A Canonical Ensemble Approach to the Fermion/Boson Random Point Processes and its Applications

H. Tamura*
Department of Mathematics, Kanazawa University,
Kanazawa 920-1192, Japan
K. R. Ito†‡
Department of Mathematics and Physics, Setsunan University,
Neyagawa, Osaka 572-8508, Japan

May 1, 2007

Abstract

We introduce the boson and the fermion point processes from the elementary quantum mechanical point of view. That is, we consider quantum statistical mechanics of canonical ensemble for a fixed number of particles which obey Bose-Einstein, Fermi-Dirac statistics, respectively, in a finite volume. Focusing on the distribution of positions of the particles, we have point processes of the fixed number of points in a bounded domain. By taking the thermodynamic limit such that the particle density converges to a finite value, the boson/fermion processes are obtained. This argument is a realization of the equivalence of ensembles, since resulting processes are considered to describe a grand canonical ensemble of points. Random point processes corresponding to para-particles of order two are discussed as an application of the formulation. A statistics of a system of composite particles at zero temperature are also considered as a model of determinantal random point processes.

1 Introduction

As special classes of random point processes, fermion point processes and boson point processes have been studied by many authors since [1, 16, 17]. Among them, [8, 6, 9, 7] made a correspondence between boson processes and locally normal states on C^*-algebra of operators on the boson Fock space. A functional integral method is used in [15] to obtain these processes

*tamurah@kenroku.kanazawa-u.ac.jp
†ito@mpg.setsunan.ac.jp
‡also at : Division of Mathematics, College of Human and Environmental Studies, Kyoto University, Kyoto 606-8501, Japan

from quantum field theories of finite temperatures. On the other hand, [23] formulated both the fermion and boson processes in a unified way in terms of the Laplace transformation and generalized them. Let $Q(R)$ be the space of all the locally finite configurations over a Polish space R and K a locally trace class integral operator on $L^2(R)$ with a Radon measure λ on R . For any nonnegative function f having bounded support and $\xi = \sum_j \delta_{x_j} \in Q(R)$, we set $< \xi, f >= \sum_j f(x_j)$. Shirai and Takahashi [23] have formulated and studied the random processes $\mu_{\alpha,K}$ which have Laplace transformations

$$E[e^{-<f,\xi>}] \equiv \int_{Q(R)} \mu_{\alpha,K}(d\xi)\, e^{-<\xi,f>} = \mathrm{Det}\big(I + \alpha\sqrt{1-e^{-f}}K\sqrt{1-e^{-f}}\big)^{-1/\alpha} \tag{1.1}$$

for the parameters $\alpha \in \{2/m; m \in \mathbb{N}\} \cup \{-1/m; m \in N\}$.

Here the cases $\alpha = \pm 1$ correspond to boson/fermion processes, respectively.

In their argument, the generalized Vere-Jones' formula[29]

$$\mathrm{Det}(1-\alpha J)^{-1/\alpha} = \sum \frac{1}{n!} \int_{R^n} \det{}_\alpha (J(x_i,x_j))_{i,j=1}^n \lambda^{\otimes n}(dx_1 \cdots dx_n) \tag{1.2}$$

has played an essential role. Here J is a trace class integral operator, for which we need the condition $||\alpha J|| < 1$ unless $-1/\alpha \in \mathbb{N}$, $\mathrm{Det}(\,\cdot\,)$ the Fredholm determinant and $\det_\alpha A$ the *α-determinant* defined by

$$\det{}_\alpha A = \sum_{\sigma \in \mathcal{S}_n} \alpha^{n-\nu(\sigma)} \prod_i A_{i\sigma(i)} \tag{1.3}$$

for a matrix A of size $n \times n$, where $\nu(\sigma)$ is the numbers of cycles in σ. The formula (1.2) is Fredholm's original definition of his functional determinant in the case $\alpha = -1$.

The purpose of the paper is to construct both the fermion and boson processes from a view point of elementary quantum mechanics in order to get simple, clear and straightforward understanding of them in the connection with physics. Let us consider the system of N free fermions/bosons in a box of finite volume V in $\mathbb{R}^d$ and the quantum statistical mechanical state of the system with a finite temperature. Giving the distribution function of the positions of all particles in terms of the square of the absolute value of the wave functions, we obtain a point process of N points in the box. As the thermodynamic limit, $N, V \to \infty$ and $N/V \to \rho$, of these processes of finite points, fermion and boson processes in $\mathbb{R}^d$ with density ρ are obtained. In the argument, we will use the generalized Vere-Jones' formula in the form:

$$\frac{1}{N!}\int \det{}_\alpha (J(x_i,x_j))_{i,j=1}^N \lambda^{\otimes N}(dx_1 \cdots dx_N) = \oint_{S_r(0)} \frac{dz}{2\pi i z^{N+1}} \mathrm{Det}(1 - z\alpha J)^{-1/\alpha}, \tag{1.4}$$

where $r > 0$ is arbitrary for $-1/\alpha \in \mathbb{N}$, otherwise r should satisfy $||r\alpha J|| < 1$. Here and hereafter, $S_r(\zeta)$ denotes the integration contour defined by the map $\theta \mapsto \zeta + r\exp(i\theta)$, where θ ranges from $-\pi$ to π, $r > 0$ and $\zeta \in \mathbb{C}$. In the terminology of statistical mechanics, we start from canonical ensemble and end up with formulae like (1.1) and (1.2) of grand canonical nature. In this sense, the argument is related to the equivalence of ensembles. The use of (1.4) makes our approach simple. The thermodynamic limit has been discussed in [11] and [18] in the contexts of local current algebras for boson and fermion gases respectively at zero temperature.

In our approach, we need neither quantum field theories nor the theory of states on the operator algebras to derive the boson/fermion processes. It is interesting to apply the method to the problems which have not been formulated in statistical mechanics on quantum field theories yet. Here, we study the system of para-fermions and para-bosons of order 2. Para statistics was first introduced by Green[10] in the context of quantum field theories. For its review, see [21]. [19] and [12, 27] formulated it within the framework of quantum mechanics of finite number of particles. See also [20]. Recently statistical mechanics of para-particles are formulated in [28, 2, 3]. However, it does not seem to be fully developed so far. We formulate here point processes as the distributions of positions of para-particles of order 2 with finite temperature and positive density through the thermodynamic limit. It turns out that the resulting processes are corresponding to the cases $\alpha = \pm 1/2$ in [23]. We also try to derive point processes from ensembles of composite particles at zero temperature and positive density in this formalism. The resulting processes also have their Laplace transforms expressed by Fredholm determinants.

This paper is organized as follows. In Section 2, the random point processes of fixed numbers of fermions as well as bosons are formulated on the base of quantum mechanics in a bounded box. Then, the theorems on thermodynamic limits are stated. The proofs of the theorems are presented in Section 3 as applications of a theorem of rather abstract form. In Sections 4 and 5, we consider the systems of para-particles and composite particles, respectively. In Appendix, we calculate complex integrals needed for the thermodynamic limits.

2 Fermion and boson processes

Consider $L^2(\Lambda_L)$ on $\Lambda_L = [-L/2, L/2]^d \subset \mathbb{R}^d$ with the Lebesgue measure on Λ_L. Let $\triangle_L$ be the Laplacian on $\mathcal{H}_L = L^2(\Lambda_L)$ satisfying periodic boundary conditions at $\partial\Lambda_L$. We deal with periodic boundary conditions in this paper, however, all the arguments except that in section 5 may be applied for other boundary conditions. Hereafter we regard $-\triangle_L$ as the quantum mechanical Hamiltonian of a single free particle. The usual factor $\hbar^2/2m$ is set at unity. For $k \in \mathbb{Z}^d$, $\varphi_k^{(L)}(x) = L^{-d/2}\exp(i2\pi k \cdot x/L)$ is an eigenfunction of $\triangle_L$, and $\{\varphi_k^{(L)}\}_{k\in\mathbb{Z}^d}$ forms an complete orthonormal system [CONS] of $\mathcal{H}_L$. In the following, we use the operator $G_L = \exp(\beta\triangle_L)$ whose kernel is given by

$$G_L(x,y) = \sum_{k\in\mathbb{Z}^d} e^{-\beta|2\pi k/L|^2}\varphi_k^{(L)}(x)\overline{\varphi_k^{(L)}(y)} \tag{2.1}$$

for $\beta > 0$. We put $g_k^{(L)} = \exp(-\beta|2\pi k/L|^2)$, the eigenvalue of G_L for the eigenfunction $\varphi_k^{(L)}$. We also need $G = \exp(\beta\triangle)$ on $L^2(\mathbb{R}^d)$ and its kernel

$$G(x,y) = \int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d} e^{-\beta|p|^2 + ip\cdot(x-y)} = \frac{\exp(-|x-y|^2/4\beta)}{(4\pi\beta)^{d/2}}.$$

Note that $G_L(x,y)$ and $G(x,y)$ are real symmetric and

$$G_L(x,y) = \sum_{k\in\mathbb{Z}^d} G(x, y+kL). \tag{2.2}$$

Let $f : \mathbb{R}^d \to [0,\infty)$ be an arbitrary continuous function whose support is compact. In the course of the thermodynamic limit, f is fixed and we assume that L is so large that Λ_L contains the support, and regard f as a function on Λ_L.

2.1 Fermion processes

In this subsection, we construct the fermion process in $\mathbb{R}^d$ as a limit of the process of N points in Λ_L. Suppose there are N identical particles which obey the Fermi-Dirac statistics in a finite box Λ_L. The space of the quantum mechanical states of the system is given by

$$\mathcal{H}^F_{L,N} = \{ A_N f \mid f \in \otimes^N \mathcal{H}_L \},$$

where

$$A_N f(x_1, \cdots, x_N) = \frac{1}{N!} \sum_{\sigma \in \mathcal{S}_N} \mathrm{sgn}(\sigma) f(x_{\sigma(1)}, \cdots, x_{\sigma(N)}) \qquad (\, x_1, \cdots, x_N \in \Lambda_L \,)$$

is anti-symmetrization in the N indices. Using the CONS $\{ \varphi^{(L)}_k \}_{k \in \mathbb{Z}^d}$ of $\mathcal{H}_L = L^2(\Lambda_L)$, we make the element

$$\Phi_k(x_1, \cdots, x_N) = \frac{1}{\sqrt{N!}} \sum_{\sigma \in \mathcal{S}_N} \mathrm{sgn}(\sigma) \varphi_{k_1}(x_{\sigma(1)}) \cdot \cdots \cdot \varphi_{k_N}(x_{\sigma(N)}) \tag{2.3}$$

of $\mathcal{H}^F_{L,N}$ for $k = (k_1, \cdots, k_N) \in (\mathbb{Z}^d)^N$. Let us introduce the lexicographic order $\prec$ in $\mathbb{Z}^d$ and put $(\mathbb{Z}^d)^N_{\precneqq} = \{ (k_1, \cdots, k_N) \in (\mathbb{Z}^d)^N \mid k_1 \precneqq \cdots \precneqq k_N \}$, then $\{ \Phi_k \}_{k \in (\mathbb{Z}^d)^N_{\precneqq}}$ forms a CONS of $\mathcal{H}^F_{L,N}$.

According to the idea of the canonical ensemble in quantum statistical mechanics, the probability density distribution of the positions of the N free fermions in the periodic box Λ_L at the inverse temperature β is given by

$$\begin{aligned} p^F_{L,N}(x_1, \cdots, x_N) &= Z_F^{-1} \sum_{k \in (\mathbb{Z}^d)^N_{\precneqq}} \Big(\prod_{j=1}^N g^{(L)}_{k_j} \Big) |\Phi_k(x_1, \cdots, x_N)|^2 \\ &= Z_F^{-1} \sum_{k \in (\mathbb{Z}^d)^N_{\precneqq}} \overline{\Phi_k(x_1, \cdots, x_N)} \big((\otimes^N G_L) \Phi_k \big)(x_1, \cdots, x_N) \end{aligned} \tag{2.4}$$

where Z_F is the normalization constant. We can define the point process of N points in Λ_L from the density (2.4). I.e., consider a map $\Lambda_L^N \ni (x_1, \cdots, x_N) \mapsto \sum_{j=1}^N \delta_{x_j} \in Q(\mathbb{R}^d)$. Let $\mu^F_{L,N}$ be the probability measure on $Q(\mathbb{R}^d)$ induced by the map from the probability measure on Λ_L^N which has the density (2.4). By $\mathrm{E}^F_{L,N}$, we denote expectation with respect to the measure $\mu^F_{L,N}$. The Laplace transform of the point process is given by

$$\begin{aligned} \mathrm{E}^F_{L,N}\big[e^{-<f,\xi>} \big] &= \int_{Q(\mathbb{R}^d)} d\mu^F_{L,N}(\xi)\, e^{-<f,\xi>} \\ &= \int_{\Lambda_L^N} \exp(-\sum_{j=1}^N f(x_j)) p^F_{L,N}(x_1, \cdots, x_N)\, dx_1 \cdots dx_N \\ &= \frac{\mathrm{Tr}_{\,\mathcal{H}^F_{L,N}}[(\otimes^N e^{-f})(\otimes^N G_L)]}{\mathrm{Tr}_{\,\mathcal{H}^F_{L,N}}[\otimes^N G_L]} \\ &= \frac{\mathrm{Tr}_{\,\otimes^N \mathcal{H}_L}[(\otimes^N \tilde{G}_L) A_N]}{\mathrm{Tr}_{\,\otimes^N \mathcal{H}_L}[(\otimes^N G_L) A_N]} \\ &= \frac{\int_{\Lambda_L^N} \det_{-1} \tilde{G}_L(x_i, x_j)\, dx_1 \cdots dx_N}{\int_{\Lambda_L^N} \det_{-1} G_L(x_i, x_j)\, dx_1 \cdots dx_N}, \end{aligned} \tag{2.5}$$

where $\tilde{G}_L$ is defined by

$$\tilde{G}_L = G_L^{1/2} e^{-f} G_L^{1/2}, \tag{2.6}$$

where e^{-f} represents the operator of multiplication by the function e^{-f}.

The fifth expression follows from $[\otimes^N G_L^{1/2}, A_N] = 0$, cyclicity of the trace and $(\otimes^N G_L^{1/2})(\otimes^N e^{-f})(\otimes^N G_L^{1/2}) = \otimes^N \tilde{G}_L$ and so on. The last expression can be obtained by calculating the trace on $\otimes^N \mathcal{H}_L$ using its CONS $\{\varphi_{k_1} \otimes \cdots \otimes \varphi_{k_N} \,|\, k_1, \cdots, k_N \in \mathbb{Z}^d\}$, where $\det_{-1}$ is the usual determinant, see eq. (1.3).

Now, let us consider the thermodynamic limit, where the volume of the box Λ_L and the number of points N in the box Λ_L tend to infinity in such a way that the densities tend to a positive finite value ρ, i.e.,

$$L,\ N \to \infty, \quad N/L^d \to \rho > 0. \tag{2.7}$$

Theorem 2.1 *The finite fermion processes $\{ \mu^F_{L,N} \}$ defined above converge weakly to the fermion process μ^F_ρ whose Laplace transform is given by*

$$\int_{Q(\mathbb{R}^d)} e^{-<f,\xi>} d\mu^F_\rho(\xi) = \mathrm{Det}\big[1 - \sqrt{1-e^{-f}}\, z_* G (1+z_* G)^{-1} \sqrt{1-e^{-f}}\big] \tag{2.8}$$

in the thermodynamic limit (2.7), where z_* is the positive number uniquely determined by

$$\rho = \int \frac{dp}{(2\pi)^d} \frac{z_* e^{-\beta|p|^2}}{1+z_* e^{-\beta|p|^2}} = (z_* G(1+z_* G)^{-1})(x,x).$$

Remark : The existence of μ^F_ρ which has the above Laplace transform is a consequence of the result of [23] we have mentioned in the introduction.

2.2 Boson processes

Suppose there are N identical particles which obey Bose-Einstein statistics in a finite box Λ_L. The space of the quantum mechanical states of the system is given by

$$\mathcal{H}^B_{L,N} = \{ S_N f \,|\, f \in \otimes^N \mathcal{H}_L \},$$

where

$$S_N f(x_1, \cdots, x_N) = \frac{1}{N!} \sum_{\sigma \in \mathcal{S}_N} f(x_{\sigma(1)}, \cdots x_{\sigma(N)}) \qquad (\ x_1, \cdots, x_N \in \Lambda_L\ )$$

is symmetrization in the N indices. Using the CONS $\{ \varphi^{(L)}_k \}_{k \in \mathbb{Z}^d}$ of $L^2(\Lambda_L)$, we make the element

$$\Psi_k(x_1, \cdots, x_N) = \frac{1}{\sqrt{N! n(k)}} \sum_{\sigma \in \mathcal{S}_N} \varphi_{k_1}(x_{\sigma(1)}) \cdot \cdots \cdot \varphi_{k_N}(x_{\sigma(N)}) \tag{2.9}$$

of $\mathcal{H}^B_{L,N}$ for $k = (k_1, \cdots, k_N) \in \mathbb{Z}^d$, where $n(k) = \prod_{l \in \mathbb{Z}^d} (\sharp\{ n \in \{1, \cdots, N\} \,|\, k_n = l \}!)$. Let us introduce the subset $(\mathbb{Z}^d)^N_\prec = \{ (k_1, \cdots, k_N) \in (\mathbb{Z}^d)^N \,|\, k_1 \prec \cdots \prec k_N \}$ of $(\mathbb{Z}^d)^N$, then $\{ \Psi_k \}_{k \in (\mathbb{Z}^d)^N_\prec}$ forms a CONS of $\mathcal{H}^B_{L,N}$.

As in the fermion's case, the probability density distribution of the positions of the N free bosons in the periodic box Λ_L at the inverse temperature β is given by

$$p^B_{L,N}(x_1, \cdots, x_N) = Z_B^{-1} \sum_{k \in (\mathbb{Z}^d)^N_{\prec}} \Big(\prod_{j=1}^N g^{(L)}_{k_j} \Big) |\Psi_k(x_1, \cdots, x_N)|^2, \tag{2.10}$$

where Z_B is the normalization constant.

We can define a point process of N points $\mu^B_{L,N}$ from the density (2.10) as in the previous section. The Laplace transform of the point process is given by

$$\mathrm{E}^B_{L,N}\big[e^{-<f,\xi>}\big] = \frac{\mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N \tilde{G}_L) S_N]}{\mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L) S_N]} = \frac{\int_{\Lambda_L^N} \det_1 \tilde{G}_L(x_i, x_j)\, dx_1 \cdots dx_N}{\int_{\Lambda_L^N} \det_1 G_L(x_i, x_j)\, dx_1 \cdots dx_N}, \tag{2.11}$$

where $\det_1$ denotes permanent, see eq. (1.3). We set

$$\rho_c = \int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d} \frac{e^{-\beta|p|^2}}{1 - e^{-\beta|p|^2}}, \tag{2.12}$$

which is finite for $d > 2$. Now, we have

Theorem 2.2 *The finite boson processes* $\{\, \mu^B_{L,N} \,\}$ *defined above converge weakly to the boson process* μ^B_ρ *whose Laplace transform is given by*

$$\int_{Q(\mathbb{R}^d)} e^{-<f,\xi>} d\mu^B_\rho(\xi) = \mathrm{Det}[1 + \sqrt{1 - e^{-f}} z_* G (1 - z_* G)^{-1} \sqrt{1 - e^{-f}}]^{-1} \tag{2.13}$$

in the thermodynamic limit (2.7) if

$$\rho = \int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d} \frac{z_* e^{-\beta|p|^2}}{1 - z_* e^{-\beta|p|^2}} = (z_* G (1 - z_* G)^{-1})(x, x) < \rho_c.$$

Remark 1 : For the existence of μ^B_ρ , we refer to [23].
Remark 2 : In this paper, we only consider the boson processes with low densities : $\rho < \rho_c$. The high density cases $\rho \geqslant \rho_c$ are related to the Bose-Einstein condensation. We need the detailed knowledge about the spectrum of $\tilde{G}_L$ to deal with these cases. It will be reported in another publication.

3 Thermodynamic limits

3.1 A general framework

It is convenient to consider the problem in a general framework on a Hilbert space $\mathcal{H}$ over $\mathbb{C}$. The proofs of the theorems of section 2 are given in the next subsection. We denote the operator norm by $||\cdot||$, the trace norm by $||\cdot||_1$ and the Hilbert-Schmidt norm by $||\cdot||_2$.

Let $\{V_L\}_{L>0}$ be a one-parameter family of Hilbert-Schmidt operators on $\mathcal{H}$ which satisfies the conditions

$$\forall L > 0 : ||\, V_L \,|| = 1, \quad \lim_{L\to\infty} ||\, V_L \,||_2 = \infty$$

and A a bounded self-adjoint operator on $\mathcal{H}$ satisfying $0 \leqslant A \leqslant 1$. Then $G_L = V_L^* V_L,\ \tilde{G}_L = V_L^* A V_L$ are self-adjoint trace class operators satisfying

$$\forall L > 0 : 0 \leqslant \tilde{G}_L \leqslant G_L \leqslant 1,\ ||G_L|| = 1 \ \text{ and } \ \lim_{L\to\infty} \mathrm{Tr}\, G_L = \infty.$$

We define $I_{-1/n} = [0, \infty)$ for $n \in \mathbb{N}$ and $I_\alpha = [0, 1/|\alpha|)$ for $\alpha \in [-1, 1] - \{0, -1, -1/2, \cdots\}$. Then the function

$$h_L^{(\alpha)}(z) = \frac{\mathrm{Tr}\,[zG_L(1 - z\alpha G_L)^{-1}]}{\mathrm{Tr}\, G_L}$$

is well defined on I_α for each $L > 0$ and $\alpha \in [-1, 1] - \{0\}$.

Theorem 3.1 *Let $\alpha \in [-1, 1] - \{0\}$ be arbitrary but fixed. Suppose that for every $z \in I_\alpha$, there exist a limit $h^{(\alpha)}(z) = \lim_{L\to\infty} h_L^{(\alpha)}(z)$ and a trace class operator K_z satisfying*

$$\lim_{L\to\infty} ||\, K_z - (1 - A)^{1/2} V_L (1 - z\alpha V_L^* V_L)^{-1} V_L^* (1 - A)^{1/2} ||_1 = 0. \tag{3.1}$$

Then, for every $\hat{\rho} \in [0, \sup_{z\in I_\alpha} h^{(\alpha)}(z))$, there exists a unique solution $z = z_* \in I_\alpha$ of $h^{(\alpha)}(z) = \hat{\rho}$. Moreover suppose that a sequence $L_1 < L_2 < \cdots < L_N < \cdots$ satisfies

$$\lim_{N\to\infty} N/\mathrm{Tr}\, G_{L_N} = \hat{\rho}. \tag{3.2}$$

Then

$$\lim_{N\to\infty} \frac{\sum_{\sigma\in\mathcal{S}_N} \alpha^{N-\nu(\sigma)} \mathrm{Tr}_{\otimes^N\mathcal{H}} [\otimes^N \tilde{G}_{L_N} U(\sigma)]}{\sum_{\sigma\in\mathcal{S}_N} \alpha^{N-\nu(\sigma)} \mathrm{Tr}_{\otimes^N\mathcal{H}} [\otimes^N G_{L_N} U(\sigma)]} = \mathrm{Det}[1 + z_* \alpha K_{z_*}]^{-1/\alpha} \tag{3.3}$$

holds. Here the operator $U(\sigma)$ on $\otimes^N\mathcal{H}$ is defined by $U(\sigma)\varphi_1 \otimes \cdots \otimes \varphi_N = \varphi_{\sigma^{-1}(1)} \otimes \cdots \otimes \varphi_{\sigma^{-1}(N)}$ for $\sigma \in \mathcal{S}_N$ and $\varphi_1, \cdots, \varphi_N \in \mathcal{H}$.

In order to prove the theorem, we prepare several lemmas under the same assumptions of the theorem.

Lemma 3.2 *$h^{(\alpha)}$ is a strictly increasing continuous function on I_α and there exists a unique $z_* \in I_\alpha$ which satisfies $h^{(\alpha)}(z_*) = \hat{\rho}$.*

Proof : From $h_L^{(\alpha)\prime}(z) = \mathrm{Tr}\,[G_L(1 - z\alpha G_L)^{-2}]/\mathrm{Tr}\, G_L$, we have $1 \leqslant h_L^{(\alpha)\prime}(z) \leqslant (1 - z\alpha)^{-2}$ for $\alpha > 0$ and $1 \geqslant h_L^{(\alpha)\prime}(z) \geqslant (1 - z\alpha)^{-2}$ for $\alpha < 0$, i.e., $\{h_L^{(\alpha)}\}_{\{L>0\}}$ is equi-continuous on I_α. By Ascoli-Arzelà's theorem, the convergence $h_L^{(\alpha)} \to h^{(\alpha)}$ is locally uniform and hence $h^{(\alpha)}$ is continuous on I_α. It also follows that $h^{(\alpha)}$ is strictly increasing. Together with $h^{(\alpha)}(0) = 0$, which comes from $h_L^{(\alpha)}(0) = 0$, we get that $h^{(\alpha)}(z) = \hat{\rho}$ has a unique solution in I_α. □

Lemma 3.3 *There exists a constant $c_0 > 0$ such that*

$$||G_L - \tilde{G}_L||_1 = \mathrm{Tr}\,[V_L^*(1-A)V_L] \leqslant c_0$$

uniformly in $L > 0$.

Proof : Since $1 - z\alpha G_L$ is invertible for $z \in I_\alpha$ and V_L is Hilbert-Schmidt, we have

$$\begin{aligned}
&\mathrm{Tr}\,[V_L^*(1-A)V_L] \\
&= \mathrm{Tr}\,[(1-z\alpha G_L)^{1/2}(1-z\alpha G_L)^{-1/2}V_L^*(1-A)V_L(1-z\alpha G_L)^{-1/2}(1-z\alpha G_L)^{1/2}] \\
&\leqslant ||1-z\alpha G_L||\mathrm{Tr}\,[(1-z\alpha G_L)^{-1/2}V_L^*(1-A)V_L(1-z\alpha G_L)^{-1/2}] \\
&= ||1-z\alpha G_L||\mathrm{Tr}\,[(1-A)^{1/2}V_L(1-z\alpha G_L)^{-1}V_L^*(1-A)^{1/2}] \\
&= (1-(\alpha\wedge 0)z)(\mathrm{Tr}\,K_z + o(1)).
\end{aligned} \tag{3.4}$$

Here we have used $|\mathrm{Tr}\,B_1 C B_2| \leqslant ||B_1||\,||B_2||\,||C||_1 = ||B_1||\,||B_2||\mathrm{Tr}\,C$ for bounded operators B_1, B_2 and a positive trace class operator C and $\mathrm{Tr}\,WV = \mathrm{Tr}\,VW$ for Hilbert-Schmidt operators W, V. □

Let us denote all the eigenvalues of G_L and $\tilde{G}_L$ in decreasing order

$$g_0(L) = 1 \geqslant g_1(L) \geqslant \cdots \geqslant g_j(L) \geqslant \cdots$$

and

$$\tilde{g}_0(L) \geqslant \tilde{g}_1(L) \geqslant \cdots \geqslant \tilde{g}_j(L) \geqslant \cdots,$$

respectively. Then we have

Lemma 3.4 *For each $j = 0, 1, 2, \cdots,$ $g_j(L) \geqslant \tilde{g}_j(L)$ holds.*

Proof: By the min-max principle, we have

$$\begin{aligned}
\tilde{g}_j(L) &= \min_{\psi_0,\cdots,\psi_{j-1}\in\mathcal{H}_L}\ \max_{\psi\in\{\psi_0,\cdots,\psi_{j-1}\}^\perp} \frac{(\psi, \tilde{G}_L\psi)}{||\psi||^2} \\
&\leqslant \min_{\psi_0,\cdots,\psi_{j-1}\in\mathcal{H}_L}\ \max_{\psi\in\{\psi_0,\cdots,\psi_{j-1}\}^\perp} \frac{(\psi, G_L\psi)}{||\psi||^2} = g_j(L).
\end{aligned}$$

□

Lemma 3.5 *For N large enough, the conditions*

$$\mathrm{Tr}\,[z_N G_{L_N}(1-\alpha z_N G_{L_N})^{-1}] = \mathrm{Tr}\,[\tilde{z}_N \tilde{G}_{L_N}(1-\alpha\tilde{z}_N\tilde{G}_{L_N})^{-1}] = N \tag{3.5}$$

determine $z_N, \tilde{z}_N \in I_\alpha$ uniquely. z_N and $\tilde{z}_N$ satisfy

$$z_N \leqslant \tilde{z}_N, \quad |\tilde{z}_N - z_N| = O(1/N) \quad \text{and} \quad \lim_{N\to\infty} z_N = \lim_{N\to\infty}\tilde{z}_N = z_*.$$

Proof : From the proof of Lemma 3.2, $H_N(z) = \mathrm{Tr}\,[zG_{L_N}(1-z\alpha G_{L_N})^{-1}] = h^{(\alpha)}_{L_N}(z)\mathrm{Tr}\,G_{L_N}$ is a strictly increasing continuous function on I_α and $H_N(0) = 0$. Let us pick $z_0 \in I_\alpha$ such that $z_0 > z_*$. Since $h^{(\alpha)}$ is strictly increasing, $h^{(\alpha)}(z_0) - h^{(\alpha)}(z_*) = \epsilon > 0$. We have

$$\frac{H_N(z_0)}{N} = \frac{\mathrm{Tr}\,G_{L_N}}{N} h_{L_N}(z_0) \to \frac{h^{(\alpha)}(z_0)}{\hat{\rho}} = 1 + \frac{\epsilon}{\hat{\rho}}, \tag{3.6}$$

which shows $H_N\big((\sup I_\alpha) - 0\big) \geqslant H_N(z_0) > N$ for large enough N. Thus $z_N \in [0, z_0) \subset I_\alpha$ is uniquely determined by $H_N(z_N) = N$.

Put $\tilde{H}_N(z) = \mathrm{Tr}\,[z\tilde{G}_{L_N}(1-z\alpha\tilde{G}_{L_N})^{-1}]$. Then by Lemma 3.4, $\tilde{H}_N$ is well-defined on I_α and $\tilde{H}_N \leqslant H_N$ there. Moreover

$$\begin{aligned} H_N(z) - \tilde{H}_N(z) &= \mathrm{Tr}\,[(1-\alpha z G_{L_N})^{-1} z (G_{L_N} - \tilde{G}_{L_N})(1-\alpha z\tilde{G}_{L_N})^{-1}] \\ &\leqslant ||(1-\alpha z G_{L_N})^{-1}||\,||(1-\alpha z\tilde{G}_{L_N})^{-1}||z\mathrm{Tr}\,[G_{L_N} - \tilde{G}_{L_N}] \\ &\leqslant C_z = \frac{zc_0}{(1-(\alpha\vee 0)z)^2} \end{aligned}$$

holds. Together with (3.6), we have

$$\frac{\tilde{H}_N(z_0)}{N} \geqslant \frac{H_N(z_0) - C_{z_0}}{N} > 1 + \frac{\epsilon}{2\hat{\rho}} - \frac{C_{z_0}}{N},$$

hence $\tilde{H}_N(z_0) > N$, if N is large enough. It is also obvious that $\tilde{H}_N$ is strictly increasing and continuous on I_α and $\tilde{H}_N(0) = 0$. Thus $\tilde{z}_N \in [0, z_0) \subset I_\alpha$ is uniquely determined by $\tilde{H}_N(\tilde{z}_N) = N$.

The convergence $z_N \to z_*$ is a consequence of $h^{(\alpha)}_{L_N}(z_N) = N/\mathrm{Tr}\,G_{L_N} \to \hat{\rho} = h^{(\alpha)}(z_*)$, the strict increasingness of $h^{(\alpha)}, h^{(\alpha)}_L$ and the pointwise convergence $h^{(\alpha)}_L \to h^{(\alpha)}$. We get $z_N \leqslant \tilde{z}_N$ from $H_N \geqslant \tilde{H}_N$ and the increasingness of $H_N, \tilde{H}_N$.

Now, let us show $|\tilde{z}_N - z_N| = O(N^{-1})$, which together with $z_N \to z_*$, yields $\tilde{z}_N \to z_*$. From

$$\begin{aligned} 0 &= N - N = H_N(z_N) - \tilde{H}_N(\tilde{z}_N) \\ &= \mathrm{Tr}\,[(1-\alpha z_N G_{L_N})^{-1}(z_N G_{L_N} - \tilde{z}_N\tilde{G}_{L_N})(1-\alpha\tilde{z}_N\tilde{G}_{L_N})^{-1}] \\ &= z_N\mathrm{Tr}\,[(1-\alpha z_N G_{L_N})^{-1}(G_{L_N} - \tilde{G}_{L_N})(1-\alpha\tilde{z}_N\tilde{G}_{L_N})^{-1}] \\ &\quad -(\tilde{z}_N - z_N)\mathrm{Tr}\,[(1-\alpha z_N G_{L_N})^{-1}\tilde{G}_{L_N}(1-\alpha\tilde{z}_N\tilde{G}_{L_N})^{-1}], \end{aligned}$$

we get

$$\begin{aligned} &\frac{\tilde{z}_N - z_N}{\tilde{z}_N}\mathrm{Tr}\,[(1-\alpha z_N G_{L_N})^{-1/2}\tilde{z}_N\tilde{G}_{L_N}(1-\alpha\tilde{z}_N\tilde{G}_{L_N})^{-1}(1-\alpha z_N G_{L_N})^{-1/2}] \\ &= z_N\mathrm{Tr}\,[(1-\alpha z_N G_{L_N})^{-1}(G_{L_N} - \tilde{G}_{L_N})(1-\alpha\tilde{z}_N\tilde{G}_{L_N})^{-1}]. \end{aligned}$$

It follows that

$$\begin{aligned} &\frac{\tilde{z}_N - z_N}{\tilde{z}_N}N = \frac{\tilde{z}_N - z_N}{\tilde{z}_N}\tilde{H}_N(\tilde{z}_N) \\ &= \frac{\tilde{z}_N - z_N}{\tilde{z}_N}\mathrm{Tr}\,[(1-\alpha z_N G_{L_N})(1-\alpha z_N G_{L_N})^{-1/2}\tilde{z}_N\tilde{G}_{L_N}(1-\alpha\tilde{z}_N\tilde{G}_{L_N})^{-1}(1-\alpha z_N G_{L_N})^{-1/2}] \end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{\tilde z_N - z_N}{\tilde z_N} ||1-\alpha z_N G_{L_N}|| \mathrm{Tr}\,[(1-\alpha z_N G_{L_N})^{-1/2} \tilde z_N \tilde G_{L_N} (1-\alpha \tilde z_N \tilde G_{L_N})^{-1} (1-\alpha z_N G_{L_N})^{-1/2}] \\
&= ||1-\alpha z_N G_{L_N}|| z_N \mathrm{Tr}\,[(1-\alpha z_N G_{L_N})^{-1} (G_{L_N} - \tilde G_{L_N})(1-\alpha \tilde z_N \tilde G_{L_N})^{-1}] \\
&\leqslant z_N ||1-\alpha z_N G_{L_N}||\, ||(1-\alpha z_N G_{L_N})^{-1}||\, \mathrm{Tr}\,[G_{L_N} - \tilde G_{L_N}]\, ||(1-\alpha \tilde z_N \tilde G_{L_N})^{-1}|| \\
&\leqslant c_0 z_0 (1-(\alpha \wedge 0) z_0)/(1-(\alpha \vee 0) z_0)^2
\end{aligned}
$$

for N large enough, because $z_N, \tilde z_N < z_0$. Thus, we have obtained $\tilde z_N - z_N = O(N^{-1})$. □

We put

$$
v^{(N)} = \mathrm{Tr}\,[z_N G_{L_N} (1-\alpha z_N G_{L_N})^{-2}] \quad \text{and} \quad \tilde v^{(N)} = \mathrm{Tr}\,[\tilde z_N \tilde G_{L_N} (1-\alpha \tilde z_N \tilde G_{L_N})^{-2}].
$$

Then we have :

Lemma 3.6 (i) $v^{(N)}, \tilde v^{(N)} \to \infty$, (ii) $\dfrac{v^{(N)}}{\tilde v^{(N)}} \to 1$.

Proof : (i) follows from the lower bound

$$
\begin{aligned}
v^{(N)} &= \mathrm{Tr}\,[z_N G_{L_N} (1-\alpha z_N G_{L_N})^{-2}] \\
&\geqslant \mathrm{Tr}\,[z_N G_{L_N} (1-\alpha z_N G_{L_N})^{-1}]\, ||1-\alpha z_N G_{L_N}||^{-1} \geqslant N(1+o(1))/(1-(\alpha \wedge 0) z_*),
\end{aligned}
$$

since $z_N \to z_*$. The same bound is also true for $\tilde v^{(N)}$.
(ii) Using

$$
\begin{aligned}
v^{(N)} &= \mathrm{Tr}\,[-z_N G_{L_N} (1-\alpha z_N G_{L_N})^{-1} + \alpha^{-1} (1-\alpha z_N G_{L_N})^{-2} - \alpha^{-1}] \\
&= -N + \alpha^{-1} \mathrm{Tr}\,[(1-\alpha z_N G_{L_N})^{-2} - 1]
\end{aligned}
$$

and the same for $\tilde v^{(N)}$, we get

$$
\begin{aligned}
|\tilde v^{(N)} - v^{(N)}| &= |\alpha^{-1} \mathrm{Tr}\,[(1-\alpha \tilde z_N \tilde G_{L_N})^{-2} - (1-\alpha z_N G_{L_N})^{-2}]| \\
&\leqslant |\alpha^{-1} \mathrm{Tr}\,[\big((1-\alpha \tilde z_N \tilde G_{L_N})^{-1} - (1-\alpha z_N \tilde G_{L_N})^{-1}\big)(1-\alpha \tilde z_N \tilde G_{L_N})^{-1}]| \\
&\quad + |\alpha^{-1} \mathrm{Tr}\,[\big((1-\alpha z_N \tilde G_{L_N})^{-1} - (1-\alpha z_N G_{L_N})^{-1}\big)(1-\alpha \tilde z_N \tilde G_{L_N})^{-1}]| \\
&\quad + |\alpha^{-1} \mathrm{Tr}\,[(1-\alpha z_N G_{L_N})^{-1}\big((1-\alpha \tilde z_N \tilde G_{L_N})^{-1} - (1-\alpha z_N \tilde G_{L_N})^{-1}\big)]| \\
&\quad + |\alpha^{-1} \mathrm{Tr}\,[(1-\alpha z_N G_{L_N})^{-1}\big((1-\alpha z_N \tilde G_{L_N})^{-1} - (1-\alpha z_N G_{L_N})^{-1}\big)]| \\
&\leqslant ||(1-\alpha \tilde z_N \tilde G_{L_N})^{-1}||\, ||(1-\alpha \tilde z_N \tilde G_{L_N})^{-1} (\tilde z_N - z_N) \tilde G_{L_N} (1-\alpha z_N \tilde G_{L_N})^{-1}||_1 \\
&\quad + ||(1-\alpha \tilde z_N \tilde G_{L_N})^{-1}||\, ||(1-\alpha z_N \tilde G_{L_N})^{-1} z_N (G_{L_N} - \tilde G_{L_N})(1-\alpha z_N G_{L_N})^{-1}||_1 \\
&\quad + ||(1-\alpha z_N G_{L_N})^{-1}||\, ||(1-\alpha \tilde z_N \tilde G_{L_N})^{-1} (\tilde z_N - z_N) \tilde G_{L_N} (1-\alpha z_N \tilde G_{L_N})^{-1}||_1 \\
&\quad + ||(1-\alpha z_N G_{L_N})^{-1}||\, ||(1-\alpha z_N \tilde G_{L_N})^{-1} z_N (G_{L_N} - \tilde G_{L_N})(1-\alpha z_N G_{L_N})^{-1}||_1 \\
&\leqslant (||(1-\alpha \tilde z_N \tilde G_{L_N})^{-1}|| + ||(1-\alpha z_N G_{L_N})^{-1}||) \\
&\quad \times \Bigg(\frac{\tilde z_N - z_N}{\tilde z_N} ||\tilde z_N \tilde G_{L_N} (1-\alpha \tilde z_N \tilde G_{L_N})^{-1}||_1\, ||(1-\alpha z_N \tilde G_{L_N})^{-1}|| \\
&\quad + z_N ||(1-\alpha z_N \tilde G_{L_N})^{-1}||\, ||G_{L_N} - \tilde G_{L_N}||_1\, ||(1-\alpha z_N G_{L_N})^{-1}|| \Bigg) = O(1).
\end{aligned}
$$

In the last step, we have used Lemmas 3.3 and 3.5. This, together with (i), implies (ii). □

Lemma 3.7

$$\lim_{N\to\infty}\sqrt{2\pi v^{(N)}}\oint_{S_1(0)}\frac{d\eta}{2\pi i\eta^{N+1}}\mathrm{Det}\big[1-\alpha z_N(\eta-1)G_{L_N}(1-\alpha z_N G_{L_N})^{-1}\big]^{-1/\alpha}=1,$$

$$\lim_{N\to\infty}\sqrt{2\pi \tilde v^{(N)}}\oint_{S_1(0)}\frac{d\eta}{2\pi i\eta^{N+1}}\mathrm{Det}\big[1-\alpha \tilde z_N(\eta-1)\tilde G_{L_N}(1-\alpha \tilde z_N \tilde G_{L_N})^{-1}\big]^{-1/\alpha}=1,$$

Proof : Put $s=1/|\alpha|$ and

$$p_j^{(N)}=\frac{|\alpha|z_N g_j(L_N)}{1-\alpha z_N g_j(L_N)}.$$

Then the first equality is nothing but proposition A.2(i) for $\alpha<0$ and proposition A.2(ii) for $\alpha>0$. The same is true for the second equality. □

Proof of Theorem 3.1 : Since the uniqueness of z_* has already been shown, it is enough to prove (3.3). The main apparatus of the proof is Vere-Jones' formula in the following form: Let $\alpha=-1/n$ for $n\in\mathbb{N}$. Then

$$\mathrm{Det}(1-\alpha J)^{-1/\alpha}=\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n}\alpha^{n-\nu(\sigma)}\mathrm{Tr}_{\otimes^n\mathcal{H}}[(\otimes^n J)U(\sigma)]$$

holds for any trace class operator J. For $\alpha\in[-1,1]-\{0,-1,-1/2,\cdots,1/n,\cdots\}$, this holds under an additional condition $||\alpha J||<1$. This has actually been proved in Theorem 2.4 of [23]. We use the formula in the form

$$\frac{1}{N!}\sum_{\sigma\in\mathcal{S}_N}\alpha^{N-\nu(\sigma)}\mathrm{Tr}_{\otimes^N\mathcal{H}}[(\otimes^N G_{L_N})U(\sigma)]=\oint_{S_{z_N}(0)}\frac{dz}{2\pi i z^{N+1}}\mathrm{Det}(1-z\alpha G_{L_N})^{-1/\alpha}\tag{3.7}$$

and in the form in which G_{L_N} is replaced by $\tilde G_{L_N}$. Here, recall that $z_N,\tilde z_N\in I_\alpha$. We calculate the right-hand side by the saddle point method.

Using the above integral representation and the property of the products of the Fredholm determinants followed by the change of integral variables $z=z_N\eta, z=\tilde z_N\eta$, we get

$$\frac{\sum_{\sigma\in\mathcal{S}_N}\alpha^{N-\nu(\sigma)}\mathrm{Tr}_{\otimes^N\mathcal{H}}[\otimes^N\tilde G_{L_N}U(\sigma)]}{\sum_{\sigma\in\mathcal{S}_N}\alpha^{N-\nu(\sigma)}\mathrm{Tr}_{\otimes^N\mathcal{H}}[\otimes^N G_{L_N}U(\sigma)]}=\frac{\oint_{S_{\tilde z_N}(0)}\mathrm{Det}(1-z\alpha\tilde G_{L_N})^{-1/\alpha}dz/2\pi i z^{N+1}}{\oint_{S_{z_N}(0)}\mathrm{Det}(1-z\alpha G_{L_N})^{-1/\alpha}dz/2\pi i z^{N+1}}$$

$$\begin{aligned}=\;&\frac{\mathrm{Det}[1-\tilde z_N\alpha G_{L_N}]^{-1/\alpha}}{\mathrm{Det}[1-z_N\alpha G_{L_N}]^{-1/\alpha}}\frac{\mathrm{Det}[1-\tilde z_N\alpha\tilde G_{L_N}]^{-1/\alpha}}{\mathrm{Det}[1-\tilde z_N\alpha G_{L_N}]^{-1/\alpha}}\frac{z_N^N}{\tilde z_N^N}\\ &\times\frac{\oint_{S_1(0)}\mathrm{Det}[1-\tilde z_N(\eta-1)\alpha\tilde G_{L_N}(1-\tilde z_N\alpha\tilde G_{L_N})^{-1}]^{-1/\alpha}d\eta/2\pi i\eta^{N+1}}{\oint_{S_1(0)}\mathrm{Det}[1-z_N(\eta-1)\alpha G_{L_N}(1-z_N\alpha G_{L_N})^{-1}]^{-1/\alpha}d\eta/2\pi i\eta^{N+1}}.\end{aligned}$$

Thus the theorem is proved if the following behaviors in $N \to \infty$ are valid:

$$
\begin{array}{ll}
\text{(a)} & \dfrac{z_N^N}{\tilde{z}_N^N} = \exp\big(- \dfrac{\tilde{z}_N - z_N}{z_N} N + o(1) \big) \\
\text{(b)} & \dfrac{\mathrm{Det}[1 - \tilde{z}_N \alpha G_{L_N}]^{-1/\alpha}}{\mathrm{Det}[1 - z_N \alpha G_{L_N}]^{-1/\alpha}} = \exp\big(\dfrac{\tilde{z}_N - z_N}{z_N} N + o(1) \big), \\
\text{(c)} & \dfrac{\mathrm{Det}[1 - \tilde{z}_N \alpha \tilde{G}_{L_N}]^{-1/\alpha}}{\mathrm{Det}[1 - \tilde{z}_N \alpha G_{L_N}]^{-1/\alpha}} \to \mathrm{Det}[1 + z_* \alpha K_{z_*}]^{-1/\alpha} \\
\text{(d)} & \dfrac{\oint_{S_1(0)} \mathrm{Det}[1 - \tilde{z}_N(\eta - 1)\alpha \tilde{G}_{L_N}(1 - \tilde{z}_N \alpha \tilde{G}_{L_N})^{-1}]^{-1/\alpha} d\eta / 2\pi i \eta^{N+1}}{\oint_{S_1(0)} \mathrm{Det}[1 - z_N(\eta - 1)\alpha G_{L_N}(1 - z_N \alpha G_{L_N})^{-1}]^{-1/\alpha} d\eta / 2\pi i \eta^{N+1}} \quad \to 1.
\end{array}
$$

In fact, (a) is a consequence of Lemma 3.5. For (b), let us define a function $k(z) = \log \mathrm{Det}[1 - z\alpha G_L]^{-1/\alpha} = -\alpha^{-1} \sum_{j=0}^{\infty} \log(1 - z\alpha g_j(L))$. Then by Taylor's formula and (3.5), we get

$$
k(\tilde{z}_N) - k(z_N) = k'(z_N)(\tilde{z}_N - z_N) + k''(\bar{z}) \frac{(\tilde{z}_N - z_N)^2}{2}
$$

$$
= \sum_{j=0}^{\infty} \frac{g_j}{1 - z_N \alpha g_j} (\tilde{z}_N - z_N) + \sum_{j=0}^{\infty} \frac{\alpha g_j^2}{(1 - \bar{z}\alpha g_j)^2} \frac{(\tilde{z}_N - z_N)^2}{2} = N \frac{\tilde{z}_N - z_N}{z_N} + \delta,
$$

where $\bar{z}$ is a mean value of z_N and $\tilde{z}_N$ and $|\delta| = O(1/N)$ by Lemma 3.5.

From the property of the product and the cyclic nature of the Fredholm determinants, we have

$$
\begin{aligned}
&\frac{\mathrm{Det}[1 - \tilde{z}_N \alpha \tilde{G}_{L_N}]}{\mathrm{Det}[1 - \tilde{z}_N \alpha G_{L_N}]} \\
&\quad = \mathrm{Det}[1 + z_* \alpha (1 - A)^{1/2} V_{L_N} (1 - z_* \alpha G_{L_N})^{-1} V_{L_N}^* (1 - A)^{1/2}] \\
&\qquad + \big\{ \mathrm{Det}[1 + \tilde{z}_N \alpha (G_{L_N} - \tilde{G}_{L_N})(1 - \tilde{z}_N \alpha G_{L_N})^{-1}] \\
&\qquad - \mathrm{Det}[1 + z_* \alpha (G_{L_N} - \tilde{G}_{L_N})(1 - z_* \alpha G_{L_N})^{-1}] \big\}.
\end{aligned}
$$

The first term converges to $\mathrm{Det}[1 + z_* \alpha K_{z_*}]$ by the assumption (3.1) and the continuity of the Fredholm determinants with respect to the trace norm. The brace in the above equation tends to 0, because of the continuity and

$$
\begin{aligned}
&||\tilde{z}_N \alpha (G_{L_N} - \tilde{G}_{L_N})(1 - \tilde{z}_N \alpha G_{L_N})^{-1} - z_* \alpha (G_{L_N} - \tilde{G}_{L_N})(1 - z_* \alpha G_{L_N})^{-1}||_1 \\
&\quad \leqslant |\tilde{z}_N - z_*| \, |\alpha| \, ||G_{L_N} - \tilde{G}_{L_N}||_1 ||(1 - \tilde{z}_N \alpha G_{L_N})^{-1}|| \\
&\quad + z_* |\alpha| \, ||G_{L_N} - \tilde{G}_{L_N}||_1 ||(1 - \tilde{z}_N \alpha G_{L_N})^{-1} - (1 - z_* \alpha G_{L_N})^{-1}|| \to 0,
\end{aligned}
$$

where we have used Lemmas 3.3 and 3.5. Thus, we get (c). (d) is a consequence of Lemma 3.6 and Lemma 3.7.

3.2 Proofs of the theorems

To prove Theorem 2.1[2.2], it is enough to show that (2.5)[(2.11)] converges to the right-hand side of (2.8) [(2.13), respectively] for every $f \in C_o(\mathbb{R}^d)$[5]. We regard $\mathcal{H}_L = L^2(\Lambda_L)$ as a closed subspace of $L^2(\mathbb{R}^d)$. Corresponding to the orthogonal decomposition $L^2(\mathbb{R}^d) = L^2(\Lambda_L) \oplus L^2(\Lambda_L^c)$, we set $V_L = e^{\beta\triangle_L/2} \oplus 0$. Let $A = e^{-f}$ be the multiplication operator on $L^2(\mathbb{R}^d)$, which can be decomposed as $A = e^{-f}\chi_{\Lambda_L} \oplus \chi_{\Lambda_L^c}$ for large L since $\mathrm{supp}\, f$ is compact. Then

$$G_L = V_L^* V_L = e^{\beta\triangle_L} \oplus 0 \quad \text{and} \quad \tilde{G}_L = V_L^* A V_L = e^{\beta\triangle_L/2} e^{-f} e^{\beta\triangle_L/2} \oplus 0$$

can be identified with those in section 2.

We begin with the following fact, where we denote

$$\square_k^{(L)} \;=\; \frac{2\pi}{L}\Big(k + \Big(-\frac{1}{2}, \frac{1}{2}\Big]^d\Big) \qquad \text{for} \quad k \in \mathbb{Z}^d.$$

Lemma 3.8 *Let $b : [0, \infty) \to [0, \infty)$ be a monotone decreasing continuous function such that*

$$\int_{\mathbb{R}^d} b(|p|)dp < \infty.$$

Define the function $b_L : \mathbb{R}^d \to [0, \infty)$ by

$$b_L(p) = b(|2\pi k/L|) \qquad \textit{if} \quad p \in \square_k^{(L)} \quad \textit{for} \quad k \in \mathbb{Z}^d.$$

Then $b_L(p) \to b(|p|)$ in $L^1(\mathbb{R}^d)$ as $L \to \infty$.

Proof : There exist positive constants c_1 and c_2 such that $b_L(p) \leqslant c_1 b(c_2|p|)$ holds for all $L \geqslant 1$ and $p \in \mathbb{R}^d$. Indeed, $c_1 = b(0)/b(2\pi\sqrt{d/(d+8)})$, $c_2 = 2/\sqrt{d+8}$ satisfy the condition, since $\inf\{\, c_1 b(c_2|p|) \,|\, p \in \square_0^{(L)} \,\} \geqslant b(0)$ for $\forall L > 1$ and $\sup\{\, c_2|p| \,|\, p \in \square_k^{(L)} \,\} \leqslant 2\pi|k|/L$ for $k \in \mathbb{Z}^d - \{0\}$. Obviously $c_1 b(c_2|p|)$ is an integrable function of $p \in \mathbb{R}^d$. The lemma follows by the dominated convergence theorem. $\square$

Finally we confirm the assumptions of theorem 3.1.

Proposition 3.9

(i) $\quad \forall L > 0 : ||V_L|| = 1, \qquad \lim_{L\to\infty} \mathrm{Tr}\, G_L/L^d = (4\pi\beta)^{-d/2}. \qquad (3.8)$

(ii) *The following convergences hold as $L \to \infty$ for each $z \in I_\alpha$:*

$$h_L^{(\alpha)}(z) = \frac{\mathrm{Tr}\,[zG_L(1 - z\alpha G_L)^{-1}]}{\mathrm{Tr}\, G_L} \;\to\; (4\pi\beta)^{d/2} \int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d} \frac{ze^{-\beta|p|^2}}{1 - z\alpha e^{-\beta|p|^2}} = h^{(\alpha)}(z), \quad (3.9)$$

$$||\sqrt{1 - e^{-f}}\big(G_L(1 - z\alpha G_L)^{-1} - G(1 - z\alpha G)^{-1}\big)\sqrt{1 - e^{-f}}||_1 \to 0. \qquad (3.10)$$

Proof : By applying the above lemma to $b(|p|) = e^{-\beta|p|^2}$ and $\tilde{b}(|p|) = ze^{-\beta|p|^2}/(1 - z\alpha e^{-\beta|p|^2})$, we have (3.8) and (3.9).

By Grüm's convergence theorem, it is enough to show

$$\sqrt{1 - e^{-f}} G_L(1 - z\alpha G_L)^{-1}\sqrt{1 - e^{-f}} \to \sqrt{1 - e^{-f}} G(1 - z\alpha G)^{-1}\sqrt{1 - e^{-f}}$$

strongly and

$$\mathrm{Tr}\,[\sqrt{1 - e^{-f}} G_L(1 - z\alpha G_L)^{-1}\sqrt{1 - e^{-f}}] = \int_{\mathbb{R}^d} (1 - e^{-f(x)})\big(G_L(1 - z\alpha G_L)^{-1}\big)(x, x)dx$$

$$\to \int_{\mathbb{R}^d} (1 - e^{-f(x)})\big(G(1 - z\alpha G)^{-1}\big)(x, x)dx = \mathrm{Tr}\,[\sqrt{1 - e^{-f}} G(1 - z\alpha G)^{-1}\sqrt{1 - e^{-f}}]$$

for (3.10). These are direct consequences of

$$\begin{aligned}
&|zG_L(1-z\alpha G_L)^{-1}(x,y) - zG(1-z\alpha G)^{-1}(x,y)| \\
&= \int \frac{dp}{(2\pi)^d} |e_L(p,x-y)\tilde{b}_L(p) - e(p,x-y)\tilde{b}(|p|)| \\
&\leqslant \int \frac{dp}{(2\pi)^d} \big(|\tilde{b}_L(p) - \tilde{b}(|p|)| + |e_L(p,x-y) - e(p,x-y)|\tilde{b}(|p|)\big) \to 0
\end{aligned}$$

uniformly in $x, y \in \operatorname{supp} f$. Here we have used the above lemma for $\tilde{b}(|p|)$ and we put $e(p,x) = e^{ip\cdot x}$ and

$$e_L(p;x) = e(2\pi k/L; x) \quad \text{if} \quad p \in \Box_k^{(L)} \quad \text{for} \quad k \in \mathbb{Z}^d. \qquad \Box$$

Thanks to (3.8), we can take a sequence $\{L_N\}_{N\in\mathbb{N}}$ which satisfies (3.2). On the relation between ρ in Theorems 2.1, 2.2 and $\hat{\rho}$ in Theorem 3.1, $\hat{\rho} = (4\pi\beta)^{d/2}\rho$ is derived from (2.7). We have the ranges of ρ in Theorem 2.2 and Theorem 2.1, since

$$\sup_{z\in I_1} h^{(1)}(z) = (4\pi\beta)^{d/2} \int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d} \frac{e^{-\beta|p|^2}}{1-e^{-\beta|p|^2}} = (4\pi\beta)^{d/2}\rho_c$$

and $\sup_{z\in I_{-1}} h^{(-1)}(z) = \infty$ from (3.9). Thus we get Theorem 2.1 and Theorem 2.2 using Theorem 3.1.

4 Para-particles

The purpose of this section is to apply the method which we have developed in the preceding sections to statistical mechanics of gases which consist of identical particles obeying para-statistics. Here, we restrict our attention to para-fermions and para-bosons of order 2. We will see that the point processes obtained after the thermodynamic limit are the point processes corresponding to the cases of $\alpha = \pm 1/2$ given in [23].

In this section, we use the representation theory of the symmetric group (cf. e.g. [13, 22, 25]). We say that $(\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{N}^n$ is a Young frame of length n for the symmetric group $\mathcal{S}_N$ if

$$\sum_{j=1}^n \lambda_j = N, \quad \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n > 0.$$

We associate the Young frame $(\lambda_1, \lambda_2, \cdots, \lambda_n)$ with the diagram of λ_1-boxes in the first row, λ_2-boxes in the second row,..., and λ_n-boxes in the n-th row. A Young tableau on a Young frame is a bijection from the numbers $1, 2, \cdots, N$ to the N boxes of the frame.

4.1 Para-bosons of order 2

Let us select one Young tableau, arbitrary but fixed, on each Young frame of length less than or equal to 2, say the tableau T_j on the frame $(N-j, j)$ for $j = 1, 2, \cdots, [N/2]$ and the tableau T_0 on the frame (N). We denote by $\mathcal{R}(T_j)$ the row stabilizer of T_j, i.e., the subgroup of $\mathcal{S}_N$ consists of those elements that keep all rows of T_j invariant, and by $\mathcal{C}(T_j)$ the column stabilizer whose elements preserve all columns of T_j.

Let us introduce the three elements

$$a(T_j) = \frac{1}{\#\mathcal{R}(T_j)} \sum_{\sigma \in \mathcal{R}(T_j)} \sigma, \qquad b(T_j) = \frac{1}{\#\mathcal{C}(T_j)} \sum_{\sigma \in \mathcal{C}(T_j)} \mathrm{sgn}(\sigma)\sigma$$

and

$$e(T_j) = \frac{d_{T_j}}{N!} \sum_{\sigma \in \mathcal{R}(T_j)} \sum_{\tau \in \mathcal{C}(T_j)} \mathrm{sgn}(\tau)\sigma\tau = c_j a(T_j) b(T_j)$$

of the group algebra $\mathbb{C}[\mathcal{S}_N]$ for each $j = 0, 1, \cdots, [N/2]$, where d_{T_j} is the dimension of the irreducible representation of $\mathcal{S}_N$ corresponding to T_j and $c_j = d_{T_j} \#\mathcal{R}(T_j) \#\mathcal{C}(T_j)/N!$. As is known,

$$a(T_j)\sigma b(T_k) = b(T_k)\sigma a(T_j) = 0 \tag{4.1}$$

hold for any $\sigma \in \mathcal{S}_N$ and $0 \leqslant j < k \leqslant [N/2]$. The relations

$$a(T_j)^2 = a(T_j), \quad b(T_j)^2 = b(T_j), \quad e(T_j)e(T_k) = \delta_{jk} e(T_j) \tag{4.2}$$

also hold. For later use, let us introduce

$$d(T_j) = e(T_j)a(T_j) = c_j a(T_j) b(T_j) a(T_j) \qquad (j = 0, 1, \cdots, [N/2]). \tag{4.3}$$

They satisfy

$$d(T_j)d(T_k) = \delta_{jk} d(T_j) \quad \text{for} \quad 0 \leqslant j, k \leqslant [N/2], \tag{4.4}$$

as is shown readily from (4.1) and (4.2). The inner product $< \cdot, \cdot >$ of $\mathbb{C}[\mathcal{S}_N]$ is defined by

$$< \sigma, \tau >= \delta_{\sigma\tau} \quad \text{for} \quad \sigma, \tau \in \mathcal{S}_N$$

and extended to all elements of $\mathbb{C}[\mathcal{S}_N]$ by sesqui-linearity.

The left representation L and the right representation R of $\mathcal{S}_N$ on $\mathbb{C}[\mathcal{S}_N]$ are defined by

$$L(\sigma)g = L(\sigma) \sum_{\tau \in \mathcal{S}_N} g(\tau)\tau = \sum_{\tau \in \mathcal{S}_N} g(\tau)\sigma\tau = \sum_{\tau \in \mathcal{S}_N} g(\sigma^{-1}\tau)\tau$$

and

$$R(\sigma)g = R(\sigma) \sum_{\tau \in \mathcal{S}_N} g(\tau)\tau = \sum_{\tau \in \mathcal{S}_N} g(\tau)\tau\sigma^{-1} = \sum_{\tau \in \mathcal{S}_N} g(\tau\sigma)\tau,$$

respectively. Here and hereafter we identify $g : \mathcal{S}_N \to \mathbb{C}$ and $\sum_{\tau \in \mathcal{S}_N} g(\tau)\tau \in \mathbb{C}[\mathcal{S}_N]$. They are extended to the representation of $\mathbb{C}[\mathcal{S}_N]$ on $\mathbb{C}[\mathcal{S}_N]$ as

$$L(f)g = fg = \sum_{\sigma,\tau} f(\sigma)g(\tau)\sigma\tau = \sum_{\sigma} \big(\sum_{\tau} f(\sigma\tau^{-1})g(\tau) \big)\sigma$$

and

$$R(f)g = g\hat{f} = \sum_{\sigma,\tau} g(\sigma)f(\tau)\sigma\tau^{-1} = \sum_{\sigma} \big(\sum_{\tau} g(\sigma\tau)f(\tau) \big)\sigma,$$

where $\hat{f} = \sum_\tau \hat{f}(\tau)\tau = \sum_\tau f(\tau^{-1})\tau = \sum_\tau f(\tau)\tau^{-1}$.

The character of the irreducible representation of $\mathcal{S}_N$ corresponding to the tableau T_j is obtained by

$$\chi_{T_j}(\sigma) = \sum_{\tau \in \mathcal{S}_N} (\tau, \sigma R(e(T_j))\tau) = \sum_{\tau \in \mathcal{S}_N} (\tau, \sigma\tau\widehat{e(T_j)}).$$

We introduce a tentative notation

$$\chi_g(\sigma) \equiv \sum_{\tau \in \mathcal{S}_N} (\tau, \sigma R(g)\tau) = \sum_{\tau,\gamma \in \mathcal{S}_N} (\tau, \sigma\tau\gamma^{-1}) g(\gamma) = \sum_{\tau \in \mathcal{S}_N} g(\tau^{-1}\sigma\tau) \tag{4.5}$$

for $g = \sum_\tau g(\tau)\tau \in \mathbb{C}[\mathcal{S}_N]$.

Let U be the representation of $\mathcal{S}_N$ (and its extension to $\mathbb{C}[\mathcal{S}_N]$) on $\otimes^N \mathcal{H}_L$ defined by

$$U(\sigma)\varphi_1 \otimes \cdots \otimes \varphi_N = \varphi_{\sigma^{-1}(1)} \otimes \cdots \otimes \varphi_{\sigma^{-1}(N)} \qquad \text{for } \ \varphi_1, \cdots, \varphi_N \in \mathcal{H}_L,$$

or equivalently by

$$(U(\sigma)f)(x_1, \cdots, x_N) = f(x_{\sigma(1)}, \cdots, x_{\sigma(N)}) \qquad \text{for } \ f \in \otimes^N \mathcal{H}_L.$$

Obviously, U is unitary: $U(\sigma)^* = U(\sigma^{-1}) = U(\sigma)^{-1}$. Hence $U(a(T_j))$ is an orthogonal projection because of $U(a(T_j))^* = U(\widehat{a(T_j)}) = U(a(T_j))$ and (4.2). So are $U(b(T_j))$'s, $U(d(T_j))$'s and $P_{2B} = \sum_{j=0}^{[N/2]} U(d(T_j))$. Note that $\mathrm{Ran}\, U(d(T_j)) = \mathrm{Ran}\, U(e(T_j))$ because of $d(T_j)e(T_j) = e(T_j), e(T_j)d(T_j) = d(T_j)$.

We refer the literatures [19, 12, 27] for quantum mechanics of para-particles. (See also [20].) The arguments of these literatures indicate that the state space of N para-bosons of order 2 in the finite box Λ_L is given by $\mathcal{H}_{L,N}^{2B} = P_{2B} \otimes^N \mathcal{H}_L$. It is obvious that there is a CONS of $\mathcal{H}_{L,N}^{2B}$ which consists of the vectors of the form $U(d(T_j))\varphi_{k_1}^{(L)} \otimes \cdots \otimes \varphi_{k_N}^{(L)}$, which are the eigenfunctions of $\otimes^N G_L$. Then, we define a point process of N free para-bosons of order 2 as in section 2 and its generating functional is given by

$$E_{L,N}^{2B}\big[e^{-<f,\xi>}\big] = \frac{\mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N \tilde{G}_L)P_{2B}]}{\mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L)P_{2B}]}.$$

Let us give expressions, which have a clear correspondence with (2.11).

Lemma 4.1

$$E_{L,N}^{2B}\big[e^{-<f,\xi>}\big] = \frac{\sum_{j=0}^{[N/2]} \sum_{\sigma \in \mathcal{S}_N} \chi_{T_j}(\sigma) \mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N \tilde{G}_L)U(\sigma)]}{\sum_{j=0}^{[N/2]} \sum_{\sigma \in \mathcal{S}_N} \chi_{T_j}(\sigma) \mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L)U(\sigma)]} \tag{4.6}$$

$$= \frac{\sum_{j=0}^{[N/2]} \int_{\Lambda_L^N} \det_{T_j}\{\tilde{G}_L(x_i, x_j)\} dx_1 \cdots dx_N}{\sum_{j=0}^{[N/2]} \int_{\Lambda_L^N} \det_{T_j}\{G_L(x_i, x_j)\} dx_1 \cdots dx_N} \tag{4.7}$$

Remark 1. $\mathcal{H}_{L,N}^{2B} = P_{2B} \otimes^N \mathcal{H}_L$ is determined by the choice of the tableaux T_j's. The spaces corresponding to different choices of tableaux are different subspaces of $\otimes^N \mathcal{H}_L$. However, they are unitarily equivalent and the generating functional given above is not affected by the choice. In fact, $\chi_{T_j}(\sigma)$ depends only on the frame on which the tableau T_j is defined.

Remark 2. $\det_T A = \sum_{\sigma\in\mathcal{S}_N} \chi_T(\sigma) \prod_{i=1}^N A_{i\sigma(i)}$ in (4.7) is called immanant, another generalization of determinant than $\det_\alpha$.

Proof : Since $\otimes^N \tilde{G}$ commutes with $U(\sigma)$ and $a(T_j)e(T_j) = e(T_j)$, we have

$$\begin{aligned}
&\mathrm{Tr}_{\otimes^N\mathcal{H}_L}\big((\otimes^N G_L)U(d(T_j))\big) = \mathrm{Tr}_{\otimes^N\mathcal{H}_L}\big((\otimes^N G_L)U(e(T_j))U(a(T_j))\big) \\
&\quad = \mathrm{Tr}_{\otimes^N\mathcal{H}_L}\big(U(a(T_j))(\otimes^N G_L)U(e(T_j))\big) = \mathrm{Tr}_{\otimes^N\mathcal{H}_L}\big((\otimes^N G_L)U(e(T_j))\big). \qquad (4.8)
\end{aligned}$$

On the other hand, we get from (4.5) that

$$\begin{aligned}
&\sum_{\sigma\in\mathcal{S}_N} \chi_g(\sigma)\mathrm{Tr}_{\otimes^N\mathcal{H}_L}\big((\otimes^N G)U(\sigma)\big) = \sum_{\tau,\sigma\in\mathcal{S}_N} g(\tau^{-1}\sigma\tau)\mathrm{Tr}_{\otimes^N\mathcal{H}_L}\big((\otimes^N G)U(\sigma)\big) \\
&= \sum_{\tau,\sigma} g(\sigma)\mathrm{Tr}_{\otimes^N\mathcal{H}_L}\big((\otimes^N G)U(\tau\sigma\tau^{-1})\big) = \sum_{\tau,\sigma} g(\sigma)\mathrm{Tr}_{\otimes^N\mathcal{H}_L}\big((\otimes^N G)U(\tau)U(\sigma)U(\tau^{-1})\big) \\
&= N!\sum_{\sigma} g(\sigma)\mathrm{Tr}_{\otimes^N\mathcal{H}_L}\big((\otimes^N G)U(\sigma)\big) = N!\mathrm{Tr}_{\otimes^N\mathcal{H}_L}\big((\otimes^N G)U(g)\big), \qquad (4.9)
\end{aligned}$$

where we have used the cyclicity of the trace and the commutativity of $U(\tau)$ with $\otimes^N G$. Putting $g = e(T_j)$ and using(4.9), the first equation is derived. The second one is obvious. □

Let ψ_{T_j} be the character of the induced representation $\mathrm{Ind}_{\mathcal{R}(T_j)}^{\mathcal{S}_N}[\mathbf{1}]$, where $\mathbf{1}$ is the representation $\mathcal{R}(T_j) \ni \sigma \to 1$, i.e.,

$$\psi_{T_j}(\sigma) = \sum_{\tau\in\mathcal{S}_N} <\tau, \sigma R(a(T_j))\tau> = \chi_{a(T_j)}(\sigma).$$

Then the determinantal form [13]

$$\begin{aligned}
\chi_{T_j} &= \psi_{T_j} - \psi_{T_{j-1}} \qquad (j = 1, \cdots, [N/2]) \\
\chi_{T_0} &= \psi_{T_0}
\end{aligned} \qquad (4.10)$$

yields the following result:

Theorem 4.2 *The finite para-boson processes defined above converge weakly to the point process whose Laplace transform is given by*

$$\mathrm{E}_\rho^{2B}\big[e^{-<f,\xi>}\big] = \mathrm{Det}\big[1 + \sqrt{1-e^{-f}}\,z_*G(1-z_*G)^{-1}\sqrt{1-e^{-f}}\big]^{-2}$$

in the thermodynamic limit, where $z_* \in (0,1)$ *is determined by*

$$\frac{\rho}{2} = \int \frac{dp}{(2\pi)^d}\frac{z_*e^{-\beta|p|^2}}{1-z_*e^{-\beta|p|^2}} = (z_*G(1-z_*G)^{-1})(x,x) < \rho_c,$$

and ρ_c *is given by (2.12).*

Proof : Using (4.10) in the expression in Lemma 4.1 and (4.9) for $g = a(T_{[N/2]})$, we have

$$\begin{aligned}
E^{2B}_{L,N}\big[e^{-<f,\xi>}\big] &= \frac{\sum_{\sigma\in\mathcal{S}_N}\psi_{T_{[N/2]}}(\sigma)\mathrm{Tr}_{\mathcal{H}_L^{\otimes N}}\big((\otimes^N\tilde{G}_L)U(\sigma)\big)}{\sum_{\sigma\in\mathcal{S}_N}\psi_{T_{[N/2]}}(\sigma)\mathrm{Tr}_{\mathcal{H}_L^{\otimes N}}\big((\otimes^N G_L)U(\sigma)\big)} \\
&= \frac{\mathrm{Tr}_{\otimes^N\mathcal{H}_L}\big((\otimes^N\tilde{G}_L)U(a(T_{[N/2]}))\big)}{\mathrm{Tr}_{\otimes^N\mathcal{H}_L}\big((\otimes^N G_L)U(a(T_{[N/2]}))\big)} \\
&= \frac{\mathrm{Tr}_{\otimes^{[(N+1)/2]}\mathcal{H}_L}\big((\otimes^{[(N+1)/2]}\tilde{G}_L)S_{[(N+1)/2]}\big)}{\mathrm{Tr}_{\otimes^{[(N+1)/2]}\mathcal{H}_L}\big((\otimes^{[(N+1)/2]}G_L)S_{[(N+1)/2]}\big)}\frac{\mathrm{Tr}_{\otimes^{[N/2]}\mathcal{H}_L}\big((\otimes^{[N/2]}\tilde{G}_L)S_{[N/2]}\big)}{\mathrm{Tr}_{\otimes^{[N/2]}\mathcal{H}_L}\big((\otimes^{[N/2]}G_L)S_{[N/2]}\big)}.
\end{aligned}$$

In the last equality, we have used

$$a(T_{[N/2]}) = \frac{\sum_{\sigma\in\mathcal{R}_1}\sigma}{\#\mathcal{R}_1}\frac{\sum_{\tau\in\mathcal{R}_2}\tau}{\#\mathcal{R}_2},$$

where $\mathcal{R}_1$ is the symmetric group of $[(N+1)/2]$ numbers which are on the first row of the tableau $T_{[N/2]}$ and $\mathcal{R}_2$ that of $[N/2]$ numbers on the second row. Then, Theorem 2.2 yields the theorem. □

4.2 Para-fermions of order 2

For a Young tableau T, we denote by T' the tableau obtained by ex
interchanging the rows and the columns of T. In another word, T' is the transpose of T. The tableau T'_j is on the frame $(\underbrace{2,\cdots,2}_{j},\underbrace{1,\cdots,1}_{N-2j})$ and satisfies

$$\mathcal{R}(T'_j) = \mathcal{C}(T_j), \qquad \mathcal{C}(T'_j) = \mathcal{R}(T_j).$$

The generating functional of the point process for N para-fermions of order 2 in the finite box Λ_L is given by

$$E^{2F}_{L,N}\big[e^{-<f,\xi>}\big] = \frac{\sum_{j=0}^{[N/2]}\mathrm{Tr}_{\otimes^N\mathcal{H}_L}\big((\otimes^N\tilde{G})U(d(T'_j))\big)}{\sum_{j=0}^{[N/2]}\mathrm{Tr}_{\otimes^N\mathcal{H}_L}\big((\otimes^N G)U(d(T'_j))\big)}$$

as in the case of para-bosons of order 2. Let us recall the relations

$$\chi_{T'_j}(\sigma) = \mathrm{sgn}(\sigma)\chi_{T_j}(\sigma), \qquad \varphi_{T'_j}(\sigma) = \mathrm{sgn}(\sigma)\psi_{T_j}(\sigma),$$

where we have denoted by

$$\varphi_{T'_j}(\sigma) = \sum_\tau < \tau, \sigma R(b(T'_j))\tau >$$

the character of the induced representation $\mathrm{Ind}^{\mathcal{S}_N}_{\mathcal{C}(T'_j)}[\,\mathrm{sgn}\,]$, where sgn is the representation $\mathcal{C}(T'_j) = \mathcal{R}(T_j) \ni \sigma \mapsto \mathrm{sgn}(\sigma)$. Thanks to these relations, we can easily translate the argument of para-bosons to that para-fermions and get the following theorem.

Theorem 4.3 *The finite para-fermion processes defined above converge weakly to the point process whose Laplace transform is given by*

$$\mathrm{E}_\rho^{2B}\big[e^{-<f,\xi>}\big] = \mathrm{Det}\big[1 - \sqrt{1-e^{-f}}\, z_* G(1+z_*G)^{-1}\sqrt{1-e^{-f}}\big]^2$$

in the thermodynamic limit, where $z_* \in (0,\infty)$ is determined by

$$\frac{\rho}{2} = \int \frac{dp}{(2\pi)^d}\frac{z_* e^{-\beta|p|^2}}{1+z_* e^{-\beta|p|^2}} = (z_*G(1+z_*G)^{-1})(x,x).$$

5 Gas of composite particles

Most gases are composed of composite particles. In this section, we formulate point processes which yield the position distributions of constituents of such gases. Each composite particle is called a "molecule", and molecules consist of "atoms". Suppose that there are two kinds of atoms, say A and B, such that both of them obey Fermi-Dirac or Bose-Einstein statistics simultaneously, that N atoms of kind A and N atoms of kind B are in the same box Λ_L and that one A-atom and one B-atom are bounded to form a molecule by the non-relativistic interaction described by the Hamiltonian

$$H_L = -\triangle_x - \triangle_y + U(x-y)$$

with periodic boundary conditions in $L^2(\Lambda_L \times \Lambda_L)$. Hence there are totally N such molecules in Λ_L. We assume that the interaction between atoms in different molecules can be neglected. We only consider such systems of zero temperature, where N molecules are in the ground state and (anti-)symmetrizations of the wave functions of the N atoms of type A and the N atoms of type B are considered. In order to avoid difficulties due to boundary conditions, we have set the masses of two atoms A and B equal. We also assume that the potential U is infinitely deep so that the wave function of the ground state has a compact support. We put

$$H_L = -\frac{1}{2}\triangle_R - 2\triangle_r + U(r) = H_L^{(R)} + H_L^{(r)},$$

where $R = (x+y)/2, \quad r = x-y$. The normalized wave function of the ground state of $H_L^{(R)}$ is the constant function $L^{-d/2}$. Let $\varphi_L(r)$ be that of the ground state of $H_L^{(r)}$. Then, the ground state of H_L is $\psi_L(x,y) = L^{-d/2}\varphi_L(x-y)$. The ground state of the N-particle system in Λ_L is, by taking the (anti-)symmetrizations,

$$\begin{aligned}\Psi_{L,N}(x_1,\cdots,x_N;y_1,\cdots,y_N) &= Z_{c\alpha}^{-1}\sum_{\sigma,\tau\in\mathcal{S}_N}\alpha^{N-\nu(\sigma)}\alpha^{N-\nu(\tau)}\prod_{j=1}^N \psi_L(x_{\sigma(j)},y_{\tau(j)}) \\ &= \frac{N!}{Z_{c\alpha}L^{dN/2}}\sum_\sigma \alpha^{N-\nu(\sigma)}\prod_{j=1}^N \varphi_L(x_j - y_{\sigma(j)}),\end{aligned} \tag{5.1}$$

where $Z_{c\alpha}$ is the normalization constant and $\alpha = \pm 1$. Recall that $\alpha^{N-\nu(\sigma)} = \mathrm{sgn}(\sigma)$ for $\alpha = -1$.

The distribution function of positions of $2N$-atoms of the system with zero temperature is given by the square of magnitude of (5.1)

$$p^{c\alpha}_{L,N}(x_1,\cdots,x_N;y_1,\cdots,y_N)=\frac{(N!)^2}{Z^2_{c\alpha}L^{dN}}\sum_{\sigma,\tau\in\mathcal{S}_N}\alpha^{N-\nu(\sigma)}\prod_{j=1}^N\varphi_L(x_j-y_{\tau(j)})\overline{\varphi_L(x_{\sigma(j)}-y_{\tau(j)})}. \quad (5.2)$$

Suppose that we are interested in one kind of atoms, say of type A. We introduce the operator φ_L on $\mathcal{H}_L=L^2(\Lambda_L)$ which has the integral kernel $\varphi_L(x-y)$. Then the Laplace transform of the distribution of the positions of N A-atoms can be written as

$$\begin{aligned} E^{c\alpha}_{L,N}\big[e^{-<f,\xi>}\big] &= \int_{\Lambda^{2N}} e^{-\sum_{j=1}^N f(x_j)}p^{c\alpha}_{L,N}(x_1,\cdots,x_N;y_1,\cdots,y_N)\,dx_1\cdots dx_N dy_1\cdots dy_N \\ &= \frac{\sum_{\sigma\in\mathcal{S}_N}\alpha^{N-\nu(\sigma)}\mathrm{Tr}_{\otimes^N\mathcal{H}}[(\otimes^N\varphi^*_L e^{-f}\varphi_L)U(\sigma)]}{\sum_{\sigma\in\mathcal{S}_N}\alpha^{N-\nu(\sigma)}\mathrm{Tr}_{\otimes^N\mathcal{H}}[(\otimes^N\varphi^*_L\varphi_L)U(\sigma)]}. \end{aligned}$$

In order to take the thermodynamic limit $N,L\to\infty, V/L^d\to\rho$, we consider a Schrödinger operator in the whole space. Let φ be the normalized wave function of the ground state of $H_r=-2\triangle_r+U(r)$ in $L^2(\mathbb{R}^d)$. Then $\varphi(r)=\varphi_L(r)$ $(\forall r\in\Lambda_L)$ holds for large L by the assumption on U. The Fourier series expansion of φ_L is given by

$$\varphi_L(r)=\sum_{k\in\mathbb{Z}^d}\Big(\frac{2\pi}{L}\Big)^{d/2}\hat\varphi\Big(\frac{2\pi k}{L}\Big)\frac{e^{i2\pi k\cdot r/L}}{L^{d/2}},$$

where $\hat\varphi$ is the Fourier transform of φ:

$$\hat\varphi(p)=\int_{\mathbb{R}^d}\varphi(r)e^{-ip\cdot r}\frac{dr}{(2\pi)^{d/2}}.$$

By φ, we denote the integral operator on $\mathcal{H}=L^2(\mathbb{R}^d)$ having kernel $\varphi(x-y)$.

Now we have the following theorem on the thermodynamic limit, where the density $\rho>0$ is arbitrary for $\alpha=-1$, $\rho\in(0,\rho^c_c)$ for $\alpha=1$ and

$$\rho^c_c=\int\frac{dp}{(2\pi)^d}\frac{|\hat\varphi(p)|^2}{|\hat\varphi(0)|^2-|\hat\varphi(p)|^2}.$$

Theorem 5.1 *The finite point processes defined above for $\alpha=\pm1$ converge weakly to the process whose Laplace transform is given by*

$$\mathrm{E}^{c\alpha}_\rho\big[e^{-<f,\xi>}\big]=\mathrm{Det}\big[1+z_*\alpha\sqrt{1-e^{-f}}\varphi(||\varphi||^2_{L^1}-z_*\alpha\varphi^*\varphi)^{-1}\varphi^*\sqrt{1-e^{-f}}\big]^{-1/\alpha}$$

in the thermodynamic limit (2.7), where the parameter z_* is the positive constant uniquely determined by

$$\rho=\int\frac{dp}{(2\pi)^d}\frac{z_*|\hat\varphi(p)|^2}{|\hat\varphi(0)|^2-z_*\alpha|\hat\varphi(p)|^2}=(z_*\varphi(||\varphi||^2_{L^1}-z_*\alpha\varphi^*\varphi)^{-1}\varphi^*)(x,x).$$

Proof : The eigenvalues of the integral operator φ_L is $\{(2\pi)^{d/2}\hat{\varphi}(2\pi k/L)\}_{k\in\mathbb{Z}^d}$. Since φ is the ground state of the Schrödinger operator, we can assume $\varphi \geqslant 0$. Hence the largest eigenvalue is $(2\pi)^{d/2}\hat{\varphi}(0) = ||\varphi||_{L^1}$. We also have

$$1 = ||\varphi||^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\hat{\varphi}(p)|^2 dp = ||\varphi_L||^2_{L^2(\Lambda_L)} = \sum_{k\in\mathbb{Z}^d} \left(\frac{2\pi}{L}\right)^d \left|\hat{\varphi}\left(\frac{2\pi k}{L}\right)\right|^2. \tag{5.3}$$

Set $V_L = \varphi_L/||\varphi||_{L^1}$ so thet

$$||V_L|| = 1, \quad ||V_L||_2^2 = L^d/||\varphi||^2_{L^1}.$$

Then Theorem 3.1 applies as follows:

For $z \in I_\alpha$, let us define functions d, d_L on $\mathbb{R}^d$ by

$$d(p) = \frac{z|\hat{\varphi}(p)|^2}{|\hat{\varphi}(0)|^2 - z\alpha|\hat{\varphi}(p)|^2}$$

and

$$d_L(p) = d(2\pi k/L) \quad \text{if} \quad p \in \square_k^{(L)} \quad \text{for} \quad k \in \mathbb{Z}^d. \tag{5.4}$$

Then

$$\int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d} d_L(p) = L^{-d}||zV_L(1 - z\alpha V_L^* V_L)^{-1} V_L^*||_1$$

and the following lemma holds:

Lemma 5.2

$$\lim_{L\to\infty} ||d_L - d||_{L^1} = 0.$$

Proof : Put

$$\hat{\varphi}_{[L]}(p) = \hat{\varphi}(2\pi k/L) \quad \text{if} \quad p \in \square_k^{(L)} \quad \text{for} \quad k \in \mathbb{Z}^d$$

and note that compactness of supp φ implies $\varphi \in L^1(\mathbb{R}^d)$ and uniform continuity of $\hat{\varphi}$. Then we have $||\,|\hat{\varphi}_{[L]}|^2 - |\hat{\varphi}|^2||_{L^\infty} \to 0$ and $||d_L - d||_{L^\infty} \to 0$. On the other hand, we get $||\,|\hat{\varphi}_{[L]}|^2||_{L^1} = ||\,|\hat{\varphi}|^2||_{L^1}$ from (5.3). It is obvious that

$$|\,||d_L||_{L^1} - ||d||_{L^1}\,| \leqslant \frac{z}{(1 - z(\alpha \vee 0))^2} \frac{||\,|\hat{\varphi}_{[L]}|^2 - |\hat{\varphi}|^2||_{L^1}}{|\varphi(0)|^2}.$$

Hence the lemma is derived by using the following fact twice:

If $f, f_1, f_2, \cdots \in L^1(\mathbb{R}^d)$ satisfy

$$||f_n - f||_{L^\infty} \to 0 \ \text{ and } \ ||\,f_n\,||_{L^1} \to ||\,f\,||_{L^1},$$

then $||f_n - f||_{L^1} \to 0$ holds.

In fact, using

$$\int_{|x|>R} |f_n(x)|\,dx = \int_{|x|>R} |f(x)|\,dx + \int_{|x|\leqslant R} (|f(x)| - |f_n(x)|)\,dx + ||\,f_n\,||_{L^1} - ||\,f\,||_{L^1},$$

we have

$$\begin{aligned}||f_n - f||_{L^1} &\leqslant \int_{|x|\leqslant R} |f_n(x) - f(x)|\,dx + \int_{|x|>R} (|f_n(x)| + |f(x)|)\,dx \\ &\leqslant 2\int_{|x|\leqslant R} |f_n(x) - f(x)|\,dx + 2\int_{|x|>R} |f(x)|\,dx + ||\,f_n\,||_{L^1} - ||\,f\,||_{L^1}.\end{aligned}$$

For any $\epsilon > 0$, we can choose R large enough to make the second term of the right hand side smaller than ϵ. For this choice of R, we set n so large that the first term and the remainder are smaller than ϵ and then $||f_n - f||_{L^1} < 3\epsilon$. □

(*Continuation of the proof of Theorem 5.1*) Using this lemma, we can show

$$h_L^{(\alpha)}(z) = \frac{\mathrm{Tr}\,[zV_L(1 - z\alpha V_L^* V_L)^{-1}V_L^*]}{\mathrm{Tr}\,V_L^* V_L} \quad \to \quad |\hat{\varphi}(0)|^2 \int_{\mathbb{R}^d} dp \frac{z|\hat{\varphi}(p)|^2}{|\hat{\varphi}(0)|^2 - z\alpha|\hat{\varphi}(p)|^2} = h^{(\alpha)}(z),$$
$$||\sqrt{1 - e^{-f}}\,\big[V_L(1 - z\alpha V_L^* V_L)^{-1}V_L^* - \varphi(||\varphi||_{L^1}^2 - z\alpha\varphi^*\varphi)^{-1}\varphi\big]\sqrt{1 - e^{-f}}||_1 \to 0,$$

as in the proof of (3.9) and (3.10). We have the conversion $\hat{\rho} = ||\varphi||_{L^1}^2 \rho$ and hence $\rho_c^c = \sup_{z\in I_1} h^{(1)}(z)/||\varphi||_{L^1}^2$. Hence the proof is completed by Theorem 3.1. □

Acknowledgements

We would like to thank Professor Y. Takahashi and Professor T. Shirai for many useful discussions. K. R. I. thanks Grant-in-Aid for Science Research (C)15540222 from MEXT.

A Complex integrals

Lemma A.1 (i) *For $0 \leqslant p \leqslant 1$ and $-\pi \leqslant \theta \leqslant \pi$,*

$$|1 + p(e^{i\theta} - 1)| \leqslant \exp\Big(-\frac{2p(1-p)\theta^2}{\pi^2}\Big)$$

holds. For $0 \leqslant p \leqslant 1$ and $-\pi/3 \leqslant \theta \leqslant \pi/3$,

$$|\log\big(1 + p(e^{i\theta} - 1)\big) - ip\theta + \frac{p(1-p)}{2}\theta^2| \leqslant \frac{4p(1-p)|\theta|^3}{9\sqrt{3}}$$

holds.

(ii) *For $p \geqslant 0$ and $-\pi \leqslant \theta \leqslant \pi$, the following inequalities hold.*

$$\begin{aligned}|1 - p(e^{i\theta} - 1)| &\geqslant \exp\Big(\frac{2p(1+p)}{1 + 4p(1+p)}\frac{\theta^2}{\pi^2}\Big) \\ |\log\big(1 - p(e^{i\theta} - 1)\big) + ip\theta - \frac{p(1+p)}{2}\theta^2| &\leqslant \frac{p(1+p)(1+2p)|\theta|^3}{6}\end{aligned}$$

Proof: (i) The first inequality follows from

$$|1+p(e^{i\theta}-1)|^2 = 1-2p(1-p)(1-\cos\theta) \tag{A.1}$$
$$\leqslant \exp(-2p(1-p)(1-\cos\theta)) \leqslant \exp(-4p(1-p)\theta^2/\pi^2),$$

where $1-\cos\theta \geqslant 2\theta^2/\pi^2$ for $\theta \in [-\pi,\pi]$ is used in the second inequality.

Put $f(\theta) = \log(1+p(e^{i\theta}-1))$, then we have $f(0)=0$,

$$\begin{aligned} f'(\theta) &= i - \frac{i(1-p)}{1-p+pe^{i\theta}}, \quad f'(0) = ip, \\ f''(\theta) &= -\frac{p(1-p)e^{i\theta}}{(1-p+pe^{i\theta})^2}, \quad f''(0) = -p(1-p) \end{aligned}$$

and

$$f^{(3)}(\theta) = -\frac{ip(1-p)e^{i\theta}(1-p-pe^{i\theta})}{(1-p+pe^{i\theta})^3}.$$

By (A.1) and $\theta \in [-\pi/3, \pi/3]$, we have $|1+p(e^{i\theta}-1)|^2 \geqslant 1-p(1-p) \geqslant 3/4$. Hence, $|f^{(3)}(\theta)| \leqslant 8p(1-p)/3\sqrt{3}$ holds. Taylor's theorem yields the second inequality.

(ii) The first inequality follows from

$$\begin{aligned} |1-p(e^{i\theta}-1)|^2 &= 1+2p(1+p)(1-\cos\theta) \\ &\geqslant \exp\Big(\frac{2p(1+p)(1-\cos\theta)}{1+4p(1+p)}\Big) \geqslant \exp\Big(\frac{4p(1+p)}{1+4p(1+p)}\frac{\theta^2}{\pi^2}\Big). \end{aligned}$$

Here we have used $1+x \geqslant e^{x/(1+a)}$ for $x \in [0,a]$ in the first inequality, which is derived from $\log(1+x) = \int_0^x dt/(1+t) \geqslant x/(1+a)$.

Put $f(\theta) = \log(1-p(e^{i\theta}-1))$. Then we have $f(0)=0$,

$$\begin{aligned} f'(\theta) &= i - \frac{i(1+p)}{1+p-pe^{i\theta}}, \quad f'(0) = -ip, \\ f''(\theta) &= \frac{p(1+p)e^{i\theta}}{(1+p-pe^{i\theta})^2}, \quad f''(0) = p(1+p) \end{aligned}$$

and

$$f^{(3)}(\theta) = \frac{ip(1+p)e^{i\theta}(1+p+pe^{i\theta})}{(1+p-pe^{i\theta})^3}.$$

Hence, we have $|f^{(3)}(\theta)| \leqslant p(1+p)(1+2p)$. Thus we get the second inequality. □

Proposition A.2 *Let $s>0$ and a collection of numbers $\{\, p_j^{(N)} \,\}_{j,N}$ satisfy*

$$p_0^{(N)} \geqslant p_1^{(N)} \geqslant p_2^{(N)} \geqslant \cdots \geqslant p_j^{(N)} \geqslant \cdots \geqslant 0, \quad \sum_{j=0}^{\infty} sp_j^{(N)} = N.$$

(i) *Moreover, if $p_0^{(N)} \leqslant 1$ and*

$$v^{(N)} \equiv \sum_{j=0}^{\infty} sp_j^{(N)}(1-p_j^{(N)}) \to \infty \quad (N \to \infty),$$

then

$$\lim_{N\to\infty}\sqrt{v^{(N)}}\oint_{S_1(0)}\frac{d\eta}{2\pi i\eta^{N+1}}\prod_{j=0}^{\infty}(1+p_j^{(N)}(\eta-1))^s=\frac{1}{\sqrt{2\pi}}$$

holds.

(ii) *If* $\{p_0^{(N)}\}$ *is bounded, then*

$$\lim_{N\to\infty}\sqrt{w^{(N)}}\oint_{S_1(0)}\frac{d\eta}{2\pi i\eta^{N+1}}\frac{1}{\prod_{j=0}^{\infty}(1-p_j^{(N)}(\eta-1))^s}=\frac{1}{\sqrt{2\pi}}$$

holds, where

$$w^{(N)}\equiv\sum_{j=0}^{\infty}sp_j^{(N)}(1+p_j^{(N)}).$$

Proof: (i) Set $\eta=\exp(ix/\sqrt{v^{(N)}})$. Then the integral is written as $\int_{-\infty}^{\infty}h_N(x)\,dx/2\pi$, where

$$h_N(x)=\chi_{[-\pi\sqrt{v^{(N)}},\pi\sqrt{v^{(N)}}]}(x)e^{-iNx/\sqrt{v^{(N)}}}\prod_{j=0}^{\infty}\left[1+p_j^{(N)}(e^{ix/\sqrt{v^{(N)}}}-1)\right]^s.$$

By Lemma A.1(i), we have

$$|h_N(x)|\leqslant\prod_{j=0}^{\infty}e^{-2sp_j^{(N)}(1-p_j^{(N)})x^2/\pi^2v^{(N)}}=e^{-2x^2/\pi^2}\in L^1(\mathbb{R}).$$

If N is so large that $|x/\sqrt{v^{(N)}}|\leqslant\pi/3$, we also get

$$\begin{aligned}
h_N(x) &= \chi_{[-\pi\sqrt{v^{(N)}},\pi\sqrt{v^{(N)}}]}(x)\exp\Big[-i\frac{Nx}{\sqrt{v^{(N)}}}+s\sum_{j=0}^{\infty}\log\big(1+p_j^{(N)}(e^{ix/\sqrt{v^{(N)}}}-1)\big)\Big]\\
&= \chi_{[-\pi\sqrt{v^{(N)}},\pi\sqrt{v^{(N)}}]}(x)\exp\Big[-i\frac{Nx}{\sqrt{v^{(N)}}}+s\sum_{j=0}^{\infty}\big(i\frac{p_j^{(N)}x}{\sqrt{v^{(N)}}}-\frac{p_j^{(N)}(1-p_j^{(N)})x^2}{2v^{(N)}}+\delta_j^{(N)}\big)\Big]\\
&= \chi_{[-\pi\sqrt{v^{(N)}},\pi\sqrt{v^{(N)}}]}(x)\exp\big(-\frac{x^2}{2}+\delta^{(N)}\big)\underset{N\to\infty}{\longrightarrow}e^{-x^2/2},
\end{aligned}$$

where

$$|\delta^{(N)}|=|\sum_{j=0}^{\infty}s\delta_j^{(N)}|\leq\sum_{j=0}^{\infty}\frac{4sp_j^{(N)}(1-p_j^{(N)})x^3}{9\sqrt{3}\sqrt{v^{(N)}}^3}=\frac{4|x^3|}{9\sqrt{3v^{(N)}}}.$$

The dominated convergence theorem yields

$$\int_{-\infty}^{\infty}h_N(x)\frac{dx}{2\pi}\underset{N\to\infty}{\longrightarrow}\int_{-\infty}^{\infty}\frac{dx}{2\pi}e^{-x^2/2}=\frac{1}{\sqrt{2\pi}}.$$

(ii) Note that $w^{(N)}\to\infty$ as $N\to\infty$. Set $\eta=\exp(ix/\sqrt{w^{(N)}})$. Then the integral is written as $\int_{-\infty}^{\infty}k_N(x)\,dx/2\pi$, where

$$k_N(x)=\chi_{[-\pi\sqrt{w^{(N)}},\pi\sqrt{w^{(N)}}]}(x)\frac{e^{-iNx/\sqrt{w^{(N)}}}}{\prod_{j=0}^{\infty}\left[1-p_j^{(N)}(e^{ix/\sqrt{w^{(N)}}}-1)\right]^s}.$$

By Lemma A.1(ii) and the boundedness of $\{p_0^{(N)}\}$, we have, with some positive constant c,

$$|k_N(x)| \leqslant \prod_{j=0}^{\infty} \exp\Big(- \frac{2sp_j^{(N)}(1+p_j^{(N)})}{1+4p_0^{(N)}(1+p_0^{(N)})} \frac{x^2}{\pi^2 w^{(N)}}\Big) \leqslant e^{-cx^2} \in L^1(\mathbb{R})$$

and

$$\begin{aligned} k_N(x) &= \chi_{[-\pi\sqrt{w^{(N)}},\pi\sqrt{w^{(N)}}]}(x) \exp\Big[-i\frac{Nx}{\sqrt{w^{(N)}}} - s\sum_{j=0}^{\infty} \log\big(1-p_j^{(N)}(e^{-ix/\sqrt{w^{(N)}}}-1)\big)\Big] \\ &= \chi_{[-\pi\sqrt{w^{(N)}},\pi\sqrt{w^{(N)}}]}(x) \exp\Big[-i\frac{Nx}{\sqrt{w^{(N)}}} - s\sum_{j=0}^{\infty} \Big(-i\frac{p_j^{(N)}x}{\sqrt{w^{(N)}}} + \frac{p_j^{(N)}(1+p_j^{(N)})x^2}{2w^{(N)}} + \delta_j^{(N)}\Big)\Big] \\ &= \chi_{[-\pi\sqrt{w^{(N)}},\pi\sqrt{w^{(N)}}]}(x) \exp\Big(-\frac{x^2}{2} + \delta^{(N)}\Big) \underset{N\to\infty}{\longrightarrow} e^{-x^2/2}, \end{aligned}$$

where

$$|\delta^{(N)}| = |\sum_{j=0}^{\infty} s\delta_j^{(N)}| \leqslant \sum_{j=0}^{\infty} \frac{p_j^{(N)}(1+p_j^{(N)})(1+2p_j^{(N)})|x^3|}{6\sqrt{w^{(N)}}^3} \leqslant \frac{(1+2p_0^{(N)})}{6\sqrt{w^{(N)}}}|x^3|.$$

The result is obtained by the dominated convergence theorem. □

References

[1] Benard, C., Macchi, O.: Detection and emission processes of quantum particles in a chaotic state. J. Math. Phys. **14** 155–167(1973)

[2] Chaturvedi, S.: Canonical partition functions for parastatistical systems of any order. Phys. Rev. E **54**, 1378–1382(1996)

[3] Chaturvedi, S., Srinivasan, V.: Grand canonical partition functions for multi level para Fermi systems of any order. Phys. Lett. A **224**, 249–252(1997)

[4] Doplicher, S., Haag, R., Roberts, J.E.: Local observables and particle statistics I. Commun. Math. Phys. **23**, 199–230(1971)

[5] Daley, D.J. Vere-Jones, D.: *An Introduction to the Theory of Point Processes.* Berlin: Springer Verlag, 1988

[6] Fichtner, K.-H.: On the position distribution of the ideal Bose gas. Math. Nachr. **151**, 59–67(1991)

[7] Freudenberg, W.: Characterization of states of infinite boson systems. II: On the existence of the conditional reduced density matrix. Commun. Math. Phys. **137**, 461–472(1991)

[8] Fichtner, K.-H. Freudenberg, W.: Point processes and the position distribution of infinite boson systems. J. Stat. Phys. **47**, 959–978(1987)

[9] Fichtner, K.-H. Freudenberg, W.: Characterization of states of infinite boson systems. I: On the construction of states of boson systems. Commun. Math. Phys. **137**, 315–357(1991)

[10] Green, H.S.: A generalized method of field quantization. Phys. Rev. **90**, 270–273(1953)

[11] Goldin, G.A. Grodnik, J. Powers, R.T. Sharp, D.H.: Nonrelativistic current algebra in the N/V limit. J. Math. Phys. **15**, 88–100(1974)

[12] Hartle, J.B., Taylor, J.R.: Quantum mechanics of paraparticles. Phys. Rev. **178**, 2043–2051(1969)

[13] James, G., Kerber, A.: *The Representation Theory of the Symmetric Group. (Encyclopedia of mathematics and its applications vol. 16)* London: Addison-Wesley Publishing, 1981

[14] Lenard, A,: States of classical statistical mechanical systems of infinitely many particles. I. Arch. Rationa. Mech. Anal. **59**, 219–139(1975)

[15] Lytvynov, E.: Fermion and boson random point processes as particle distributions of infinite free Fermi and Bose gases of finite density. Rev. Math. Phys. **14**, 1073–1098(2002)

[16] Macchi, O.: The coincidence approach to stochastic point processes. Adv. Appl. Prob. **7**, 83–122(1975)

[17] Macchi, O.: The fermion process–a model of stochastic point process with repulsive points. In: *Transactions of the Seventh Prague Conference on Information Theory, Statistical Decision Functions, Random Processes and of the Eighth European Meeting of Statisticians Vol. A* Prague 1974. Dordrecht: Reidel Publishing, 1977, pp. 391–398

[18] Menikoff, R.: The Hamiltonian and generating functional for a nonrelativistic local current algebra. J. Math. Phys. **15**, 1138–1152(1974)

[19] Messiah, A.M.L., Greenberg, O.W.: Symmetrization postulate and its experimental foundation. Phys. Rev. **136**, B248–B267(1964)

[20] Ohnuki, Y., Kamefuchi, S.: Wavefunctions of identical particles. Ann. Phys. **51**, 337–358(1969)

[21] Ohnuki, Y., Kamefuchi, S.: *Quantum field theory and parastatistics.* Berlin: Springer-Verlag 1982

[22] Sagan, B.E.; *The Symmetric Group.* New York: Springer-Verlag, 1991

[23] Shirai, T., Takahashi, Y.: Random point fields associated with certain Fredholm determinants I: fermion, Poisson and boson point processes. J. Funct. Anal. **205**, 414–463(2003)

[24] Simon, B.: *Trace ideals and their applications.* London Mathematical Society Lecture Note Series Vol.35 Cambridge: Cambridge University Press, 1979

[25] Simon, B.: *Representations of Finite and Compact Groups.* A.M.S. Providence, 1996

[26] Soshnikov, A.: Determinantal random point fields. Russian Math. Surveys **55**, 923–975(2000)

[27] Stolt, R.H., Taylor, J.R.: Classification of paraparticles. Phys. Rev. D**1**, 2226–2228(1970)

[28] Suranyi, P.: Thermodynamics of parabosonic and parafermionic systems of order two. Phys. Rev. Lett. **65**, 2329–2330(1990)

[29] Vere-Jones, D.: A generalization of permanents and determinants. Linear Algebra Appl. **111**, 119–124(1988)