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Semi-dynamic traffic assignment model with mode and route choices under stochastic travel times

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Abstract. Transportation network conditions vary significantly during the course of a day. In many urban areas, public transit and (private) automobiles constitute the actual forms of transportation that use such networks. Public transportation by rail is more reliable than by automobiles or buses; therefore, ordinary static and deterministic traffic assignment models with combined mode and route choices may not be suitable to assess a transportation network that includes public railways. Moreover, within-day dynamics and reliability need to be incorporated in such a model. In this paper, we use a semi-dynamic traffic assignment model that considers within-day dynamics by improving the static traffic assignment model. In addition, stochastic travel times are incorporated into the model. Thus, we propose a semi-dynamic traffic assignment model with mode choice between public transit and automobiles, route choice with stochastic travel times, and an accompanying computing algorithm. This model enables us to assess within-day dynamics of transportation networks and travel time reliability of public railways.

Keywords: semi-dynamic traffic assignment model, stochastic travel times, mode choice, route choice, non-linear complementarity problem

1. Introduction

Traffic conditions in most cities vary significantly during the course of a day. A static traffic assignment model may not be able to adequately represent time-varying congestion phenomena in transportation network analyses. In theory, a (full) dynamic traffic assignment (DTA) model in continuous time is preferable. However, for practical applications, a continuous-time DTA is not necessarily reasonable, considering limited computational resources and the lack of detailedness in the origin–destination (OD) data. Thus, in this study, we adopt a semi-dynamic approach in which a day is divided into several periods. Static network equilibrium is reached in each period, but the flow propagation between periods is considered.

A flow that cannot reach its destination within the period of departure is called “residual flow” in this study. The dynamics in our semi-dynamic approach are expressed as the propagation of residual flow to the next period. A discrete-time dynamic model and a semi-dynamic model are similar to each other, except that in the semi-dynamic model, static network equilibrium is reached in each period, which is relatively long, i.e., 15–90 min. Most travel demands reach their destinations within the period in which they depart given the duration of the period is relatively long. In such a case, representing the transportation system in each period by static traffic equilibrium is acceptable. The ordinary static traffic equilibrium technique and algorithm can be applied to the semi-dynamic model. Furthermore, the semi-dynamic model has less computational cost and more applicability, although it cannot describe traffic dynamics in as much detail as a discrete-time dynamic traffic assignment model. Models with the following characteristics are classified under the “semi-dynamic approach” in this study: 1) the duration of a period is sufficiently long so that most travel demands reach the destination within the departure period (i.e., in the period in which they depart); 2) static network equilibrium is reached in each period, but flow propagation between periods is considered; and 3) the ordinary static traffic equilibrium modeling and algorithm can basically be applied. Such a semi-dynamic approach is often used in traffic simulations. In this study, we focus on a semi-dynamic traffic assignment model with user equilibrium, rather than on simulations. Fujita *et al.* (1988, 1989), Miyagi and Makimura (1991), and Akamatsu *et al.* (1998) have developed semi-dynamic user equilibrium models, as mentioned in the next section.

In public transportation, the function of railways is very different from that of buses. The travel time for railways or a light rail transit (LRT) system is more reliable than that for automobiles, buses, or others constituting road traffic. When assessing the effect of railways, the reliability of travel times should be considered, otherwise, their effect may be underestimated. For an exact evaluation, first, we should not only treat public transportation and automobile users simultaneously and consistently under within-day dynamics, but also consider the uncertainty of travel time for automobile users versus the reliability (or punctuality) of railways. In this study, a semi-dynamic traffic assignment model with mode choice between public transportation and automobiles and route choice under stochastic travel times is formulated as a non-linear complementarity problem. Next, an algorithm is proposed to solve the model. The model is applied to the Kanazawa transportation network in consideration of the planned introduction of an LRT system, and the applicability and validity of the model are examined.

2. Semi-Dynamic Traffic Assignment Model with Mode Choice

In our semi-dynamic approach, a day is divided into several periods. In addition, the duration of each period is not as short as the length of the time interval for ordinary discrete-time DTA models. The semi-dynamic approach presupposes that a majority of OD flows reach their destination within the departure period. The semi-dynamic approach is a compromise between the static and the dynamic network equilibrium approaches. In the semi-dynamic approach, enhanced usability and reduction of computational cost are the focus rather than a detailed description of dynamic queues or congestion. Therefore, a semi-dynamic approach should be adopted only when detailed dynamics are not necessarily required.

In our semi-dynamic model, residual flow plays an important role in describing the semi-dynamics (or flow propagation) between periods. Existing semi-dynamic traffic assignment models can be classified into three categories: the demand modification approach, the link flow approach, and the queue approach, by the formulation of residual flow. Fujita *et al.* (1988) and Miyagi and Makimura (1991) proposed the demand modification approach. In their models, residual flow propagation is represented by modifying or changing the OD demand in the current and next periods, and traffic equilibrium in each period is formulated as an optimization problem with elastic demand. This means that the computational cost in each period is approximately equal to the ordinary static traffic assignment model with elastic demand. Thus, it is relatively easy to apply the demand modification approach to large networks; however, the approach does not necessarily model flow dynamics appropriately. Fujita *et al.* (1989) proposed the link flow approach to treat residual flow on each link directly. However, their formulation includes the variational inequality problem, which entails a far greater computational cost. Akamatsu *et al.* (1998) adopted a vertical queue approach and propagated residual flow as a vertical queue. This is classified into the queue approach in this study. The model of Akamatsu *et al.* (1998) is formulated as an optimization problem.

In this study, the semi-dynamic model of Fujita *et al.* (1988), i.e., the demand modification approach, is used because it is the easiest to operate and is inexpensive in terms of computational cost. This approach should be applied to cases in which considerable detail and accurate flow dynamics are not of paramount importance. Assuming that users select the route with the minimum cost, the residual flow can be defined as

$$r_{it} = \frac{d_{it}\tau_{it}}{l} \tag{1}$$

where

r_{it} = residual flow between an OD pair i ($\in I$) in period t ($\in T$),

τ_{it} = travel time on the minimum total cost route between an OD pair i in period t,

d_{it} = demand between an OD pair i in period t,
l = duration of each period.

If we assume that the demand departs uniformly during the period, the departure rate of demand (the number of automobile users) is d_{it}/l. Users are traveling on the network for the time of the travel time on the minimum total cost because they take the route with the minimum cost if he reaches his destination within the period. The amount $d_{it}\tau_{it}/l$ is the residual flow, which has not reached the destination and is traveling on the network at the end of the period. As mentioned above, the duration of each period is relatively long (15–90 min) in the semi-dynamic model. It is presupposed in this study that $r_{it} < d_{it}$ for any OD pair and period, and $l > \max[\tau_{it}]$. That is, the duration of each period should be longer than the maximum travel times on the routes used in this study. A large network may require long l. This implies that the semi-dynamic traffic assignment model should not be applied for modeling detailed traffic conditions. Moreover, for that purpose, continuous- or discrete-time dynamic traffic assignment models should to be adopted.

Because all the residual flow does not reach its destination within the period, it affects travel times in both the current and the next periods. In the demand modification approach, the residual flow is divided into two: one is added to the OD demand in the current period and the other is added to the next period. We can adopt any ratio, but in this study, for simplicity, we suppose that half of the residual flow is added to the current period and the other half is added to the next period. Revising Eq. 2 in the next paragraph enables us to choose any ratio. Even if the same half of the residual flow is added, the effect on traffic congestion is generally different between the current and the next periods. However, this depends on the level of traffic congestions and other factors in each period.

To model this, a modified OD demand is introduced. The modified demand is defined as

$$q_{it} = \frac{1}{2} r_{it-1} + d_{it} - \frac{1}{2} r_{it} \tag{2}$$

where

q_{it} = modified demand of OD pair i in period t,
d_{it} = (original) demand of OD pair i in period t,
r_{it} = residual flow for OD pair i in period t.

As shown in Eq. 2, the modified demand includes half of the residual flow from the previous period, while half of the residual flow is subtracted from the current demand and is added to the next demand. The modified demand q_{it} is a function of τ_{it} because r_{it-1} in Eq. 2 is given from period $t-1$ and can be regarded as a constant in period t. Let $g_{it}(\tau_{it})$ denote the modified demand function, that is, $q_{it} = g_{it}(\tau_{it})$. Using this modified demand, the semi-dynamic traffic assignment model can be expressed as a static user equilibrium formulation with elastic (modified) demand, because half of the residual flow is subtracted from the current modified demand. The semi-dynamic user equilibrium in period t can be formulated as the following minimization problem:

$$\min_{\mathbf{x},\mathbf{q}} . L_t = \sum_{a \in A} \int_0^{x_a} t_a(w) dw - \sum_{i \in I} \int_0^{q_{it}} g_{it}^{-1}(v)\, dv \tag{3}$$

s.t. $$q_{it} = \sum_{j \in J_i} f_{ijt} \qquad \forall i \in I \tag{4}$$

$$x_{at} = \sum_{i \in I} \sum_{j \in J_i} \delta_{aij} f_{ijt} \quad \forall a \in A \tag{5}$$

$$f_{ijt} \geq 0 \qquad \forall i \in I, j \in J_i \tag{6}$$

where

x_{at} = flow through link a ($\in A$) in period t,

$t_a(\cdot)$ = travel time function on link a ($\in A$),

f_{ijt} = flow through route j ($\in J_i$) between OD pair i in period t,

δ_{aij} = link–route incidence variable, and

$g_{it}^{-1}(v)$ = inverse of modified demand function

Safwat and Magnanti (1988) and Fernandez *et al.* (1994) introduced mode choice into static network equilibrium models. Using the demand modification approach created by Fujita *et al.* (1988), Ujii *et al.* (2003) developed a semi-dynamic traffic assignment model that offered the option of using a toll expressway. In their model, a binary choice on whether the expressway is taken is incorporated using a binary logit model. Their semi-dynamic model is based on deterministic flows (and travel times) and is formulated as an optimization problem. In our study, mode choice is incorporated into the semi-dynamic traffic assignment model using the logit model. Furthermore, unlike Ujii et al. (2003), stochastic flows (and travel times) are considered. As explained later, our model is formulated as a complementarity problem because of the asymmetry of modes, and an algorithm with the relaxation technique is presented. First, travel costs including public transit fares as well as time and monetary costs are defined as follows:

$$v_{ijt}^{c} = \upsilon c_{ijt}^{c} + \xi \tag{7}$$

$$v_{ijt}^{tr} = \upsilon (c_{ijt}^{tr} + w_{ijt} + \iota_i) + s_{ij} \tag{8}$$

where

$v_{ijt}{}^{c}$ = automobile travel cost
$v_{ijt}{}^{tr}$ = public transit travel cost
$c_{ijt}{}^{c}$ = automobile travel time
$c_{ijt}{}^{tr}$ = public transit travel time
w_{ijt} = total waiting time
All above variables are defined for route *j* between an OD pair *i* in period *t*
ι_i = access and egress time for public transit between an OD pair *i*
s_{ij} = total public transit fares for route *j* between an OD pair *i*

υ = monetary value of time

ξ = maintenance cost of automobile

The logit model adopted describes a binary choice between public transit and (private) automobiles. The mode choice is given as

$$y_{it}^{c} = q_{it} \frac{1}{1 + \exp\{-\theta(\lambda_{it}^{tr} - \lambda_{it}^{c})\}} \tag{9}$$

where

y_{it}^{c} = number of automobile users between an OD pair *i* in period *t*,

λ_{it}^{c} = minimum (monetary) cost for automobile users between an OD pair *i* in period *t*,

λ_{it}^{tr} = minimum (monetary) cost for public transit users between an OD pair *i* in period *t*,

and

θ = positive parameter.

Clearly, $y_{it}^{tr} = q_{it} - y_{it}^{c}$, where y_{it}^{tr} denotes the number of public transit users between an OD pair *i* in period *t*. After the mode is chosen, a route is chosen by selecting the minimum cost route between road routes or public transit routes.

The expected minimum travel time between an OD pair is a function of τ_{it}^{c} and τ_{it}^{tr} where τ_{it}^{m} is the travel time on the minimum generalized cost route of mode *m* between an OD pair *i* in period *t*, and is given as $-\ln[\exp(-\theta\tau_{it}^{c}) + \exp(-\theta\tau_{it}^{tr})]/\theta$. In the semi-dynamic model, the residual flow propagates and affects the traffic state in the next period. The residual flow is given using the travel time on the minimum monetary cost route, rather than the minimum monetary cost itself, as shown in Eq. 1. The expected minimum monetary cost includes public fares and other expenses as shown in Eqs. 7 and 8, and it is not appropriate for calculating the residual flow. In this study, the expected travel time on the minimum monetary cost route of both automobile and public transit users between an OD pair is given as follows:

$$\tau_{it} = -\frac{1}{\theta}\ln\left(\exp(-\theta\tau_{it}^{c}) + \exp(-\theta\tau_{it}^{tr})\right) \tag{10}$$

The travel times τ_{it}^{c} and τ_{it}^{tr} can be given by λ_{it}^{c} and λ_{it}^{tr} as shown in Eqs. 7 and 8; namely, τ_{it}^{c} and τ_{it}^{tr} are functions of λ_{it}^{c} and λ_{it}^{tr}. Therefore, using Eq. 2, the modified demand is given as

$$q_{it} = \tilde{h}_{it}\left(\tau_{it}(\lambda_{it}^{c}, \lambda_{it}^{tr})\right) = h_{it}(\lambda_{it}^{c}, \lambda_{it}^{tr}) = \frac{\tau_{it-1}d_{it-1}}{2l} + d_{it} - \frac{\tau_{it}d_{it}}{2l}. \tag{11}$$

$\tilde{h}_{it}(\cdot)$ and $h_{it}(\cdot,\cdot)$ are substantially identical, but the variable of $\tilde{h}_{it}(\cdot)$ is τ only while those of $h_{it}(\cdot,\cdot)$ is λ_{it}^{c} and λ_{it}^{tr}. Therefore, the shape of the functions is different. The function $h_{it}(\lambda_{it}^{c}, \lambda_{it}^{tr})$ yields the modified demand for the semi-dynamic model with mode choice when λ_{it-1}^{c} and λ_{it-1}^{tr} are obtained from the previous period.

The automobile routes and bus routes are different, and the influence of buses and private automobiles on the network is asymmetric; therefore, we cannot formulate the semi-dynamic traffic assignment model with mode choice using an optimization problem when buses are introduced. Thus, in this study, a non-linear complementarity problem is formulated, unlike the model of Ujii *et al.* (2003) with the optimization problem. The semi-dynamic traffic assignment model with mode and route choices is given by the complementarity problem described by the following equations:

$$f_{ijt}^m\left(c_{ijt}^m - \lambda_{it}^m\right) = 0 \qquad \forall i \in I, j \in J_i^m, m \in M \tag{12}$$

$$f_{ijt}^m \geq 0,\ c_{ijt}^m - \lambda_{it}^m \geq 0 \qquad \forall i \in I, j \in J_i^m, m \in M \tag{13}$$

$$y_{it}^m\left(\lambda_{it}^m + \frac{1}{\theta}\ln y_{it}^m - \kappa_{it}\right) = 0 \qquad \forall i \in I, m \in M \tag{14}$$

$$y_{it}^m \geq 0,\ \ \lambda_{it}^m + \frac{1}{\theta}\ln y_{it}^m - \kappa_{it} \geq 0 \qquad \forall i \in I, m \in M \tag{15}$$

$$q_{it}\left(q_{it} - h_{it}(\lambda_{it}^c, \lambda_{it}^{tr})\right) = 0 \qquad \forall i \in I \tag{16}$$

$$q_{it} \geq 0,\ q_{it} - h_{it}(\lambda_{it}^c, \lambda_{it}^{tr}) \geq 0 \qquad \forall i \in I \tag{17}$$

$$\lambda_{it}^m\left(y_{it}^m - \sum_{j \in J_i^m} f_{ijt}^m\right) = 0 \qquad \forall i \in I, m \in M \tag{18}$$

$$\lambda_{it}^m \geq 0,\ y_{it}^m - \sum_{j \in J_i^m} f_{ijt}^m \geq 0 \qquad \forall i \in I, m \in M \tag{19}$$

$$\kappa_{it}\left(q_{it} - y_{it}^c - y_{it}^{tr}\right) = 0 \qquad \forall i \in I \tag{20}$$

$$\kappa_{it} \geq 0,\ q_{it} - y_{it}^c - y_{it}^{tr} \geq 0 \qquad \forall i \in I \tag{21}$$

where

f_{ijt}^m = flow on route *j* of mode *m* ($\in M = \{c, tr\}$) between an OD pair *i* in period *t*,

c_{ijt}^m = (monetary) cost on route *j* of mode *m* between an OD pair *i* in period *t*,

J_j^m = set of routes of mode *m* between an OD pair *i*, and

κ_{it} = multiplier related to the minimum travel cost between an OD pair *i* in period *t*.

The flow conservations are originally expressed as equations, but the complementarity form is used for consistent formulation. In the vector form, the above complementarity problem is expressed as

$$\text{Find. } \mathbf{z} = \left(\mathbf{f}_t, \mathbf{q}_t, \mathbf{y}_t, \boldsymbol{\kappa}_t, \boldsymbol{\lambda}_t\right)^T \tag{22}$$

$$\text{s.t. } \left\langle \mathbf{z}, \boldsymbol{\Phi}(\mathbf{z}) \right\rangle = \mathbf{0},\ \ \mathbf{z} \geq \mathbf{0},\ \ \boldsymbol{\Phi}(\mathbf{z}) \geq \mathbf{0}. \tag{23}$$

where

$$\Phi(\mathbf{z}) = \begin{pmatrix} \mathbf{c}(\mathbf{f}_t) - \boldsymbol{\Gamma}^T \boldsymbol{\lambda}_t \\ \boldsymbol{\lambda}_t^c + \frac{1}{\theta}\ln \mathbf{y}_t^c - \boldsymbol{\kappa}_t \\ \boldsymbol{\lambda}_t^{tr} + \frac{1}{\theta}\ln \mathbf{y}_t^{tr} - \boldsymbol{\kappa}_t \\ \mathbf{q}_t - \mathbf{h}_t(\boldsymbol{\lambda}_t) \\ \mathbf{y}_t - \boldsymbol{\Gamma}\mathbf{f}_t \\ \mathbf{q}_t - \mathbf{y}_t^c - \mathbf{y}_t^{tr} \end{pmatrix} \tag{24}$$

$$\mathbf{y}_t = \begin{pmatrix} \mathbf{y}_t^c \\ \mathbf{y}_t^{tr} \end{pmatrix} \tag{25}$$

$$\boldsymbol{\lambda}_t = \begin{pmatrix} \boldsymbol{\lambda}_t^c \\ \boldsymbol{\lambda}_t^{tr} \end{pmatrix} \tag{26}$$

$\mathbf{c}(\mathbf{f}_t)$ = vector-valued function of $c_{ijt}^m(\mathbf{f}_t)$

$\mathbf{h}_t(\boldsymbol{\lambda}_t)$ = vector-valued function of $h_{it}(\boldsymbol{\lambda}_t)$

$\ln \mathbf{y}_t^m$ = vector of $\ln y_{it}^m$

$\mathbf{f}_t$ = vector of f_{ijt}^m

$\boldsymbol{\lambda}_t^c$ = vector of λ_{it}^c

$\boldsymbol{\lambda}_t^{tr}$ = vector of λ_{it}^{tr}

$\mathbf{y}_t^c$ = vector of y_{it}^c

$\mathbf{y}_t^{tr}$ = vector of y_{it}^{tr}

$\mathbf{q}_t$ = vector of q_{it}

$\boldsymbol{\kappa}_t$ = vector of κ_{it}

All the above variables are defined between an OD pair i in period t.

$\boldsymbol{\Gamma}$ = OD–route incidence matrix

$\langle \mathbf{x}, \mathbf{y} \rangle$ = inner product of $\mathbf{x}$ and $\mathbf{y}$

^{T} = vector or matrix transpose

3. Stochastic Travel Times

In this study, travel time reliability is examined to accurately evaluate the effect of punctual railway public transit. Several studies using network equilibrium models with stochastic travel times have been conducted, such as Mirchandani and Soroush (1987), Watling (2002), Lo and Tung (2003), Yin *et al.* (2004), Lo *et al.* (2006), Watling (2006), Nakayama (2007), and Lam *et al.* (2008). In this study, we describe the traffic state semi-dynamically and with a reasonable approximation rather than using detailed dynamics, and each link travel time is assumed to follow an independent probability distribution for the sake of simplicity. Both the mean and variance of each link travel time are functions of the link flow.

The travel time of buses in public transportation is influenced by road traffic flows, unlike railways, where the travel time is constant. Buses stop at bus stops, and thus the bus travel time is definitely longer than (private) automobile travel time. Kawakami and Takada (1990) presented $c^b = 1.5c^c$, where c^c and c^b are automobile and bus travel times, respectively, as a relationship between automobile and bus travel times. It is unclear whether the variance in bus travel time is greater than that for private automobiles. Bus drivers may drive at a constant rate, and thus bus travel time would not fluctuate much. We do not focus on this aspect to any significant degree in this paper. For simplicity, it is assumed that the variance in bus travel time is equal to that in private automobile travel time. In this study, mean bus travel time and the variance of travel time on link a are given as

$$\mathrm{E}[C_{at}^b] = \varsigma\,\mathrm{E}[C_{at}^c] \tag{27}$$

$$\mathrm{Var}[C_{at}^b] = \mathrm{Var}[C_{at}^c] \tag{28}$$

where

$\mathrm{E}[\cdot]$ = expected value,

$\mathrm{Var}[\cdot]$ = variance,

C_{at}^c = random variable for automobile travel time on link a in period t,

C_{at}^b = random variable for bus travel time on link a in period t,

ς = positive constant parameter ($\varsigma \geq 1.0$).

The variance of railway travel time is assumed to be 0. Under the above stochastic travel times, Eqs. 7 and 8 are revised as

$$V_{ijt}^c = \tau C_{ijt}^c + \xi \tag{29}$$

$$V_{ijt}^{tr} = \tau(C_{ijt}^{tr} + w_{ijt} + \iota_i) + s_{ij} \tag{30}$$

where

V_{ijt}^m = random variable of the cost on route j via mode m between an OD pair i in period t,

C_{ijt}^m = random variable of the travel time on route j via mode m between an OD pair i in period t.

Public transit routes comprise buses and LRT in this study, and C_{ijt}^{tr} consists of C_{at}^b and c_{at}^{lrt}, where c_{at}^{lrt} is a fixed travel time on LRT link a in period t. In this study, we assume that link travel time is independent. Then,

$$\mathrm{Var}[C_{at}^{c}] = \sum_{a\in A^c} \delta_{aij} \mathrm{Var}[C_{at}^{c}] \tag{31}$$

where C_{at}^{c} denotes the random variable of automobile travel time on link a in period t. In this study, the expected disutility $\mathrm{E}[u(V_{ijt})]$ is formulated as

$$\bar{u}_{ijt}^{m} = \mathrm{E}[V_{ijt}^{m}] + \gamma \mathrm{Var}[V_{ijt}^{m}] \tag{32}$$

where

$\bar{u}_{ijt}^{m}$ = expected disutility of travel for mode m on route j between an OD pair i in period t,

γ = risk attitude parameter.

Given the stochastic travel times, mode choice is expressed by the binary logit model with the minimum disutility $\min[\bar{u}_{ijt}^{m} \mid \forall j \in J_i]$. A route is selected as the minimum disutility route within the road routes or public transit routes.

Let $\tilde{u}_{at}^{c} = \mathrm{E}[C_{at}^{c}] + \gamma \mathrm{Var}[C_{at}^{c}]$. As described above, the mean and variance of each link travel time are functions of the link flows. Therefore, $\tilde{u}_{at}^{c}$ is a function of the link flow, and $\tilde{u}_{at}^{c} = \tilde{u}_{a}^{c}(x_{at}^{c}, \mathbf{f}_{t}^{tr})$, where x_{at}^{c} is the automobile flow on link a in period t and $\mathbf{f}_{t}^{tr}$ is the vector of public transit route flows in period t because the link flow is the sum of the automobile and bus flows.

4. Model Formulation and Algorithm using Stochastic Travel Times

By substituting the expected disutility given by Eq. 32 for the travel costs in the complementarity problem, i.e., Eqs. 22–24 in Section 2, we obtain the following semi-dynamic traffic assignment model with mode and route choices under stochastic travel times:

$$\text{Find. } \mathbf{z} = \left(\mathbf{f}_t, \mathbf{q}_t, \mathbf{y}_t, \boldsymbol{\kappa}_t, \boldsymbol{\lambda}_t\right)^T \tag{33}$$

$$\text{s.t. } \left\langle \mathbf{z}, \hat{\boldsymbol{\Phi}}(\mathbf{z}) \right\rangle = \mathbf{0},\ \mathbf{z} \geq \mathbf{0},\ \hat{\boldsymbol{\Phi}}(\mathbf{z}) \geq \mathbf{0}. \tag{34}$$

where

$$
\widehat{\mathbf{\Phi}}(\mathbf{z}) = \begin{pmatrix} \overline{\mathbf{u}}(\mathbf{f}_t) - \mathbf{\Gamma}^T \boldsymbol{\lambda}_t \\ \boldsymbol{\lambda}_t^c + \dfrac{1}{\theta} \ln \mathbf{y}_t^c - \boldsymbol{\kappa}_t \\ \boldsymbol{\lambda}_t^{tr} + \dfrac{1}{\theta} \ln \mathbf{y}_t^{tr} - \boldsymbol{\kappa}_t \\ \mathbf{q}_t - \mathbf{h}_t(\boldsymbol{\lambda}_t) \\ \mathbf{y}_t - \mathbf{\Gamma}\mathbf{f}_t \\ \mathbf{q}_t - \mathbf{y}_t^c - \mathbf{y}_t^{tr} \end{pmatrix} \tag{35}
$$

$\overline{\mathbf{u}}(\mathbf{f}_t)$ = vector-valued function whose components give $\overline{u}_{ijt}^{m}$.

A relaxation technique (Dafermos, 1982) is adopted to compute the model proposed in this study. Convergence of the relaxation algorithm is also discussed in Dafermos (1982). Let

$$
\boldsymbol{\omega}_t = \left(\mathbf{x}_t^c, \mathbf{f}_t^{tr}, \mathbf{q}_t, \mathbf{y}_t\right)^T \tag{36}
$$

where

$\mathbf{x}_t^c$ = vector of automobile link flows in period t.

Public transit users take public transit links, which are established by public transit companies. In contrast, automobile users are able to choose any link on the network. Asymmetric interaction between automobile flows and public transit flows on the network prevents us from formulating an optimization problem. To compute the complementarity problem, a relaxation problem as an optimization problem is formulated, and the relaxation problem is solved iteratively. In the relaxation problem, the public transit travel times are fixed and $\tilde{u}_{at}^c = u_{at}^c(x_{at}^c)$, that is, $\mathbf{f}_t^{tr}$ is fixed (or given from the previous step) in $\tilde{u}_a^c(x_{at}^c, \mathbf{f}_t^{tr})$. The relaxation problem in this study is formulated as the following optimization problem:

$$
\min . \vec{L}_t = \sum_{a\in A} \int_0^{x_{at}} u_{at}^c(w)\,dw + \xi \sum_{i\in I} y_{it}^c + \sum_{i\in I} \sum_{j\in J_i^{tr}} \overline{u}_{ijt}^{tr} f_{ijt}^{tr} + \frac{1}{\theta} \sum_{i\in I} \sum_{m\in M} y_{it}^m \ln y_{it}^m - \sum_{i\in I} \int_0^{q_{it}} h_{it}^{-1}(w)\,dw \tag{37}
$$

s.t.

$$
q_{it} = y_{it}^c + y_{it}^{tr} \qquad \forall i \in I \tag{38}
$$

$$
x_{at} = x_{at}^c + \sum_{i\in I} \sum_{j\in J_i^{tr}} \delta_{aij} f_{ijt}^{tr} \qquad \forall a \in A \tag{39}
$$

$$
x_{at}^c = \sum_{i\in I} \sum_{j\in J_i^c} \delta_{aij} f_{ijt}^c \qquad \forall a \in A \tag{40}
$$

$$
y_{it}^c = \sum_{j\in J_i^c} f_{ijt}^c \qquad \forall i \in I \tag{41}
$$

$$y_{it}^{tr} = \sum_{j \in J_i^{tr}} f_{ijt}^{tr} \qquad \forall i \in I \tag{42}$$

$$f_{ijt}^{c}, f_{ijt}^{tr} \geq 0 \qquad \forall i \in I,\ \forall j \in J_i^{m} \tag{43}$$

where

$$u_{at}^{c} = \mathrm{E}[C_{at}^{c}] + \gamma \mathrm{Var}[C_{at}^{c}], \text{ and} \tag{44}$$

$h_{it}^{-1}(\cdot)$ = inverse of h_{it}.

This relaxation problem seems to be an extension of the minimization problem given in Eqs. 3–6. It easily confirms that the Karush-Kuhn-Tucker condition of the above optimization problem is the complementarilty condition of Eq. 34 under the condition that $\mathbf{f}_t^{tr}$ is fixed.

As stated above, using the previous residual flows (or the travel time on the minimum expected disutility in the previous period), traffic equilibrium in each period is calculated in an ascending order. The algorithm to calculate $\boldsymbol{\omega}_t$ in period t is as follows:

Step 1. Initial solution

An initial solution $\boldsymbol{\omega}_t^1$ is given. Set iteration k = 1. (For example, at the initial solution, the residual flows or the travel time on the minimum expected disutility in the previous period are given and the function h_{it} is set. The minimum expected disutility is given under free-flow travel times, and $\mathbf{q}_t$ is provided. If t = 1, no residual flows are given. At the initial solution, no user between an OD pair chooses public transit, $\mathbf{f}_t^{tr,1} = \mathbf{0}$, and all automobile users between an OD pair choose the minimum expected disutility route under free-flow travel times. Then, $\mathbf{x}_t^{c,1}, \mathbf{f}_t^{tr,1}$, and $\mathbf{y}_t^1$ as well as $\mathbf{d}_t^1$ are all set.)

Step 2. Solve the relaxation problem

Step 2-1 Initial solution

Set initial solution $\boldsymbol{\omega}_t^{k,1} = \boldsymbol{\omega}_t^k$, and set n = 1.

Step 2-2 Initial solution

Calculate $u_a^{c,k,n}$ $(\forall a \in A)$, $\lambda_{it}^{c,k,n}$, and $\lambda_{it}^{tr,k}$ $(\forall i \in I)$, where $u_a^{c,k,n}$ is the automobile disutility on link a in sub-iteration n and iteration k. The fixed $\bar{u}_{ijt}^{tr,k-1}$ is used to solve the above relaxation problem during iteration k.

Step 2-3 Search direction

Calculate the search direction $\boldsymbol{\mu}^{k,n}$ whose components are as follows:

$$\boldsymbol{\mu}^{k,n} = \begin{pmatrix} \boldsymbol{\chi}_t^{k,n} - \mathbf{x}_t^{c,k,n} \\ \boldsymbol{\varphi}_t^{k,n} - \mathbf{f}_t^{tr,k,n} \\ \mathbf{h}_t(\boldsymbol{\lambda}^{k,n}) - \mathbf{q}_t^{k,n} \\ \boldsymbol{\psi}_t^{c} - \mathbf{y}_t^{c,k,n} \\ \boldsymbol{\psi}_t^{tr} - \mathbf{y}_t^{tr,k,n} \end{pmatrix}$$

where $\boldsymbol{\chi}^{k,n}$ denotes road link flows of the all-or-nothing assignment to the minimum disutility route based on $u_a^{c,k,n}$, $\boldsymbol{\varphi}^{k,n}$ denotes public transit route flows of all-or-nothing assignment to the minimum disutility route based on $\bar{u}_{ijt}^{tr,k,n}$, and $\boldsymbol{\psi}_t^{c}$ and $\boldsymbol{\psi}_t^{tr}$ are the vectors whose components are as follows:

$$\psi_{it}^{c,k,n} = \frac{h_{it}(\lambda_{it}^{tr,k}, \lambda_{it}^{c,k,n})}{1 + \exp\{-\theta(\lambda_{it}^{tr,k} - \lambda_{it}^{c,k,n})\}}$$

$$\psi_{it}^{tr,k,n} = h_{it}(\lambda_{it}^{tr,k}, \lambda_{it}^{c,k,n}) \left[1 - \frac{q_{it}^{k,n}}{1 + \exp\{-\theta(\lambda_{it}^{tr} - \lambda_{it}^{c})\}} \right].$$

<u>Step 2-4 Line Search</u>

Minimize $\hat{L}_t(\zeta) = \tilde{L}_t(\boldsymbol{\omega}_t^{k,n} + \zeta\, \boldsymbol{\mu}^{k,n})$ by a line search to obtain $\zeta\,(0 \le \zeta \le 1)$. Then, set $\boldsymbol{\omega}^{k,n+1} = \boldsymbol{\omega}^{k,n} + \zeta\, \boldsymbol{\mu}^{k,n}$ and $n = n + 1$.

<u>Step 2-5 Convergence verification</u>

If convergence is reached, go to Step 3; otherwise, go to Step 2-2.

<u>Step 3. Convergence verification</u>

If $\left| \boldsymbol{\omega}_t^{k} - \boldsymbol{\omega}_t^{k-1} \right| \le \varepsilon$, end; otherwise, calculate $\bar{u}_{ijt}^{tr,k}$, update $u_{at}^{c,k}(x_{at}^{c})$, that is, $\mathbf{f}_t^{tr}$ in $\tilde{u}_a^{c}(x_{at}^{c}, \mathbf{f}_t^{tr})$ is replaced by $\mathbf{f}_t^{tr,k}$, and set $k = k + 1$ and go to Step 2, where ε represents the convergence level.

5. Application of the Model

In this section, we apply the semi-dynamic user equilibrium model with mode and route choices under stochastic travel times to the Kanazawa transportation network, and examine the applicability of the proposed model. A plan to introduce an LRT line is currently being discussed in Kanazawa. The exact effect of introducing LRT will be examined and presented in a later study; the applicability of the model is the focus of this paper. Figure 1 shows the Kanazawa transportation network with roads, bus routes, and an LRT line. The network consists of 178 nodes and 489 links. The bus routes share roads with automobiles; however, the LRT has an exclusive railway. The public transit network in Kanazawa is not very dense, and only a single representative route is considered as a public transit route set. The duration of a period is set at 60 min, and three periods during the morning peak time are considered: period 1 is 6:00–7:00 AM, period 2 is 7:00–8:00 AM, and period 3 is 8:00–9:00 AM. The OD matrix is derived from a personal trip survey within the Kanazawa urban area that was conducted previously, and is shown in Table 1. For simplicity, the total access/egress time and waiting time for public transit is set to 10 min, and the public transit fare is set to 200 Japanese yen (flat rate). The value of time υ is set to 40 (Japanese yen/min). The parameter for the mean bus travel time in Eq. 27, ς, is set to 1.5. We assume that travel time variation is caused by link flow fluctuations, and that link flows are normally distributed. We observed various links on the network, and found that Var[X_a] = 42 E[X_a] represents the relationship between the mean and variance of link flows (Nakayama *et al.*, 2006). The mean and variance of link travel times are expressed as functions of the mean link flows because of Var[X_a] = 42 E[X_a]. E[$t_a(X_a)$] and Var[$t_a(X_a)$] are derived using a standard BPR-type cost-flow performance function $t_a = t_{a0}\,[1 + 0.15(x_a/\nu_a)^4]$ and Var[X_a] = 42 E[X_a], where t_{a0} denotes the free-flow travel time on link a and ν_a the capacity on link a, using the moment generating function of normal distribution (Papoulis, 1965). In the example, x_a in the above formulation is regarded as the mean link flow. Let T_a denote $t_a(X_a)$ and x_a denote E[X_a]. $T_a = t_a(X_a)$ and $X_a \sim$ N[x_a, $42x_a$], where N[x, y] is the normal distribution, the mean and variance of which are x and y, respectively. According to the method of Nakayama *et al.* (2006), the mean and variance of link a are derived as

$$\mathrm{E}[T_a] = t_{a0}\left(1 + \frac{0.15}{\nu_a^{\,4}}\left[x_a^{\,4} + 252x_a^{\,3} + 3\left(42x_a\right)^2\right]\right) \tag{45}$$

$$\mathrm{Var}[T_a] = 7.56\,t_{a0}^{\,2}\,\frac{x_a^{\,4}}{\nu_a^{\,8}}\left[42^2\left(576 + 48x_a\right) + 882x_a^{\,2} + 2x_a^{\,3}\right] \tag{46}$$

As described above, for simplicity, the link travel times are mutually independent. A computationally efficient consideration of the correlation between the link travel times is a subject for future study.

We do not have sufficient data to identify the risk attitude parameter γ in Eq. 32. We solved the model result with risk attitude parameter settings of 0.0, 1.0, and 2.0. The risk parameter of 0.0 represents risk neutral, while 1.0 and 2.0 represent risk averse. The convergence condition was $\max_m\left[\left|\omega_{mt}^{k} - \omega_{mt}^{k-1}\right| \big/ \omega_{mt}^{k-1}\right] \le 10^{-4}$, where ω_{mt}^{k} is the mth component of $\boldsymbol{\omega}_t^k$. The computational time for reaching the convergence condition calculated using an ordinary personal computer (CPU: Intel Core i7 2.80 GHz) is 6 min 44 sec in period 1; 9 min 11 sec in period 2; and 9 min 20 sec in period 3. Figure 2 illustrates the split rates of automobile users (the automobile split rate) for each risk attitude parameter in each period. The observed automobile split rate during the morning peak time in Kanazawa is 80.11%. As the attitude becomes more risk averse, the split rate of automobile users decreases in each period. This is because the LRT is punctual and has no variance of its travel time, and therefore, some automobile users switch to public transit. In period 1, travel demand is less than that in other periods, and the roads are not so congested. Therefore, the automobile split rate is the highest during this period with each risk attitude. Table 2 shows the mean disutilities of automobile and public transit for each risk attitude parameter in each period, where the mean is taken from among OD pairs. The mean disutilities of both automobile and public transit increase as the attitude becomes more risk averse, as seen from Eq. 32. The difference between the automobile and public transit users decreases as the risk attitude becomes more risk averse. Table 3 shows the total residual flows with each risk attitude and the total of all original OD demands in each period. The total residual flow represents the sum of residual flows among OD pairs. The residual flow rate in Table 3 is the total residual flow divided by the total OD demands. In period 1, travel demand is low and the residual flow, which is propagated to period 2, is small. The residual flows increase as the roads become congested. The residual flows in periods 2 and 3 are larger than those in period 1. Thus, the results show that the model proposed is applicable to an actual general transportation network.

6. Conclusions

Transportation network conditions vary greatly during the course of a day. In many urban areas, public transit and (private) automobiles constitute the actual forms of transportation. Public transportation by rail is more reliable than by automobiles or buses; therefore, ordinary static and deterministic traffic assignment models with combined mode and route choices may not be suitable to assess a transportation network when railway public transit is also present.

We adopted a semi-dynamic approach to consider within-day dynamics of a transportation network. The semi-dynamic models had the following characteristics: 1) the duration of a period is sufficiently long so that most demands reach their destination within the departure period; 2) static network equilibrium is reached in each period; 3) the residual flow is propagated to the next period; and 4) the ordinary static traffic equilibrium modeling and algorithm can basically be applied. In theory, a continuous-time dynamic traffic assignment model is preferable. However, a discrete-time dynamic traffic assignment model is more applicable for computation and is able to describe detailed and accurate traffic dynamics if sufficient dynamic OD data are provided. In reality, detailed and accurate dynamic OD data are rarely compiled. In such a case, a continuous- or discrete-time dynamic traffic assignment model is not necessarily efficient, that is, the description of traffic dynamics is too detailed and accurate for the detailedness or accuracy of the dynamic OD data. Thus, a simpler and more computationally reasonable approach is required. An appropriate model in such a case is the semi-dynamic approach.

In this paper, we introduced stochastic travel times to examine reliability of public transportation by railway. Stochastic travel times are incorporated into the semi-dynamic traffic assignment model. Hence, we formulated a semi-dynamic traffic assignment model with combined mode and route choices given stochastic travel times, and proposed a relaxation algorithm to compute the model solution.

We applied the model to the Kanazawa transportation network in three periods: 6:00–7:00 AM, 7:00–8:00 AM, and 8:00–9:00 AM. The computational time for each period was less than 10 min using the relaxation algorithm, and the model seems practically applicable—at least for the Kanazawa transportation network or for same size or smaller networks. The resultant automobile/public transit split rate is close to the observed value, and the model showed a measure of validity. Thus, the model facilitates transportation planning and formulation of policies.

In future work, departure time choice and correlation between link travel times should be incorporated. The estimation of risk attitude and variance of link travel time should also be further developed. A method of constructing a semi-dynamic OD matrix is another important issue.

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Figures:

Table list:

Table 1 Semi-dynamic OD matrix of Kanazawa transportation network

	Number of OD pairs [i)]	Travel demand [ii)]
6:00–7:00 AM	485	10445
7:00–8:00 AM	1985	74683
8:00–9:00 AM	1906	64530

i) The number of OD pairs whose demands are greater than 0.

ii) The total number of users for both automobile and public transit who depart within each period.

Table 2 Mean disutilities of automobile and public transit users in each period

	6:00–7:00AM			7:00–8:00AM			8:00–9:00AM		
Risk attitude	0.0	1.0	2.0	0.0	1.0	2.0	0.0	1.0	2.0
Auto	449.01	462.67	466.86	802.33	1127.14	1259.93	999.37	1334.95	1632.86
Public Transit	1003.21	1002.05	1002.01	1283.94	1535.40	1643.97	1449.11	1740.93	2025.92
Difference	554.20	539.38	535.15	481.61	408.25	384.04	449.74	405.98	393.06

Table 3 Total residual flow and original demand in each period

	6:00–7:00AM			7:00–8:00AM			8:00–9:00AM		
Risk attitude	0.0	1.0	2.0	0.0	1.0	2.0	0.0	1.0	2.0
Total residual flow	1,018	1,022	1,030	16,482	17,250	18,210	17,121	19,138	19,964
Residual flow rate	0.098	0.098	0.099	0.221	0.231	0.244	0.265	0.297	0.309
Total original OD demand	10,445			74,683			64,530		