Asymptotic mean square stability analysis for a stochastic delay differential equation

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Asymptotic Mean Square Stability Analysis for a Stochastic Delay Differential Equation

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Abstract: For a stochastic delay differential equation, the effects of noise and time delay are discussed in the sense of mean square stability. Neither time delay nor noise play bad roles for the differential equations and both of them are ubiquitous in nature. The so-called domain subdivision approach is taken to study the stability regions in terms of the parameters of a given equation and the Ito formula is employed to deal with the fluctuation noise. An interesting result demonstrated in this paper shows that noise with appropriate power could reduce the influence of time delay.

Keywords: Stochastic systems, Time delay, Stability.

1. INTRODUCTION

Delays and noises in feedback loops can be seen in real systems such as human-machine systems, biomedical systems, process control, remote control and robots. The retardation comes from transportation lags, and conduction or communication times, etc. [1]. Two famous models, the neural control of stick balancing at the fingertip and the study of the fluctuations in the center of pressure during quiet standing, were discussed by many researchers[2]. In general, time delay is known as a bad factor to the stability of the control system, but the study of human balancing shows that noise may help the stability of the control system. The challenge of this paper focuses on the relationship between noise and time delay, and the contribution of the noise to stabilizability in the sense of mean square stability.

In this paper, the proof of the main result of [4] is corrected. In [3] and [4], the authors showed that the influence of time delay can be reduced by noise for a scalar stochastic delay system with particular parameters. It means that the existence of noise may relax the stability condition. They use the so-called domain subdivision approach together with the Ito's formula to derive the stability regions in terms of the parameters of a given equation and to deal with the fluctuation noise. We also demonstrate a result which shows that appropriate noise power can reduce the influence of time delay and overpowered loses its stability.

2. TIME DELAY AND NOISE

Consider a scalar system

$$\dot{x}(t) = ax(t) + bu(t),\ a > 0,\ b > 0, \tag{1}$$

where $u(t)$ is the control input. It is well known that the system is stabilized by $u(t) = -px(t)$ with the proportional gain p if and only if $a - bp < 0$. It is also known that noise and delay disturb the stability condition. In this section, we review the effects of noise and delay, respectively.

Several sufficient conditions for exponentially p-th moment stable were given for stochastic delay differential equations[5]. Both delay and noise are taken as unkindly to the stability in the results of [5]. Following the results of [1], an end-fixed inverted pendulum system was studied in [6] and their result shows that the stochastic system can be stabilized by noise in almost surely stable. In the sense of mean square stable[7], the solution of a stochastic system will be discussed in section 3. Our result shows that noise still can reduce the influence of time delay even if it cannot help stability when there is no delay in the stochastic system in mean square asymptotically stable.

2.1 Influence of time delay

When we use time delayed feedback

$$u(t) = -px(t-\tau) \tag{2}$$

for the system (1), the resulting delayed system is given by

$$\begin{cases} \dot{x}(t) = ax(t) - bpx(t-\tau), \\ x(\theta) = \phi(\theta), \quad \theta \in [-\tau, 0], \end{cases} \tag{3}$$

where τ denotes the time delay, $\phi(\theta) : [-\tau, 0] \mapsto \mathbb{R}$ is a continuous function.

Definition 1: The equilibrium solution $x(t) \equiv 0$ of (3) is said to be **exponential stable** if there exist $\alpha > 0$ and $\beta > 0$ such that for all ϕ, the solution satisfies

$$\|x(t, \phi)\| \le \alpha \|\phi\| \exp(-\beta t). \tag{4}$$

It is well known that the equilibrium solution $x \equiv 0$ of (3) is exponentially stable if and only if all the infinitely many characteristic roots of the characteristic equation

$$\lambda = a - bp \exp(-\lambda\tau) \tag{5}$$

have negative real parts. Clearly, a pure imaginary characteristic root $\lambda = j\omega, \omega > 0$ encloses the stability region

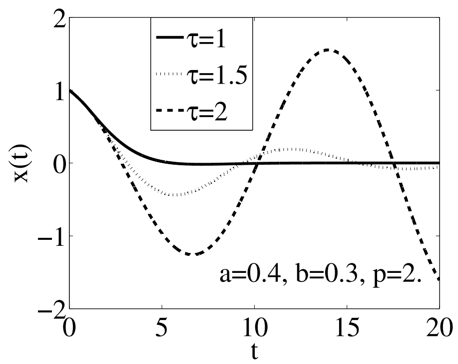

Fig. 1 Trajectories $x(t)$ of (3) for the initial state $x(t) = 1, -\tau \leq t \leq 0$.

in the parameter space. To derive the boundary, by substituting $\lambda = j\omega$ into (5), we have

$$a - bp\cos(\omega\tau) = 0,\ \ \omega - bp\sin(\omega\tau) = 0. \quad (6)$$

The stability boundaries (6) form ω-parameterized curves in the parameter plane (a, b) for the change of ω for fixed $\tau > 0$ and $p > 0$, which are derived from

$$\begin{cases} a = bp,\ b \in (0, +\infty) & \text{for } \omega = 0, \\ a = \omega\dfrac{\cos(\omega\tau)}{\sin(\omega\tau)},\ b = \dfrac{\omega}{p\sin(\omega\tau)} & \text{for } \omega > 0. \end{cases} \quad (7)$$

By analyzing the sign of $\frac{\mathrm{d}}{\mathrm{d}b}\mathrm{Re}\lambda\big|_{\lambda=\mathrm{j}\omega}$ together with the implicit differentiation of the characteristic function with respect to the parameters a and b in (7), we can check if some characteristic root crosses the imaginary axis from left to right (toward instability), or right to left (toward stability). The stability region shrinking with increasing of time delay as shown in Fig. 1. This technique is called the domain-subdivision method (also called the D-decomposition method)[8].

2.2 Effect of noise

When the feedback gain p is affected by noise, the system (1) is

$$u(t) = -(p + \xi(t))x(t), \quad (8)$$

where $\xi(t)$ denotes the Gaussian white noise, and at least formally, $\sigma\dot{w}(t) = \xi(t)$. Then, the resulting feedback system is

$$\mathrm{d}x(t) = (a - bp)x(t)\mathrm{d}t - \sigma bx(t)\mathrm{d}w(t). \quad (9)$$

Definition 2: The equilibrium solution $x(t) \equiv 0$ of (9) is said to be **almost surely stable (stable with probability 1)**, if

$$\Pr\left\{\lim_{\|x(0)\|\to 0}\sup_{t\geq 0}\|x(t; t_0, \phi)\| = 0\right\} = 1. \quad (10)$$

The solution for the initial state $x(0) = x_0$ is

$$x(t) = x_0\exp\left\{\left[a - bp - \frac{(\sigma b)^2}{2}\right]t - \sigma bw(t)\right\}. \quad (11)$$

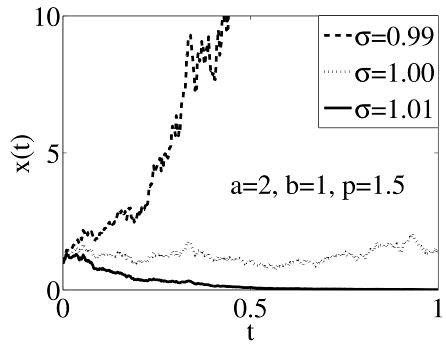

Fig. 2 Trajectories $x(t)$ of the stochastic system (9) for different σ.

Since (11) is equivalent to

$$\frac{\ln x(t) - \ln x_0}{t} = \left[a - bp - \frac{(\sigma b)^2}{2}\right] - b\sigma\frac{\omega(t)}{t}, \quad (12)$$

the stochastic system (9) is almost surely stable if and only if $a - bp - (\sigma b)^2/2 < 0$. Hence, even if $a - bp > 0$, (9) can be stabilized by noise, as shown in Fig. 2. A sufficient stability condition for a stochastic system with no delay has been given by Ushida[6].

On the other hand, by using the Ito's formula, we obtain

$$\begin{aligned}\mathrm{d}x^2(t) =& \big(2(a - bp) + (\sigma b)^2\big)x^2(t)\mathrm{d}t \\ & - 2\sigma bx^2(t)\mathrm{d}w(t).\end{aligned} \quad (13)$$

Furthermore, we have

$$\mathrm{E}\left\{x^2(t)\right\} = x_0^2\exp\left\{(2(a - bp) + (\sigma b)^2)t\right\}. \quad (14)$$

Hence, the stochastic system (9) is exponentially mean square stable if $a - bp + (\sigma b)^2/2 < 0$. This example shows that the system cannot be stabilized by noise in the sense of moment stability but almost sure stability.

3. STABILITY CONDITION

As we reviewed in the previous section, stability conditions have been investigated for time delay and noise individually. In this section, we discuss mean square stability of a stochastic delay feedback system.

The system (1) and control

$$u(t) = -(p + \xi(t))x(t - \tau) \quad (15)$$

results in a stochastic delay differential equation

$$\begin{aligned}\mathrm{d}x(t) =& \left[ax(t) - bpx(t - \tau)\right]\mathrm{d}t \\ & - \sigma bx(t - \tau)\mathrm{d}w(t).\end{aligned} \quad (16)$$

Definition 3: The equilibrium solution $x(t) \equiv 0$ of (16) is said to be the following.

1) **mean square stable**, if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\mathrm{E}\left\{\|x(t, \phi)\|^2\right\} < \epsilon, \quad (17)$$

whenever $\|\phi\| < \delta$.

2) **asymptotically mean square stable**, if it is mean square stable and for any initial function ϕ,

$$\lim_{t\to\infty} \mathrm{E}\left\{\|x(t,\phi)\|^2\right\} = 0. \tag{18}$$

Theorem 1: The system (16) is asymptotically mean square stable if $2a - 2bp + \sigma^2 b^2 \neq 0$ and all solutions of the characteristic equation

$$F(\lambda) := 2\lambda - 2a + 2bpe^{-\lambda\tau} - \sigma^2 b^2 e^{-2\lambda\tau} = 0 \tag{19}$$

satisfy Re $\lambda < 0$.

Proof: Using the Ito's formula, we obtain

$$\begin{aligned} \mathrm{d}x^2(t) = &\left[2ax^2(t) - 2bpx(t)x(t-\tau)\right. \\ &\left.+\sigma^2 b^2 x^2(t-\tau)\right]\mathrm{d}t \\ &- 2\sigma bx(t)x(t-\tau)\mathrm{d}w(t). \end{aligned} \tag{20}$$

To integrate (20) from 0 to t and take the expectation, since $\mathrm{E}\left\{\int_0^t 2\sigma bx(s)x(s-\tau)\mathrm{d}w(s)\right\} = 0$, we have

$$\begin{aligned} \mathrm{dE}\left\{x^2(t)\right\} = &\left[2a\mathrm{E}\left\{x^2(t)\right\} - 2bp\mathrm{E}\left\{x(t)x(t-\tau)\right\}\right. \\ &\left.+\sigma^2 b^2 \mathrm{E}\left\{x^2(t-\tau)\right\}\right]\mathrm{d}t. \end{aligned} \tag{21}$$

By differentiating (21), we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}\left\{x^2(t)\right\} = &2a\mathrm{E}\left\{x^2(t)\right\} - 2bp\mathrm{E}\left\{x(t)x(t-\tau)\right\} \\ &+ \sigma^2 b^2 \mathrm{E}\left\{x^2(t-\tau)\right\}. \end{aligned} \tag{22}$$

If (22) has a steady-state solution $K^* := \mathrm{E}\left\{x^2(t)\right\} = \mathrm{E}\left\{x(t)x(t-\tau)\right\} = \mathrm{E}\left\{x(t-\tau)x(t-\tau)\right\}$, K^* must satisfy

$$0 = 2aK^* - 2bpK^* + \sigma^2 b^2 K^*. \tag{23}$$

Hence, if $2a - 2bp + \sigma^2 b^2 \neq 0$, the steady-state solution must be zero, i.e., $K^* = 0$. With this condition and the existence of the steady-state solution K^*, the system (16) becomes mean square asymptotically stable, i.e.,

$$\lim_{t\to+\infty} \mathrm{E}\left\{x^2(t)\right\} = K^* = 0. \tag{24}$$

Next, when we assume that there exist nontrivial solutions of the form $\mathrm{E}\left\{x(t)x(s)\right\} = ce^{\lambda(t+s)}$ with a constant c, we obtain the characteristic function (19) from (22). ■

Remark 1: When $\tau = 0$, the characteristic equation (19) has a single solution $\lambda = a - bp + \frac{\sigma^2}{2}b^2$. Hence, if

$$a < pb - \frac{\sigma^2}{2}b^2, \tag{25}$$

the system (16) is asymptotically mean square stable when $\tau = 0$.

Theorem 2: All solutions of the characteristic function (19) satisfy Re $\lambda < 0$ if the following conditions are held.

$$p\sqrt{\tau} > 2\sigma, \tag{26}$$

$$(a, b) \in \Omega_0 \cap \Omega_\omega \tag{27}$$

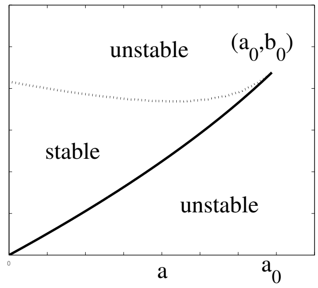

Fig. 3 The stability region of the intersection of (28) and (29). The solid line (35) and the dotted line (30)(31).

where

$$\Omega_0 = \left\{(a,b) \mid 0 < a < pb - \frac{\sigma^2}{2}b^2\right\} \tag{28}$$

$$\Omega_\omega = \left\{(a(\omega), b) \mid b < b(\omega) \text{ for } \omega \in [0, \omega_0/\tau)\right\} \tag{29}$$

where

$$a(\omega) = \omega\cot(2\omega\tau) + \frac{p}{4\sigma^2}\frac{p - Q(\omega)}{\cos^2(\omega\tau),} \tag{30}$$

$$b(\omega) = \frac{1}{2\sigma^2}\frac{p - Q(\omega)}{\cos(\omega\tau),} \tag{31}$$

$$Q(\omega) = \sqrt{p^2 - 4\sigma^2\omega\cot(\omega\tau)}, \tag{32}$$

and ω_0 is a minimal positive solution of

$$4\omega\sigma^2\cos^2(\omega\tau)\cot(2\omega\tau) + p(p - Q(\omega)) = 0. \tag{33}$$

The stability region given by Theorem 2 is depicted in Fig. 3.

Proof: When the stability is violated by changing parameters a, b, and so on, there exists at least one solution λ of (19) on the imaginary axis. That is, there exists $\omega \geq 0$ satisfying $F(j\omega) = 0$. To separate the real and imaginary parts of $F(j\omega) = 0$, we obtain

$$\begin{cases} \sigma^2 b^2 \sin(2\omega\tau) - 2bp\sin(\omega\tau) + 2\omega = 0 \\ 2a - 2bp\cos(\omega\tau) + \sigma^2 b^2\cos(2\omega\tau) = 0. \end{cases} \tag{34}$$

Then, the pair $(a(\omega), b(\omega))$ satisfying (34) provides the boundary between the stable region and the unstable one.

When (34) has a solution $\omega = 0$, it is equivalent to

$$a = pb - \frac{\sigma^2}{2}b^2,\ b \in (0, +\infty). \tag{35}$$

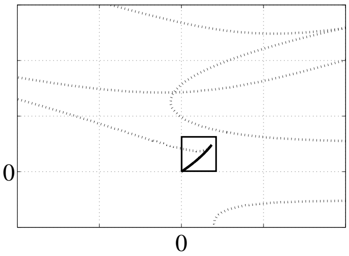

Fig. 4 Boundaries of domain subdivision formed by (34). The central rectangle represents Fig. 3.

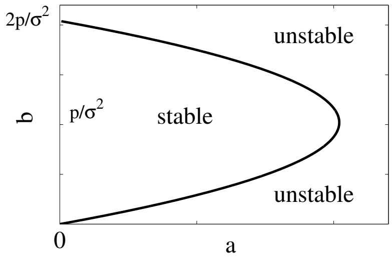

Fig. 5 The stability region Ω_0 (28) for $\omega = 0$.

Hence, if the characteristic function (19) does not have the solution $\lambda = 0$, (a, b) never satisfy (35). In addition, (35) separates two regions Re $\lambda < 0$ and Re $\lambda > 0$ in the complex plane. To determine which direction corresponds to Re $\lambda < 0$, we will check the sign of $\mathrm{Re}\frac{\mathrm{d}\lambda}{\mathrm{d}a}\Big|_{\lambda=0}$. To do so, by differentiating (19) by a, we obtain

$$\frac{\mathrm{d}\lambda}{\mathrm{d}a} - 1 - bp\tau e^{-\lambda\tau}\frac{\mathrm{d}\lambda}{\mathrm{d}a} + \sigma^2 b^2 \tau e^{-2\lambda\tau}\frac{\mathrm{d}\lambda}{\mathrm{d}a} = 0. \quad (36)$$

From this, we have

$$\mathrm{Re}\frac{\mathrm{d}\lambda}{\mathrm{d}a} = \mathrm{Re}\frac{1}{1 - bp\tau e^{-\lambda\tau} + \sigma^2 b^2 \tau e^{-2\lambda\tau}}. \quad (37)$$

Since under (26)

$$1 - p\tau b + \sigma^2\tau b^2 \geq 1 - \frac{p^2}{4\sigma^2}\tau > 0, \quad (38)$$

we obtain

$$\mathrm{Re}\frac{\mathrm{d}\lambda}{\mathrm{d}a}\Big|_{\lambda=0} = \frac{1}{\sigma^2\tau b^2 - p\tau b + 1} > 0. \quad (39)$$

Hence, we can conclude that (25) specifies the region Re $\lambda < 0$. Together with $a > 0$, the set Ω_0 (28) will provide a condition for (a, b) so that the characteristic function (19) has solutions Re $\lambda < 0$.

Next, we consider case where (34) has a solution $\lambda = j\omega$ with $\omega \in (0, \frac{\pi}{2\tau}), (\frac{\pi}{2\tau}, \frac{\pi}{\tau}), (\frac{\pi}{\tau}, \frac{3\pi}{2\tau}), \cdots$. Then, by solving (34) with respect to a and b, we have

$$\begin{aligned} b &= \frac{p\sin(\omega\tau) \pm \sqrt{p^2\sin^2(\omega\tau) - 2\omega\sigma^2\sin(2\omega\tau)}}{\sigma^2\sin(2\omega\tau)} \\ &= \frac{p + Q'(\omega)}{2\sigma^2\cos(\omega\tau)} \quad (40) \\ a &= bp\cos(\omega\tau) - \frac{bp\sin(\omega\tau) - \omega}{\sin(2\omega\tau)}\cos(2\omega\tau) \\ &= bp\cos(\omega\tau) - \frac{bp\cos(2\omega\tau)}{2\cos(\omega\tau)} + \omega\cot(2\omega\tau) \\ &= bp\left(\cos(\omega\tau) - \frac{2\cos^2(\omega\tau) - 1}{2\cos(\omega\tau)}\right) + \omega\cot(2\omega\tau) \\ &= p\frac{p + Q'(\omega)}{4\sigma^2\cos^2(\omega\tau)} + \omega\cot(2\omega\tau), \quad (41) \end{aligned}$$

where

$$Q' = \begin{cases} \sqrt{p^2 - 4\omega\sigma^2\cot(\omega\tau)} & (42) \\ -\sqrt{p^2 - 4\omega\sigma^2\cot(\omega\tau)} & (43) \end{cases}$$

The pair (a, b) satisfying (40) and (41) forms the stability boundaries. However, the boundary formed by (a, b) satisfying (40) and (41) with Q' (42) does not intersect Ω_0. Therefore, we ignore (42) hereafter. Hence, we consider (a, b) with (43), that is, we have (30) and (31). Here, when we regard a as a function of ω, (30) is a monotonic decreasing function. Hence, there exists a solution ω_0 satisfying

$$0 = \omega\cot(2\omega\tau) + \frac{p}{4\sigma^2}\frac{p - Q(\omega)}{\cos^2(\omega\tau)}, \quad (44)$$

equivalently, (33). Then, $a > 0$ for $\omega < \omega_0/\tau$ and $a \leq 0$ for $\omega \geq \omega_0/\tau$. Hence, we do not need consider the solution $\lambda = j\omega$ for $\omega \geq \omega_0/\tau$.

Since $\lim_{\omega\to 0}\omega\cot(\omega\tau) = \tau^{-1}$, letting $\omega \to 0$ in (30), we obtain the limit value of a and b as

$$a_0 = \frac{1}{2\tau} + \frac{p}{4\sigma^2}\left(p - \sqrt{p^2 - 4\sigma^2\tau^{-1}}\right), \quad (45)$$

$$b_0 = \frac{1}{2\sigma^2}\left(p - \sqrt{p^2 - 4\sigma^2\tau^{-1}}\right). \quad (46)$$

Since they also satisfy (35), we can regard (a_0, b_0) as an intersection of (35) and (30). (46) gives the upper bound of a.

When the characteristic function (19) have no solution $\lambda = j\omega$ for $\omega \in (0, \omega_0/\tau]$, a and b have different value form (30) and (31) for $\omega \in (0, \omega_0/\tau]$. In addition, the curve formed by (30) and (31) as ω changes in the interval $\omega \in (0, \omega_0/\tau]$ separates two regions Re $\lambda < 0$ and Re $\lambda > 0$ in the complex plane. To determine which direction corresponds to Re $\lambda < 0$, we will check the sign of $\mathrm{Re}\frac{\mathrm{d}\lambda}{\mathrm{d}b}\Big|_{\lambda=0}$. To do so, by differentiating (19) by b, we obtain

$$\begin{aligned} &\frac{\mathrm{d}\lambda}{\mathrm{d}b} + p\tau e^{-\lambda\tau} - bp\tau e^{-\lambda\tau}\frac{\mathrm{d}\lambda}{\mathrm{d}b} \\ &\quad - \sigma^2 b\tau e^{-2\lambda\tau} + \sigma^2 b^2\tau e^{-2\lambda\tau}\frac{\mathrm{d}\lambda}{\mathrm{d}b} = 0. \quad (47) \end{aligned}$$

From the above, we have

$$\mathrm{Re}\frac{\mathrm{d}\lambda}{\mathrm{d}b} = \mathrm{Re}\frac{-pe^{-\lambda\tau} + \sigma^2 b e^{-2\lambda\tau}}{1 - bp\tau e^{-\lambda\tau} + \sigma^2 b^2\tau e^{-2\lambda\tau}}. \quad (48)$$

By substituting $\lambda = j\omega$ into (48), we obtain

$$\begin{aligned} \mathrm{Re}\frac{\mathrm{d}\lambda}{\mathrm{d}b}\Big|_{\lambda=j\omega} &= \mathrm{Re}\frac{-pe^{-\omega\tau j} + \sigma^2 b e^{-2\omega\tau j}}{1 - bp\tau e^{-\omega\tau j} + \sigma^2 b^2\tau e^{-2\omega\tau j}} \\ &= \mathrm{Re}\frac{n_r(\omega) + jn_i(\omega)}{d_r(\omega) + jd_i(\omega)} \quad (49) \end{aligned}$$

where

$$\begin{aligned} n_r(\omega) &= -p\cos(\omega\tau) + \sigma^2 b\cos(2\omega\tau) \\ n_i(\omega) &= p\sin(\omega\tau) - \sigma^2 b\sin(2\omega\tau) \\ d_r(\omega) &= 1 - bp\tau\cos(\omega\tau) + \sigma^2 b^2\tau\cos(2\omega\tau) \\ d_i(\omega) &= bp\tau\sin(\omega\tau) - \sigma^2 b^2\tau\sin(2\omega\tau). \quad (50) \end{aligned}$$

Furthermore, we have

$$\mathrm{Re}\frac{\mathrm{d}\lambda}{\mathrm{d}b}\bigg|_{\lambda=j\omega} = \frac{f(\omega)}{d_r^2(\omega)+d_i^2(\omega),} \tag{51}$$

where

$$\begin{aligned} f(\omega) =& n_r(\omega)d_r(\omega) + n_i(\omega)d_i(\omega) \\ =& \sigma^2 b\cos(2\omega\tau) - p(1+2\sigma^2 b^2\tau)\cos(\omega\tau) \\ & + (\sigma^4 b^2 + p^2)\tau b. \end{aligned} \tag{52}$$

Since $f(\omega) > 0$ due to Lemma 1 in Appendix,

$$\mathrm{Re}\frac{\mathrm{d}\lambda}{\mathrm{d}b}\bigg|_{\lambda=j\omega} > 0. \tag{53}$$

Hence, we conclude that $b < b(\omega)$ specifies the region Re $\lambda < 0$. The set Ω_ω (29) will provide a condition for (a, b) so that the characteristic function (19) has solutions Re $\lambda < 0$. ■

4. NUMERICAL EXAMPLES

To illustrate our result, we will apply the Euler-Maruyama method [9] to the stochastic delay differential equation (20) with an initial function $\phi = 1, -\tau \le t \le 0$, and parameters $a = 0.4, b = 0.3, p = 2, \tau = 2, \sigma = 1.4$. These parameters satisfy conditions in Theorem 2 as follows. $2a - 2bp + \sigma^2 b^2 = -0.2236 \neq 0$, and $p\sqrt{\tau} = 2.8284 > 2\sigma = 2.8$, and satisfies the conditions in Theorem 2 as

$$0 < a = 0.4 < a_0 = 0.8633, \tag{54}$$

$$0.2248 < b = 0.3 < 0.3694. \tag{55}$$

The Euler-Maruyama method gives us an approximation of true solution of (20) as the Markov chain $x^2(t_j)$ defined by

$$\begin{aligned} & x^2(t_j) \\ =& x^2(t_{j-1}) + \big[ax^2(t_{j-1}) - bpx(t_{j-1})x(t_{j-m}) \\ & + \sigma^2 b^2 x^2(t_{j-m})\big]\,\delta - 2\sigma b x(t_{j-1})x(t_{j-m})\Delta w(t_j) \end{aligned} \tag{56}$$

where $\delta := t_j - t_{j-1}$ is the small time interval, $m = \tau/\delta$ and $\Delta w(t_j) = w(t_j) - w(t_{j-1}) = \sqrt{\delta}N(0, 1)$ which is generated from a discretized Brownian path. A typical simulation result by the method is shown in Fig. 6. Although the system (20) is unstable when $\sigma = 0$, $x^2(t)$ approaches to 0 when $\sigma = 1.4$. However, when the power of noise exceeds the critical value $\sigma_{\mathrm{cr}} = \frac{p}{2}\sqrt{\tau}$, the stochastic system will be unstable. The critical value is derived from the equation $bp - \frac{\sigma^2}{2}b^2 - a = -\frac{3p}{4\sigma^2}\sqrt{p^2 - 4\sigma^2\tau^{-1}} = 0$ which is obtained by substituting (30) into (35), and letting $\omega \to 0$. In the example, the critical value is $\sigma_{\mathrm{cr}} = \sqrt{2} \approx 1.414$. When σ exceeds the critical value, the boundaries cannot be determined.

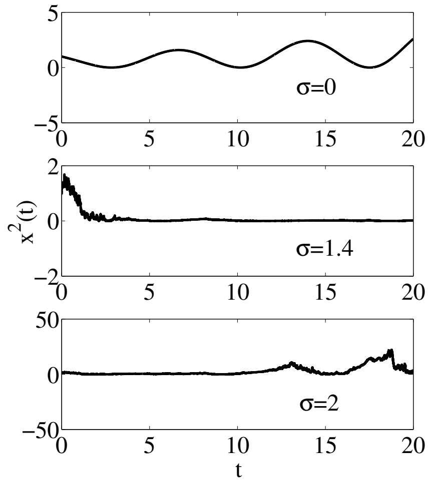

Fig. 6 Trajectories $x^2(t)$ of (56) for different σ.

5. CONCLUSION

In this paper, we have shown that noise with appropriate power may enhance the stability of a time delayed system. Our results show that the delayed systems can be stabilized by noise if the power of noise is appropriate.

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APPENDIX

Lemma 1: When $0 < \omega\tau < \omega_0\tau < \pi/2$,

$$f(\omega) > 0 \text{ for } b(\omega) = \frac{1}{2\sigma^2}\frac{p - Q(\omega)}{\cos(\omega\tau)}. \tag{57}$$

Proof: We introduce a new variable $y = \omega\tau$. Then,

$$\begin{aligned} f(y) =& \sigma^2(2\cos^2 y - 1)b - p(1 + 2\sigma^2 b^2\tau)\cos y \\ & + \sigma^4\tau b^3 + \tau p^2 b, \end{aligned} \tag{58}$$

$$b(y) = \frac{1}{2\sigma^2}\frac{p - Q(y)}{\cos(y)}, \tag{59}$$

$$Q(y) = \sqrt{p^2 - 4\sigma^2\omega\cot(y)/\tau}, \tag{60}$$

By differentiating $f(y)$ to y with $b(y)$, we obtain

$$\begin{aligned} \frac{\mathrm{d}f(y)}{\mathrm{d}y} =& \sigma^2(2\cos^2 y - 1)\frac{\mathrm{d}b}{\mathrm{d}y} - 4\sigma^2 b\cos y\sin y \\ & + p(1 + 2\sigma^2\tau b^2)\sin y - 4\sigma^2 p\tau b\cos y\frac{\mathrm{d}b}{\mathrm{d}y} \\ & + 3\sigma^4\tau b^2\frac{\mathrm{d}b}{\mathrm{d}y} + \tau p^2\frac{\mathrm{d}b}{\mathrm{d}y}, \end{aligned} \tag{61}$$

where

$$\frac{\mathrm{d}b(y)}{\mathrm{d}y} = \frac{1}{\tau Q(y)}\frac{\sin y\cos y - y}{\sin^2 y\cos y} + \frac{p - Q(y)}{\cos^2 y}\frac{\sin y}{2\sigma^2}. \tag{62}$$

Note the condition (26), we derive

$$\begin{aligned}
&\frac{\mathrm{d}f(y)}{\mathrm{d}y}2Q(y)\tau\sigma^2\sin^3 y\cos^4 y\\
=&4\sigma^4\sin^2 y\cos^6 y+2\sigma^2p^2\tau\sin^4 y\cos^4 y\\
&+2\sigma^4\sin^2 y\cos^4 y+4\sigma^2p^2\tau\sin^4 y\cos^2 y\\
&+4p^2\tau^2\sin^4 y\cos^2 y+\sigma^2p^2\tau\sin^2 y\cos^4 y\\
&+5\sigma^4y^2\cos^2 y+4\sigma^2p^2\tau y\sin^5 y\cos y\\
&+8\sigma^2p^2\tau y\sin^3 y\cos y-4\sigma^4 y\sin y\cos^5 y\\
&-8\sigma^4 y\sin^3 y\cos^5 y-4\sigma^4 y\sin^3 y\cos^3 y\\
&-12\sigma^4y^2\sin^2 y\cos^2 y-8\sigma^4 y\sin y\cos^3 y\\
&+4\sigma^2\tau pQ(y)\sin^2 y\cos^4 y+3\sigma^2\tau pQ(y)y\sin y\cos y\\
&+4\sigma^2\tau pQ(y)y\sin^3 y\cos^3 y+3\tau^2p^3Q(y)\sin^4 y\\
&-\sigma^2\tau pQ(y)\sin^4 y\cos^2 y-4\sigma^2\tau pQ(y)y\sin y\cos^4 y\\
&-3\sigma^2\tau pQ(y)\sin^2 y\cos^2 y-\tau^2p^3Q(y)\sin^4 y\cos^2 y\\
&-9\sigma^2\tau pQ(y)y\sin^3 y\cos y\\
>&\sigma^4\cos(y)H_1(y)+p\tau Q(y)\sin(y)H_2(y), \qquad (63)
\end{aligned}$$

where

$$\begin{aligned}
H_1(y)=&4\sin^2 y\cos^5 y+8\sin^4 y\cos^3 y+6\sin^2 y\cos^3 y\\
&+16\sin^4 y\cos y+5y^2\cos y+16y\sin^5 y\\
&+32y\sin^3 y-4y\sin y\cos^4 y-8y\sin^3 y\cos^4 y\\
&-4y\sin^3 y\cos^2 y-12y^2\sin^2 y\cos y\\
&-8y\sin y\cos^2 y, \qquad (64)\\
H_2(y)=&4\sin y\cos^4 y+3y\cos y+4y\sin^2 y\cos^3 y\\
&+8\sin^3 y+\sin^5 y\cos^2 y-\sin^3 y\cos^2 y\\
&-4y\cos^4 y-3\sin y\cos^2 y-9y\sin^2 y\cos y. \qquad (65)
\end{aligned}$$

When $0<y<\omega\tau<\pi/2$, $\sin y>0$, $\cos y>0$, $H_1(y)>0$ and $H_2(y)>0$ (we also show that $H_1(y)>0$ and $H_1(y)>0$ in Fig. 7). Hence

$$\frac{\mathrm{d}f(y)}{\mathrm{d}y}>0. \qquad (66)$$

Using (46) and (52), we have

$$\lim_{\omega\to 0} f(\omega)=(\sigma^2b_0-p)(\sigma^2\tau b_0^2-p\tau b_0+1)=0. \qquad (67)$$

Hence, $f(\omega)>(67)=0$. ■

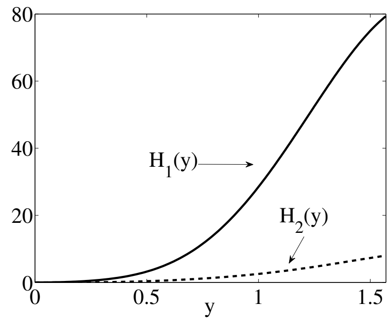

Fig. 7 $H_1(y)>0, H_2(y)>0$ for $y\in(0,\pi/2)$.

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