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著者	Matsubayashi Akira, Ueno Shuichi
journal or publication title	Networks
volume	33
number	1
page range	71-77
year	1999-01-15
URL	http://hdl.handle.net/2297/3642

Small Congestion Embedding of Graphs into Hypercubes

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Abstract

We consider the problem of embedding graphs into hypercubes with minimal congestion. Kim and Lai [2] showed that for a given N-vertex graph G and a hypercube it is NP-complete to determine whether G is embeddable in the hypercube with unit congestion, but G can be embedded with unit congestion in a hypercube of dimension $6\lceil \log N \rceil$ if the maximum degree of a vertex in G is no more than $6\lceil \log N \rceil$. Bhatt, Chung, Leighton, and Rosenberg [1] showed that every N-vertex binary tree can be embedded in a hypercube of dimension $\lceil \log N \rceil$ with $O(1)$ congestion. In this paper we extend the results above and show the following:

- Every N-vertex graph G can be embedded with unit congestion in a hypercube of dimension $2\lceil \log N \rceil$ if the maximum degree of a vertex in G is no more than $2\lceil \log N \rceil$.
- Every N-vertex binary tree can be embedded in a hypercube of dimension $\lceil \log N \rceil$ with congestion at most 5.

The former answers a question posed by Kim and Lai [2]. The latter is the first result that shows a simple embedding of a binary tree into an optimal sized hypercube with explicit small congestion of 5. This partially answers a question posed by Bhatt, Chung, Leighton, and Rosenberg [1]. The embeddings proposed here are quite simple and can be constructed in polynomial time.

1 Introduction

The problem of efficiently implementing parallel algorithms on parallel machines has been studied as the graph embedding problem, which is to embed the communication graph underlying a parallel algorithm within the processor interconnection graph for a parallel machine with minimal communication overhead. It is well-known that the dilation and/or congestion of the embedding are lower bounds on the communication delay, and the problem of embedding a guest graph within a host graph with minimal dilation and/or congestion has been extensively studied.

We consider minimal congestion embeddings of graphs in hypercubes, which are well-known as one of the most popular processor interconnection graphs for parallel machines. It was pointed out by Kim and Lai [2] that minimal congestion embeddings are very important for a hypercube that uses circuit switching for node-to-node communication such as Intel iPSC/2 [4].

Let G be a graph and let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. We denote by $\Delta(G)$ the maximum degree of a vertex in G. A tree T is said to be *binary* if $\Delta(T) \leq 3$. An *embedding* $\langle \phi, \rho \rangle$ of a graph G into a graph H is defined by a one-to-one mapping $\phi : V(G) \to V(H)$, together with a mapping ρ that maps each edge $(u, v) \in E(G)$ onto a path $\rho(u, v)$ in H that connects $\phi(u)$ and $\phi(v)$. ϕ and ρ are called the labeling and routing of an embedding $\langle \phi, \rho \rangle$, respectively. The *dilation* of an edge $e \in E(G)$ under $\langle \phi, \rho \rangle$ is the length of the path $\rho(e)$. The dilation of an embedding $\langle \phi, \rho \rangle$ is the maximum dilation of an edge in G. The *congestion* of an edge $e' \in E(H)$ under $\langle \phi, \rho \rangle$ is the number of edges e in G such that $\rho(e)$ contains e'. The congestion of an embedding $\langle \phi, \rho \rangle$ is the maximum congestion of an edge in H. The *n-cube* (*n-dimensional cube*) $Q(n)$ is the graph with 2^n vertices labeled 0 through $2^n - 1$ such that two vertices are joined by an edge if and only if their labels in the binary representation differ by exactly one bit. We assume that the bits are numbered 0 through $n-1$. An edge (u, v) in $Q(n)$ is called an *i-edge* (*i-dimensional edge*) if the labels of u and v in the binary representation differ in the ith bit $(0 \leq i \leq n-1)$. It is well-known that $Q(n)$ is n-connected.

Kim and Lai [2] showed that for a given N-vertex graph G and a hypercube it is NP-complete to determine whether G is embeddable in the hypercube with unit congestion, but G can be embedded with unit congestion in $Q(6\lceil \log N \rceil)$ if $\Delta(G) \leq 6\lceil \log N \rceil$. They posed the question of whether G can be embedded with unit congestion in a hypercube of dimension less than $6\lceil \log N \rceil$. We answer the question by proving the following theorem.

Theorem 1 *Every N-vertex graph G can be embedded with unit congestion in $Q(2\lceil \log N \rceil)$ if $\Delta(G) \leq 2\lceil \log N \rceil$.*

The basic idea of the embedding is quite simple. We adopt a plain labeling of vertices and a simple routing for edges, and the embedding can be constructed in polynomial time. We do not know whether G can be embedded with unit congestion in a hypercube of dimension less than $2\lceil \log N \rceil$. However, we can show that some graphs can be embedded with unit congestion in hypercubes of asymptotically smaller dimensions. More precisely, we can easily show by combining the results of Saad and Shultz [5] and Valiant [6] that every N-vertex tree T with $\Delta(T) \leq 4$ can be embedded with unit congestion in a hypercube of dimension $\log N + O(1)$, and every N-vertex planar graph G with $\Delta(G) \leq 4$ can be embedded with unit congestion in a hypercube of dimension $\log N + 2 \log \log N + O(1)$.

Bhatt, Chung, Leighton, and Rosenberg [1] showed that every N-vertex binary tree can be embedded in $Q(\lceil \log N \rceil)$ with dilation and congestion both $O(1)$. Although their embedding is optimal to within a constant factor, there is much room for reducing the dilation and/or congestion. They posed the question of finding a simple embedding of binary trees into hypercubes with smaller dilation and/or congestion. Monien and Sudborough [3] partially answer the question by proving that every N-vertex binary tree can be embedded in $Q(\lceil \log N \rceil)$ with dilation at most 5. We also partially answer the question by proving the following theorem.

Theorem 2 *Every N-vertex binary tree can be embedded in $Q(\lceil \log N \rceil)$ with congestion at most 5.*

Theorem 2 is the first result that shows a simple embedding of a binary tree into an optimal sized hypercube with explicit small congestion of 5. The embedding is quite simple. We use the postorder labeling of vertices and a greedy (shortest path) routing for edges, and the embedding can be constructed in polynomial time. It is interesting that such a simple embedding guarantees a small congestion of 5. We do not know an N-vertex binary tree that cannot be embedded in $Q(\lceil \log N \rceil)$ with unit congestion except $K_{1,3}$ (a complete bipartite graph). The authors verified that every N-vertex binary tree except $K_{1,3}$ can be embedded in $Q(\lceil \log N \rceil)$ with unit congestion if $N \leq 16$. In this connection, based on some conjecture, Wagner [7] mentioned a heuristic algorithm which would embed every N-vertex binary tree into $Q(\lceil \log N \rceil)$ with dilation and congestion both at most 2.

The paper is organized as follows. We prove Theorems 1 and 2 in Sections 2 and 3, respectively. In Section 4, we conclude with remarks on dilations of our embeddings and some other remarks.

2 Proof of Theorem 1

Let $V(G) = \{0, 1, \ldots, N-1\}$ and $n = \lceil \log N \rceil$. We assume that $\Delta(G) \leq 2n$. We construct an embedding $\langle \phi_1, \rho_1 \rangle$ of G into $Q(2n)$ with unit congestion. We define the labeling ϕ_1 in Section 2.1. In Section 2.2, we consider an arc coloring of a digraph associated with G. We define the routing ρ_1 in Section 2.3 based on the results in Section 2.2. We analyze the congestion of embedding $\langle \phi_1, \rho_1 \rangle$ in Section 2.4.

2.1 Labeling ϕ_1

The labeling $\phi_1 : V(G) \to V(Q(2n))$ is defined as follows. For each $u \in V(G)$, $\phi_1(u) = u(2^n+1)$. That is, the binary representation of $\phi_1(u)$ is the concatenation of two copies of the binary representation of u.

2.2 Arc Coloring

In this section, we consider an arc coloring of a digraph associated with G which will be used to define routing ρ_1. The associated digraph D of G is the digraph obtained from G by replacing each edge e of G by two oppositely oriented arcs with the same ends as e. We denote the vertex set and arc set of D by $V(D)$ and $A(D)$, respectively. We denote an arc a by $[u, v]$ if u is the tail of a, and v is its head. Let $\Gamma^+_D(u)$ denote the set of arcs with tail u, and $d^+_D(u) = |\Gamma^+_D(u)|$. Let $\Gamma^-_D(u)$ denote the set of arcs with head u, and $d^-_D(u) = |\Gamma^-_D(u)|$. Since $\Delta(G) \leq 2n$, $d^+_D(u) \leq 2n$ and $d^-_D(u) \leq 2n$ for any $u \in V(D)$.

We construct a coloring C of the arcs of D with two colors $\{0, 1\}$ such that both of the following two conditions are satisfied. We denote by $C[u, v]$ the color of an arc $[u, v]$ assigned by C. Define that $X^0_C(w) = \{[w, x] \mid [w, x] \in \Gamma^+_D(w), C[w, x] = 0\}$, and $X^1_C(w) = \{[w, y] \mid [w, y] \in \Gamma^+_D(w), C[w, y] = 1\}$.

Condition 1 *For each edge $(u, v) \in E(G)$, $C[u, v] = 0$ if and only if $C[v, u] = 1$.*

Condition 2 *For any vertex $u \in V(D)$, $|X^0_C(u)| \leq n$ and $|X^1_C(u)| \leq n$*

Lemma 3 *There exists a 2-arc coloring of D satisfying Conditions 1 and 2.*

Proof It is well-known that G has an orientation D' such that $|d^+_{D'}(u) - d^-_{D'}(u)| \le 1$ for any $u \in V(D')$. It follows that $d^+_{D'}(u) \le n$ and $d^-_{D'}(u) \le n$ for any $u \in V(D')$ since $\Delta(G) \le 2n$. Moreover, for each $(u, v) \in E(G)$, exactly one of the associated arcs $[u, v]$ and $[v, u]$ of D is contained in $\Gamma^+_{D'}(u) \cup \Gamma^-_{D'}(u)$. Thus, $|\Gamma^+_D(u) \cap \Gamma^+_{D'}(u)| \le n$ and $|\Gamma^+_D(u) - \Gamma^+_{D'}(u)| = |\Gamma^-_D(u) \cap \Gamma^-_{D'}(u)| \le n$ for any $u \in V(D)$. For each vertex $u \in V(D)$, we assign color 0 to the arcs in $\Gamma^+_D(u) \cap \Gamma^+_{D'}(u)$, and color 1 to the arcs in $\Gamma^+_D(u) - \Gamma^+_{D'}(u)$. The resulting 2-arc coloring of D satisfies Conditions 1 and 2. □

2.3 Routing ρ_1

For two vertices w and w' of G, let $m(w, w')$ be the vertex of $Q(2n)$ labeled with $w2^n + w'$. There exists a 2-arc coloring C of D satisfying Conditions 1 and 2 by Lemma 3. For a vertex $w \in V(G)$, suppose that $X^0_C(w) = \{[w, x_1], [w, x_2], \ldots, [w, x_k]\}$, and $X^1_C(w) = \{[w, y_1], [w, y_2], \ldots, [w, y_l]\}$, where $k = |X^0_C(w)|$ and $l = |X^1_C(w)|$. $k \le n$ and $l \le n$ since C satisfies Condition 2. Let $Q^0_w(n)$ and $Q^1_w(n)$ be the n-dimensional subcubes of $Q(2n)$ induced by the vertices $\{w2^n + i \mid 0 \le i \le 2^n - 1\}$ and the vertices $\{i2^n + w \mid 0 \le i \le 2^n - 1\}$, respectively. Notice that $\phi_1(w) \in V(Q^0_w(n)) \cap V(Q^1_w(n))$ and that $m(w, w') \in V(Q^0_w(n)) \cap V(Q^1_{w'}(n))$. Since $Q^0_w(n)$ is n-connected, there exist k vertex-disjoint paths P_i in $Q^0_w(n)$ connecting $\phi_1(w)$ and $m(w, x_i)$ $(1 \le i \le k)$. Define that $P[w, x_i] = P_i$ $(1 \le i \le k)$. Also, since $Q^1_w(n)$ is n-connected, there exist l vertex-disjoint paths P'_j in $Q^1_w(n)$ connecting $\phi_1(w)$ and $m(y_j, w)$ $(1 \le j \le l)$. Define that $P[w, y_j] = P'_j$ $(1 \le j \le l)$.

Now we define the routing ρ_1. Let (u, v) be an edge of G. We may assume that $C[u, v] = 0$ and $C[v, u] = 1$ since C satisfies Condition 1. Define the path $\rho_1(u, v)$ connecting $\phi_1(u)$ and $\phi_1(v)$ in $Q(2n)$ as the concatenation of $P[u, v]$ connecting $\phi_1(u)$ and $m(u, v)$ in $Q^0_u(n)$ and $P[v, u]$ connecting $\phi_1(v)$ and $m(u, v)$ in $Q^1_v(n)$.

Notice that the embedding $\langle \phi_1, \rho_1 \rangle$ defined above can be constructed in polynomial time.

2.4 Congestion of $\langle \phi_1, \rho_1 \rangle$

Lemma 4 *The congestion of $\langle \phi_1, \rho_1 \rangle$ is one.*

Proof It suffices to show that $P[u, v]$ and $P[s, t]$ are edge-disjoint for any distinct arcs $[u, v], [s, t] \in A(D)$.

Case 1 $C[u, v] \ne C[s, t]$. We may assume without loss of generality that $C[u, v] = 0$ and $C[s, t] = 1$. Since $Q^0_u(n)$ and $Q^1_s(n)$ are edge-disjoint, and $P[u, v]$ and $P[s, t]$ are contained in $Q^0_u(n)$ and $Q^1_s(n)$, respectively, $P[u, v]$ and $P[s, t]$ are edge-disjoint.

Case 2 $C[u, v] = C[s, t]$. We assume that $C[u, v] = C[s, t] = 0$. The proof for the case when $C[u, v] = C[s, t] = 1$ can be accomplished by a similar argument, and is omitted.

Case 2.1 $u \ne s$. Since $Q^0_u(n)$ and $Q^0_s(n)$ are vertex-disjoint, and $P[u, v]$ and $P[s, t]$ are contained in $Q^0_u(n)$ and $Q^0_s(n)$, respectively, $P[u, v]$ and $P[s, t]$ are edge-disjoint.

Case 2.2 $u = s$. Since $[u, v], [u, t] \in X^0_C(u)$, $P[u, v]$ and $P[u, t]$ are edge-disjoint by definition.

□

This completes the proof of Theorem 1.

3 Proof of Theorem 2

Let T be an N-vertex binary tree and $n = \lceil \log N \rceil$. We construct an embedding $\langle \phi_2, \rho_2 \rangle$ of T into $Q(n)$ with congestion at most 5. We define $\langle \phi_2, \rho_2 \rangle$ in Section 3.1. In Section 3.2, we show some lemmas on the postorder numbering. In Section 3.3, we analyze the congestion of $\langle \phi_2, \rho_2 \rangle$ based on the results of Section 3.2.

3.1 Embedding $\langle \phi_2, \rho_2 \rangle$

The embedding we propose here is quite simple. We choose a vertex of T with degree at most two as the root of T, and we suppose that T is a rooted tree. Without loss of generality, we assume that for each vertex u of T, the number of left descendants of u (i.e., the number of vertices of left subtree rooted at u) is not less than that of right descendants of u. Give each vertex of T a number from 0 through $N-1$ according to the postorder numbering of T so that the left most leaf has the number 0.

We define the labeling $\phi_2 : V(T) \to V(Q(n))$ as follows. For each $u \in V(T)$, $\phi_2(u)$ is the vertex of $Q(n)$ labeled with the postorder number of u.

We define the routing ρ_2 as follows. Let (u, v) be an edge of T, and $\phi_2(u) < \phi_2(v)$. The path $\rho_2(u, v)$ connecting $\phi_2(u)$ and $\phi_2(v)$ in $Q(n)$ starts at $\phi_2(u)$, passes through i-edges in the increasing order of i such that the binary representations of $\phi_2(u)$ and $\phi_2(v)$ differ in the ith bit. Thus, ρ_2 is a greedy (shortest path) routing for edges.

Notice that the embedding $\langle \phi_2, \rho_2 \rangle$ defined above can be constructed in polynomial time.

In what follows, for each $u \in V(T)$, we denote the postorder number of u and $\phi_2(u)$ simply by u. In addition, if we denote an edge of T by (u, v), we assume that $u < v$.

3.2 Properties of Postorder Numbering

The following lemmas on the postorder numbering will be used in the next section to analyze the congestion of $\langle \phi_2, \rho_2 \rangle$.

Lemma 5 *For any distinct edges $(u, v), (s, t) \in E(T)$ $(u \le s)$, $u < s < t \le v$ or $u < v \le s < t$.*

Proof Since the vertices of T are labeled according to the postorder numbering, each $y \in V(T)$ is adjacent to at most one vertex with a label more than y. Thus, $u \ne s$ and we may assume that $u < s$. Define that $I = \{x \in V(T) \mid u < x < v\}$. I is the set of right descendants of v if u is the left child of v, and I is the empty set if u is the right child of v. It follows that any $x \in I$ is adjacent only to vertices of $I \cup \{v\}$. Thus, if $s \in I$ then $t \in I \cup \{v\}$. This means that $u < s < t \le v$. If $s \notin I$, we have $u < v \le s < t$ by the assumption that $u < s$ and the definition of I. □

Lemma 6 *For any distinct edges $(u, v), (s, t) \in E(T)$ $(u < s < t \le v)$, $t - s \le s - u + 1$.*

Proof Since $u < s < t \le v$, u is the left child of v and both s and t are right descendants of v. If s is the right child of t then $t - s = 1$ and the lemma is immediate. Thus, we assume that s is the left child of t. Let m_L and m_R be the numbers of left descendants and right descendants of t, respectively, and let w be the vertex with the minimum postorder number in the descendants of s. It follows that

$$w - u \ge 1. \tag{1}$$

Since $m_L - 1$ is the number of descendants of s and $m_L \ge m_R$,

$$s - w = m_L - 1 \ge m_R - 1. \tag{2}$$

Since s is the left child of t,

$$m_R = t - s - 1. \tag{3}$$

From (1), (2), and (3), we have $t - s \leq s - u + 1$, as desired. $\square$

3.3 Congestion of $\langle \phi_2, \rho_2 \rangle$

In this section, we show that the congestion of $\langle \phi_2, \rho_2 \rangle$ is no more than 5. We will prove this by a series of lemmas. Let $\mathrm{bit}(m, k)$ denote the number (0 or 1) in the kth bit ($k \geq 0$) in the binary representation of a non-negative integer m. For each edge $(u, v) \in E(T)$ and an integer k ($0 \leq k \leq n-1$), define that $\mathrm{dir}((u, v), k) = \mathrm{bit}(v, k) - \mathrm{bit}(u, k)$. If some paths in $Q(n)$ contain an edge $d \in E(Q(n))$ then the paths are said to *share* d. We can easily see the following lemma from the definition of ρ_2.

Lemma 7 *For any distinct edges $(u, v), (s, t) \in E(T)$, $\rho_2(u, v)$ and $\rho_2(s, t)$ share a k-edge in $Q(n)$ if and only if the following three conditions are satisfied.*

Condition 3 $\mathrm{dir}((u, v), k) \neq 0$ *and* $\mathrm{dir}((s, t), k) \neq 0$.

Condition 4 *If $k < n - 1$, the $(n - k - 1)$-bit strings consisting of the $(k + 1)$st bit through the $(n - 1)$st bit in the binary representations of u and s are identical.*

Condition 5 *If $k > 0$, the k-bit strings consisting of the 0th bit through the $(k - 1)$st bit in the binary representations of v and t are identical.* $\square$

Lemma 8 *For any distinct edges $(u, v), (s, t) \in E(T)$ such that*

$$u < s < t < v \text{ and } \mathrm{dir}((u, v), k) = \mathrm{dir}((s, t), k), \tag{4}$$

if $\rho_2(u, v)$ and $\rho_2(s, t)$ share a k-edge in $Q(n)$ then

$$t - s \leq 2^k, \text{ and} \tag{5}$$

$$v - u > 2^{k+1}. \tag{6}$$

Proof We have $\mathrm{bit}(u, k) = \mathrm{bit}(s, k) \neq \mathrm{bit}(v, k) = \mathrm{bit}(t, k)$ from (4) and Lemma 7 (Condition 3). Thus, $s - u < 2^k$ and $v - t \geq 2^{k+1}$ by Lemma 7 (Conditions 4 and 5). Therefore, we have (5) by Lemma 6, and (6) since $u < t$. $\square$

Lemma 9 *For any distinct edges $(u, v), (s, t) \in E(T)$ such that*

$$u < s < t = v, \tag{7}$$

if $\rho_2(u, v)$ and $\rho_2(s, t)$ share a k-edge in $Q(n)$ then

$$t - s \leq 2^k. \tag{8}$$

Proof Since $t = v$, $\mathrm{bit}(u, k) = \mathrm{bit}(s, k) \neq \mathrm{bit}(v, k) = \mathrm{bit}(t, k)$ by Lemma 7 (Condition 3). Therefore, $s - u < 2^k$ by Lemma 7 (Condition 4). By Lemma 6, we have (8). $\square$

Lemma 10 *For any distinct edges* $(u,v),(s,t) \in E(T)$ *such that*

$$u < s < t < v \text{ and } \operatorname{dir}((u,v),k) \neq \operatorname{dir}((s,t),k), \tag{9}$$

if $\rho_2(u,v)$ *and* $\rho_2(s,t)$ *share a* k*-edge in* $Q(n)$ *then*

$$t - s \leq 2^{k+1}. \tag{10}$$

Proof $s - u < 2^{k+1}$ by Lemma 7 (Condition 4). Thus, we have (10) by Lemma 6. □

Lemma 11 *For any distinct edges* $(u,v),(s,t) \in E(T)$ *such that*

$$u < v \leq s < t, \tag{11}$$

if $\rho_2(u,v)$ *and* $\rho_2(s,t)$ *share a* k*-edge in* $Q(n)$ *then*

$$v - u < 2^{k+1}. \tag{12}$$

Proof $s - u < 2^{k+1}$ by Lemma 7 (Condition 4). Since $v \leq s$, we have (12). □

Lemma 12 *Any distinct edges* $(u,v),(s,t) \in E(T)$ $(u < s)$ *satisfy exactly one of (4), (7), (9), and (11).*

Proof Immediate from Lemma 5. □

Lemma 13 *For any distinct edges* $(u,v),(s,t) \in E(T)$ $(u < s)$ *such that* $\rho_2(u,v)$ *and* $\rho_2(s,t)$ *share a* k*-edge in* $Q(n)$*,* (u,v) *and* (s,t) *satisfy either (4) or (7) if and only if* $\operatorname{dir}((u,v),k) = \operatorname{dir}((s,t),k) \neq 0$*, and* (u,v) *and* (s,t) *satisfy either (9) or (11) if and only if* $\operatorname{dir}((u,v),k) = 1$ *and* $\operatorname{dir}((s,t),k) = -1$*.*

Proof We first show the necessities. If (u,v) and (s,t) satisfy either (4) or (7) then $\operatorname{dir}((u,v),k) = \operatorname{dir}((s,t),k) \neq 0$ from the proofs of Lemmas 8 and 9. If (u,v) and (s,t) satisfy (9) then $\operatorname{dir}((u,v),k) = 1$ and $\operatorname{dir}((s,t),k) = -1$ by Lemma 7 (Conditions 3 and 4). Assume that (u,v) and (s,t) satisfy (11). If $k < n-1$ then the $(n-k-1)$-bit strings consisting of the $(k+1)$st bit through the $(n-1)$st bit in the binary representations of u, v, and s are identical by Lemma 7 (Condition 4). Thus, $\operatorname{dir}((u,v),k) = 1$ and $\operatorname{dir}((s,t),k) = -1$ by Lemma 7 (Condition 3).

We next show the sufficiencies. Assume that $\operatorname{dir}((u,v),k) = \operatorname{dir}((s,t),k) \neq 0$. It follows from Lemma 12 that (u,v) and (s,t) satisfy exactly one of (4), (7), (9), and (11). If (u,v) and (s,t) satisfy (9) or (11), then it follows from the necessities that $\operatorname{dir}((u,v),k) = 1$ and $\operatorname{dir}((s,t),k) = -1$, a contradiction. Thus, (u,v) and (s,t) satisfy either (4) or (7). Assume that $\operatorname{dir}((u,v),k) = 1$ and $\operatorname{dir}((s,t),k) = -1$. It follows from Lemma 12 that (u,v) and (s,t) satisfy exactly one of (4), (7), (9), and (11). If (u,v) and (s,t) satisfy (4) or (7), then it follows from the necessities that $\operatorname{dir}((u,v),k) = \operatorname{dir}((s,t),k) \neq 0$, a contradiction. Thus, (u,v) and (s,t) satisfy either (9) or (11). □

For distinct edges $e_1, e_2, \ldots,$ and e_l in T, suppose that $\rho_2(e_1), \rho_2(e_2), \ldots,$ and $\rho_2(e_l)$ share a k-edge $d \in E(Q(n))$. If $\operatorname{dir}(e_1,k) = \operatorname{dir}(e_2,k) = \ldots = \operatorname{dir}(e_l,k) \neq 0$ then $\rho_2(e_1), \rho_2(e_2), \ldots,$ and $\rho_2(e_l)$ are said to share d in the *same direction.*

Lemma 14 *For any distinct edges* (u,v)*,* (s,t)*, and* (w,x) *in* T *which are a matching,* $\rho_2(u,v)$*,* $\rho_2(s,t)$*, and* $\rho_2(w,x)$ *do not share an edge in the same direction.*

Proof We may assume without loss of generality that $u < s < w$. Assume that $\rho_2(u, v)$ and $\rho_2(s, t)$ share a k-edge $e \in E(Q(n))$ in the same direction. Since (u, v) and (s, t) are a matching of T, we have $u < s < t < v$ from Lemma 13. Thus, it follows from Lemma 8 that $t - s \leq 2^k$. If $\rho_2(s, t)$ and $\rho_2(w, x)$ share e in the same direction, we have $s < w < x < t$ from Lemma 13, and it follows from Lemma 8 that $t - s > 2^{k+1}$, a contradiction. □

Lemma 15 *For any distinct edges (u, v), (s, t), and (w, x) in T which are incident to a vertex, $\rho_2(u, v)$, $\rho_2(s, t)$, and $\rho_2(w, x)$ do not share an edge in the same direction.*

Proof Suppose that $\rho_2(u, v)$, $\rho_2(s, t)$, and $\rho_2(w, x)$ share an edge in the same direction. Then we have $v = t = x$ by Lemma 13. Therefore $u < v$, $s < v$ and $w < v$. This is a contradiction, however, since each $y \in V(T)$ is adjacent to at most two vertices with labels less than y by the definition of the postorder numbering. □

Let d be a k-edge of $Q(n)$. We define that

$$\begin{aligned} H^+(d) &= \{e \mid e \in E(T), \mathrm{dir}(e, k) = 1, \rho_2(e) \text{ contains } d\}, \\ H^-(d) &= \{e \mid e \in E(T), \mathrm{dir}(e, k) = -1, \rho_2(e) \text{ contains } d\}. \end{aligned}$$

Lemma 16 $|H^+(d)| \leq 3$ *and* $|H^-(d)| \leq 3$ *for any* $d \in E(Q(n))$. *That is, the congestion of* $\langle \phi_2, \rho_2 \rangle$ *is at most* 6.

Proof Suppose that d is a k-edge $(0 \leq k \leq n - 1)$. If all edges in $H^+(d)$ are incident to a vertex then $|H^+(d)| \leq 2$ by Lemma 15. We next consider the case that there are edges $(u, v), (s, t) \in H^+(d)$ $(u < s)$ which are a matching of T. Then we have $u < s < t < v$ by Lemma 13, and it follows from Lemma 8 that

$$v - u > 2^{k+1}. \tag{13}$$

Suppose that there exists an edge $(w, x) \in H^+(d) - \{(u, v), (s, t)\}$. By Lemma 14, (w, x) is adjacent to (u, v) or (s, t).

If (w, x) is adjacent to (u, v) then $x = v$ from Lemma 13. Thus we have $x - w \leq 2^k$ by Lemma 9 and (13). Since $t < v = x$, it follows from Lemma 13 that $w < s < t < x$ for (w, x) and (s, t). Thus, we have $x - w > 2^{k+1}$ from Lemma 8, which is a contradiction. Therefore, (w, x) is adjacent to (s, t), and $x = t$ from Lemma 13. In addition, (w, x) is the only edge in $H^+(d)$ adjacent to (s, t) by Lemma 15. Thus we conclude $|H^+(d)| \leq 3$.

Similarly, we can show that $|H^-(d)| \leq 3$. □

Lemma 17 *The congestion of* $\langle \phi_2, \rho_2 \rangle$ *is at most* 5.

Proof $|H^+(d)| \leq 3$ and $|H^-(d)| \leq 3$ for any $d \in E(Q(n))$ by Lemma 16. If $|H^+(d)| \leq 2$ and $|H^-(d)| \leq 2$ for any $d \in E(Q(n))$ then the lemma is immediate.

Suppose first that $|H^+(d)| = 3$ for a k-edge $d \in E(Q(n))$. Then $H^+(d)$ contains non-adjacent two edges from the proof of Lemma 16. Let (u, v) be one of such edges which satisfies (13). Then, we have $v - u > 2^{k+1}$. Let (s, t) be an edge in $H^-(d)$. It follows from Lemma 13 that we have either $u < s < t < v$ or $u < v \leq s < t$ for (u, v) and (s, t). However, if $u < v \leq s < t$ then $v - u < 2^{k+1}$ from Lemma 11, which is a contradiction. Thus, $u < s < t < v$ and we have

$$t - s \leq 2^{k+1} \tag{14}$$

by Lemma 10. Suppose (w, x) and (y, z) are any distinct edges in $H^-(d)$ $(w < y)$. We have $x - w \leq 2^{k+1}$ from (14). It follows that $x = z$, for otherwise $w < y < z < x$ from Lemma 13, and we have $x - w > 2^{k+1}$ by Lemma 8, which is a contradiction. Therefore, $|H^-(d)| \leq 2$ by Lemma 15.

Suppose next that $|H^-(d)| = 3$ for a k-edge $d \in E(Q(n))$. Then there exists an edge $(s, t) \in H^-(d)$ such that $t - s > 2^{k+1}$. Let (u, v) be an edge in $H^+(d)$. It follows from Lemma 13 that we have either $u < s < t < v$ or $u < v \leq s < t$ for (u, v) and (s, t). However, if $u < s < t < v$ then $t - s \leq 2^{k+1}$ from Lemma 10, which is a contradiction. Thus, $u < v \leq s < t$ and we have

$$v - u < 2^{k+1} \tag{15}$$

by Lemma 11. Suppose (w, x) and (y, z) are any distinct edges in $H^+(d)$ $(w < y)$. We have $x - w < 2^{k+1}$ from (15). It follows that $x = z$, for otherwise $w < y < z < x$ from Lemma 13, and we have $x - w > 2^{k+1}$ by Lemma 8, which is a contradiction. Therefore, $|H^+(d)| \leq 2$ by Lemma 15.

Thus, we conclude that the congestion of $\langle \phi_2, \rho_2 \rangle$ is at most 5. □

This completes the proof of Theorem 2.

4 Concluding Remarks

Although $\langle \phi_1, \rho_1 \rangle$ may have a large dilation, we can also construct an embedding of G into $Q(2n)$ with dilation at most $2n + 2$ and unit congestion using a more sophisticated routing. It should be noted that the dilation of $\langle \phi_2, \rho_2 \rangle$ is at most the diameter of the hypercube since ρ_2 is a shortest path routing.

Our analysis of the congestion of $\langle \phi_2, \rho_2 \rangle$ is tight possible. That is, there exist binary trees for which the congestion of $\langle \phi_2, \rho_2 \rangle$ is exactly 5. For the tree shown in Figure 1, the image paths of five bold edges by ρ_2 share $(10000, 10100) \in E(Q(6))$. This is also true when we choose any vertex in the right subtree (represented as the gray triangle) as the root. Moreover, the same situation occurs if the root is not in the right subtree. Thus the congestion of $\langle \phi_2, \rho_2 \rangle$ for the tree is independent of the choice of the root.

Acknowledgment

The authors are grateful to Professor Y. Kajitani for his encouragement. The research is a part of CAD21 project at TIT.

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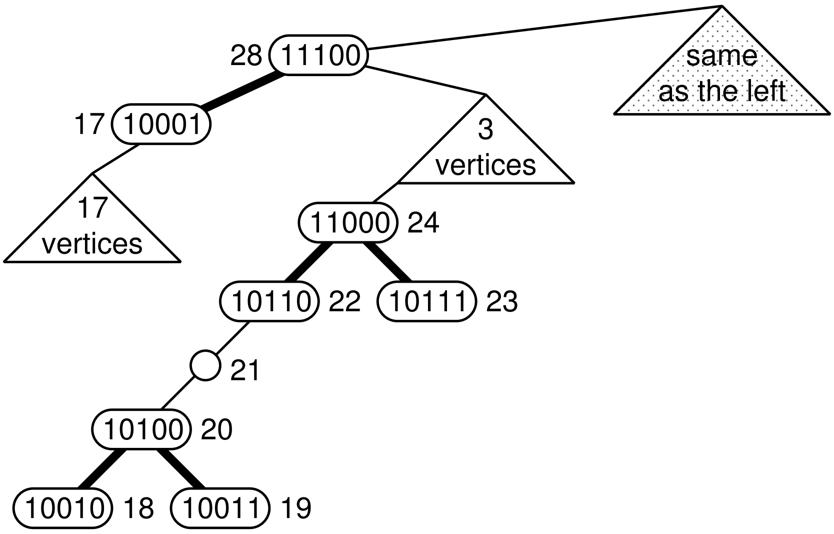

Figure 1: An example with 58 vertices of binary trees for which the congestion of $\langle \phi_2, \rho_2 \rangle$ is 5.

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