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著者	Kanjin Yuichi
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HARDY'S INEQUALITIES FOR HERMITE AND LAGUERRE EXPANSIONS

Dedicated to Professor Satoru Igari on his 60th birthday

YUICHI KANJIN

ABSTRACT

The well-known inequality of Hardy for Fourier coefficients of functions $f(t) \sim \sum_{n=-\infty}^{\infty} b_n e^{int}$ in the real Hardy space is $\sum_{n=-\infty}^{\infty} |b_n|/(|n|+1) < \infty$. We shall establish analogues of this inequality for the Hermite function expansions and also for the Laguerre function expansions.

1. *Introduction*

Hardy's inequality says that if $f(t) \sim \sum_{n=-\infty}^{\infty} b_n e^{int}$ is in $\operatorname{Re} H^1$, then

$$\sum_{n=-\infty}^{\infty} \frac{|b_n|}{|n|+1} < \infty,$$

where $\operatorname{Re} H^1$ is the real Hardy space consisting of the boundary values of the real parts of functions in the Hardy space H^1 on the unit disk in the plane. The aim of this paper is to establish analogues of this inequality for the Hermite function expansions and also for the Laguerre function expansions.

Let $\mathscr{H}_n(x)$ be the Hermite function defined by

$$\mathscr{H}_n(x) = \{\pi^{1/2} 2^n \Gamma(n+1)\}^{-1/2} H_n(x)\, e^{-x^2/2},$$

where $H_n(x)$ is the Hermite polynomial of degree n given by

$$H_n(x) = (-1)^n \exp(x^2)(d/dx)^n \exp(-x^2).$$

Then the system $\{\mathscr{H}_n\}_{n=0}^{\infty}$ is complete orthonormal on the real line $\mathbb{R}$ with respect to the ordinary Lebesgue measure dx (compare [**6**, 5.7]). This system leads to the formal expansion of a function $f(x)$ on $\mathbb{R}$:

$$f(x) \sim \sum_{n=0}^{\infty} c_n(f)\, \mathscr{H}_n(x),$$

where $c_n(f) = \int_{-\infty}^{\infty} f(x)\, \mathscr{H}_n(x)\, dx$ is the nth Hermite–Fourier coefficient of $f(x)$.

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Let $\mathscr{L}_n^\alpha(x)$, $\alpha > -1$, be the Laguerre function defined by

$$\mathscr{L}_n^\alpha(x) = \left\{\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}\right\}^{1/2} L_n^\alpha(x)\, e^{-x/2}\, x^{\alpha/2},$$

where $L_n^\alpha(x) = (n!)^{-1} x^{-\alpha} e^x (d/dx)^n \{x^{n+\alpha} e^{-x}\}$ is the Laguerre polynomial of degree n and of order α. Then $\{\mathscr{L}_n^\alpha\}_{n=0}^\infty$ is a complete orthonormal system on the interval $(0, \infty)$ with respect to dx (compare [**6**, 5.7]). We have the formal expansion

$$g(x) \sim \sum_{n=0}^{\infty} c_n^\alpha(g)\, \mathscr{L}_n^\alpha(x)$$

of a function $g(x)$ on $(0, \infty)$, where $c_n^\alpha(g) = \int_0^\infty g(x)\, \mathscr{L}_n^\alpha(x)\, dx$ is the nth Laguerre–Fourier coefficient.

Let $H^1(\mathbb{R})$ be the real Hardy space consisting of the boundary values of the real parts of functions in the Hardy space $H^1(\mathbb{R}_+^2)$ on the upper half plane $\mathbb{R}_+^2$. In $H^1(\mathbb{R})$, we consider the norm induced by $H^1(\mathbb{R}_+^2)$. For a function $g(x)$ on $(0, \infty)$, we define the extension $\tilde{g}(x)$ of $g(x)$ to $\mathbb{R}$ by $\tilde{g}(x) = g(x)$ if $0 < x$, and $\tilde{g}(x) = 0$ if $x \leqslant 0$. We define

$$H^1(0, \infty) = \{g(x) \text{ on } (0, \infty);\ \tilde{g}(x) \in H^1(\mathbb{R})\},$$
$$\|g\|_{H^1(0,\infty)} = \|\tilde{g}\|_{H^1(\mathbb{R})}.$$

Our main results of this paper are as follows.

THEOREM. (i) *There exists a constant C such that*

$$\sum_{n=0}^{\infty} \frac{|c_n(f)|}{(n+1)^{29/36}} \leqslant C \|f\|_{H^1(\mathbb{R})} \tag{1}$$

for $f(x) \sim \sum_{n=0}^\infty c_n(f)\, \mathscr{H}_n(x)$ in $H^1(\mathbb{R})$.

(ii) *Let $\alpha \geqslant 0$. Then there exists a constant C such that*

$$\sum_{n=0}^{\infty} \frac{|c_n^\alpha(g)|}{n+1} \leqslant C \|g\|_{H^1(0,\infty)} \tag{2}$$

for $g(x) \sim \sum_{n=0}^\infty c_n^\alpha(g)\, \mathscr{L}_n^\alpha(x)$ in $H^1(0, \infty)$.

The proof of the Theorem will be given in the next section. The atomic decomposition characterization of Hardy spaces will play an essential role in the proof. Also, the transplantation theorem (see (9) below) of R. Askey [**1**, p. 401, line 14] for Laguerre coefficients will be of use in the proof of part (ii) of the Theorem. The deduction of Paley's theorem from the Theorem will be discussed in Section 3 with a remark.

For a historical survey on Hardy's inequality and Paley's theorem, we may refer to [**2**, p. 398, Comments]. The exposition of the role of the atomic decomposition characterization in the classial harmonic analysis is in R. R. Coifman and G. Weiss [**3**]. We shall also consult J. Garcia-Cuerva and J. L. Rubio de Francia [**4**] for the Hardy space theory.

2. *Proof of the Theorem*

An H^1 atom is a function $a(x)$, $x \in \mathbb{R}$, supported in an interval $(b, b+h)$ satisfying $|a(x)| \leqslant h^{-1}$ and $\int a(x)\, dx = 0$. The space $H^1(\mathbb{R})$ is characterized in terms of atoms (compare [**4**, Chapter 3]): $f(x)$ belongs to $H^1(\mathbb{R})$ if and only if $f(x) = \sum_{j=0}^\infty \lambda_j a_j(x)$, where every $a_j(x)$ is an H^1 atom and $\sum_{j=0}^{\infty}|\lambda_j| < \infty$. The $H^1(\mathbb{R})$ norm of $f(x)$ is equivalent to $\inf\sum_{j=0}^{\infty}|\lambda_j|$, the infimum being taken over all decompositions. The space $H^1(0,\infty)$ is also characterized as follows [**4**, Lemma 7.40]: $g(x)$ belongs to $H^1(0,\infty)$ if and only if $g(x) = \sum_{j=0}^{\infty}\lambda_j a_j(x)$, where every $a_j(x)$ is an H^1 atom satisfying $\operatorname{supp} a(x) \subset (0,\infty)$ and $\sum_{j=0}^{\infty}|\lambda_j| < \infty$.

We shall first give a lemma. Let $\{\phi_n\}_{n=0}^{\infty}$ be an orthonormal system on an interval (c,d). For a function $f(x)$ on (c,d), we consider the formal expansion

$$f(x) \sim \sum_{n=0}^{\infty} b_n(f)\,\phi_n(x),$$

where $b_n(f) = \int_c^b f(x)\,\phi_n(x)\,dx$ is the nth Fourier coefficient with respect to the system $\{\phi_n\}$.

LEMMA. *Suppose that*

$$\max_{c<x<d}\left|\frac{d}{dx}\phi_n(x)\right| \leqslant Kn^{\delta}, \quad \delta > -\tfrac{1}{2},$$

with a constant K independent of n. Then there exist constants C_1 and C_2 satisfying the following.

(i) *For an H^1 atom $a(x)$ with* $\operatorname{supp} a(x) \subset (c,d)$,

$$|b_n(a)| \leqslant C_1 n^{\delta}\|a\|_2^{-2} \tag{3}$$

and

$$\sum_{n=0}^{\infty}\frac{|b_n(a)|}{(n+1)^{(\delta+2)/3}} \leqslant C_2. \tag{4}$$

(ii) *Let $f(x)$ be a function of the form $f(x) = \sum_{j=0}^{\infty}\lambda_j a_j(x)$, where every $a_j(x)$ is an H^1 atom with* $\operatorname{supp} a(x) \subset (c,d)$ *and $\sum_{j=0}^{\infty}|\lambda_j| < \infty$. Then*

$$\sum_{n=0}^{\infty}\frac{|b_n(f)|}{(n+1)^{(\delta+2)/3}} \leqslant C_2\sum_{j=0}^{\infty}|\lambda_j|. \tag{5}$$

Proof. Let $a(x)$ be an H^1 atom with support contained in $(b, b+h) \subset (c,d)$ such that $|a(x)| \leqslant h^{-1}$. Since

$$b_n(a) = \int_c^d a(x)\,\phi_n(x)\,dx - \phi_n(b)\int_c^d a(x)\,dx = \int_b^{b+h} a(x)\,\{\phi_n(x)-\phi_n(b)\}\,dx,$$

it follows from Schwarz's inequality and the mean value theorem that

$$\begin{aligned}|b_n(a)| &\leqslant \|a\|_2\left\{\int_b^{b+h}(x-b)^2\,dx\right\}^{1/2}\max_{c<x<d}\left|\frac{d}{dx}\phi_n(x)\right|\\ &\leqslant C_1 n^{\delta}\|a\|_2\,h^{3/2},\end{aligned}$$

where $C_1 = K/3^{1/2}$. By the fact $h \leqslant \|a\|_2^{-2}$, we have the inequality (3).

For simplicity, we put $\gamma = \|a\|_2^{6/(2\delta+1)}$ and $\sigma = (\delta+2)/3$. It follows from (3) that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{|b_n(a)|}{(n+1)^{\sigma}} &= \left\{\sum_{n\leqslant\gamma} + \sum_{n>\gamma}\right\}\frac{|b_n(a)|}{(n+1)^{\sigma}}\\ &\leqslant C_1\|a\|_2^{-2}\sum_{n\leqslant\gamma}\frac{n^{\delta}}{(n+1)^{\sigma}} + \|a\|_2\left\{\sum_{n>\gamma}\frac{1}{(n+1)^{2\sigma}}\right\}^{1/2}\\ &\leqslant C_1'(\|a\|_2^{-2}\gamma^{(2\delta+1)/3} + \|a\|_2\,\gamma^{-(2\delta+1)/6}) \leqslant C_2,\end{aligned}$$

where C_1' and C_2 are positive constants independent of $a(x)$.

Since $\|a\|_1 \leqslant 1$, the series $\sum_{j=0}^{\infty} \lambda_j a_j(x)$ converges in $L^1(c, d)$. We see that (4) implies (5).

We now come to the proof of the Theorem. By virtue of the Lemma and the atomic decomposition characterization, it is enough to show, for part (i) of the Theorem, that

$$\left| \frac{d}{dx} \mathscr{H}_n(x) \right| \leqslant C n^{5/12} \tag{6}$$

for $x \in \mathbb{R}$. Here and below, C denotes a positive constant which may differ at each different occurrence. It follows from the equations

$$H_n(x) = 2xH_{n-1}(x) - 2(n-1)\, H_{n-2}(x) \quad \text{and} \quad (d/dx)\, H_n(x) = 2nH_{n-1}(x)$$

(compare [**6**, (5.5.8), (5.5.10)]) that

$$(d/dx)\, \mathscr{H}_n(x) = (n/2)^{1/2}\, \mathscr{H}_{n-1}(x) + ((n+1)/2)^{1/2}\, \mathscr{H}_{n+1}(x).$$

By the inequality

$$|\mathscr{H}_n(x)| \leqslant C n^{-1/12} \tag{7}$$

(compare [**5**, (2.23)]), we have (6), which completes the proof of part (i) of the Theorem.

We turn to the proof of part (ii) of the Theorem. Similarly, it suffices to show that

$$\left| \frac{d}{dx} \mathscr{L}^{\alpha}_n(x) \right| \leqslant Cn, \quad x > 0. \tag{8}$$

But we shall be able to prove this inequality only in the case that $\alpha \geqslant 2$ or $\alpha = 0$. The case $0 < \alpha < 2$ will be treated by the transplantation theorem for Laguerre coefficients between α and $\alpha+2$ [**1**, p. 401, line 14]:

$$|c^{\alpha}_n(f)| \leqslant C\left(|c^{\alpha+2}_{n-1}(f)| + \sum_{j=n}^{\infty} |c^{\alpha+2}_j(f)| \left(\frac{n}{j}\right)^{\alpha/2} j^{-1} \right), \quad n = 1, 2, \ldots. \tag{9}$$

To prove (8), we use the equation $(d/dx)\, L^{\alpha}_n(x) = -L^{\alpha+1}_{n-1}(x)$ (compare [**6**, (5.1.14)]). It follows that

$$\frac{d}{dx} \mathscr{L}^{\alpha}_n(x) = \tau^{\alpha}_n \left(-L^{\alpha+1}_{n-1}(x)\, e^{-x/2}\, x^{\alpha/2} + \frac{\alpha}{2} L^{\alpha}_n(x)\, e^{-x/2}\, x^{\alpha/2-1} + (-\tfrac{1}{2})\, L^{\alpha}_n(x)\, e^{-x/2}\, x^{\alpha/2} \right),$$

where $\tau^{\alpha}_n = \{\Gamma(n+1)/\Gamma(n+\alpha+1)\}^{1/2}$. By the inequality

$$|\mathscr{L}^{\alpha}_n(x)| \leqslant C, \quad x > 0, \quad \alpha \geqslant 0 \tag{10}$$

(compare [**5**, (2.9)]), we have

$$\begin{aligned} \left| \frac{d}{dx} \mathscr{L}^{\alpha}_n(x) \right| &\leqslant C\left(\tau^{\alpha}_n |L^{\alpha+1}_{n-1}(x)|\, e^{-x/2}\, x^{\alpha/2} + \frac{\alpha}{2} \tau^{\alpha}_n |L^{\alpha}_n(x)|\, e^{-x/2}\, x^{\alpha/2}\, x^{-1} + 1 \right) \\ &\leqslant C(A+B+1), \quad \text{say}. \end{aligned}$$

We divide the matter into two cases, $nx \geqslant 1$ and $0 < nx < 1$. For $nx \geqslant 1$, we have, by (10) and $\tau^{\alpha}_n \leqslant Cn^{-\alpha/2}$,

$$A \leqslant \tau^{\alpha}_n |L^{\alpha+1}_{n-1}(x)|\, e^{-x/2}\, x^{(\alpha+1)/2}\, x^{-1/2} \leqslant Cn(nx)^{-1/2} \leqslant Cn,$$

$$B \leqslant \frac{\alpha}{2} \tau^{\alpha}_n |L^{\alpha}_n(x)|\, e^{-x/2}\, x^{\alpha/2}\, x^{-1} \leqslant Cn(nx)^{-1} \leqslant Cn.$$

Thus we have (8) for $nx \geqslant 1$ when $\alpha \geqslant 0$. For $0 < nx < 1$, we use the inequality $|L_n^\alpha(x)| \leqslant Cn^\alpha$, $0 < nx < 1$ [**6**, (7.6.8)]. It follows that $A \leqslant Cn^{-\alpha/2}n^{\alpha+1}x^{\alpha/2} \leqslant Cn(nx)^{\alpha/2} \leqslant Cn$ when $\alpha \geqslant 0$, and that $B \leqslant Cn^{-\alpha/2}n^\alpha x^{\alpha/2}x^{-1} \leqslant Cn(nx)^{\alpha/2-1} \leqslant Cn$ when $\alpha \geqslant 2$. Note that the term B does not appear when $\alpha = 0$. We conclude that (8) holds when $\alpha \geqslant 2$ or $\alpha = 0$. Therefore we have the desired inequality (2) when $\alpha \geqslant 2$ or $\alpha = 0$.

We shall interpolate between $\alpha = 0$ and $\alpha = 2$ by using (9). Let $0 < \alpha < 2$ and $g(x) \in H^1(0,\infty)$. By (10), we note that $|c_0^\alpha(g)| \leqslant C\|g\|_{L^1(0,\infty)} \leqslant C\|g\|_{H^1(0,\infty)}$. It follows from (9) that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{|c_n^\alpha(g)|}{n+1} &= |c_0^\alpha(g)| + \sum_{n=1}^{\infty}\frac{|c_n^\alpha(g)|}{n+1}\\ &\leqslant C\|g\|_{H^1(0,\infty)} + C\left(\sum_{n=1}^{\infty}\frac{|c_{n-1}^{\alpha+2}(g)|}{n+1} + \sum_{n=1}^{\infty}\frac{1}{n+1}\sum_{j=n}^{\infty}|c_j^{\alpha+2}(g)|\left(\frac{n}{j}\right)^{\alpha/2}j^{-1}\right).\end{aligned}$$

We treat the last sum. Since $\{\sum_{n=1}^{j} n^{\alpha/2}(n+1)^{-1}\}j^{-\alpha/2} \leqslant C$ for $\alpha > 0$, it follows that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n+1}\sum_{j=n}^{\infty}|c_j^{\alpha+2}(g)|\left(\frac{n}{j}\right)^{\alpha/2}j^{-1} &= \sum_{j=1}^{\infty}\left\{\sum_{n=1}^{j}\frac{n^{\alpha/2}}{n+1}\right\}j^{-\alpha/2}j^{-1}|c_j^{\alpha+2}(g)|\\ &\leqslant C\sum_{j=1}^{\infty}j^{-1}|c_j^{\alpha+2}(g)|.\end{aligned}$$

Therefore we have

$$\sum_{n=0}^{\infty}\frac{|c_n^\alpha(g)|}{n+1} \leqslant C\left(\|g\|_{H^1(0,\infty)} + \sum_{n=0}^{\infty}\frac{|c_n^{\alpha+2}(g)|}{n+1}\right)$$

for $\alpha > 0$, which completes the proof of part (ii) of the Theorem since inequality (2) with $\alpha \geqslant 2$ or $\alpha = 0$ has been proved.

3. *Paley type theorem*

The following inequalities, (11) and (12), with respect to the Laguerre function system follow from Paley's theorem (compare [**7**, Vol. II, Theorem (5.1)]) with respect to general systems $\{\phi_n\}$ of functions orthonormal and uniformly bounded, $|\phi_n(x)| \leqslant C$.

Let $\alpha \geqslant 0$. For $g(x) \sim \sum_{n=0}^{\infty} c_n^\alpha(g)\,\mathscr{L}_n^\alpha(x) \in L^p(0,\infty)$, $1 < p \leqslant 2$,

$$\sum_{n=0}^{\infty}|c_n^\alpha(g)|^p\,(n+1)^{p-2} \leqslant C\|g\|_{L^p(0,\infty)}^p. \tag{11}$$

For $\{b_n\}_{n=0}^{\infty}$ with $\sum_{n=0}^{\infty}|b_n|^q\,(n+1)^{q-2}$, $2 \leqslant q < \infty$, there exists a function

$$g(x) \sim \textstyle\sum_{n=0}^{\infty} b_n\,\mathscr{L}_n^\alpha(x) \in L^q(0,\infty)$$

such that

$$\|g\|_{L^q(0,\infty)}^q \leqslant C\sum_{n=0}^{\infty}|b_n|^q\,(n+1)^{q-2}. \tag{12}$$

Also, by applying Paley's theorem to the Hermite function system, we have the same inequalities as (11) and (12), where we substitute $c_n^\alpha(g)$, $g(x) \in L^p(0,\infty)$, by $c_n(f)$, $f(x) \in L^p(\mathbb{R})$, respectively. But, for the Hermite function system, we can obtain sharper inequalities by interpolating Hardy's inequality (1) and Parseval's equation.

Proposition. (i) *Let $1 < p \leqslant 2$. Then there exists a constant C such that*

$$\sum_{n=0}^{\infty} |c_n(f)|^p (n+1)^{29(p-2)/36} \leqslant C\|f\|^p_{L^p(\mathbb{R})} \tag{13}$$

for $f(x) \sim \sum_{n=0}^{\infty} c_n(f)\, \mathscr{H}_n(x) \in L^p(\mathbb{R})$.

(ii) *Let $2 \leqslant q < \infty$. Then there exists a constant C such that if $\{b_n\}_{n=0}^{\infty}$ satisfies $\sum_{n=0}^{\infty} |b_n|^q (n+1)^{29(q-2)/36} < \infty$, then*

$$\|f\|^q_{L^q(\mathbb{R})} \leqslant C \sum_{n=0}^{\infty} |b_n|^q (n+1)^{29(q-2)/36}, \tag{14}$$

where $f(x) \sim \sum_{n=0}^{\infty} b_n\, \mathscr{H}_n(x) \in L^q(\mathbb{R})$.

Proof. For $\kappa > 0$, we denote by l^p_κ, $p \geqslant 1$, the space of sequences $\{b_n\}_{n=0}^{\infty}$ such that $\|\{b_n\}\|_p = \{\sum_{n=0}^{\infty} |b_n|^p (n+1)^{-2\kappa}\}^{1/p} < \infty$. We define an operator T_κ of $L^p(\mathbb{R})$ to l^p_κ by $T_\kappa f = \{c_n(f)(n+1)^\kappa\}_{n=0}^{\infty}$. We consider the case $\kappa = 29/36$. Then by (1), we see that T_κ is a bounded linear operator of $H^1(\mathbb{R})$ to l^1_κ, which implies that T_κ is of weak type $(H^1(\mathbb{R}), l^1_\kappa)$. Parseval's equation says that T_κ is an isometry of $L^2(\mathbb{R})$ to l^2_κ; in particular, T_κ is of weak type $(L^2(\mathbb{R}), l^2_\kappa)$. We follow line by line the proof [**4**, pp. 308–310] of the interpolation theorem between H^1 space and L^p space. We conclude that T_κ is a bounded operator of $L^p(\mathbb{R})$ to l^p_κ, $1 < p \leqslant 2$, which means (13).

The inequality (14) is obtained by a standard duality argument (compare [**7**, Vol. II, proof of Theorem (5.1)]).

Lastly, we give a remark. Instead of the interpolation theorem between H^1 space and L^1 space, we may use the interpolation theorem between weak-L^p spaces. But it will become clear in the sequel that the inequality (15) below, obtained by this method, is weaker than our inequality (13).

We now consider l^p_κ, T_κ with $\kappa = 11/12$. Parseval's equation implies that T_κ is of weak type $(L^2(\mathbb{R}), l^2_\kappa)$. Moreover, we see that T_κ is of weak type $(L^1(\mathbb{R}), l^1_\kappa)$. For, we put $\mathcal{O}_t = \{n; |c_n(f)(n+1)^\kappa| > t\}$ for $t > 0$, where $f(x) \in L^1(\mathbb{R})$. It follows from (7) that $t < |c_n(f)(n+1)^\kappa| < Cn^{-1/12} n^\kappa \|f\|_{L^1(\mathbb{R})} = Cn^{10/12}\|f\|_{L^1(\mathbb{R})}$. Thus we have

$$\{t/\|f\|_{L^1(\mathbb{R})}\}^{12/10} < n.$$

This gives $\sum_{n\in\mathcal{O}_t} (n+1)^{-2\kappa} \leqslant C\{t/\|f\|_{L^1(\mathbb{R})}\}^{12(-2\kappa+1)/10} = C\|f\|_{L^1(\mathbb{R})}/t$, which means that T_κ is of weak type $(L^1(\mathbb{R}), l^1_\kappa)$. By the Marcinkiewicz interpolation theorem, we see that T_κ is a bounded operator of $L^p(\mathbb{R})$ to l^p_κ, $1 < p < 2$, that is,

$$\sum_{n=0}^{\infty} |c_n(f)|^p (n+1)^{11(p-2)/12} \leqslant C\|f\|^p_{L^p(\mathbb{R})}. \tag{15}$$

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Department of Mathematics
College of Liberal Arts
Kanazawa University
Kanazawa 920-11
Japan