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A Design Method for Cascade Form Digital Nyquist Filters with Zero Intersymbol Interference

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SUMMARY

In data communication systems, a Nyquist waveform shaping filter is an important element. The Nyquist filter not only restricts the bandwidth of the data signal, but also realizes zero intersymbol interference where the time response intersects zero at equal intervals except at one point. This paper discusses a design method of a digital Nyquist waveform shaping filter for sampled signals. In particular, the method proposes realization of overall zero intersymbol interference when an identical transfer function is used for both the transmitter and receiver filters. In this method, the transfer function coefficients are used as approximate variables. They are divided into coefficient x_t for time response approximation and x_f that approximates the frequency response. The condition for zero intersymbol interference is given as the linear equation for x_t. Therefore, the approximation variable x_t for realization of zero intersymbol interference is obtained from solution of the linear equation. On the other hand, the frequency response is optimized with the iterative approximation due to the observation that the relation between x_f and the approximation function is nonlinear. The proposed method is applicable to both the FIR filter and the IIR filter for which the conventional method is not applicable. Further, the effect of the quantization errors in the multiplier coefficients and in the internal signal on the intersymbol interference is analyzed, and an estimation formula is obtained.

1. Introduction

In data communication, the Nyquist waveform shaping filter (hereafter referred to as Nyquist filter) is important [1, 2]. This filter realizes zero intersymbol interference for which the time response crosses zero at equal intervals except at one point. This filter also restricts the data signal bandwidth. With recent advances in digital electronics, digital Nyquist filter development has also been studied [3 - 5]. Several design methods have been proposed to realize zero intersymbol interference for both the FIR (Finite Impulse Response) filter and the IIR (Infinite Impulse Response) filter [3, 6, 7].

On the other hand, in data communication systems, transmitter filters are used for limiting the transmitted data bandwidth, and receiver filters are used for removing the unwanted wave components generated by channel noise and demodulation. There are two ways pertinent to designing Nyquist filters: either the transmitter filter or the receiver filter is fixed and the other is used for waveform shaping; both filters are used for waveform shaping. For the latter, CCITT recommended a method using identical filters for both the transmitter and receiver filters in systems such as 4800 bps modems [8]. In this paper, the design method is described for these cascaded Nyquist filters.

There are several design methods reported for cascaded Nyquist filters realizing zero intersymbol interference in FIR filters [9, 10]. However, no extensive reports have been made on IIR filters. This paper describes a design method valid for both FIR and IIR filters for realizing zero intersymbol interference [11].

2. Linear Equations for Zero Intersymbol Interference in Cascaded Nyquist Filters

2.1 FIR filter

In general, the transfer function of the FIR filter is given by

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$$H(z)=\sum_{n=0}^{N} h_n z^{-n}\ , \qquad z=e^{j2\pi f/f_s} \tag{1}$$

where f_s is the sampling frequency. In the discussions below, f_s = 1 Hz except in section 5. Let the cascaded form $H^2(z)$ of $H(z)$ be

$$H^2(z)=\sum_{n=0}^{2N} h_n^* z^{-n} \tag{2}$$

Then h_n^* is expressed as the autocorrelation of h_n:

$$h_n^* = \sum_{m=0}^{n} h_m h_{n-m} \tag{3}$$

If K designates the sampling point at which the time response is maximum, the equally spaced zero crossing condition except at one point in h_n^* is

$$h_{K+iM}^* = 0\ , \quad i = \pm 1\ ,\ \pm 2\ ,\ \cdots \tag{4a}$$

$$M = f_s / 2 f_N \quad : \text{integer} \tag{4b}$$

where f_N is the Nyquist frequency [1]. This paper treats only the case where M is an integer. The condition in Eq. (4) becomes in general a simultaneous quadratic equation system for h_n. However, it is possible to transform Eq. (4) into linear equation system by assuming a sub-set of the coefficient set $\{h_n\}$ as the variable and the remainder as constants. For the existence of this subset of coefficients, the following lemma holds:

Lemma 1.

$$h_n^* \neq 0\ , \quad n \neq K + iM\ , \quad i = \pm 1\ ,\ \pm 2 \tag{5}$$

When the sub-set of the coefficients for transforming Eq. (4) into linear equations is x_t,

(1) $((K))_M = 0$ or $((2N-K))_M = 0$ (6)

x_t does not exist

(2) $((K))_M \neq 0$ and $((2N-K))_M \neq 0$ (7)

(i) $M = 2$ x_t does not exist

(ii) $M = 3$

(a) $((K))_M = 1$ and $((2N-K))_M = 1$

K = even number x_t exists

K = odd number x_t does not exist for an arbitrary N

(b) $((K))_M \neq 1$ or $((2N-K))_M \neq 1$ (9)

x_t exists.

(iii) $4 \leq M$ x_t exists

where $((\ ))_M$ is the residual operation with modulus M. A proof of Lemma 1 is given in Appendix 1.

2.2 IIR filter

Let $H(z)$ be the transfer function for an IIR filter. Then the necessary and sufficient condition for its impulse response to cross zero at equal intervals, except at one point, becomes available [7]. Hence, in this paper, the zero intersymbol interference condition is formulated so that the transfer function $H^2(z)$ for the cascaded transmitter and receiver satisfies this condition. Considering the necessary and sufficient condition given in [7], $H(z)$ is expressed in direct form as follows:

$$H(z)=\frac{\sum_{n=0}^{N_n} a_n z^{-n}}{1+\sum_{n=1}^{N_d} b_n z^{-nM}} \tag{10}$$

Further, let

$$H^2(z)=\frac{\sum_{n=0}^{2N_n} a_n^* z^{-n}}{1+\sum_{n=1}^{2N_d} b_n^* z^{-n}} \tag{11}$$

Then

$$a_n^* = \sum_{m=0}^{n} a_m a_{n-m} \tag{12a}$$

$$b_n^* = \sum_{m=1}^{n-1} b_m b_{n-m}\ , \quad 1 \leq n \tag{12b}$$

The condition for zero intersymbol interference for the impulse response for a filter with the transfer function $H^2(z)$ is

$$a_{K-iM}^* = 0\ , \quad i = 1,\ 2,\ \cdots \tag{13a}$$

$$a_K^* \neq 0 \tag{13b}$$

$$a_{K+iM}^* = a_K^* b_i^*\ , \quad i = 1,\ 2,\ \cdots \tag{13c}$$

From Eq. (12), Eq. (13) becomes, in general, a simultaneous fourth-order equation system for a_n and b_n. In a manner similar to the FIR filter, it is possible to select a subset of the coefficients x_t that transforms the equations to a linear system. For this process, the following lemma holds:

Lemma 2.

(1) $K + 2N_d M < 2N_n$

For $N = N_n$ and $h_n = a_n$, Lemma 1 holds.

(2) $K + 2N_d M \geqq 2N_n$

For $N = N_n$ and $h_n = a_n$, Lemma 1 holds. Where the conditions $((2N-K))_M \neq 0$ in (1), $(2N-K))_M \neq 0$, $((2N-K))_M = 1$ and $((2N-K))_M \neq 1$ in (2) in Lemma 1 can be removed. An abridged proof is given in Appendix 3.

3. Approximation Method

3.1 Algorithm

As described in the previous section, the approximation for the time response is carried out through solving linear equations with the appropriate approximation variable x_t. Hence, the method proposed in [12] is basically applicable as a simultaneous approximation in the frequency and time domains. The approximate algorithm proposed in this paper is summarized in the following.

(1) Since, in general, the relation between the frequency response and the approximation variable x_f is nonlinear, the frequency response is approximated through the iterative Chebyshev approximation method [13].

(2) The time domain approximation is to obtain x_t through solving the linear equations with constants x_f obtained at each step of the iterative approximation described above. By evaluating the frequency response with x_f and x_t obtained above, the condition for zero intersymbol interference is always satisfied in the frequency response approximation process.

The process in (2) can be expressed mathematically as follows. First, the transfer function $H(z)$ can be expressed in general with x_t and x_f as

$$H(z,\ x_t,\ x_f) \tag{14}$$

If we let the linear equation with variable x_t

$$A x_t = c \tag{15}$$

then the elements of the matrix A and the vector c are made of the elements of x_f. From Eq. (15), let

$$x_t = A^{-1} c \tag{16}$$

Then, the transfer function of Eq. (14) can be expressed with an approximation variable x_f only as in

$$H(z,\ A^{-1}c,\ x_f) \tag{17}$$

Therefore, use of the function form (17) automatically satisfies the condition for zero intersymbol interference.

3.2 Flow chart

Figure 1 shows an approximation flow chart.

(1) Estimate initial guess for $H(z, x_t, x_f)$

This paper uses a method for approximating the ideal response in the time domain. This is because the calculation is complicated in the frequency domain where amplitude and phase must be approximated.

Let the transfer function $H_i(z)$ have an amplitude response interpolated with a cosine function in the transition band and a linear phase response:

$$|H_i(e^{j\omega})| = 1, \qquad 0 \leqq \omega \leqq \omega_1 \tag{18a}$$

$$= \frac{1}{2}\left\{1 + \cos\pi\left(\frac{\omega-\omega_1}{\omega_2-\omega_1}\right)\right\}, \ \omega_1 < \omega < \omega_2 \tag{18b}$$

$$= 0, \qquad \omega_2 \leqq \omega \leqq \pi \tag{18c}$$

Further, the rolloff rate is defined by

$$\rho = (\omega_2 - \omega_1)/4\pi f_N \tag{19}$$

If the impulse response for the filter with the transfer function $\sqrt{H_i(z)}$ is $\hat{h}_n$, this is an ideal impulse response for the filter with the transfer function $H(z)$. For the approximation of the initial value, the conventional time domain approximation methods can be used [14].

(2) Iterative Chebyshev approximation

The approximation problem in the frequency domain can be formulated as follows:

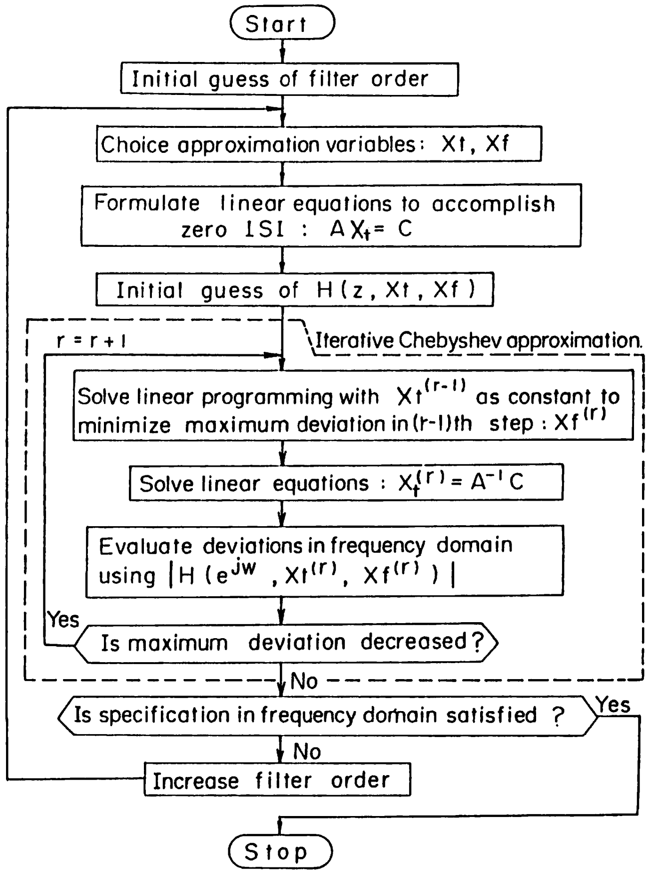


Fig. 1. Approximation flow chart.

Minimize δ in

$$\left| \frac{H(e^{j\omega}, A^{-1}c, x_f)}{H(e^{j0}, A^{-1}c, x_f)} \right| < \delta, \quad \omega_2 \leqq \omega \leqq \pi \qquad (20)$$

The local linear programming with the first derivative is used for obtaining $x_f^{(r)}$ minimizing the maximum deviation at the $(r-1)$-th step [13]. In this case, x_t is treated as a constant in terms of $x_t^{(r-1)}$ obtained in the $(r-1)$-th step. The frequency response at the r-th step can be evaluated from the transfer function $H(z, x_t^{(r)}, x_f^{(r)})$ using $x_f^{(r)}$ and $x_t^{(r)}$ obtained through solving the linear equations with coefficients $x_t^{(r)}$.

Since the denominator coefficient of the transfer function is contained in x_t, the denominator function expression in the iterative approximation is arbitrary. Therefore, by the expression of the product of the first- or second-order factors in the denominator function, an iterative approximation is always possible while stability is checked.

4. Analysis of Intersymbol Interference Due to Quantization Error

In the actual operation of the digital filter, quantization errors are generated in the coefficients and the internal operations. An equation is derived that estimates the intersymbol interference caused by the quantization errors. The direct form is chosen as the $H(z)$ circuit configuration.

4.1 Intersymbol interference due to quantization errors in the coefficient

Let

$$G(z) = H^2(z) \qquad (21)$$

and $H(z)$ is expressed as

$$H(z) = P(z)/Q(z) \qquad (22)$$

where $P(z)$ and $Q(z)$ are polynomials in z^{-1}. It is assumed that errors $\Delta G(z)$, $\Delta P(z)$, and $\Delta Q(z)$ are generated in $G(z)$, $P(z)$, and $Q(z)$ due to quantizing the coefficients of $P(z)$ and $Q(z)$:

$$G(z)+\Delta G(z)=\left(\frac{P(z)+\Delta P(z)}{Q(z)+\Delta Q(z)}\right)^2 \quad (23)$$

In the region where the quantization error of the coefficients is sufficiently small, $\Delta G(z)$ in Eq. (23) can be expressed approximately by

$$\Delta G(z) \fallingdotseq \frac{2}{Q(z)}(H(z)\Delta P(z)-G(z)\Delta Q(z)) \quad (24)$$

Intersymbol interference due to quantization of the coefficients is defined as

$$\mathrm{ISI}(Q^c) \triangleq \left\{ \sum_{\substack{n=0 \\ \neq n_0}}^{\infty} \Delta g_{i+nM}^2 \right\}^{1/2} \Big/ g_K \quad (25a)$$

$$i+n_0 M=K, \quad 0 \leqq i \leqq M-1 \quad (25b)$$

Here, g_n and Δg_n are the impulse responses for filters with transfer functions $G(z)$ and $\Delta G(z)$. Next, $\Delta G(z)$ is broken down to the following parallel configuration of the functions with low sampling frequency:

$$\Delta G(z)=\sum_{i=0}^{M-1} z^{-i} \Delta G_i(z^M) \quad (26)$$

Since in Eq. (24), auto-correlations for the coefficients of $\Delta P(z)$ and $\Delta Q(z)$ are considered in general to be very small, their amplitude responses become flat [15]. Hence, the $\Delta G(z)$ amplitude response can be determined by $H(z)/Q(z)$ and $G(z)/Q(z)$. These are almost zero at

$$\pi/M \leqq \omega \leqq \pi \quad (27)$$

from the band-limiting condition of the Nyquist filter. Since $\Delta G(z)$ also has the band-limiting response, the following relation holds [16]:

$$|\Delta G_i(e^{jM\omega})|=\frac{1}{M}|\Delta G(e^{j\omega})|, \quad 0 \leqq \omega \leqq \frac{\pi}{M} \quad (28)$$

On the other hand, Parseval's relation holds between the impulse response Δg_{i+nM} of the filter with transfer function $\Delta G_i(z^M)$ and the latter itself [17]. Based on these factors,

$$\sum_{n=0}^{\infty} \Delta g_{i+nM}^2=\frac{M}{2\pi}\int_{-\pi}^{\pi}\frac{1}{M^2}|\Delta G(e^{j\omega})|^2 d\omega \quad (29)$$

From Eqs. (25) and (29)

$$\mathrm{ISI}(Q^c)=\left\{\frac{1}{M}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}|\Delta G(e^{j\omega})|^2 d\omega\right\}^{1/2} \Big/ g_K \quad (30)$$

Since the coefficient distributions of $\Delta P(z)$ and $\Delta Q(z)$ are considered unrelated to each other, we obtain

$$|\Delta G(e^{j\omega})|^2 \fallingdotseq \frac{\Delta_c^2}{3}\left\{(N_n+1)\left|\frac{H(e^{j\omega})}{Q(e^{j\omega})}\right|^2+N_d\left|\frac{G(e^{j\omega})}{Q(e^{j\omega})}\right|^2\right\} \quad (31)$$

where $\Delta_c = 2^{-t_c}$ and t_c is the number of bits below the decimal point. From Eqs. (30) and (31),

$$\mathrm{ISI}(Q^c) \fallingdotseq \left\{M\frac{\Delta_c^2}{3}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[(N_n+1)\left|\frac{H(e^{j\omega})}{Q(e^{j\omega})}\right|^2 + N_d\left|\frac{G(e^{j\omega})}{Q(e^{j\omega})}\right|^2\right]d\omega\right\}^{1/2} \quad (32)$$

where the following relation is used:

$$g_K \fallingdotseq 1/M \quad (33)$$

In Eq. (32) the amplitude responses for $H(z)$ and $G(z)$ are determined only by the rolloff rate and are considered fixed for a given rolloff rate. Hence, the parameters to determine $\mathrm{ISI}(Q^c)$ are orders N_n and N_d for the transfer functions and the denominator function $Q(z)$.

In the case of FIR filters, let $Q(z) = 1$ and $N_d = 0$ in Eq. (32). Further, considering

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|H(e^{j\omega})|^2 d\omega \fallingdotseq \frac{1}{M} \quad (34)$$

we obtain a simple estimation equation

$$\mathrm{ISI}(Q^c) \fallingdotseq \sqrt{\frac{N_n+1}{3}}\cdot\Delta_c \quad (35)$$

4.2 Intersymbol interference due to quantization of internal signals

The quantization error in the internal

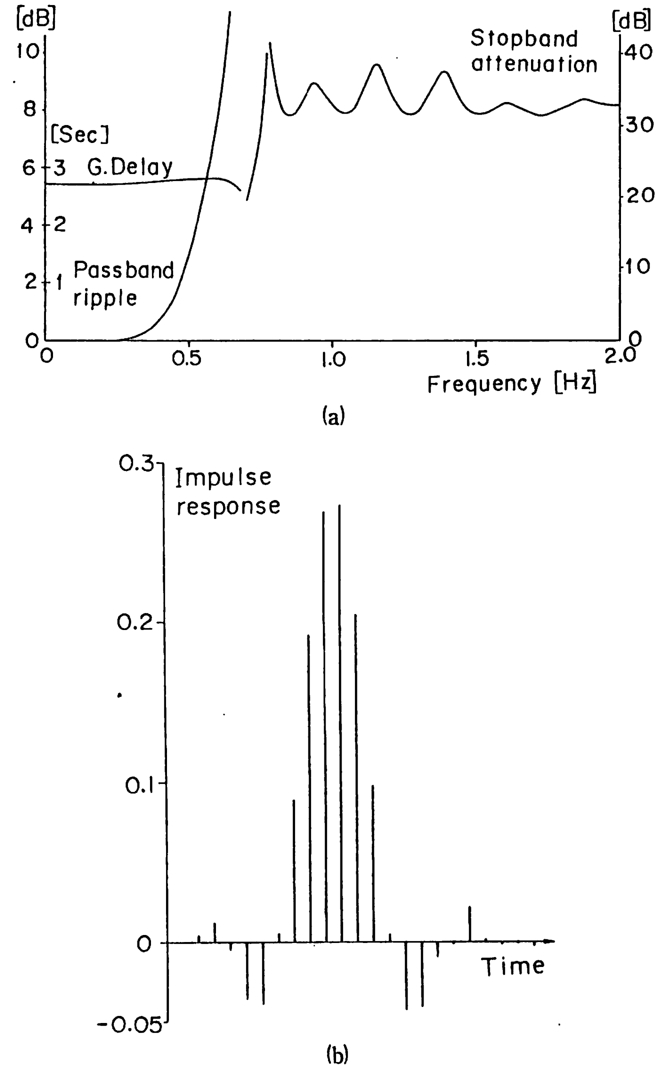


Fig. 2. (a) $H(z)$ frequency responses for $H(z)$. (b) impulse response: h_n.

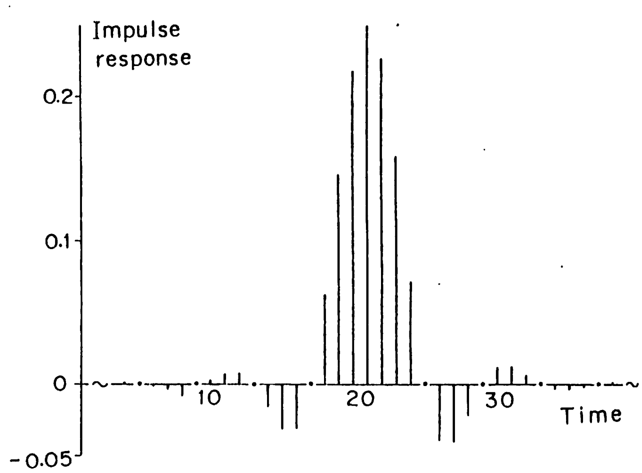


Fig. 3. Impulse response: h_n^*.

signals, or the intersymbol interference ISI(Q^d) caused by rounding off any error generated after multiplication can be obtained in a manner similar to ISI(Q^c) as follows:

$$\mathrm{ISI}(Q^d) \fallingdotseq \left\{ M \frac{\Delta_d^2}{12} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} [S^{-2} N_d |G(e^{j\omega})|^2 + (N_n + 1 + S^{-2} N_d) |H(e^{j\omega})|^2 + (N_n + 1)] \, d\omega \right\}^{1/2} \quad (36)$$

Scaling factor S is given by

$$S = \min_{\omega} \{ |Q(e^{j\omega})| \} \quad (37)$$

5. Design Examples

Design example 1.

Circuit structure	FIR filter
Sampling frequency (f_s)	4 Hz
Nyquist frequency (f_N)	0.5 Hz, $M = f_s / 2f_N = 4$
Rolloff rate (ρ)	50%
Order (N_n)	23rd

Considering Lemma 1, waveform center K was chosen at 21. Further, under the conditions in Eq. (5) the approximation variable x_t that transforms the zero intersymbol interference condition to the linear equation and the approximation variable x_f made of remaining coefficients are chosen as

$$x_t = (h_1, h_2, h_6, h_9, h_{10}, h_{13}, h_{14}, h_{17}, h_{18}, h_{21})$$

$$x_f = (h_0, h_3, h_4, h_5, h_7, h_8, h_{11}, h_{12}, h_{15}, h_{16}, h_{19}, h_{20}, h_{22}, h_{23})$$

Figure 2(a) shows the frequency response for $H(z)$ and Fig. 2(b) shows the impulse response h_n. Figure 3 shows the impulse response h_n^* corresponding to $H^2(z)$.

From Fig. 3, it was found that zero intersymbol interference is realized in the impulse response h_n^*. Although the passband amplitude response and the group delay characteristic are obtained as a result of time domain approximation, they are close to ideal responses. Figure 4 shows the ISI(Q^c)

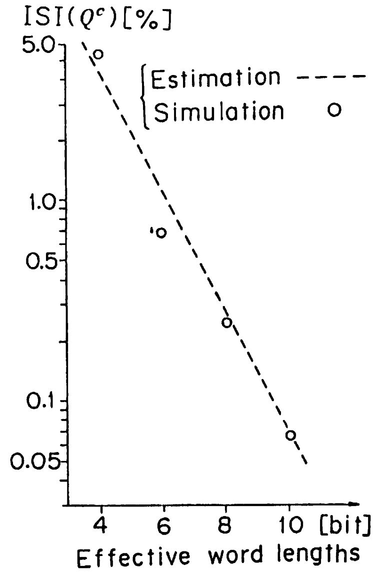


Fig. 4. Estimation and simulation for ISI(Q^c).

vs. the coefficient word length. Here, the effective word length is the number of bits below the decimal point, when the maximum value of the coefficient is normalized to unity. The dotted line is obtained from the estimation equation (35) whereas o's are found by an actual simulation. The results confirm that the estimation equation (35) is effective.

Design example 2.

Circuit structure IIR filter

$$H(z) = \sum_{n=0}^{15} a_n z^{-n} / (1 + b_1 z^{-4}), \quad z = e^{j2\pi f/4} \quad (38)$$

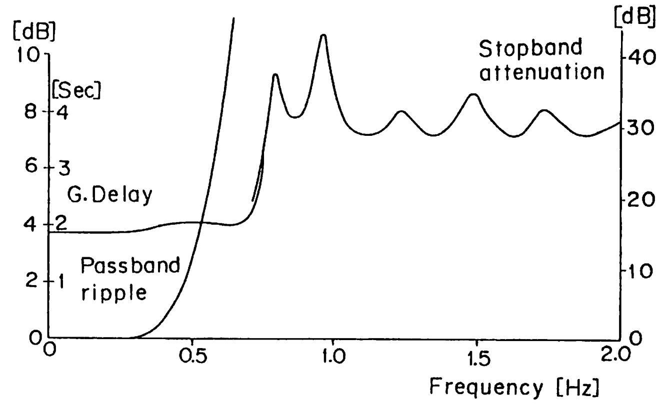
Other conditions are identical to those for Design example 1. The approximate variables x_t and x_f are chosen as follows:

$$x_t = (a_1, a_3, a_5, a_7, a_9, a_{13})$$

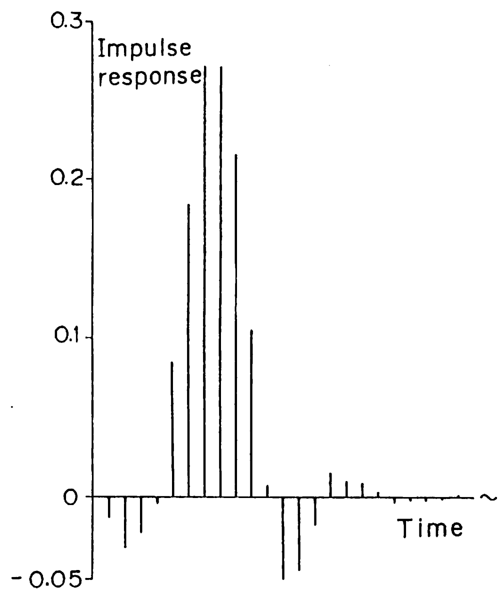
$$x_f = (a_0, a_2, a_4, a_6, a_8, a_{10}, a_{11}, a_{12}, a_{14}, a_{15}, b_1)$$

Design results are shown in Figs. 5 to 7.

Although the impulse response h_n^* in Fig. 6 is somewhat asymmetric, the zero intersymbol interference condition is satisfied. The stopband attenuations in Fig. 2(a) and Fig. 5(a) are almost equal. It is clear that the desired frequency response is realized with



(a)



(b)

Fig. 5. (a) Frequency responses for $H(z)$. (b) Impulse response: h_n.

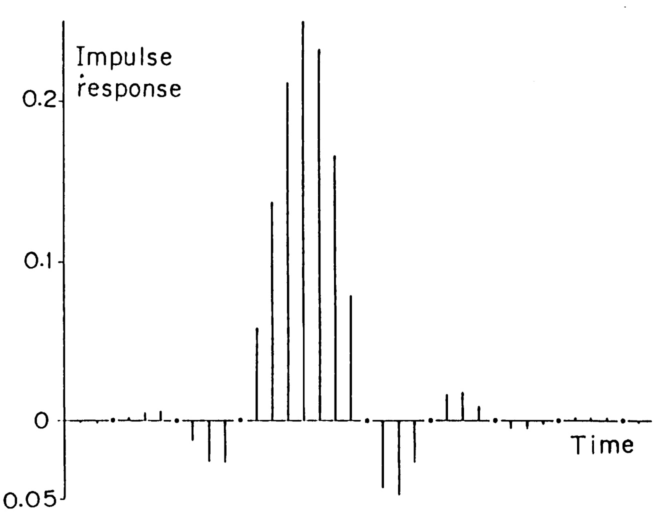


Fig. 6. Impulse response: h_n^*.

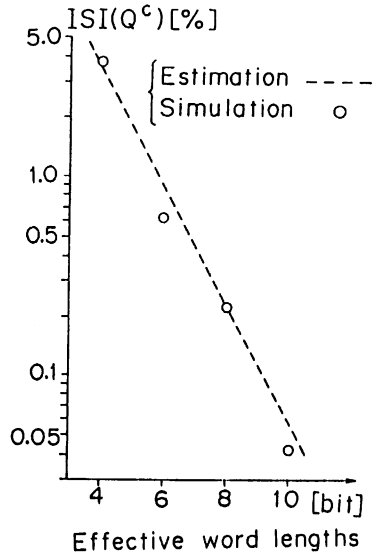


Fig. 7. Estimation and simulation for ISI(Q^c).

an IIR transfer function of lower order than an FIR structure.

6. Conclusions

A method is reported for designing a cascaded Nyquist filter with zero intersymbol interference, particularly when the same transfer function is used in both the transmitter and receiver. The proposed method is applicable to both IIR and FIR filters.

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APPENDIX

1. Proof of Lemma 1

(1) From Eq. (6)

$$h_0 = 0 \quad \text{or} \quad h_N = 0 \tag{A1}$$

From this,

$$h_1^* = 0 \quad \text{or} \quad h_{N-1}^* = 0 \tag{A2}$$

Hence, the condition (5) is not satisfied.

(2) - (i) From Eq. (7), for $M = 2$,

$$((K))_M = 1 \quad \text{and} \quad ((2N-K))_M = 1 \tag{A3}$$

holds. Further, considering Eq. (4) and

$$h_0^* \neq 0 \quad \text{and} \quad h_{2N}^* \neq 0 \tag{A4}$$

we have

$$\left.\begin{aligned} h_n = 0,\ n &= K - iM,\ i = 1,\ 2,\ \cdots,\ \left[\frac{K}{M}\right] \\ &= N + 1 - iM,\ i = 1,\ 2,\ \cdots,\ \left[\frac{2N-K}{M}\right] \end{aligned}\right\} \tag{A5}$$

where for a real R, $[R]$ is the maximum integer not exceeding R. On the other hand,

$$h_K^* = \sum_{n=0}^{K} h_n h_{K-n} \tag{A6}$$

From Eq. (A5) for an odd N, only if

$$K = N \tag{A7}$$

we have

$$h_K^* = h_0 h_N \neq 0 \tag{A8}$$

However, in this case from Eq. (A5)

$$h_{K\pm 1}^* = 0 \tag{A9}$$

holds, and the condition in Eq. (5) is not satisfied. For other K and even N, h_K^* becomes zero from Eq. (A5) and the condition in Eq. (5) is not satisfied. From the above, x_t does not exist for $M = 2$.

(2) - (ii)

Four sub-sets are defined here.

$\{h_n\}_1$: This consists of h_1 and h_{N-1} being uniquely zero when $((K))_M = 1$ and $((2N-K))_M = 1$

$\{h_n\}_2$: This consists of h_n values that are not mutually related via product in the expressions of h_{K+iM} ($i = 0, \pm 1, \pm 2, \ldots$), however, $\{h_n\}_1$ is not contained.

$\{h_n\}_3$: This consists of $h_{K/2}$ obtained from $h_K^* \neq 0$ for even K.

$\{\tilde{h}_n\}$: The sum of $\{h_n\}_1$, $\{h_n\}_2$ and $\{h_n\}_3$.

(a) Each sub-set can be set as follows:

$$\left.\begin{aligned} \{h_n\}_1 &= (h_1,\ h_{N-1}) \\ \{h_n\}_2 &= (h_0,\ h_3,\ \cdots,\ h_{\left(\left[\frac{N}{3}\right]-1\right)3}) \\ \{h_n\}_3 &= (h_{K/2}), \quad K = \text{even} \\ &= \text{empty}, \quad K = \text{odd} \end{aligned}\right\} \tag{A10}$$

$\{h_n\}_2$ in Eq. (A10) is not unique, but the maximum number of elements in $\{h_n\}_2$ is $[N/3]$. If the number of elements in $\{\tilde{h}_n\}$ is $\mathcal{N}[\{\tilde{h}_n\}]$, then from Eq. (A10)

$$\left.\begin{aligned} \mathcal{N}[\{\tilde{h}_n\}] &= \left[\frac{N}{3}\right] + 3, \quad K = \text{even} \\ &= \left[\frac{N}{3}\right] + 2, \quad K = \text{odd} \end{aligned}\right\} \tag{A11}$$

On the other hand, the number of equations in Eq. (4a) is

$$\mathcal{N}[EQ] = \frac{2N-2}{3} \tag{A12}$$

Therefore, a sufficient condition for selecting the elements of $\{\tilde{h}_n\}$ for x_t is

$$\mathcal{N}[\{\tilde{h}_n\}] \geqq N[EQ] \tag{A13}$$

For N satisfying Eq. (8) the equality of Eq. (A13) holds for K = even. There exists an N for which Eq. (A13) is not satisfied for odd K (e.g., $M = 3$, $K = 7$, $N = 7$).

(b) For an odd K, each sub-set is set as follows. First the condition in Eq. (9) is divided into Eqs. (A14) and (A15):

$$((K))_M = 1 \quad \text{or} \quad ((2N-K))_M = 1 \tag{A14}$$

$$((K))_M \neq 1 \quad \text{and} \quad ((2N-K))_M \neq 1 \tag{A15}$$

Individual sub-sets are

$$\left.\begin{array}{l}\{h_n\}_1 = (h_1 \text{ or } h_{N-1}),\ \text{empty} \\ \{h_n\}_2 = \left(h_0,\ h_3,\ \cdots,\ h_{\left[\frac{N}{3}\right]_3}\right) \\ \{h_n\}_3 = \text{empty}\end{array}\right\} \quad \text{(A16)}$$

The first and second terms in $\{h_n\}_1$ in Eq. (A16) correspond to Eqs. (A14) and (A15). Then the number of elements in $\{\tilde{h}_n\}$ and the number of equations in Eq. (4a) are

$$\mathcal{N}(\{\tilde{h}_n\}) = \left[\frac{N}{3}\right]+2,\quad \left[\frac{N}{3}\right]+1 \qquad \text{(A17)}$$

$$\mathcal{N}(EQ) = \frac{2N-3}{3},\quad \frac{2N-4}{3} \qquad \text{(A18)}$$

For N satisfying Eq. (9), Eq. (A13) holds. Since for K = even $h_{K/2}$ can be added to the elements of $\{\tilde{h}_n\}$, the condition in Eq. (A13) can also be satisfied.

(2) - (iii)

Individual sub-sets are chosen as follows:

$$\left.\begin{array}{l}\{h_n\}_1 = (h_1,\ h_{N-1}) \\ \qquad \text{where, } ((K))_M = 1,\ ((2N-K))_M = 1 \\ \{h_n\}_2 = \left(h_0,\ h_M,\ \cdots,\ h_{\left[\frac{N}{M}\right]_M}\right) \\ \{h_n\}_3 = (h_{K/2}),\ K:\text{even},\ = \text{empty},\ K:\text{odd}\end{array}\right\} \quad \text{(A19)}$$

The general formulas for the number of elements in $\{\tilde{h}_n\}$ and the number of equations in Eq. (4a) can be given for each combination of N, M, and K. Although not all are described here, it can easily be shown that each one satisfies the condition in Eq. (A13). (End of proof.)

2. Examples for x_t Selection in an FIR Filter

An example is shown here for N = 11, M = 4, and K = 9. The conditions for Eqs. (4) and (5) are given by

$$h_1^* = 2h_0h_1 = 0 \qquad \text{(A20a)}$$

$$h_5^* = 2(h_0h_5 + h_1h_4 + h_2h_3) = 0 \qquad \text{(A20b)}$$

$$h_{13}^* = 2(h_2h_{11} + h_3h_{10} + h_4h_9 + h_5h_8 + h_6h_7) = 0 \qquad \text{(A20c)}$$

$$h_{17}^* = 2(h_6h_{11} + h_7h_{10} + h_8h_9) = 0 \qquad \text{(A20d)}$$

$$h_{21}^* = 2h_{10}h_{11} = 0 \qquad \text{(A20e)}$$

$$h_j^* \neq 0,\quad 0 \leqq j \leqq 22 \text{ and } j \neq 9 \pm 4,\ \pm 8,\ +12 \qquad \text{(A21)}$$

From Eqs. (A20a), (A20e), and (A21), $h_1 = h_{10} = 0$ and they become the elements of $\{h_n\}_1$; $\{h_n\}_2$ consists of coefficients not included in the same product in the expressions of Eq. (A20) and h_9^*. For instance, it is possible to choose

$$\{h_n\}_2 = (h_0,\ h_3,\ h_7,\ h_{11}),\ (h_2,\ h_5,\ h_6,\ h_9) \qquad \text{(A22)}$$

Since K is odd, $\{h_n\}_3$ becomes empty; x_t can be constructed from three elements selected from $\{h_n\}_1$ in addition to $\{h_n\}_2$. For instance,

$$x_t = (h_0,\ h_1,\ h_3,\ h_7,\ h_{10}),\ (h_1,\ h_2,\ h_5,\ h_6,\ h_{10}) \qquad \text{(A23)}$$

Difficulty in choosing x_t is determined by M as discussed in Lemma 1 and is independent from the order of the filter; x_t choice is always possible for an M larger than 4. However, further study is needed in regard to choosing x_t effective for amplitude response approximation.

3. Abridged Proof of Lemma 2

Since $b_i^2 (i = 1, 2, \ldots, N_d)$ is used in Eq. (13c), it is not possible to select the denominator coefficients as the x_t elements. If the numerator coefficients a_n and N_n are regarded as h_n and N in Lemma 1, the subsequent proof can easily be derived from that of Lemma 1. However, if Eq. (13c) is considered, the condition for $((2N-K))_M$ can be removed under the condition of (2). (End of proof.)