha > 0$.*
(b) *Let $2N = p$. Then D is attained for $\alpha > \frac{1}{B}$ and not attained for $\alpha \leq \frac{1}{B}$.*
(c) *Let $2N < p$. Then D is attained for $\alpha \geq \alpha_* = \alpha_*(N)$ and not attained for $\alpha < \alpha_*$.*

As is mentioned above, the attainability of the supremum value $d_{N,\beta}$ associated with the Trudinger-Moser type inequality defined by (1.3) is closely related with the behavior of the functional $J_{N,\beta}$ given in (1.4). If $N \geq 3$, then the balance between the first and the second term of $J_{N,\beta}$ satisfies $2N > \frac{N^2}{N-1}$, thus Corollary 1.5 yields the existence of the maximizer for $J_{N,\beta}$ for any $\beta > 0$. On the other hand, since $2N = \frac{N^2}{N-1}$ holds for $N = 2$, thus $J_{N,\beta}$ possesses a maximizer for $\beta > \frac{2}{B}$ and does not possess any maximizer for $\beta \leq \frac{2}{B}$. These results suggest the close relationship between Proposition 1.1 and Corollary 1.5.

This paper is organized as follows. Section 2 is devoted to preliminary facts for proofs of Theorem 1.2–Theorem 1.4. We show Theorem 1.2, Theorem 1.3 and Theorem 1.4 in Section 3, 4 and 5, respectively. In the appendix, we prove some auxiliary facts used throughout the paper.

Throughout the paper, $\|\cdot\|_{p,\Omega}$ denotes the standard $L^p(\Omega)$-norm. We occasionally use the abbreviation $\|\cdot\|_p$. We write B_R and B_R^c as the ball in $\mathbb{R}^N$ with radius R centered at the origin and its complement, respectively. ω_{N-1} denotes the surface area of the $(N-1)$-dimensional unit sphere in $\mathbb{R}^N$. We pass to subsequences freely.

2 Preliminaries

The proofs of Theorem 1.2–Theorem 1.4 are based on the fundamental existence theorem together with the careful estimates of D in terms of suitable family of comparison functions. We introduce these facts in this section. Let $I_\alpha(u) := \|u\|_N^N + \alpha\|u\|_p^p$.

2.1 Existence and nonexistence

The following proposition is a key fact for the proof of Theorem 1.2 to Theorem 1.4.

Proposition 2.1.
Let $\gamma \leq N$ and $\alpha > \alpha_*$, where

$$\alpha_* = \alpha(N,p,\gamma) := \inf_{\substack{u \in H^{1,N}(\mathbb{R}^N), \\ \|u\|_{H^{1,N}_\gamma}=1}} \left(\frac{1-\|u\|_N^N}{\|u\|_p^p} \right).$$

Then D is attained.

In the rest of this subsection, we are devoted to the proof of Proposition 2.1.

Let $(u_n)_{n\in\mathbb{N}} \subset H^{1,N}(\mathbb{R}^N)$ be a sequence satisfying $\|u_n\|_{H^{1,N}_\gamma} = 1$ and $u_n \rightharpoonup u$ weakly in $H^{1,N}(\mathbb{R}^N)$ as $n \to \infty$. Let us introduce values defined by

$$\begin{aligned}
\nu_0 &:= \lim_{R\to\infty}\lim_{n\to\infty}\left(\|u_n\|_{N,B_R}^N + \alpha\|u_n\|_{p,B_R}^p\right),\\
\nu_\infty &:= \lim_{R\to\infty}\lim_{n\to\infty}\left(\|u_n\|_{N,B_R^c}^N + \alpha\|u_n\|_{p,B_R^c}^p\right),\\
\eta_0 &:= \lim_{R\to\infty}\lim_{n\to\infty}\|u_n\|_{N,B_R}^N, \quad \eta_\infty := \lim_{R\to\infty}\lim_{n\to\infty}\|u_n\|_{N,B_R^c}^N,\\
\mu_0 &:= \lim_{R\to\infty}\lim_{n\to\infty}\|\nabla u_n\|_{N,B_R}^N, \quad \mu_\infty := \lim_{R\to\infty}\lim_{n\to\infty}\|\nabla u_n\|_{N,B_R^c}^N
\end{aligned}$$

by taking subsequences if necessary. For $R > 0$, take $h_R \in C^\infty([0,\infty))$ satisfying

$$\begin{cases}
h_R(r) = 1 \quad \text{for } 0 \leq r \leq R,\\
0 \leq h_R(r) \leq 1 \quad \text{for } R \leq r \leq R+1,\\
h_R(r) = 0 \quad \text{for } r \geq R+1,\\
|h_R'(r)| \leq 2 \quad \text{for } r \geq 0.
\end{cases}$$

We define cut-off functions ϕ_R^0 and ϕ_R^∞ by

$$\phi_R^0(x) := h_R(|x|) \quad \text{and} \quad \phi_R^\infty(x) := 1 - h_R(|x|),$$

and let $u_{n,R}^* := u_n \phi_R^*$, where $* = 0$ or ∞.

Lemma 2.2. *Let $* = 0$ or ∞. Then there hold*

$$(i)\ \nu_* = \lim_{R\to\infty}\lim_{n\to\infty} \left(\|u_{n,R}^*\|_N^N + \alpha\|u_{n,R}^*\|_p^p\right),$$
$$(ii)\ \eta_* = \lim_{R\to\infty}\lim_{n\to\infty} \|u_{n,R}^*\|_N^N,$$
$$(iii)\ \mu_* = \lim_{R\to\infty}\lim_{n\to\infty} \|\nabla u_{n,R}^*\|_N^N.$$

Proof. We first show (ii). By the definition of ϕ_R^0, we see

$$\int_{B_R} |u_n|^N dx \le \int_{\mathbb{R}^N} |u_{n,R}^0|^N dx \le \int_{B_{R+1}} |u_n|^N dx.$$

Hence, taking $\lim_{R\to\infty}\lim_{n\to\infty}$, we obtain $\eta_0 = \lim_{R\to\infty}\lim_{n\to\infty} \|u_{n,R}^0\|_N^N$. Similarly, we have

$$\int_{B_{R+1}^c} |u_n|^N dx \le \int_{\mathbb{R}^N} |u_{n,R}^\infty|^N dx \le \int_{B_R^c} |u_n|^N dx,$$

and then passing to the limits $\lim_{R\to\infty}\lim_{n\to\infty}$ yields $\eta_\infty = \lim_{R\to\infty}\lim_{n\to\infty} \|u_{n,R}^\infty\|_N^N$. Since we can verify (i) in the same way as above, we omit its proof. Finally, we prove (iii). By the definition of ϕ_R^∞, we have

$$\int_{B_{R+1}^c} |\nabla u_n|^N dx \le \int_{\mathbb{R}^N} |\phi_R^\infty|^N |\nabla u_n|^N dx \le \int_{B_R^c} |\nabla u_n|^N dx,$$

and then

$$\lim_{R\to\infty}\lim_{n\to\infty} \int_{B_R^c} |\nabla u_n|^N dx = \lim_{R\to\infty}\lim_{n\to\infty} \int_{\mathbb{R}^N} |\phi_R^\infty|^N |\nabla u_n|^N dx.$$

On the other hand, by the mean value theorem, we see

$$\|\nabla u_{n,R}^\infty\|_N^N = \int_{\mathbb{R}^N} |\phi_R^\infty \nabla u_n + u_n \nabla \phi_R^\infty|^N dx = \int_{\mathbb{R}^N} |\phi_R^\infty|^N |\nabla u_n|^N dx + \mathcal{R}_{n,R},$$

where

$$\mathcal{R}_{n,R} := N \int_{\mathbb{R}^N} |\phi_R^\infty \nabla u_n + \theta u_n \nabla \phi_R^\infty|^{N-2} (\phi_R^\infty \nabla u_n + \theta u_n \nabla \phi_R^\infty) \cdot u_n \nabla \phi_R^\infty dx$$

with some $0 < \theta < 1$. From Hölder's inequality and $\|\nabla \phi_R^\infty\|_\infty \le 2$, we obtain

$$\begin{aligned}
|\mathcal{R}_{n,R}| &\le N \int_{\mathbb{R}^N} \left(|\phi_R^\infty \nabla u_n| + |u_n \nabla \phi_R^\infty|\right)^{N-1} |u_n \nabla \phi_R^\infty| dx \\
&\le N \left(\int_{\mathbb{R}^N} \left(|\phi_R^\infty \nabla u_n| + |u_n \nabla \phi_R^\infty|\right)^N dx\right)^{\frac{N-1}{N}} \left(\int_{\mathbb{R}^N} |u_n \nabla \phi_R^\infty|^N dx\right)^{\frac{1}{N}} \\
&\le N \left(\|\nabla u_n\|_N + \|\nabla \phi_R^\infty\|_\infty \|u_n\|_N\right)^{N-1} \|\nabla \phi_R^\infty\|_\infty \|u_n\|_{N,A(R,R+1)} \le 2\cdot 3^{N-1} N \|u_n\|_{N,A(R,R+1)},
\end{aligned}$$

where $A(R, R+1) := \{x \in \mathbb{R}^N \mid R < |x| < R+1\}$. Since $\|u_n\|_{N,A(R,R+1)} \to \|u\|_{N,A(R,R+1)}$ as $n \to \infty$ by the compactness, we see that $\lim_{R\to\infty}\lim_{n\to\infty} \mathcal{R}_{n,R} = 0$. As a result, we have $\mu_\infty = \lim_{R\to\infty}\lim_{n\to\infty} \|\nabla u_{n,R}^\infty\|_N^N$. In a same way as above, we can show $\mu_0 = \lim_{R\to\infty}\lim_{n\to\infty} \|\nabla u_{n,R}^0\|_N^N$. Thus Lemma 2.2 is proved. □

Let $(u_n)_{n\in\mathbb{N}} \subset H^{1,N}(\mathbb{R}^N)$ be a sequence satisfying $u_n \rightharpoonup u$ weakly in $H^{1,N}(\mathbb{R}^N)$ as $n \to \infty$. We call $(u_n)_{n\in\mathbb{N}}$ a normalized vanishing sequence (NVS) if $(u_n)_{n\in\mathbb{N}}$ satisfies $\|u_n\|_{H^{1,N}_\gamma} = 1$, $u = 0$ in $H^{1,N}(\mathbb{R}^N)$ and $\nu_0 = 0$. A (NVS) consisting of functions to be non-negative, radially symmetric and non-increasing in the radial direction is called a radially symmetric normalized vanishing sequence (RNVS). Let us also introduce a value d_{NVL} called a normalized vanishing limit defined by

$$d_{NVL} = d_{NVL}(N, p, \gamma, \alpha) := \sup_{(u_n)_{n\in\mathbb{N}} : (RNVS)} \lim_{n\to\infty} \left(\|u_n\|_N^N + \alpha\|u_n\|_p^p\right).$$

The following Lemma is a key for the proof of Proposition 2.1. In fact, we can calculate $d_{NVL} = 1$ which is a threshold so that the vanishing phenomenon for $(u_n)_{n\in\mathbb{N}}$ occurs.

Lemma 2.3.
There holds $d_{NVL} = 1$.

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a (RNVS). We first claim that $\lim_{R\to\infty} \lim_{n\to\infty} \|u_n\|^p_{p,B_R^c} = 0$. To show this, we need a decay estimate of u_n. Indeed, since u_n is non-negative, radially symmetric and non-increasing in the radial direction, by notating $\tilde{u}_n(|x|) := u_n(x)$, we see for any $r > 0$,

$$1 \geq \|u_n\|_N^N = \omega_{N-1} \int_0^\infty \tilde{u}_n(s)^N s^{N-1} ds \geq \omega_{N-1} \tilde{u}_n(r)^N \int_0^r s^{N-1} ds = \frac{\omega_{N-1}}{N} \tilde{u}_n(r)^N r^N,$$

and then $\tilde{u}_n(r) \leq (\frac{N}{\omega_{N-1}})^{\frac{1}{N}} r^{-1}$. Hence, we obtain

$$\begin{aligned}\|u_n\|^p_{p,B_R^c} &= \omega_{N-1} \int_R^\infty \tilde{u}_n(r)^p r^{N-1} dr \\ &\leq \omega_{N-1} \left(\frac{N}{\omega_{N-1}}\right)^{\frac{p}{N}} \int_R^\infty r^{-p+N-1} dr = \frac{\omega_{N-1}}{p-N} \left(\frac{N}{\omega_{N-1}}\right)^{\frac{p}{N}} R^{-(p-N)},\end{aligned}$$

which implies $\lim_{R\to\infty} \lim_{n\to\infty} \|u_n\|^p_{p,B_R^c} = 0$. Hence, it holds $\nu_\infty = \lim_{R\to\infty} \lim_{n\to\infty} \|u_n\|^N_{N,B_R^c}$. Then since $\nu_0 = 0$ and $\|u_n\|_N \leq 1$, we have

$$\lim_{n\to\infty} \left(\|u_n\|_N^N + \alpha\|u_n\|_p^p\right) = \nu_0 + \nu_\infty = \nu_\infty = \lim_{R\to\infty} \lim_{n\to\infty} \|u_n\|^N_{N,B_R^c} \leq 1,$$

which gives $d_{NVL} \leq 1$.

Next, we show the converse inequality. Take $\phi \in H^{1,N}(\mathbb{R}^N)$ to be non-negative, radially symmetric and non-increasing in the radial direction, and assume ϕ satisfies $\|\phi\|_N = \|\nabla\phi\|_N = 1$. By scaling, let $\psi_n(x) := \frac{1}{n}\phi(\frac{x}{n})$ for $n \in \mathbb{N}$. Then we see $\|\psi_n\|_N = \|\phi\|_N = 1$ and $\|\nabla\psi_n\|_N = \frac{1}{n}$, and then $\psi_n \rightharpoonup \psi$ weakly in $H^{1,N}(\mathbb{R}^N)$ as $n \to \infty$ for some $\psi \in H^{1,N}(\mathbb{R}^N)$. It turns out that $\psi = 0$ in $H^{1,N}(\mathbb{R}^N)$ since $\|\nabla\psi\|_N \leq \lim_{n\to\infty} \|\nabla\psi_n\|_N = 0$. Moreover, defining $w_n := \frac{\psi_n}{\|\psi_n\|_{H^{1,N}_\gamma}}$, we have $\|w_n\|_{H^{1,N}_\gamma} = 1$ and

$$w_n = \frac{\psi_n}{\left(1 + \left(\frac{1}{n}\right)^\gamma\right)^{\frac{1}{\gamma}}} \rightharpoonup 0 \quad \text{weakly in } H^{1,N}(\mathbb{R}^N)$$

as $n \to \infty$. From the compactness, we obtain for any $R > 0$,

$$\lim_{n\to\infty} \left(\|w_n\|^N_{N,B_R} + \alpha\|w_n\|^p_{p,B_R}\right) = 0,$$

and then it holds $\nu_0 = 0$. Thus $(w_n)_{n\in\mathbb{N}}$ is a (RNVS). Since $\|\psi_n\|_N = 1$ and $\psi_n \rightharpoonup 0$ weakly in $H^{1,N}(\mathbb{R}^N)$ as $n \to \infty$, by the compactness again, we see for any $R > 0$,

$$1 = \lim_{n\to\infty} \int_{B_R} |\psi_n|^N dx + \lim_{n\to\infty} \int_{B_R^c} |\psi_n|^N dx = \lim_{n\to\infty} \int_{B_R^c} |\psi_n|^N dx,$$

and then

$$1 = \lim_{R\to\infty} \lim_{n\to\infty} \int_{B_R^c} |\psi_n|^N dx.$$

On the other hand, we have

$$\lim_{n\to\infty} \|\psi_n\|_{H^{1,N}_\gamma} = \lim_{n\to\infty} \left(1 + \left(\frac{1}{n}\right)^\gamma\right)^{\frac{1}{\gamma}} = 1.$$

To sum-up, we see

$$\lim_{n\to\infty} \left(\|w_n\|_N^N + \alpha\|w_n\|_p^p\right) = \nu_0 + \nu_\infty = \nu_\infty \geq \lim_{R\to\infty} \lim_{n\to\infty} \|w_n\|^N_{N,B_R^c} = \lim_{R\to\infty} \lim_{n\to\infty} \frac{\|\psi_n\|^N_{N,B_R^c}}{\|\psi_n\|^N_{H^{1,N}_\gamma}} = 1,$$

which gives $d_{NVL} \geq 1$. Thus Lemma 2.3 is proved. □

In what follows, let $(u_n)_{n\in\mathbb{N}} \subset H^{1,N}(\mathbb{R}^N)$ be a maximizing sequence for D, and assume $\gamma \leq N$ and $\alpha > \alpha_*$. Note that $\alpha > \alpha_*$ is equivalent to $D > 1$. Moreover, by virtue of the radially symmetric rearrangement, we can assume that $(u_n)_{n\in\mathbb{N}}$ are non-negative, radially symmetric and non-increasing in the radial direction.

Lemma 2.4. *There hold*

(i) $D = \nu_0 + \nu_\infty$,

(ii) $1 = (\eta_0 + \eta_\infty)^{\frac{\gamma}{N}} + (\mu_0 + \mu_\infty)^{\frac{\gamma}{N}}$.

Proof. (i) For any $R > 0$, we see

$$\|u_n\|_N^N + \alpha\|u_n\|_p^p = \left(\|u_n\|^N_{N,B_R} + \alpha\|u_n\|^p_{p,B_R}\right) + \left(\|u_n\|^N_{N,B_R^c} + \alpha\|u_n\|^p_{p,B_R^c}\right),$$

and then taking $\lim\limits_{R\to\infty} \lim\limits_{n\to\infty}$ yields $D = \nu_0 + \nu_\infty$.

(ii) Since $\|u_n\|_{H^{1,N}_\gamma} = 1$, we see

$$1 = \left(\int_{B_R} |u_n|^N dx + \int_{B_R^c} |u_n|^N dx\right)^{\frac{\gamma}{N}} + \left(\int_{B_R} |\nabla u_n|^N dx + \int_{B_R^c} |\nabla u_n|^N dx\right)^{\frac{\gamma}{N}},$$

and then passing to the limits $\lim\limits_{R\to\infty} \lim\limits_{n\to\infty}$ yields

$$1 = (\eta_0 + \eta_\infty)^{\frac{\gamma}{N}} + (\mu_0 + \mu_\infty)^{\frac{\gamma}{N}}.$$

□

Lemma 2.5. *It holds*

$$(\eta_0^{\frac{\gamma}{N}} + \mu_0^{\frac{\gamma}{N}}, \nu_0) = (1, D) \text{ and } (\eta_\infty^{\frac{\gamma}{N}} + \mu_\infty^{\frac{\gamma}{N}}, \nu_\infty) = (0, 0).$$

Proof. First we show $\eta_0^{\frac{\gamma}{N}} + \mu_0^{\frac{\gamma}{N}} = 0$ or $= 1$. We argue by the contradiction. Assume on the contrary, we have

$$0 < \eta_0^{\frac{\gamma}{N}} + \mu_0^{\frac{\gamma}{N}} < 1. \tag{2.1}$$

Now we show, under (2.1),

$$0 < \eta_\infty^{\frac{\gamma}{N}} + \mu_\infty^{\frac{\gamma}{N}} < 1 \tag{2.2}$$

holds. Indeed, if $\eta_\infty^{\frac{\gamma}{N}} + \mu_\infty^{\frac{\gamma}{N}} = 0$, then we have $\eta_\infty = \mu_\infty = 0$ and $1 = \eta_0^{\frac{\gamma}{N}} + \mu_0^{\frac{\gamma}{N}}$ in view of (ii) of Lemma 2.4, which contradicts (2.1). On the other hand, if $\eta_\infty^{\frac{\gamma}{N}} + \mu_\infty^{\frac{\gamma}{N}} = 1$, then again together with (ii) of Lemma 2.4,

$$1 = (\eta_0 + \eta_\infty)^{\frac{\gamma}{N}} + (\mu_0 + \mu_\infty)^{\frac{\gamma}{N}} \geq \eta_\infty^{\frac{\gamma}{N}} + \mu_\infty^{\frac{\gamma}{N}} = 1,$$

thus we obtain $\eta_0 = \mu_0 = 0$, which implies

$$\eta_0^{\frac{\gamma}{N}} + \mu_0^{\frac{\gamma}{N}} = 0,$$

a contradiction to (2.1).

By the definition of D, we see

$$\begin{aligned} D &\geq \frac{\|u_{n,R}^*\|_N^N}{\|u_{n,R}^*\|_{H_\gamma^{1,N}}^N} + \alpha \frac{\|u_{n,R}^*\|_p^p}{\|u_{n,R}^*\|_{H_\gamma^{1,N}}^p} \\ &= \frac{1}{\|u_{n,R}^*\|_{H_\gamma^{1,N}}^N} \left(\|u_{n,R}^*\|_N^N + \alpha \|u_{n,R}^*\|_p^p + \alpha \left(\frac{1}{\|u_{n,R}^*\|_{H_\gamma^{1,N}}^{p-N}} - 1 \right) \|u_{n,R}^*\|_p^p \right), \end{aligned} \tag{2.3}$$

and then taking $\lim\limits_{R\to\infty} \lim\limits_{n\to\infty}$ yields

$$D(\eta_*^{\frac{\gamma}{N}} + \mu_*^{\frac{\gamma}{N}})^{\frac{N}{\gamma}} \geq \nu_* + \left(\frac{1}{(\eta_*^{\frac{\gamma}{N}} + \mu_*^{\frac{\gamma}{N}})^{\frac{p-N}{\gamma}}} - 1 \right) (\nu_* - \eta_*), \tag{2.4}$$

where we have used Lemma 2.2. Then since $\nu_* \geq \eta_*$ to be seen by the definition of ν_* and η_*, and $0 < \eta_*^{\frac{\gamma}{N}} + \mu_*^{\frac{\gamma}{N}} < 1$, we obtain

$$D(\eta_*^{\frac{\gamma}{N}} + \mu_*^{\frac{\gamma}{N}})^{\frac{N}{\gamma}} \geq \nu_*. \tag{2.5}$$

Moreover, using (2.5), $\frac{\gamma}{N} \leq 1$ and Lemma 2.4, we have

$$D = D\left((\eta_0 + \eta_\infty)^{\frac{\gamma}{N}} + (\mu_0 + \mu_\infty)^{\frac{\gamma}{N}} \right)^{\frac{N}{\gamma}} \geq D\left((\eta_0^{\frac{\gamma}{N}} + \mu_0^{\frac{\gamma}{N}})^{\frac{N}{\gamma}} + (\eta_\infty^{\frac{\gamma}{N}} + \mu_\infty^{\frac{\gamma}{N}})^{\frac{N}{\gamma}} \right) \geq \nu_0 + \nu_\infty = D, \tag{2.6}$$

which implies $D(\eta_*^{\frac{\gamma}{N}} + \mu_*^{\frac{\gamma}{N}})^{\frac{N}{\gamma}} = \nu_*$.

Then since $0 < \eta_*^{\frac{\gamma}{N}} + \mu_*^{\frac{\gamma}{N}} < 1$, from (2.4), we obtain $\nu_* \leq \eta_*$, and then $D = \nu_0 + \nu_\infty \leq \eta_0 + \eta_\infty \leq 1$. This is a contradiction to $D > 1$. As a result, it holds either $\eta_0^{\frac{\gamma}{N}} + \mu_0^{\frac{\gamma}{N}} = 0$ or 1.

Now we assume

$$\eta_0^{\frac{\gamma}{N}} + \mu_0^{\frac{\gamma}{N}} = 0 \tag{2.7}$$

and derive a contradiction. Under (2.7), we see

$$1 = (\eta_0 + \eta_\infty)^{\frac{\gamma}{N}} + (\mu_0 + \mu_\infty)^{\frac{\gamma}{N}} = \eta_\infty^{\frac{\gamma}{N}} + \mu_\infty^{\frac{\gamma}{N}}.$$

Moreover, since $\eta_0^{\frac{\gamma}{N}} + \mu_0^{\frac{\gamma}{N}} = 0$, we have $\frac{1}{\|u_{n,R}^0\|_{H_\gamma^{1,N}}^{p-N}} - 1 > \frac{1}{2}$ for large $R > 0$ and $n \in \mathbb{N}$. For such R and n, it follows from (2.3) that

$$D\|u_{n,R}^0\|_{H_\gamma^{1,N}}^N \geq \|u_{n,R}^0\|_N^N + \alpha\|u_{n,R}^0\|_p^p + \frac{\alpha}{2}\|u_{n,R}^0\|_p^p.$$

Passing to the limits $\lim_{R\to\infty}\lim_{n\to\infty}$ to this relation, we obtain

$$D(\eta_0^{\frac{\gamma}{N}} + \mu_0^{\frac{\gamma}{N}})^{\frac{N}{\gamma}} \geq \nu_0 + \frac{1}{2}(\nu_0 - \eta_0) \geq \nu_0,$$

which implies $\nu_0 = 0$ since $\eta_0^{\frac{\gamma}{N}} + \mu_0^{\frac{\gamma}{N}} = 0$. Then we see $D = \nu_0 + \nu_\infty = \nu_\infty$ and $(u_n)_{n\in\mathbb{N}}$ is a (RNVS) since $\nu_0 = 0$. Hence, from Lemma 2.3, we obtain

$$D = \lim_{n\to\infty}\left(\|u_n\|_N^N + \alpha\|u_n\|_p^p\right) \leq d_{NVL} = 1,$$

which is a contradiction to $D > 1$.

Thus it holds

$$\eta_0^{\frac{\gamma}{N}} + \mu_0^{\frac{\gamma}{N}} = 1. \tag{2.8}$$

First we show

$$\eta_\infty^{\frac{\gamma}{N}} + \mu_\infty^{\frac{\gamma}{N}} = 0 \tag{2.9}$$

irrelevant to $\frac{\gamma}{N} \leq 1$ or > 1. Indeed, if $\frac{\gamma}{N} \leq 1$, then we have

$$1 = \left((\eta_0 + \eta_\infty)^{\frac{\gamma}{N}} + (\mu_0 + \mu_\infty)^{\frac{\gamma}{N}}\right)^{\frac{N}{\gamma}} \geq (\eta_0^{\frac{\gamma}{N}} + \mu_0^{\frac{\gamma}{N}})^{\frac{N}{\gamma}} + (\eta_\infty^{\frac{\gamma}{N}} + \mu_\infty^{\frac{\gamma}{N}})^{\frac{N}{\gamma}} = 1 + (\eta_\infty^{\frac{\gamma}{N}} + \mu_\infty^{\frac{\gamma}{N}})^{\frac{N}{\gamma}},$$

which implies (2.9). If $\frac{\gamma}{N} > 1$, then we have

$$1 = (\eta_0 + \eta_\infty)^{\frac{\gamma}{N}} + (\mu_0 + \mu_\infty)^{\frac{\gamma}{N}} \geq \eta_0^{\frac{\gamma}{N}} + \eta_\infty^{\frac{\gamma}{N}} + \mu_0^{\frac{\gamma}{N}} + \mu_\infty^{\frac{\gamma}{N}} = 1 + \eta_\infty^{\frac{\gamma}{N}} + \mu_\infty^{\frac{\gamma}{N}},$$

again (2.9).

Then we see $\frac{1}{\|u_{n,R}^\infty\|_{H_\gamma^{1,N}}^{p-N}} - 1 > \frac{1}{2}$ for large $R > 0$ and $n \in \mathbb{N}$. For such R and n, by (2.3), we have

$$D\|u_{n,R}^\infty\|_{H_\gamma^{1,N}}^N \geq \|u_{n,R}^\infty\|_N^N + \alpha\|u_{n,R}^\infty\|_p^p + \frac{\alpha}{2}\|u_{n,R}^\infty\|_p^p.$$

Taking $\lim_{R\to\infty}\lim_{n\to\infty}$ to this relation yields

$$D(\eta_\infty^{\frac{\gamma}{N}} + \mu_\infty^{\frac{\gamma}{N}})^{\frac{N}{\gamma}} \geq \nu_\infty + \frac{1}{2}(\nu_\infty - \eta_\infty) \geq \nu_\infty,$$

which implies $\nu_\infty = 0$ since $\eta_\infty^{\frac{\gamma}{N}} + \mu_\infty^{\frac{\gamma}{N}} = 0$. Then we obtain $D = \nu_0 + \nu_\infty = \nu_0$. Thus Lemma 2.5 is proved. □

We are now in the position to complete the proof of Proposition 2.1.

Proof of Proposition 2.1. We claim that $D = \|u\|_N^N + \alpha\|u\|_p^p$. To show this, we need to prove $\lim_{n\to\infty} \|u_n\|_N^N = \|u\|_N^N$. Indeed, for any $R > 0$, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^N} \left(|u_n|^N - |u|^N\right) dx\right| \le \left|\int_{B_R} \left(|u_n|^N - |u|^N\right) dx\right| + \int_{B_R^c} |u_n|^N dx + \int_{B_R^c} |u|^N dx \\ &=: (A) + (B) + (C).\end{aligned}$$

It is obvious that $\lim_{R\to\infty} (C) = 0$. Moreover, we have $\lim_{n\to\infty} (A) = 0$ since the embedding $H^{1,N}(B_R) \hookrightarrow L^N(B_R)$ is compact. Finally, we obtain $\lim_{R\to\infty} \lim_{n\to\infty} (B) = \eta_\infty = 0$ by Lemma 2.5. Thus $\lim_{n\to\infty} \|u_n\|_N^N = \|u\|_N^N$ is obtained. Then we see

$$\begin{aligned}&D - (\|u\|_N^N + \alpha\|u\|_p^p) = \|u_n\|_N^N + \alpha\|u_n\|_p^p - (\|u\|_N^N + \alpha\|u\|_p^p) + o(1) \\ &= \|u_n\|_N^N - \|u\|_N^N + \alpha(\|u_n\|_p^p - \|u\|_p^p) + o(1) = o(1),\end{aligned}$$

where we have used the compactness of $\{u \in H^{1,N}(\mathbb{R}^N) \,|\, u \text{ is radial}\} \hookrightarrow L^p(\mathbb{R}^N)$ with $p > N$. As a result, we obtain $D = \|u\|_N^N + \alpha\|u\|_p^p$. Thus it remains to prove $\|u\|_{H_\gamma^{1,N}} = 1$, which implies that u is a maximizer for D. Since $0 < \|u\|_{H_\gamma^{1,N}} \le \lim_{n\to\infty} \|u_n\|_{H_\gamma^{1,N}} = 1$, we see

$$\begin{aligned}D &\ge I_\alpha\left(\frac{u}{\|u\|_{H_\gamma^{1,N}}}\right) = \frac{\|u\|_N^N}{\|u\|_{H_\gamma^{1,N}}^N} + \alpha\frac{\|u\|_p^p}{\|u\|_{H_\gamma^{1,N}}^p} = \frac{1}{\|u\|_{H_\gamma^{1,N}}^N}\left(\|u\|_N^N + \alpha\frac{\|u\|_p^p}{\|u\|_{H_\gamma^{1,N}}^{p-N}}\right) \\ &\ge \frac{1}{\|u\|_{H_\gamma^{1,N}}^N}\left(\|u\|_N^N + \alpha\|u\|_p^p\right) = \frac{1}{\|u\|_{H_\gamma^{1,N}}^N} D,\end{aligned}$$

which gives $\|u\|_{H_\gamma^{1,N}} \ge 1$, and then it holds $\|u\|_{H_\gamma^{1,N}} = 1$. Thus Proposition 2.1 is proved. □

We end with the following assertion which claims a converse of Proposition 2.1.

Proposition 2.6.
D is not achieved if $\alpha < \alpha_*$.

Proof. First note that

$$\alpha < \alpha_* = \inf_{\substack{u\in H^{1,N}(\mathbb{R}^N),\\ \|u\|_{H_\gamma^{1,N}}=1}} \left(\frac{1 - \|u\|_N^N}{\|u\|_p^p}\right) \le \left(\frac{1 - \|u\|_N^N}{\|u\|_p^p}\right)$$

for all $u \in H^{1,N}(\mathbb{R}^N)$ with $\|u\|_{H_\gamma^{1,N}} = 1$, hence we have $1 > \|u\|_N^N + \alpha\|u\|_p^p$ for all $u \in H^{1,N}(\mathbb{R}^N)$ with $\|u\|_{H_\gamma^{1,N}} = 1$. On the other hand, $D \ge d_{NVL} = 1$ holds in view of Lemma 2.3. These facts show that no admissible u achieves D. □

2.2 A family of comparison functions

Let $u^\lambda(x) := \lambda u(\lambda x)$ for $\lambda > 0$ and $u \in H^{1,N}(\mathbb{R}^N)$. It is easy to see that

$$\|u^\lambda\|_p^p = \lambda^{p-N}\|u\|_p^p, \quad \|\nabla u^\lambda\|_N^N = \lambda^N\|\nabla u\|_N^N. \tag{2.10}$$

Now take any $v \in H^{1,N}(\mathbb{R}^N)$ with $\|v\|_{H_\gamma^{1,N}} = 1$ and $t \in (0, 1)$. We define a curve v_t passing v in the following way:

(1) First, for v and t above, let

$$w_t := t^{\frac{1}{\gamma}} \frac{v}{\|v\|_N}. \tag{2.11}$$

It is easy to see that

$$\|w_t\|_N^\gamma = t \in (0,1).$$

(2) Next, for $L(\lambda) := \|\nabla w_t^\lambda\|_N^\gamma + \|w_t^\lambda\|_N^\gamma = \lambda^\gamma \|\nabla w_t\|_N^\gamma + \|w_t\|_N^\gamma$, note that $L(0) = \|w_t\|_N^\gamma = t < 1$ and $L(\lambda) \to \infty$ as $\lambda \uparrow \infty$. Hence there exists $\lambda = \lambda(t)$ satisfying $L(\lambda(t)) = 1$, namely,

$$\|\nabla w_t^{\lambda(t)}\|_N^\gamma + \|w_t^{\lambda(t)}\|_N^\gamma = 1.$$

Particularly, (2.10) and (2.11) yield

$$1 = \|\nabla w_t^{\lambda(t)}\|_N^\gamma + \|w_t^{\lambda(t)}\|_N^\gamma = \lambda(t)^\gamma t \frac{\|\nabla v\|_N^\gamma}{\|v\|_N^\gamma} + t,$$

which implies

$$\lambda(t) = \left(\frac{1-t}{t}\right)^{\frac{1}{\gamma}} \frac{\|v\|_N}{\|\nabla v\|_N}.$$

(3) Finally, we put

$$\begin{aligned} v_t(x) &:= w_t^{\lambda(t)}(x) = \lambda(t) w_t(\lambda(t)x) \\ &= \frac{(1-t)^{\frac{1}{\gamma}}}{\|\nabla v\|_N} v\left(\left(\frac{1-t}{t}\right)^{\frac{1}{\gamma}} \frac{\|v\|_N}{\|\nabla v\|_N} x\right). \end{aligned} \tag{2.12}$$

Then we see that

$$\|v_t\|_N^\gamma = t \in (0,1), \quad \|v_t\|_{H_\gamma^{1,N}} = 1, \tag{2.13}$$

$$\|v_t\|_p^p = t^{\frac{N}{\gamma}} (1-t)^{\frac{p-N}{\gamma}} \frac{\|v\|_p^p}{\|\nabla v\|_N^{p-N} \|v\|_N^N} =: t^{\frac{N}{\gamma}} (1-t)^{\frac{p-N}{\gamma}} B(v). \tag{2.14}$$

Moreover, noting the fact that $\|\nabla v\|_N^\gamma = 1 - t$ since $t = \|v\|_N^\gamma (\in (0,1))$ and $\|v\|_{H_\gamma^{1,N}} = 1$, we have, from (2.12),

$$v_{\|v\|_N^\gamma}(x) = v(x). \tag{2.15}$$

Consequently, v_t is a curve in $H^{1,N}(\mathbb{R}^N)$ passing v satisfying (2.13) and (2.14). We denote v_t simply by v if no confusion occurs. Moreover, we have

$$I_\alpha(v_t) = \|v_t\|_N^N + \alpha \|v_t\|_p^p = t^{\frac{N}{\gamma}} + \alpha B(v) t^{\frac{N}{\gamma}} (1-t)^{\frac{p-N}{\gamma}} =: f_\alpha(t; v) \tag{2.16}$$

and

$$\begin{aligned} f'_\alpha(t; v) &= \frac{N}{\gamma} t^{\frac{N}{\gamma}-1} \left[1 + \alpha B(v)(1-t)^{\frac{p-N}{\gamma}-1} \left(1 - \frac{p}{N} t\right)\right] \\ &=: \frac{N}{\gamma} t^{\frac{N}{\gamma}-1} \left[1 + \alpha B(v) h(t)\right], \end{aligned} \tag{2.17}$$

where

$$h(t) := (1-t)^{\frac{p-N}{\gamma}-1} \left(1 - \frac{p}{N} t\right).$$

3 Proof of Theorem 1.2

Let $\gamma > p - N$ and $\alpha > 0$. Take any $v \in H^{1,N}(\mathbb{R}^N)$ with $\|v\|_{H^{1,N}_\gamma} = 1$. Then from (2.17), we have

$$f'_\alpha(t; v) = \frac{N}{\gamma} t^{\frac{N}{\gamma}-1} \left[1 + \alpha B(v) \frac{1}{(1-t)^{1-\frac{p-N}{\gamma}}} \left(1 - \frac{p}{N} t\right) \right] \downarrow -\infty$$

as $t \uparrow 1$, thus obtain

$$D_\alpha \geq f_\alpha(t; v) > f_\alpha(1; v) = 1 \tag{3.1}$$

for t sufficiently close to 1. This fact together with (1.5) yields $\alpha > \alpha_*$, hence $\alpha_* = 0$ follows. Proposition 2.1 together with (3.1) leads the existence of a maximizer associated with D_α. □

4 Proof of Theorem 1.3

Let $\gamma = p - N$ and $\alpha > \frac{N}{B(p-N)}$. This assumption together with the definition of B implies that there exists v such that

$$B(v) \in \left(\frac{N}{\alpha(p-N)}, B \right), \quad \|v\|_{H^{1,N}_\gamma} = 1.$$

Then by (2.17), we find that

$$f'_\alpha(1; v) = \frac{N}{p-N} \left[1 + \alpha B(v) \left(1 - \frac{p}{N}\right) \right] < 0.$$

Hence we obtain $D_\alpha \geq f_\alpha(t; v) > f_\alpha(1; v) = 1$ for t sufficiently close to 1. This fact together with (1.5) yields

$$\alpha > \alpha_*, \tag{4.1}$$

hence

$$\alpha_* \leq \frac{N}{B(p-N)} \tag{4.2}$$

follows. Proposition 2.1 together with (4.1) leads the existence of the maximizer associated with D_α.

Next let $\alpha = \frac{N}{B(p-N)}$. We will derive a contradiction by assuming that D_α is attained by a function v_0 with $v_0 \in H^{1,N}(\mathbb{R}^N)$ satisfying $\|v_0\|_{H^{1,N}_\gamma} = 1$. Then by noting the fact that $t \mapsto f_\alpha(t; v_0)$ takes its maximum at $t = \|v_0\|_N^\gamma$ in view of (2.15), we get

$$f'_\alpha(\|v_0\|_N^\gamma; v_0) = 0. \tag{4.3}$$

We next show that the function v_0 becomes a maximizer for B. To this end, we use the scaling $v_0^\lambda(x) := \lambda v_0(\lambda x)$ for $\lambda > 0$. Note that $\frac{v_0^\lambda}{\|v_0^\lambda\|_{H^{1,N}_\gamma}}$ is a curve passing v_0 (for $\lambda = 1$) and there holds

$$0 = \frac{d}{d\lambda} I_\alpha \left(\frac{v_0^\lambda}{\|v_0^\lambda\|_{H^{1,N}_\gamma}} \right) \Bigg|_{\lambda=1} \tag{4.4}$$

since v_0 attains a supremum value D_α.

By the scaling, we see $\|v_0^\lambda\|_p = \lambda^{1-\frac{N}{p}}\|v_0\|_p$, $\|v_0^\lambda\|_N = \|v_0\|_N$ and $\|\nabla v_0^\lambda\|_N = \lambda\|\nabla v_0\|_N$, and then we have

$$I_\alpha\left(\frac{v_0^\lambda}{\|v_0^\lambda\|_{H_\gamma^{1,N}}}\right) = \frac{\|v_0\|_N^N}{(\|v_0\|_N^\gamma + \lambda^\gamma\|\nabla v_0\|_N^\gamma)^{\frac{N}{\gamma}}} + \alpha\frac{\lambda^{p-N}\|v_0\|_p^p}{(\|v_0\|_N^\gamma + \lambda^\gamma\|\nabla v_0\|_N^\gamma)^{\frac{p}{\gamma}}}.$$

Thus a direct computation yields

$$\begin{aligned}\frac{d}{d\lambda}I_\alpha\left(\frac{v_0^\lambda}{\|v_0^\lambda\|_{H_\gamma^{1,N}}}\right) &= -N\lambda^{\gamma-1}\|v_0\|_N^N\|\nabla v_0\|_N^\gamma(\|v_0\|_N^\gamma + \lambda^\gamma\|\nabla v_0\|_N^\gamma)^{-\frac{N}{\gamma}-1}\\ &+ \alpha(p-N)\lambda^{p-N-1}\|v_0\|_p^p(\|v_0\|_N^\gamma + \lambda^\gamma\|\nabla v_0\|_N^\gamma)^{-\frac{p}{\gamma}}\\ &- \alpha p\lambda^{p-N+\gamma-1}\|v_0\|_p^p\|\nabla v_0\|_N^\gamma(\|v_0\|_N^\gamma + \lambda^\gamma\|\nabla v_0\|_N^\gamma)^{-\frac{p}{\gamma}-1}.\end{aligned}$$

Hence, recalling $\|v_0\|_{H_\gamma^{1,N}} = 1$, we obtain

$$\begin{aligned}&\frac{d}{d\lambda}I_\alpha\left(\frac{v_0^\lambda}{\|v_0^\lambda\|_{H_\gamma^{1,N}}}\right)\Bigg|_{\lambda=1} = -N\|v_0\|_N^N\|\nabla v_0\|_N^\gamma + \alpha(p-N)\|v_0\|_p^p - \alpha p\|v_0\|_p^p\|\nabla v_0\|_N^\gamma\\ &\le -N\|v_0\|_N^N\|\nabla v_0\|_N^\gamma + \alpha(p-N)\|v_0\|_p^p = N\|v_0\|_N^N\|\nabla v_0\|_N^\gamma\left(-1 + \frac{\alpha(p-N)\|v_0\|_p^p}{N\|\nabla v_0\|_N^\gamma\|v_0\|_N^N}\right).\end{aligned}$$

Then this relation together with (4.4) and the definition of B implies

$$1 \le \frac{\alpha(p-N)\|v_0\|_p^p}{N\|\nabla v_0\|_N^\gamma\|v_0\|_N^N} \le \frac{\alpha(p-N)B\|v_0\|_N^N\|\nabla v_0\|_N^{p-N}}{N\|\nabla v_0\|_N^\gamma\|v_0\|_N^N} = 1 \tag{4.5}$$

since $\gamma = p-N$ and $\alpha = \frac{N}{B(p-N)}$. Thus the second equality in (4.5) shows that the value B is attained by v_0. Therefore, since $\alpha = \frac{N}{B(p-N)}$ and $B = B(v_0)$, we have, by (2.17),

$$\begin{aligned}f'_\alpha(\|v_0\|_N^\gamma; v_0) &= \frac{N}{p-N}\|v_0\|_N^{2N-p}\left[1 + \frac{N}{p-N}\left(1 - \frac{p}{N}\|v_0\|_N^{p-N}\right)\right]\\ &= \frac{pN}{(p-N)^2}\|v_0\|_N^{2N-p}(1 - \|v_0\|_N^{p-N}) > 0,\end{aligned}$$

which contradicts (4.3). This fact together with (4.2) implies $\alpha_* = \frac{N}{B(p-N)}$, since otherwise $D_{\frac{N}{B(p-N)}}$ is achieved in view of Proposition 2.1. Therefore Proposition 2.6 yields the nonexistence of maximizers for $\alpha < \frac{N}{B(p-N)}$. □

5 Proof of Theorem 1.4

Let V be a Gagliardo-Nirenberg-Sobolev (GNS) maximizer with $\|V\|_{H_\gamma^{1,N}} = 1$ (see Proposition A.1 for the existence) and, for $t \in [0,1]$, let V_t be a curve defined by (2.12) with $v = V$. The scale invariance of $B(\cdot)$ yields $B(V) = B(V_t)$. We denote $B(V)$ simply by B. Let $f_\alpha(V) := \max_{t\in[0,1]} f_\alpha(t;V) = \max_{t\in[0,1]} I_\alpha(V_t)$, where $f_\alpha(t;V)$ is defined in (2.16).

We need more facts on the behavior of a function $f_\alpha(t;V)$ to prove Theorem 1.4. In appendix B, we will prove that, for any $\alpha > \alpha_0 := \frac{N}{B\gamma}\left(\frac{p}{p-N-\gamma}\right)^{\frac{p-N}{\gamma}-1}$, there exists a smallest solution $t_{1,\alpha}$ of $f'_\alpha(t;V) = 0$, namely,

$$1 + \alpha B(1-t)^{\frac{p-N}{\gamma}-1}\left(1 - \frac{p}{N}t\right) = 0,$$

see (B.5). Moreover, $f_\alpha(t;V)$ takes a local maximum at $t_{1,\alpha}$ (see (B.7)) and there holds

$$t_{1,\alpha} \in \left(\frac{N}{p}, \frac{N+\gamma}{p}\right), \tag{5.1}$$

see (B.9).

Also we shall show in the appendix B that there exists $\alpha_\dagger > \alpha_0$ such that

$$f_\alpha(t_{1,\alpha};V) = 1 \text{ holds for } \alpha = \alpha_\dagger \tag{5.2}$$

(see Proposition B.2) and the following:

Lemma 5.1.
(a) *It holds that* $f_\alpha(V) = 1$ *if* $\alpha \le \alpha_\dagger$ *and* $f_\alpha(V) = f_\alpha(t_{1,\alpha};V) > 1$ *if* $\alpha > \alpha_\dagger$.
(b) *There holds* $f_\alpha(t;V) < 1$ *for* $t \in [0,1)$ *if* $\alpha < \alpha_\dagger$.

Proposition 5.2. *There holds*

$$\frac{N}{B\gamma}\left(\frac{p}{p-N-\gamma}\right)^{\frac{p-N}{\gamma}-1} < \alpha_\dagger < \left[\left(\frac{p}{N}\right)^{\frac{N}{\gamma}} - 1\right]\frac{1}{B}\left(\frac{p}{p-N-\gamma}\right)^{\frac{p-N}{\gamma}}. \tag{5.3}$$

We start with the following fact:

Lemma 5.3. *For* $\alpha > 0$, $D_\alpha = 1$ *is equivalent to* $f_\alpha(V) \le 1$.

Proof. It is easy to see that $f_\alpha(V) = \max_{t\in[0,1]} f_\alpha(t;V) = \max_{t\in[0,1]} I_\alpha(V_t) \le D_\alpha = 1$ if $D_\alpha = 1$. We show the converse. Let

$$f_\alpha(V) \le 1. \tag{5.4}$$

Let $u \in H^{1,N}(\mathbb{R}^N)$ be a function satisfying $\|u\|_{H^{1,N}_\gamma} = 1$ and let $t_u := \|u\|_N^\gamma$. Then since V is a maximizer for the functional $B(\cdot)$, we obtain

$$\begin{aligned} I_\alpha(u) &= f_\alpha(t_u;u) = t_u^{\frac{N}{\gamma}} + \alpha B(u) t_u^{\frac{N}{\gamma}}(1-t_u)^{\frac{p-N}{\gamma}} \\ &\le t_u^{\frac{N}{\gamma}} + \alpha B(V) t_u^{\frac{N}{\gamma}}(1-t_u)^{\frac{p-N}{\gamma}} = f_\alpha(t_u;V) \le f_\alpha(V) \le 1 \end{aligned}$$

in view of (5.4), hence $D_\alpha = \sup_{u\in H^{1,N}(\mathbb{R}^N),\, \|u\|_{H^{1,N}_\gamma}=1} I_\alpha(u) \le 1$. On the other hand, $D_\alpha \ge d_{NVL} = 1$ holds by Lemma 2.3, thus we obtain $D_\alpha = 1$. □

Proposition 5.4.
(a) *There holds* $D_\alpha = 1$ *and* D_α *is not achieved if* $\alpha < \alpha_\dagger$.
(b) *There holds* $D_\alpha = 1$ *and* D_α *is achieved by* $V_{t_{1,\alpha}}$ *if* $\alpha = \alpha_\dagger$.
(c) *There holds* $D_\alpha > 1$ *and* D_α *is achieved if* $\alpha > \alpha_\dagger$.

Proof. (a) Let $\alpha < \alpha_\dagger$. Lemma 5.1 (a) yields $f_\alpha(V) = \max_{t\in[0,1]} f_\alpha(t;V) = 1$, thus $D_\alpha = 1$ holds by virtue of Lemma 5.3. For any $w \in H^{1,N}(\mathbb{R}^N)$ with $\|w\|_{H^{1,N}_\gamma} = 1$, let $t_w := \|w\|_N^\gamma$. Then we see that

$$I_\alpha(w) = f_\alpha(t_w;w) < 1$$

in view of Lemma 5.1 (b). Hence no w can achieve $D_\alpha = 1$.

(b) Let $\alpha = \alpha_\dagger$. Then by Lemma 5.1 (a), we have

$$f_\alpha(V) = 1. \tag{5.5}$$

Now note that for any u with $\|u\|_{H^{1,N}_\gamma} = 1$ and $t_u := \|u\|_N^\gamma$, we see that

$$\begin{aligned} I_\alpha(u) = f_\alpha(t_u; u) &= t_u^{\frac{N}{\gamma}} + \alpha B(u) t_u^{\frac{N}{\gamma}} (1 - t_u)^{\frac{p-N}{\gamma}} \\ &\le t_u^{\frac{N}{\gamma}} + \alpha B(V) t_u^{\frac{N}{\gamma}} (1 - t_u)^{\frac{p-N}{\gamma}} = f_\alpha(t_u; V) \le f_\alpha(V). \end{aligned}$$

This together with (5.5) yields $I_\alpha(u) \le 1$, and thus $D_{\alpha_\dagger} \le 1$. This fact together with (5.2) implies that $D_{\alpha_\dagger} = 1$. Moreover, (5.2) shows that $D_{\alpha_\dagger} = 1$ is achieved by $V_{t_{1,\alpha_\dagger}}$ (see (2.12) for definition).

(c) Let $\alpha > \alpha_\dagger$. Then Lemma 5.1 (a) yields $f_\alpha(V) > 1$. This fact together with $D_\alpha \ge f_\alpha(V)$, (1.5) and Proposition 2.1 yield the conclusion. □

Now we clarify the relationship between critical numbers $\alpha_\dagger$ and α_*.

Proposition 5.5.
There holds $\alpha_\dagger = \alpha_*$.

Proof. Let $\alpha < \alpha_\dagger$. By Proposition 5.4 (a), we have $D_\alpha = 1$. Hence for every $u \in H^{1,N}(\mathbb{R}^N)$ with $\|u\|_{H^{1,N}_\gamma} = 1$, there holds $1 \ge \|u\|_N^N + \alpha \|u\|_p^p$, thus $\frac{1 - \|u\|_N^N}{\|u\|_p^p} \ge \alpha$. This fact together with the definition $\alpha_* = \inf_{\|u\|_{H^{1,N}_\gamma} = 1} \frac{1 - \|u\|_N^N}{\|u\|_p^p}$ yields $\alpha_\dagger \le \alpha_*$.

Next let $\alpha > \alpha_\dagger$. Note that Proposition 5.4 (c) implies $D_\alpha > 1$, thus there exists $u \in H^{1,N}(\mathbb{R}^N)$ with $\|u\|_{H^{1,N}_\gamma} = 1$ such that $1 < \|u\|_N^N + \alpha \|u\|_p^p$, namely, $\frac{1 - \|u\|_N^N}{\|u\|_p^p} < \alpha$. This implies $\alpha_* = \inf_{\|u\|_{H^{1,N}_\gamma} = 1} \frac{1 - \|u\|_N^N}{\|u\|_p^p} < \alpha$, hence $\alpha_\dagger \ge \alpha_*$ follows. □

We are now in the position to prove Theorem 1.4.

Proof of Theorem 1.4. The first assertion follows from Proposition 5.4 and Proposition 5.5. Proposition 5.2 and Proposition 5.5 yield (1.6). Also it is easy to see that (1.7) follows from the first inequality of (1.6). Now we will prove (1.8).

In the following, we regard α_* as a function of γ and denote it by $\alpha_*(\gamma)$. Proposition 5.2 yields

$$\liminf_{\gamma \uparrow p-N} \alpha_*(\gamma) \ge \lim_{\gamma \uparrow p-N} \frac{N}{B\gamma} \left(\frac{p}{p - N - \gamma} \right)^{\frac{p-N}{\gamma} - 1} = \frac{N}{B(p-N)}. \tag{5.6}$$

First we note that

$$\alpha := \limsup_{\gamma \uparrow p-N} \alpha_*(\gamma) < \infty. \tag{5.7}$$

Indeed, for any $\gamma \in (\frac{1}{2}(p-N), p-N)$, we see

$$f_\alpha\left(\frac{1}{2}; V\right) = \left(\frac{1}{2}\right)^{\frac{N}{\gamma}} \left[1 + \alpha B \left(\frac{1}{2}\right)^{\frac{p-N}{\gamma}} \right] \ge \left(\frac{1}{2}\right)^{\frac{2N}{p-N}} \left(1 + \frac{1}{4} \alpha B \right). \tag{5.8}$$

This allows us to give $D_{\alpha_0} \ge f_{\alpha_0}\left(\frac{1}{2}; V\right) > 1$ for $\alpha_0 := \frac{5}{B}(2^{\frac{2N}{p-N}} - 1)$. Then since $\alpha > \alpha_*(\gamma)$ is equivalent to $D_\alpha > 1$ in view of (1.5), we have $\alpha_0 > \alpha_*(\gamma)$ for all $\gamma \in (\frac{1}{2}(p-N), p-N)$, and hence (5.7) follows.

Now we are in the position to give a proof of (1.8). It follows from (5.6) and (5.7) that for any $\gamma_n \uparrow p - N$, there exists $\hat{\alpha}$ such that

$$\alpha_n := \alpha_*(\gamma_n) \to \hat{\alpha} \geq \frac{N}{B(p-N)}. \tag{5.9}$$

Let us denote the smallest solution of

$$0 = 1 + \alpha_n B(1-t)^{\frac{p-N}{\gamma_n}-1}\left(1 - \frac{p}{N}t\right)$$

by $t_n (:= t_{1,\alpha_n})$ (see (B.5)). From this relation, we have

$$t_n = \frac{N}{p}\left(1 + \frac{1}{\alpha_n B(1-t_n)^{\frac{p-N}{\gamma_n}-1}}\right). \tag{5.10}$$

Now by noting $t_n < \frac{N+\gamma_n}{p}$ by virtue of (5.1), we obtain

$$\left(\frac{\gamma_n}{p}\right)^{\frac{p-N}{\gamma_n}-1}\left(\frac{p-N}{\gamma_n}-1\right)^{\frac{p-N}{\gamma_n}-1} < (1-t_n)^{\frac{p-N}{\gamma_n}-1} (< 1).$$

This relation yields

$$\lim_{n\to\infty} (1-t_n)^{\frac{p-N}{\gamma_n}-1} = 1,$$

which together with (5.10) and (5.9) implies

$$t_n = \frac{N}{p}\left(1 + \frac{1}{\hat{\alpha}B}\right) + o(1)$$

as $n \to \infty$. Note that α_n satisfies

$$1 = f_{\alpha_n}(t_n; V) = t_n^{\frac{N}{\gamma_n}}\left(1 + \alpha_n B(1-t_n)^{\frac{p-N}{\gamma_n}}\right)$$

(see (5.2)). Then by taking $n \to \infty$ to this relation, we obtain $\hat{\alpha}$ satisfies

$$1 = \left[\frac{N}{p}\left(1 + \frac{1}{\alpha B}\right)\right]^{\frac{N}{p-N}}\left[1 + \alpha B\left\{1 - \frac{N}{p}\left(1 + \frac{1}{\alpha B}\right)\right\}\right]. \tag{5.11}$$

Now we show the uniqueness of the solution α of (5.11). From (5.11), we have

$$1 = \left(\frac{N}{p}\right)^{\frac{N}{p-N}}\frac{p-N}{p}\left(1 + \frac{1}{\alpha B}\right)^{\frac{N}{p-N}}(1 + \alpha B) =: \left(\frac{N}{p}\right)^{\frac{N}{p-N}}\frac{p-N}{p}f(x), \tag{5.12}$$

where we put $x = \frac{1}{\alpha B}$ and $f(x) = (1+x)^{\frac{N}{p-N}}\left(1 + \frac{1}{x}\right)$. It is easy to see that $f(x)$ takes a minimum at $x_0 = \frac{p-N}{N}$ and $f\left(\frac{p-N}{N}\right) = \left(\frac{p}{N}\right)^{\frac{N}{p-N}}\frac{p}{p-N}$. This implies that the minimum of the right-hand side of (5.12) is 1. Hence (5.11) has a unique solution $\alpha = \frac{1}{Bx_0} = \frac{N}{B(p-N)}$, and thus $\hat{\alpha} = \frac{N}{B(p-N)}$ holds. These arguments show that $\alpha_n + o(1) = \hat{\alpha} = \frac{N}{B(p-N)}$, which implies $\lim_{\gamma\uparrow p-N} \alpha_*(\gamma) = \frac{N}{B(p-N)}$. This completes the proof. □

A Attainability of B

In this section, we give a proof of the following fact used throughout the paper. The proof is essentially the same one as in Weinstein [18] (in [18], the case $N = 2$ is treated). Let $N \geq 2$ and $B := \sup_{H^{1,N}(\mathbb{R}^N)\setminus\{0\}} \frac{\|v\|_p^p}{\|\nabla v\|_N^{p-N}\|v\|_N^N}$.

Proposition A.1.
There exists a maximizer V associated with B satisfying $\|V\|_{H^{1,N}_\gamma} = 1$ for any $N \geq 2$.

Proof. For any $w \in H^{1,N}(\mathbb{R}^N)$, $\mu > 0$ and $\lambda > 0$, let $w^{\mu,\lambda}(x) := \mu w(\lambda x)$. Then there hold

$$\|w^{\mu,\lambda}\|_N^N = \frac{\mu^N}{\lambda^N}\|w\|_N^N, \quad \|\nabla w^{\mu,\lambda}\|_N^N = \mu^N\|\nabla w\|_N^N, \quad B(w^{\mu,\lambda}) = B(w). \tag{A.1}$$

Let (u_n) be a maximizing sequence associated with B. Without loss of generality, we can assume that u_n is a radially symmetric function. By choosing $\mu_n := \frac{1}{\|\nabla u_n\|_N}$ and $\lambda_n := \frac{\|u_n\|_N}{\|\nabla u_n\|_N}$ and by letting $v_n := u_n^{\mu_n,\lambda_n}$, we see that, from (A.1),

$$(v_n) \text{ is a maximizing sequence associated with } B \text{ and } \|\nabla v_n\|_N = \|v_n\|_N = 1. \tag{A.2}$$

Then there exists $v \in H^{1,N}(\mathbb{R}^N)$ such that $v_n \rightharpoonup v$ weakly in $H^{1,N}(\mathbb{R}^N)$, especially,

$$\|v\|_N \leq \liminf_{n\to\infty}\|v_n\|_N = 1, \quad \|\nabla v\|_N \leq \liminf_{n\to\infty}\|\nabla v_n\|_N = 1 \tag{A.3}$$

in view of (A.2). Again (A.2) and the compact embedding $H^{1,N}_{\mathrm{rad}}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ yield

$$B = \frac{\|v_n\|_p^p}{\|\nabla v_n\|_N^{p-N}\|v_n\|_N^N} + o(1) = \|v_n\|_p^p + o(1) = \|v\|_p^p,$$

thus $v \neq 0$. Then since

$$\frac{B}{\|\nabla v\|_N^{p-N}\|v\|_N^N} = \frac{\|v\|_p^p}{\|\nabla v\|_N^{p-N}\|v\|_N^N} \leq B,$$

we obtain $\|\nabla v\|_N^{p-N}\|v\|_N^N \geq 1$ and finally

$$\|\nabla v\|_N = \|v\|_N = 1 \tag{A.4}$$

follows from (A.3). Hence we get $v_n \to v$ strongly in $H^{1,N}(\mathbb{R}^N)$ and v is a maximizer. Let $V(\cdot) := \left(\frac{1}{2}\right)^{\frac{1}{\gamma}} v(\cdot)$. Then (A.1) together with (A.4) implies that V is a maximizer associated with B satisfying $\|V\|_{H^{1,N}_\gamma} = 1$. □

B Proof of Lemma 5.1 and Proposition 5.2

Throughout this section, we use notation in §2.2.

B.1 The sign-changing condition for $f'_\alpha(t;v)$

Recall that $f'_\alpha(t;v)=0$ is equivalent to

$$1+\alpha B(v)h(t)=0, \tag{B.1}$$

where $h(t)=(1-t)^{\frac{p-N}{\gamma}-1}\left(1-\frac{p}{N}t\right)$, see (2.17).

First we consider the condition for α under which (B.1) has a solution t. By noting

$$h'(t)=-(1-t)^{\frac{p-N}{\gamma}-2}\frac{p-N}{\gamma N}\left[(N+\gamma)-pt\right]$$

and

$$t_0:=\frac{N+\gamma}{p}\in(0,1)$$

we have, under $\gamma<p-N$, $h(t)$ is strictly decreasing for $t\in(0,t_0)$, takes global minimum at $t=t_0$ and is strictly increasing for $t\in(t_0,1)$. Particularly, since

$$\min_{t\in[0,1]}h(t)=h(t_0)=-\frac{\gamma}{N}\left(\frac{p-N-\gamma}{p}\right)^{\frac{p-N}{\gamma}-1},$$

the condition for the existence of the solution for (B.1), $\alpha B(v)h(t_0)\le -1$, is given by

$$\alpha\ge\frac{N}{B(v)\gamma}\left(\frac{p}{p-N-\gamma}\right)^{\frac{p-N}{\gamma}-1}=:\alpha_0(v). \tag{B.2}$$

From now on, we assume the condition (B.2). As is stated above, $\alpha B(v)h(t)=-1$ has a unique solution $t=t_0=\frac{N+\gamma}{p}$ if $\alpha=\alpha_0(v)$ and exactly two solutions $t_{1,\alpha}(v)$, $t_{2,\alpha}(v)$ satisfying

$$0<t_{1,\alpha}(v)<t_0<t_{2,\alpha}(v)<1 \tag{B.3}$$

if $\alpha>\alpha_0(v)$.

First assume that $\alpha=\alpha_0(v)=\frac{N}{B(v)\gamma}\left(\frac{p}{p-N-\gamma}\right)^{\frac{p-N}{\gamma}-1}$. Then we know that $f'_\alpha(t;v)=0$ has a unique solution $t=\frac{N+\gamma}{p}$ and $f_\alpha(t;v)$ is strictly increasing for $t\in(0,1)$ with $t\neq\frac{N+\gamma}{p}$. Particularly, we obtain

$$f_\alpha(t;v)<f_\alpha(1;v)=1 \text{ for all } t\in(0,1). \tag{B.4}$$

Next we assume $\alpha>\alpha_0(v)$. Then the above analysis shows that $f'_\alpha(t;v)=0$, namely,

$$1+\alpha B(v)(1-t)^{\frac{p-N}{\gamma}-1}\left(1-\frac{p}{N}t\right)=0 \tag{B.5}$$

possesses exactly two solutions $t_{1,\alpha}(v)$ and $t_{2,\alpha}(v)$ satisfying $0<t_{1,\alpha}(v)<t_0<t_{2,\alpha}(v)<1$. Moreover, $f_\alpha(t;v)$ is

$$\begin{aligned}&\text{strictly increasing for } t\in(0,t_{1,\alpha}(v)), \text{ strictly decreasing for } t\in(t_{1,\alpha}(v),t_{2,\alpha}(v))\\ &\text{and strictly increasing for } t\in(t_{2,\alpha}(v),1).\end{aligned} \tag{B.6}$$

Particularly, we obtain that

$$f_\alpha(t;v) \text{ takes local maximum at } t=t_{1,\alpha}(v). \tag{B.7}$$

Since $t_{1,\alpha}(v)$ satisfies $f'_\alpha(t_{1,\alpha}(v); v) = 0$, i.e.,

$$-1 = \alpha B(v)(1 - t_{1,\alpha}(v))^{\frac{p-N}{\gamma}-1}\left(1 - \frac{p}{N}t_{1,\alpha}(v)\right), \tag{B.8}$$

we have $t_{1,\alpha}(v) > \frac{N}{p} \in (0, 1)$ and, since $t_{1,\alpha}(v) < t_0 = \frac{N+\gamma}{p} \in (0, 1)$ by (B.3), we have

$$t_{1,\alpha}(v) \in \left(\frac{N}{p}, \frac{N+\gamma}{p}\right) \subset (0, 1). \tag{B.9}$$

This relation yields

$$\begin{aligned} D_\alpha &\geq \max_{t\in[0,1]} f_\alpha(t; v) \geq f_\alpha(t_{1,\alpha}(v); v) = t_{1,\alpha}(v)^{\frac{N}{\gamma}} + \alpha B(v)t_{1,\alpha}(v)^{\frac{N}{\gamma}}(1 - t_{1,\alpha}(v))^{\frac{p-N}{\gamma}} \\ &> \left(\frac{N}{p}\right)^{\frac{N}{\gamma}}\left[1 + \alpha B(v)\left(\frac{p-N-\gamma}{p}\right)^{\frac{p-N}{\gamma}}\right]. \end{aligned} \tag{B.10}$$

Lemma B.1.
There holds $\frac{\partial}{\partial\alpha}t_{1,\alpha}(v) < 0$ *for* $\alpha > \alpha_0(v)$.

Proof. By (B.8), we see that $t_{1,\alpha}(v)$ satisfies

$$0 = 1 + \alpha B(v)(1 - t_{1,\alpha}(v))^{\frac{p-N}{\gamma}-1}\left(1 - \frac{p}{N}t_{1,\alpha}(v)\right).$$

By using the implicit function theorem and by differentiating both sides by α, we obtain

$$0 = B(v)(1 - t_{1,\alpha}(v))^{\frac{p-N}{\gamma}-2}\left[(1 - t_{1,\alpha}(v))\left(1 - \frac{p}{N}t_{1,\alpha}(v)\right) - \alpha\frac{\partial t_{1,\alpha}(v)}{\partial\alpha}\frac{p-N}{N\gamma}(N + \gamma - pt_{1,\alpha}(v))\right],$$

which implies

$$\frac{\partial t_{1,\alpha}(v)}{\partial\alpha} = \frac{\gamma N}{p-N}\frac{1}{\alpha}(1 - t_{1,\alpha}(v))\left(1 - \frac{p}{N}t_{1,\alpha}(v)\right)\frac{1}{N + \gamma - pt_{1,\alpha}(v)}.$$

Now by noting $t_{1,\alpha}(v) \in (0, 1)$ and (B.9), we see that the above relation yields $\frac{\partial t_{1,\alpha}(v)}{\partial\alpha} < 0$. □

Proposition B.2.
For any $v \in H^{1,N}(\mathbb{R}^N)$ *with* $\|v\|_{H^{1,N}_\gamma} = 1$, *there exists* $\alpha_\dagger(v) > \alpha_0(v)$ *such that*

$$\begin{aligned} &f_\alpha(t_{1,\alpha}(v); v) < 1 \text{ if } \alpha_0(v) < \alpha < \alpha_\dagger(v), \quad f_\alpha(t_{1,\alpha}(v); v) = 1 \text{ if } \alpha = \alpha_\dagger(v), \\ &f_\alpha(t_{1,\alpha}(v); v) > 1 \text{ if } \alpha > \alpha_\dagger(v). \end{aligned}$$

Proof. Take any $v \in H^{1,N}(\mathbb{R}^N)$ with $\|v\|_{H^{1,N}_\gamma} = 1$. First we show

$$\alpha \mapsto f_\alpha(t_{1,\alpha}(v); v) \text{ is monotone increasing for } \alpha > \alpha_0(v). \tag{B.11}$$

Recalling that $t_{1,\alpha}(v)$ is a solution of (B.5), we see

$$\alpha B(v)(1 - t_{1,\alpha}(v))^{\frac{p-N}{\gamma}} = \frac{1 - t_{1,\alpha}(v)}{\frac{p}{N}t_{1,\alpha}(v) - 1}.$$

This yields

$$f_\alpha(t_{1,\alpha}(v); v) = \frac{p-N}{p}\frac{t_{1,\alpha}(v)^{\frac{N}{\gamma}+1}}{t_{1,\alpha}(v) - \frac{N}{p}},$$

which implies

$$\frac{\partial}{\partial\alpha} f_\alpha(t_{1,\alpha}(v);v) = \frac{p-N}{p} \frac{t_{1,\alpha}(v)^{\frac{N}{\gamma}}}{\left(t_{1,\alpha}(v) - \frac{N}{p}\right)^2} \frac{N}{\gamma} \frac{\partial t_{1,\alpha}(v)}{\partial\alpha} \left(t_{1,\alpha}(v) - \frac{N+\gamma}{p}\right).$$

Then (B.9) and Lemma B.1 imply that $\frac{\partial}{\partial\alpha} f_\alpha(t_{1,\alpha}(v);v) > 0$, hence $\alpha \mapsto f_\alpha(t_{1,\alpha}(v);v)$ is monotone increasing for $\alpha > \alpha_0(v)$.

Also since $t \mapsto f_\alpha(t;v)$ is monotone increasing for $t < t_{1,\alpha}(v)$ and $\frac{N}{p} < t_{1,\alpha}(v)$ in view of (B.6) and (B.9) respectively, we obtain

$$f_\alpha(t_{1,\alpha}(v);v) \geq f_\alpha\left(\frac{N}{p};v\right) = \|u_{\frac{N}{p}}\|_N^N + \alpha\|u_{\frac{N}{p}}\|_p^p \to \infty$$

as $\alpha \uparrow \infty$. Also we see that $f_\alpha(t_{1,\alpha}(v);v) < 1$ for any α close to $\alpha_0(v)$ in view of (B.4). To sum-up the above facts, we have the desired conclusion. □

For $v \in H^{1,N}(\mathbb{R}^N)$ with $\|v\|_{H^{1,N}_\gamma} = 1$, let

$$f_\alpha(v) := \max_{t\in[0,1]} f_\alpha(t;v).$$

Proof of Lemma 5.1. (a) The relation (B.6) yields

$$f_\alpha(v) = \max(f_\alpha(1;v), f_\alpha(t_{1,\alpha}(v);v)) = \max(1, f_\alpha(t_{1,\alpha}(v);v))$$

and this together with Proposition B.2 yield the conclusion when $\alpha > \alpha_0(v)$. Also for $\alpha \leq \alpha_0(v)$, since $f'_\alpha(t;v) \geq 0$ for $t \in (0,1)$, we obtain $f_\alpha(v) = f_\alpha(1;v) = 1$.

(b) First let $\alpha_0(v) < \alpha < \alpha_\dagger(v)$. For $t \in [0, t_{2,\alpha}(v)]$, by (B.6), we have $f_\alpha(t;v) \leq f_\alpha(t_{1,\alpha}(v);v)$. This relation and the assumption $\alpha < \alpha_\dagger(v)$ together with Proposition B.2 imply $f_\alpha(t;v) < 1$ for $t \in [0, t_{2,\alpha}(v)]$. For $t \in [t_{2,\alpha}(v), 1)$, (B.6) directly leads $f_\alpha(t;v) < f_\alpha(1;v) = 1$. Next when $\alpha \leq \alpha_0$, we see

$$f'_\alpha(t;v) \begin{cases} > 0 & \text{for } t \in (0,1) \quad \text{if } \alpha < \alpha_0, \\ > 0 & \text{for } t \in (0,1) \setminus \{t_0\} \quad \text{if } \alpha = \alpha_0, \end{cases}$$

which implies $f_\alpha(v) < f_\alpha(1;v) = 1$. □

Proof of Proposition 5.2. Let us denote $\alpha_\dagger(V)$ and $t_{1,\alpha}(V)$ by $\alpha_\dagger$ and $t_{1,\alpha}$, respectively. The first inequality in (5.3), namely,

$$\alpha_0 := \frac{N}{B\gamma}\left(\frac{p}{p-N-\gamma}\right)^{\frac{p-N}{\gamma}-1} < \alpha_\dagger$$

is a part of the statement of Proposition B.2 with $v = V$.

Next we show the second inequality in (5.3), namely,

$$\alpha_\dagger < \alpha_\sharp := \left[\left(\frac{p}{N}\right)^{\frac{N}{\gamma}} - 1\right] \frac{1}{B} \left(\frac{p}{p-N-\gamma}\right)^{\frac{p-N}{\gamma}}.$$

Note that there holds

$$\begin{aligned} f_{\alpha_\sharp}(V) &\geq f_{\alpha_\sharp}\left(\frac{N}{p};V\right) = \left(\frac{N}{p}\right)^{\frac{N}{\gamma}}\left[1+\alpha_\sharp B\left(\frac{p-N}{p}\right)^{\frac{p-N}{\gamma}}\right] \\ &> \left(\frac{N}{p}\right)^{\frac{N}{\gamma}}\left[1+\alpha_\sharp B\left(\frac{p-N-\gamma}{p}\right)^{\frac{p-N}{\gamma}}\right] = 1, \end{aligned} \tag{B.12}$$

where the last equality in (B.12) is obtained from the definition of $\alpha_\sharp$. Thus Lemma 5.1 (a) yields $\alpha_\sharp > \alpha_\dagger$. This completes the proof. □

Acknowledgments

The authors would like to express their hearty thanks to the referee for his/her valuable comments.

References

[1] S. Adachi and K. Tanaka, *A scale-invariant form of Trudinger-Moser inequality and its best exponent*, Proc. Amer. Math. Soc. **1102** (1999), 148–153.

[2] D. M. Cao, *Nontrivial solution of semilinear elliptic equation with critical exponent in* $\mathbb{R}^2$, Comm. Partial Differential Equations **17** (1992), 407–435.

[3] L. Carleson and S.-Y. A. Chang, *On the existence of an extremal function for an inequality of J. Moser*, Bull. Sci. Math. (2) **110** (1986), 113–127.

[4] M. Flucher, *Extremal functions for the Trudinger-Moser inequality in 2 dimensions*, Comm. Math. Helv. **67** (1992), 471–479.

[5] M. Ishiwata, *Existence and nonexistence of maximizers for variational problems associated with Trudinger-Moser type inequalities in* $\mathbb{R}^N$, Math. Ann. **351** (2011), 781–804.

[6] H. Kozono, T. Sato and H. Wadade, *Upper bound of the best constant of a Trudinger-Moser inequality and its application to a Gagliardo-Nirenberg inequality*, Indiana Univ. Math. J. **55** (2006), 1951–1974.

[7] Y. Li and B. Ruf, *A sharp Trudinger-Moser type inequality for unbounded domains in* $\mathbb{R}^n$, Indiana Univ. Math. J. **57** (2008), 451–480.

[8] K. C. Lin, *Extremal functions for Moser's inequality*, Trans. Amer. Math. Soc. **348** (1996), 2663–2671.

[9] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1970), 1077–1092.

[10] S. Nagayasu and H. Wadade, *Characterization of the critical Sobolev space on the optimal singularity at the origin*, J. Funct. Anal. **258** (2010), 3725–3757.

[11] T. Ogawa, *A proof of Trudinger's inequality and its application to nonlinear Schrodinger equation*, Nonlinear Anal. **14**, (1990) 765–769.

[12] T. Ogawa and T. Ozawa, *Trudinger type inequalities and uniqueness of weak solutions for the nonlinear Schrodinger mixed problem*, J. Math. Anal. Appl. **155**, (1991) 531–540.

[13] T. Ozawa, *Characterization of Trudinger's inequality*, J. Inequal. Appl. **1** (1997), 369–374.

[14] T. Ozawa, *On critical cases of Sobolev's inequalities*, J. Funct. Anal. **127** (1995), 259–269.

[15] S. I. Pohozaev, *The Sobolev embedding in the case $pl = n$*, In : Proceedings of the Technical Scientific Conference on Advances of Scientific Research 1964–1965. Mathematics Section, pp.158–170, Moskov. Energetics Inst., Moscow, (1965).

[16] B. Ruf, *A sharp Trudinger-Moser type inequality for unbounded domains in $\mathbb{R}^2$*, J. Funct. Anal. **219** (2005), 340–367.

[17] N. S. Trudinger, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473–483.

[18] M. I. Weinstein, *Nonlinear Schrodinger equations and sharp interpolation estimates*, Commun. Math. Phys. **87** (1982/1983), 567–576.

[19] V. I. Yudovich, *Some estimates connected with integral operators and with solutions of elliptic equations*, Dok. Akad. Nauk SSSR **138** (1961), 804–808.