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Regularity Aspects of a Higher-Order Variational Approach to the Denoising and Inpainting of Images with TV-type Energies

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Abstract

This thesis deals with a certain class of variational problems of higher order that stem from applications in mathematical image processing. The main intention is to study the regularity behavior of minimizers of integral functionals on Sobolev spaces $W^{m,1}(\Omega)$ ($m \in \mathbb{N}$, Ω an open and bounded subset of $\mathbb{R}^n$) with differentiable energy densities of linear growth approximating the TV-case. Building upon results that were given by Bildhauer, Fuchs, Tietz and Weickert in the first-order case ($m = 1$), we treat existence of weakly differentiable, relaxed, dual as well as of classically differentiable solutions under suitable conditions on the model. Our considerations are supplemented with a detailed study of the lower-dimensional cases $n = 1, 2$, as well as with a "coupling model" which offers an alternative approach to the higher-order case.

Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit einer bestimmten Klasse von Variationsproblemen höherer Ordnung, die von Anwendungen in der mathematischen Bildverarbeitung herrühren. Das Hauptaugenmerk liegt dabei auf der Untersuchung des Regularitätsverhaltens der Minimierer von Integral-Funktionalen auf Sobolev-Räumen $W^{m,1}(\Omega)$ ($m \in \mathbb{N}$, Ω eine offene und beschränkte Teilmenge von $\mathbb{R}^n$) mit differenzierbaren Energiedichten von linearem Wachstum, die den TV-Fall approximieren. Aufbauend auf Resultaten, die von Bildhauer, Fuchs, Tietz und Weickert im Falle erster Ordnung ($m = 1$) erbracht wurden, behandeln wir die Existenz von schwach differenzierbaren, relaxierten, dualen sowie von klassisch differenzierbaren Lösungen unter jeweils hinreichenden Voraussetzungen an das Modell. Unsere Betrachtungen werden ergänzt durch eine eingehendere Analyse der niederdimensionalen Fälle $n = 1, 2$, sowie eines "Kopplungsmodells", das einen alternativen Zugang zum Fall höherer Ordnung bietet.

List of Symbols

Function Spaces

$BV^m(\Omega)$	space of functions of m-th order bounded variation, Section 1.1
$C^k_{(0)}(\Omega)$	k-times continuously differentiable functions (with compact support)
$C^{k,\alpha}(\Omega)$	k-times differentiable functions with locally α-Hölder-continuous derivatives, Appendix A
F_ε	regularized TV-density, see p. 7
$\mathcal{H}^\varepsilon$	ε-dimensional Hausdorff measure
$\mathcal{L}^n$	Lebesgue's measure on $\mathbb{R}^n$
$L^p(\Omega)$	Lebesgue p-space, Appendix A
$\mathcal{M}(\Omega)$	space of Radon measures on Ω, Appendix A
Φ_μ	a certain μ-elliptic density function, see p. 6
$W^{k,p}_{(\mathrm{loc})}(\Omega)$	(local) Sobolev space of differentiability order k and integrability exponent p, Appendix A
$\lvert\nabla u\rvert(\Omega)$, $\int_\Omega \lvert\nabla u\rvert$	total variation of $u \in BV(\Omega)$
$\lvert\cdot\rvert$	Euclidean norm on $\mathbb{R}^n$
$\lVert\cdot\rVert_{p;\Omega}$	standard norm on $L^p(\Omega)$, Appendix A
$\lVert\cdot\rVert_{k,p;\Omega}$	standard norm on $W^{k,p}(\Omega)$, Appendix A

General Notation

DF, D^2F	first and second order total derivative of F
$D_{\alpha_1\ldots\alpha_m}u$	m-th partial distributional derivative of u in coordinate directions x_{α_1},...,x_{α_m}
$\partial_i u$	classical partial derivative with respect to x_i
$\nabla^m u$	tensor of m-th-order partial (distributional) derivatives of u, Appendix A
$\mathbb{N}$, $\mathbb{N}_0$	$\{1,2,3,...\}$, $\{0,1,2,...\}$
$\otimes$	tensor product
$\mathrm{sgn}(x)$	sign function
$\frac{\mu}{\nu}$	Radon-Nikodym derivative of measures $\mu \ll \nu$
$\mu = \mu^a \cdot \mathcal{L}^n + \mu^s$	Lebesgue decomposition of the measure μ
$\fint$	mean value integral, $\fint_\Omega f\,\mathrm{d}\mu := \frac{1}{\mu(\Omega)}\int_\Omega f\,\mathrm{d}\mu$
$\cdot$, $:$	componentwise dot product of vectors, tensors
$\rightrightarrows$	uniform convergence of functions
$*$	convolution of functions

Introduction

Application Background: TV-Denoising of Images

Driven by technological progress, the mathematical branch of image analysis has undergone a rapid development during the last three decades, gaining more and more the attention not only of computer scientists and applied mathematicians, but also that of "pure theorists". This is surely promoted by the opportunity to fathom new fields of application for classical theories from mathematical analysis (with the Calculus of Variations and the theory of Partial Differential Equations leading the way), often linking interesting new theoretical aspects with the gratifying prospect of a practical use for the obtained results. Of particular note in this context is e.g. the theory of BV functions (i.e. functions of bounded variation, see e.g. [1], [2]), whose development was promoted by De Giorgi and Caccioppoli in the context of the minimal surface problem around the midst of the 20th century and which nowadays finds widespread application in the modeling of digital images and total variation (TV-) denoising algorithms. This already brings us to the underlying topic of this thesis.

The history of "TV-denoising" has its roots in the seminal work [3] of Rudin, Osher and Fatemi that was published in 1992 and in which the authors are concerned with the following problem: assume that we are given an image that was recorded with some optical device such as a camera. Due to technical reasons, the occurrence of artifacts is practically unavoidable and finding ways to restore the actual image from the defective data is the fundamental task of image processing. Mathematically, we may model a (black and white) image as a function $f : \Omega \to [0, 1]$ that maps every point x in the image domain $\Omega \subset \mathbb{R}^2$ (think e.g. of the rectangular piece of cardboard a customary photograph is printed on) to a gray value $f(x)$ between zero and one; $f(x) = 0$ indicating a black and $f(x) = 1$ indicating a white point. Let f_0 denote the "clean" image, and assume that the artifacts are incorporated by a distortion function $n : \Omega \to \mathbb{R}$, so that the actually observed image f is given by the superposition $f = f_0 + n$. If n has a certain statistical profile (e.g. if the values of n are normally distributed), one speaks of a (Gaussian) noise and any process that restores f from f_0 is called *denoising*. At this point, Variational and PDE methods come into play. The idea is to reconstruct f_0 as the minimizer of a suitable functional that penalizes wild oscillations of the data. To this purpose, we consider a variational problem of the general form

$$\begin{cases} \text{minimize } \mathcal{F}[u] \text{ subject to the constraint} \\ \fint_\Omega (u-f)^2 \, \mathrm{d}x = \sigma, \end{cases}$$

where σ is the variance of the distribution of the values $n(x)$. $\mathcal{F}$ acts on a suitable function class for which a natural choice in the context of variational methods would e.g. be some Sobolev space (see Appendix B). Usually, it is advantageous to consider the related free problem (also known as Tikhonov problem) which reads as

$$\text{minimize } I[u] := \mathcal{F}[\nabla u] + \lambda \int_\Omega (u-f)^2 \, \mathrm{d}x, \tag{1}$$

where the real valued parameter λ now plays the role of a Lagrangian multiplier.

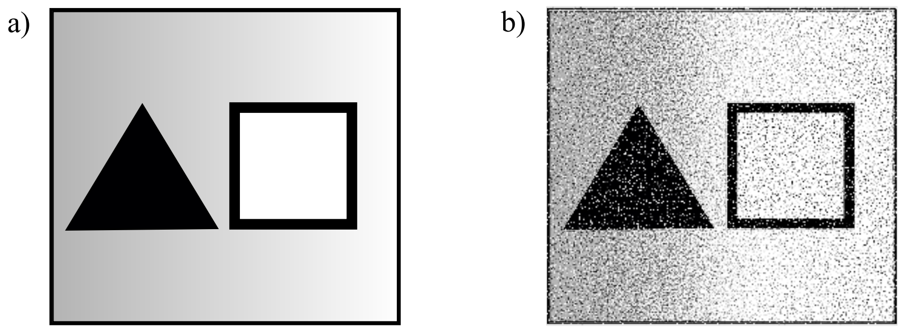

Fig. 1: An example image a) without and b) with a noisy data corruption.

So far we have neither fixed the class of functions u in which we want to determine the solution of (1), nor have we specified the regularizing functional $\mathcal{F}$. A realistic image f, having sharp contours and edges, will surely not give a classically, yet not even a weakly differentiable function (cf. the example image in Figure 1 a)). In their work [3], the authors therefore propagated the space $BV(\Omega)$ of functions of bounded variation (see Section 1.1) as the correct framework for mathematical imaging tasks, which was already proposed by Rudin in his 1987 doctoral thesis (see [4]). BV-functions can be characterized as functions with distributional derivatives in form of a finite Radon measure. Maybe the simplest example is the Heaviside function (i.e. the characteristic function of the interval $[0,\infty)$), whose distributional derivative is a Dirac measure δ_0 of mass 1 concentrated in the point $x=0$. As this example shows, BV-functions can have jump-type discontinuities which enables them to reproduce discontinuous image features, while on the other hand they are still in some sense "smooth enough" to allow a meaningful treatment with analytical methods. Now the idea of Rudin, Osher and Fatemi in [3] was to consider (1) with the choice $\mathcal{F}[u] := |\nabla u|(\Omega)$

(i.e. $\mathcal{F}[u]$ is the total variation of u) and thereby the variational problem

$$\text{minimize } I[u] = |\nabla u|(\Omega) + \lambda \int_\Omega (u-f)^2 \,\mathrm{d}x \quad \text{in } BV(\Omega). \tag{2}$$

The numerical results which could be produced with this approach were so convincing, that TV-denoising and related concepts soon became a major field of investigation in image processing and (2) is today simply referred to as the "ROF-model". In an abstract form, this is also the basis of the thesis at hand where, broadly speaking, we consider a generalized variant of problem (1), in which the gradient ∇u is replaced with a higher derivative. To understand the intention behind this approach, we have to look at a certain phenomenon which frequently leads to a degradation of the solutions of (2) and is called *staircasing effect*. This may occur whenever the data function f_0 is smooth on a subregion $\Omega_0 \subset \Omega$ of the image domain. If then f is afflicted with a noise, it may happen that, on Ω_0, the solution u of (2) displays a "blocky" pattern of piecewise constant sections, resembling a staircase (therefore the name). This effect has been studied rigorously in the one dimensional case in [5] and numerical examples for its occurrence can be found e.g. in [6], [7], [8] and [9]. Figure 2 below depicts an idealized visualization of this phenomenon.

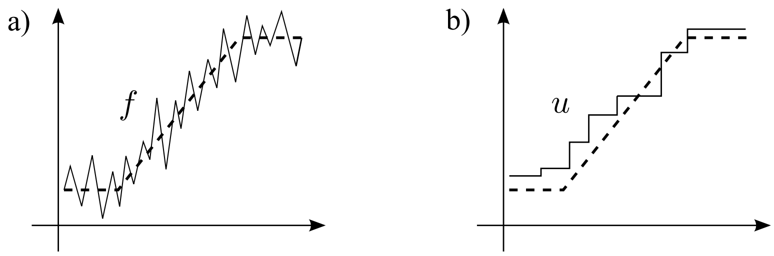

Fig. 2: The staircasing effect. a) Noisy data function (the dashed line marks the piecewise affine "clean" signal). b) The denoised signal features a staircase-like pattern.

This poses the problem, to avoid "staircasing" while at the same time preventing the data from "oversmoothing", for which one possible solution was found in form of higher-order models such as

$$\text{minimize } I_m[u] = |\nabla^m u|(\Omega) + \lambda \int_\Omega (u-f)^2 \,\mathrm{d}x \tag{3}$$

(for another approach, based on "TV-H^1-interpolation", see e.g. [10]). In (3), $m \in \{2,3,4,...\}$ and $\nabla^m u = \left(D_{\alpha_1\ldots\alpha_m} u\right)_{\alpha_i=1,2}$ stands for the tensor of all m-th order partial derivatives of u (in the sense of distributions). We say that "*u is of m-th order bounded variation*" (or $u \in BV^m(\Omega)$ for short), if the tensor $\nabla^{m-1}u$ lies in the class $BV\big(\Omega, \mathbb{R}^{2^{m-1}}\big)$ (for notational simplicity, we consider $\nabla^k u(x)$ as an element of the Euclidean space $\mathbb{R}^{2^k}$), see Section 1.1 for the details. Numerical experiments then confirm that the solutions of (3) are free from the undesired staircasing since the unpleasant artifacts are relocated to the

$(m-1)$-th derivative and therefore become less visible to the observer. As representatives of many others we mention in this regard the contributions [6], [11], [12], [13], [14] [15], [16], [17], [18], [19] and [20]. In alternative to the simple model (3), extensive research in this field over the last years gave rise to a number of varying approaches to the higher-order (TV-) case (some of which are specially geared to certain applications or with regard to computational performance). We particularly mention the "infimal convolution" model from [21] which initialized the study of the higher-order case, the concept of "total generalized variation" as developed in [22], the "coupling model" from [23] and the combined first an second order approach from [24]. We also noticed that the study of merely theoretical aspects of such higher-order models seems to enjoy a rising popularity as e.g. the works [20], [25], [26] and, more recently, [27] document.

Next to denoising, another prominent example of a classical task in image processing comes from the so called *inpainting* problem. Here, the defectiveness of the data is not caused by the interference of a noisy signal, but is the result of a partial loss of image information on some parts of the domain Ω. Think e.g. of a section $D \subset \Omega$ (for *deficiency set*) that has been cut out from the picture (cf. Figure 3). In this case, f is a map from the difference set $\Omega - D$ to $[0,1]$ and the objective is to find a sensible extrapolation of f to the points of D where it is yet undefined (therefore, some authors favor the term *image completion* over the term image inpainting). There exist many different techniques related to the inpainting problem, we recommend the survey [28] for an overview. However, in the context of this thesis, the variational approach using the ROF-model (2) that was introduced in [29] is of central significance. The idea is simply as follows: in (2), replace the data fitting term $\lambda \int_\Omega (u-f)^2 \,\mathrm{d}x$ with $\lambda \int_{\Omega - D} (u-f)^2 \,\mathrm{d}x$ and solve

$$\text{minimize } I[u] = |\nabla u|(\Omega) + \lambda \int\limits_{\Omega - D} (u-f)^2 \,\mathrm{d}x \quad \text{in } BV(\Omega). \tag{4}$$

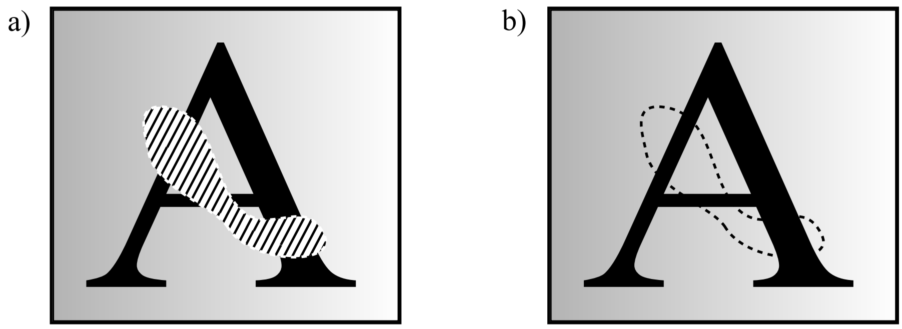

Fig. 3: The inpainting problem: a) On the deficiency set D (dashed region) no image data are available. b) Inpainting algorithms strive to fill in the missing parts.

Since in applications noise and other types of data corruption often go hand in hand, one of the main advantages of this combined denoising-inpainting model is its robustness to noise, which Chan and Shen in [29] even declare as one of the fundamental principles any reasonable inpainting technique should obey to.

Interestingly, in some situations (namely if there is a larger gap in the image data), the numerical results e.g. from [13], [24] and [30] indicate that even for the inpainting problem itself (i.e. detached from the staircasing effect) it is advantageous to consider higher-order variants of (4), i.e.

$$\text{minimize } I_m[u] = |\nabla^m u|(\Omega) + \lambda \int\limits_{\Omega - D} (u-f)^2 \, \mathrm{d}x \quad \text{in } BV^m(\Omega). \tag{5}$$

This variational problem can be regarded as the basic object of research of this thesis, which will be formulated with all details in the following section.

Concretization of the Problem, General Assumptions

Until now, the image domain Ω was assumed to be a subset of $\mathbb{R}^2$. However, some applications (such as e.g. MRI in medical imaging) can lead to three or even higher dimensional data (cf. e.g. the monograph [31]). Therefore we may allow Ω to be an open and bounded subset of some Euclidean space $\mathbb{R}^n$ with at least Lipschitz smooth boundary (see e.g. [32] for an explanation of this terminology). The imposed smoothness condition on $\partial\Omega$ and the boundedness are sufficient for the application of many fundamental theorems such as e.g. Sobolev's embedding theorems (cf. [32]) and does not represent a severe restriction from the viewpoint of applications (note that in particular the "standard" domain of image processing, i.e. a rectangle in $\mathbb{R}^2$, has this property). As far as the inpainting problem is concerned, we will assume that the deficiency set $D \subset \Omega$ is at least Lebesgue ($\mathcal{L}^n$-) measurable, satisfying $0 \le \mathcal{L}^n(D) < \mathcal{L}^n(\Omega)$. The higher-order setting will also require in some cases the Lipschitz-smoothness of the boundary $\partial(\Omega - D)$ (cf. Theorem 1.2.1). As above, the image data is given by a function f defined on $\Omega - D$, where we replace the condition $f(x) \in [0,1]$ with the more general assumption $f \in L^\infty(\Omega)$ (i.e. f is an essentially bounded $\mathcal{L}^n$-measurable real valued function). For the general existence results in the first part of this thesis, we could also allow vector valued data $f \in L^\infty(\Omega, \mathbb{R}^N)$ ($N \in \mathbb{N}$), corresponding to a multi-channel image (think e.g. of a color picture), but in order to safe us from another index, we prefer to formulate our results merely for the scalar case. We emphasize in this connection that for the regularity results from the second and third part the assumption $N = 1$ is *not* optional. Now on the basis of (5), we define the underlying variational problem on the function class $W^{m,1}(\Omega) \cap L^q(\Omega - D)$ for some $m \in \mathbb{N}$ and $q \in [1, \infty)$ as

$$\begin{aligned} \text{minimize } I[u] := \int\limits_\Omega F(\nabla^m u) \, \mathrm{d}x + \lambda \int\limits_{\Omega - D} \phi\big(|u - f|\big) \, \mathrm{d}x \\ \text{in } W^{m,1}(\Omega) \cap L^q(\Omega - D). \end{aligned} \tag{V}$$

Here, $F : \mathbb{R}^{nm} \to [0,\infty)$ denotes a strictly convex energy density of class C^2, satisfying the following set of conditions ($Z \in \mathbb{R}^{nm}$):

$$F(0) = 0 \text{ and } F(-Z) = F(Z), \tag{F1}$$
$$|\nabla F(Z)| \le \nu_1, \tag{F2}$$
$$F(Z) \ge \nu_2|Z| - \nu_3, \tag{F3}$$

for some positive constants ν_1, ν_2, ν_3, and the so called fidelity function $\phi : [0,\infty) \to \mathbb{R}$ is at least of class C^1 and should have the properties

$$\phi \text{ is strictly convex and } \phi(0) = 0 \text{ as well as } \phi'(0) = 0, \tag{$\phi1$}$$
$$c_1 t^q \le \phi(t) \le c_2\big(1 + t^q\big) \text{ for some } q \in [1,\infty),\ c_1, c_2 > 0. \tag{$\phi2$}$$

We would like to remind the reader of the fact that the strict convexity of F implies that the second derivative is a positive semi-definite bilinear form, i.e.

$$D^2F(Z)(Y,Y) \ge 0 \quad \text{for all } Y, Z \in \mathbb{R}^{nm},$$

which is important since we will often employ the *Cauchy-Schwarz inequality* in the form

$$D^2F(Z)(X,Y) \le D^2F(Z)(X,X) \cdot D^2F(Z)(Y,Y), \quad X, Y, Z \in \mathbb{R}^{nm}.$$

Moreover, we note that by the strict convexity of ϕ, $\phi(0) = 0$, ($\phi2$) and the mean value theorem we have

$$\phi'(t) \ge \frac{\phi(t)}{t} \ge c_1 t^{q-1},$$

which together with Lemma 2.2 on p. 156 of [33] yields the estimate

$$c_1' t^{q-1} \le \phi'(t) \le c_2'\big(1 + t^{q-1}\big) \tag{$\phi3$}$$

for the first derivative of ϕ. The reader has probably already noticed that the requirement $F \in C^2$ prohibits the choice $F(Z) = |Z|$, meaning that (5) is not included as a special case of (V). This has the following background: since in this thesis we are mainly interested in the analytical study of regularity properties of solutions to (V) with methods from the Calculus of Variations and elliptic PDE's, it is not sensible to choose a non-differentiable density such as $|\cdot|$. However, our conditions on F allow for some well known smooth approximations of the TV-density, most prominently the regularized TV-density

$$F(Z) = F_\varepsilon(Z) := \sqrt{\varepsilon^2 + |Z|^2} \quad \text{for some small } \varepsilon > 0$$

as well as

$$F(Z) = \Phi_\mu(|Z|) := \int_0^{|Z|}\int_0^s (1+t)^{-\mu}\,\mathrm{d}t\,\mathrm{d}s,\ r \ge 0,\ \mu \in [1,\infty), \tag{6}$$

which is the standard example of a so called μ-elliptic density. This is how a density function F is called which in addition to (F1)-(F3) also satisfies the ellipticity estimate

$$\nu_4\big(1+|Z|\big)^{-\mu}|X|^2 \leq D^2F(Z)\big(X,X\big) \leq \nu_5\frac{1}{1+|Z|}|X|^2, \tag{F$_\mu$}$$

where ν_4, ν_5 are positive constants, $X, Z \in \mathbb{R}^{n^m}$ and the exponent μ can be chosen from the interval $(1, \infty)$. In the boundary case $\mu = 1$, condition (F$_\mu$) corresponds to the notion of $L\log L$-growth (see e.g. [34]). As we will see in the second part of this thesis, the notion of μ-ellipticity plays an important role in connection with the solvability of problem (V) (cf. also the works [35], [36], [37] and [38]). Note further that we have the explicit formulas

$$\begin{cases} \Phi_\mu(r) = \dfrac{1}{\mu-1}r + \dfrac{1}{\mu-1}\dfrac{1}{\mu-2}(r+1)^{-\mu+2} - \dfrac{1}{\mu-1}\dfrac{1}{\mu-2},\ \mu \neq 2, \\ \Phi_2(r) = r - \ln(1+r),\ r \geq 0 \end{cases} \tag{7}$$

from which it is easily seen that $\mu \cdot \Phi_\mu\big(|\cdot|\big) \rightrightarrows |\cdot|$ as $\mu \to \infty$. Hence $\mu \cdot \Phi_\mu$ can be used to approximate the TV-density just as well as F_ε.

Contents and Aim of this Thesis

Let us begin with a short account of some earlier results concerning the first-order case $m = 1$. This is the subject of numerous publications of Bildhauer, Fuchs, Tietz and Weickert that were mainly concerned with regularity properties of minimizers to relaxed and dual formulations of (V). We should mention in this connection the works [35], [36], [37] and [39] that stand at the beginning of their research in this direction and which contain various results for the case $m = 1$, $n = 2$ and $\mathcal{L}^n(D) = 0$ (i.e. pure denoising is considered) as well as for related models of nearly linear growth, proving existence, higher integrability and regularity for minimizers of (V) with μ-elliptic densities. Another important contribution, despite not being directly related to the denoising problem, is the maximum principle from [40], which states that if the data function f is (essentially) bounded, then $\|u\|_\infty \leq \|f\|_\infty$ holds for the solution of (V), provided $m = 1$ and $F(Z) = g(|Z|)$ for some real valued function $g : [0,\infty) \to [0,\infty)$, if vector valued data f are considered. This result constitutes the basis for the study of the vector valued and higher dimensional generalizations that Tietz considered in his Thesis [41]. It is one of the major drawbacks of the higher-order model that there is no comparable boundedness principle, as e.g. the counterexample from [6] on p. 213 shows. More recent are the works [42] which may serve as a synopsis of the widespread topic, [38] which treats the solvability of (V) in the Sobolev class for $m = 1$, [23] where the authors investigate analytical properties of a decoupled version of the problem (V) for $m = n = 2$ with superlinear densities (cf. also Part III of the thesis at hand), [43] where a certain class of anisotropic densities for vector valued images is considered, [44] and finally [45], where the "maximum principle" for the first-order case is established even if f is only of class L^2.

As the reader may guess from the title of this work, we are not primarily interested in questions concerning the performance and applicability of the proposed model in practice, but instead in its examination from the viewpoint of elliptic regularity theory (cf. e.g. the monograph [46]). The various benefits that the use of higher-order methods provides for the solution of the denoising/inpainting problem have been revealed in a large number of publications, where the reader will also find many numerical examples that justify its usefulness; we refer to the list of citations at the top of p. 4.

Our further considerations are divided into three parts, the first of which treats general aspects of the variational problem (V) concerning the existence of generalized solutions in the class BV^m and an analysis of the convex dual problem. It is initialized by a chapter in which we collect some basic properties of the function class BV^m. Of particular note is the density result from Section 1.2, which proves that the class of smooth functions $C^\infty(\overline{\Omega})$ is a dense subspace of $W^{m,1}(\Omega) \cap L^q(\Omega - D)$ (with respect to the norm topology) as well as of $BV^m(\Omega) \cap L^q(\Omega - D)$ (with respect to the area strict topology). This will be useful for the study of the relaxed and the dual problem in the second chapter, where we follow basic ideas from [36] and [37] to examine the set of generalized minimizers of problem (V) and establish their connection to the solution of the convex dual problem via a so called duality formula. This approach becomes necessary due to the non-reflexivity of the Sobolev space $W^{m,1}(\Omega)$, which renders an application of the "direct method of the Calculus of Variations", relying on the weak compactness principle, impossible.

The second part is devoted to the examination of regularity properties of the solutions of (V). By its very nature, this requires to add some further constraints to our model. We start with the easiest possible case in Chapter 3, where we set $m = n = 1$. Although this one dimensional model may appear trivial at first glance, its study will shed a light on some aspects of the general case, especially the connection between the solvability of (V) in the Sobolev class $W^{m,1}(\Omega)$ and the ellipticity parameter μ from (F_μ). Namely, we will construct an example which proves that, for values $\mu > 2$, problem (V) does not admit a solution in the Sobolev class, in general. Following up on this, we take a look at the two-dimensional specialization $m = n = 2$ in Chapter 4; without doubt the setting with the highest relevance to possible applications in image analysis. The framework of μ-ellipticity will here allow us to prove Sobolev regularity of generalized minimizers (i.e. the existence of a solution of (V)) with the optimal bound $\mu < 2$ in the case of pure denoising ($\mathcal{L}^n(D) = 0$), and $\mu < \frac{3}{2}$ in the general setting. An application of the blow-up technique by Evans and Gariepy (see [47]) will yield that this minimizer is even locally Hölder continuous away from a set of Hausdorff dimension zero on the interior of Ω in Section 4.3.

The third and last part finally discusses a different approach to the higher-order case that was already proposed and analyzed in [23] for densities F of superlinear growth. The basis for this is a certain coupling procedure, where an additional set of tensor valued variables is introduced in order to replace the higher-order derivatives. The resulting variational problem then involves only first-order terms, and the associated differential equations are of second order which constitutes an enormous simplification from the viewpoint of numerics and computation. Following the general scheme of the thesis, we discuss the relaxation and the dual problem, followed by a study of regularity properties for suitable μ-elliptic coupling-densities. In the last section we will be able to deduce partial interior Hölder regularity of the corresponding minimizers by an adaption of the method described in [36] by Bildhauer and Fuchs, based on results by Frehse and Seregin (cf. [48], [49]).

In a short Appendix, the reader will find a roundup of some fundamental results from the theory of function spaces and convex functions of a measure that will hopefully facilitate the understanding of the text, as well as the execution of a rather technical auxiliary calculation in connection with the well known difference quotient technique.

With some few exceptions (concerning only minor adjustments or elementary observations), all results of this thesis are published and can be found in the papers [50], [51], [52], and [53].

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Part I

The Higher-Order Model: General Aspects

Chapter 1

Functions of Higher-Order Bounded Variation

The natural domain of an m-th order variational problem of linear growth is the Sobolev space $W^{m,1}(\Omega)$, which consists of all $L^1(\Omega)$-functions with distributional (partial) derivatives up to order m in L^1. However, due to the nonreflexivity of this space, bounded sets fail to be precompact in its weak topology, which means that there is no direct method to prove the existence of a solution for this class of problems. Therefore, one has to resort to a suitable relaxed formulation of the underlying problem, which can e.g. consist in an extension of the associated functional to a larger class of functions where a minimum is attained owing to an appropriate compactness principle. In the first-order context ($m = 1$), this is found in the space $BV(\Omega)$ of functions of bounded variation, i.e. those $L^1(\Omega)$-functions whose first-order distributional derivatives are Radon measures of finite total mass. We refer to the monographs [1] and [2] for a detailed introduction to this topic. Since we are interested in functionals that involve m-th derivatives, we have to consider a higher-order equivalent of the space $BV(\Omega)$ which we denote by $BV^m(\Omega)$ ($m = 1, 2, ...$). This space consists of all $W^{m-1,1}(\Omega)$-functions whose $(m-1)$-th partial distributional derivatives are BV-functions. Note that $BV^1(\Omega)$ coincides with the classical $BV(\Omega)$. In the first section of this chapter, we collect some basic properties of the spaces $BV^m(\Omega)$ which, for the most part, follow directly from the corresponding results in the first-order case. The second section treats a specific approximation result for functions in the class $BV^m(\Omega) \cap L^q(\Omega - D)$, which will be needed for the further discussion of the higher-order inpainting problem. This result is published in [50] (see also [51] for the more general version that is presented here).

1.1 The Space $BV^m(\Omega)$

Let $\Omega \subset \mathbb{R}^n$ $(n = 1, 2, ...)$ be an open set. We define

$$BV^m(\Omega) := \left\{u \in W^{m-1,1}(\Omega) \,:\, D_{\alpha_1 \dots \alpha_m} u \in BV(\Omega),\ \alpha_i = 1, ..., n\right\}$$

and call it the space of *functions of m-th order bounded variation*, which together with the norm

$$\|u\|_{BV^m(\Omega)} := \|u\|_{W^{m-1,1}(\Omega)} + |\nabla^m u|(\Omega)$$

becomes a Banach space. Here $|\nabla^m u|(\Omega)$ denotes the total variation of the tensor field

$$\nabla^{m-1} u(x) = \left(D_{\alpha_1 \dots \alpha_{m-1}} u(x)\right)_{\alpha_1, \dots, \alpha_{m-1}=1}^n,$$

as defined by

$$|\nabla^m u|(\Omega) := \sup \left(\sum_{\alpha_1, \dots, \alpha_m = 1}^{n} \int_{\Omega} D_{\alpha_1 \dots \alpha_{m-1}} u \cdot \partial_{\alpha_m} \varphi_{\alpha_1, \dots, \alpha_m} \,\mathrm{d}x \right),$$

where the supremum is taken over all $\varphi \in C_0^1\left(\Omega.\mathbb{R}^{n^m}\right)$ with $\|\varphi\|_\infty = 1$.

Obviously, $W^{m,1}(\Omega)$ is a subspace of $BV^m(\Omega)$ and $\|u\|_{BV^m(\Omega)} = \|u\|_{m,1;\Omega}$ holds for all $u \in W^{m,1}(\Omega)$. Function spaces of this type have e.g. been considered in [54] (and particularly in [55], where the notion $HB(\Omega)$ (*"Hessien bornée"="bounded Hessian"*) has been used for the case $m = 2$). The norm topology of $BV^m(\Omega)$ is generally too restrictive for most applications, which is due to the fact that the set of smooth functions $C^\infty(\Omega) \cap BV^m(\Omega)$ is not dense with respect to $\|\cdot\|_{BV^m}$. This is easily seen from the fact that the subspace $W^{m,1}(\Omega) \supset C^\infty(\Omega) \cap BV^m(\Omega)$ is complete and therefore closed in $(BV^m(\Omega), \|\cdot\|_{BV^m})$. However, since quite often approximation by smooth functions is a useful tool, this gives reason to equip BV^m with another topology that is induced by the following metric: for $u, v \in BV^m(\Omega)$ we define

$$\begin{aligned} \rho(u, v) :=& \|u - v\|_{m-1,1;\Omega} + \big||\nabla^m u|(\Omega) - |\nabla^m v|(\Omega)\big| \\ &+ \left| \int_\Omega \sqrt{1 + |\nabla^m u|^2} - \int_\Omega \sqrt{1 + |\nabla^m v|^2} \right|, \end{aligned} \tag{1.1}$$

where for the convex function $F : \mathbb{R} \to \mathbb{R},\ t \mapsto \sqrt{1 + t^2}$ the term $\int_\Omega F(|\nabla u|)$ is defined according to [54], see also Appendix B. Convergence with respect to this metric is called *area strict convergence* owing to the presence of the surface-area term in the last summand. If this term is canceled out from the definition of ρ, then the induced convergence is simply referred to as *strict convergence*. The following result is obtained as a corollary from a more general result by Demengel and Temam (see Theorem 2.2 and Remark 2.1 in [54]):

Lemma 1.1.1
Let $\Omega \subset \mathbb{R}^n$ be an open subset with Lipschitz boundary. Then, for any $u \in BV^m(\Omega)$ and $\varepsilon > 0$ there exists $\varphi \in C^\infty(\overline{\Omega}) \cap BV^m(\Omega)$ such that

$$\rho(u, \varphi) < \varepsilon.$$

Remark 1.1.1 *a) In [54] it is actually proved that the above result is true if $\sqrt{1+t^2}$ is replaced with any other convex function of linear growth and ∇^m is replaced with any other linear differential operator with constant coefficients.*

b) Demengel and Temam actually require $\partial\Omega$ to be of class C^1. However, the proof of Lemma 1.1.1 does not seem to make use of this stronger regularity assumption and works for Lipschitz boundaries as well.

The next result concerns the existence of boundary traces of BV^m-functions. It is a simple consequence of the corresponding result in the first-order case ($m = 1$), see Theorem 3.87 and Theorem 3.88 in [1]:

Lemma 1.1.2
Let $\Omega \subset \mathbb{R}^n$ be open and bounded with Lipschitz boundary, and let $u \in BV^m(\Omega)$. Then there exist $(m-1)$ linear operators $\mathcal{T}_1, ..., \mathcal{T}_{m-1}$ which map u to $\mathcal{T}_k u \in L^1\big(\partial\Omega, \mathbb{R}^{n^k}\big)$ such that for $\mathcal{H}^{n-1}$-almost all $x \in \partial\Omega$ it holds

$$\lim_{r\downarrow 0} \fint_{B_r(x)\cap\Omega} |\nabla^k u(y) - \mathcal{T}_k u(x)| \, \mathrm{d}y = 0, \quad i \in \{1, ..., m-1\}.$$

Furthermore, there is a constant $C > 0$ which depends only on Ω such that $\|\mathcal{T}_k u\|_{L^1(\partial\Omega)} \le C\|u\|_{BV^m(\Omega)}$ for all $k \in \{1, ..., m-1\}$ and the operators $\mathcal{T}_k$ are continuous even with respect to the area-strict topology of $BV^m(\Omega)$.

Proof. This immediately follows from an application of Theorem 3.87 and Theorem 3.88 from [1] to each of the functions $\nabla^k u \in BV(\Omega, \mathbb{R}^{n^k})$, $k \in \{1, ..., m-1\}$. □

Remark 1.1.2 (See Remark 2.2 in [54])
If Ω is bounded with Lipschitz boundary, then the approximating functions $\varphi \in C^\infty(\Omega) \cap BV^m(\Omega)$ from Lemma 1.1.1 in [54] can be chosen to satisfy $\mathcal{T}_k\varphi = \mathcal{T}_k u$ for all $k = 1, ..., m-1$.

Another useful feature of BV-functions is their extension property from Ω to $\mathbb{R}^n$, provided the domain Ω is a so called *extension domain*, which is in particular the case for our choice of Ω (see Proposition 3.21 in [1]):

Lemma 1.1.3
Let $\Omega \subset \mathbb{R}^n$ be open and bounded with Lipschitz boundary. Then, for any $u \in BV^m(\Omega)$ there exists $\widetilde{u} \in BV^m(\mathbb{R}^n)$ such that $u = \widetilde{u}$ on Ω and $|\nabla^m \widetilde{u}|(\partial\Omega) = 0$.

Proof. By Lemma 1.1.1 (see Remark 1.1.2) there is a function $\varphi \in C^\infty(\overline{\Omega}) \cap BV^m(\Omega) \subset W^{m,1}(\Omega)$ with $\mathcal{T}_k\varphi = \mathcal{T}_k u$, $k = 1, ..., m-1$. Hence we can apply

Stein's extension theorem (see Theorem 5 on p. 181 in [56]) to obtain a function $v \in W^{m,1}(\mathbb{R}^n)$ with $v = \varphi$ on Ω. We claim that the function

$$\widetilde{u}(x) := \begin{cases} u(x) & \text{if } x \in \Omega, \\ v(x) & \text{if } x \in \mathbb{R}^n - \overline{\Omega} \end{cases}$$

has the desired properties. First we note that due to $\mathcal{T}_k(u-\varphi) = 0$ for $k = 1, ..., m-2$, each of the functions $\nabla^k(u-\phi)$ lies in the space $\overset{\circ}{W}{}^{1,1}\big(\Omega, \mathbb{R}^{n^k}\big)$ (cf. Theorem 2, p. 275 in [57]). Hence we can extend $u - \varphi$ by zero outside of Ω and

$$\widetilde{v}(x) := \begin{cases} u(x) - \varphi(x) & \text{if } x \in \Omega, \\ 0 \quad \text{if } x \in \mathbb{R}^n - \overline{\Omega} \end{cases}$$

is consequently an element of the Sobolev space $W^{m-1,1}(\Omega)$. But then it follows that $\widetilde{u} = v + \widetilde{v}$ lies in $W^{m-1,1}(\mathbb{R}^n)$, too. Now Corollary 3.89 in [1] applied to $\nabla^{m-1}u$ and $\nabla^{m-1}v$ yields $\widetilde{u} \in BV^m(\mathbb{R}^n)$ together with

$$|\nabla^m \widetilde{u}|(\partial\Omega) = \int_{\partial\Omega} |\mathcal{T}_{m-1}u - \mathcal{T}_{m-1}v| \,\mathrm{d}\mathcal{H}^{n-1} = 0.$$

□

Next we consider embeddings of BV^m into L^p-spaces. As before, the results of the following theorem are straight consequences of the corresponding first-order embedding theorem, see Theorem 3.47, Corollary 3.49 in [1]:

Lemma 1.1.4
Let $\Omega \subset \mathbb{R}^n$ be open and bounded with Lipschitz boundary, and $u \in BV^m(\Omega)$.

a) If $n = 1$, then $u \in L^\infty(\Omega)$.

b) If $n > m$, then $u \in L^p(\Omega)$ for every $1 \le p \le \frac{n}{n-m}$.

c) If $n \ge 2$ and $n = m$, then $u \in L^p(\Omega)$ for every $1 \le p < \infty$.

d) If $n < m$, then $u \in C^{m-n-1,1}(\overline{\Omega})$.

Proof. By the BV-embedding Theorem (see Theorem 3.49 in [1]), it follows that $\nabla^{m-1}u \in L^\infty\big(\Omega, \mathbb{R}^{n^{(m-1)}}\big)$ if $n = 1$ and $\nabla^{m-1}u \in L^{n/(n-1)}\big(\Omega, \mathbb{R}^{n^{(m-1)}}\big)$ if $n > 1$. Thus $u \in W^{m-1,\frac{n}{n-1}}(\Omega)$ and the results b)-d) follow from Sobolev's embedding Theorem (see Theorem 4.12 in [32]). □

The following compactness result is the key tool for proving the existence of minimizers in the class $BV^m(\Omega)$:

Lemma 1.1.5
Let $\Omega \subset \mathbb{R}^n$ be open and bounded with Lipschitz boundary, and let $(u_k)_{k=1}^{\infty}$ be a sequence of $BV^m(\Omega)$-functions such that

$$\|u_k\|_{BV^m(\Omega)} \leq M$$

for some constant $M > 0$. Then there is a subsequence $(u_{k_l})_{l=1}^{\infty}$ and a function $u \in BV^m(\Omega)$ such that

$$\|u - u_{k_l}\|_{W^{m-1,1}(\Omega)} \to 0 \text{ for } l \to \infty \text{ and } \|u\|_{BV^m(\Omega)} \leq M.$$

Proof. By Lemma 1.1.1, there is a sequence $(\varphi_k)_{k=1}^{\infty} \subset C^{\infty}(\overline{\Omega}) \cap BV^m(\Omega)$ such that

$$\|u_k - \varphi_k\|_{W^{m-1,1}(\Omega)} \leq 1/k \quad \text{as well as} \quad \|\varphi_k\|_{BV^m(\Omega)} \leq M + 1/k.$$

Since by the Rellich-Kondrachov Theorem (see Theorem 6.3 in [32]) $W^{m,1}(\Omega)$ is compactly embedded into $W^{m-1,1}(\Omega)$, there exists a subsequence $(\varphi_{k_l})_{l=1}^{\infty}$ which converges to a function $u \in W^{m-1,1}(\Omega)$ as $l \to \infty$. In particular,

$$\nabla^{m-1}\varphi_{k_l} \xrightarrow{l\to\infty} \nabla^{m-1}u \text{ in } L^1\big(\Omega, \mathbb{R}^{n^{m-1}}\big)$$

and therefore $\nabla^{m-1}u \in BV\big(\Omega, \mathbb{R}^{n^{m-1}}\big)$ together with

$$|\nabla^m u|(\Omega) \leq \liminf_{l\to\infty} \int_{\Omega} |\nabla^m \varphi_{k_l}| \, \mathrm{d}x$$

by Propositions 3.6 and 3.13 in [1]. Thus $u \in BV^m(\Omega)$ and

$$\|u\|_{BV^m(\Omega)} \leq \liminf_{l\to\infty} \|\varphi_{k_l}\|_{BV^m(\Omega)} \leq M.$$

□

We end this section with a higher-order variant of the famous Poincaré inequality, which will be useful throughout the following:

Lemma 1.1.6
Let $\Omega \subset \mathbb{R}^n$ be open and bounded with Lipschitz boundary, $m \in \mathbb{N}$, $1 \leq p < \infty$ and let $D \subset \Omega$ be a measurable subset with $0 \leq \mathcal{L}^n(D) < \mathcal{L}^n(\Omega)$.

a) There is a constant $C > 0$, depending only on Ω, D, m, n and p such that for all $u \in W^{m,p}(\Omega)$

$$\|u\|_{W^{m,p}(\Omega)} \leq C \cdot \big(\|\nabla^m u\|_{L^p(\Omega)} + \|u\|_{L^1(\Omega - D)}\big).$$

b) There is a constant $C > 0$, depending only on Ω, D, m and n such that for all $u \in BV^m(\Omega)$

$$\|u\|_{BV^m(\Omega)} \leq C \cdot \Big(|\nabla^m u|(\Omega) + \|u\|_{L^1(\Omega - D)}\Big).$$

Proof. Ad a). From Theorem 1.1.15 and Corollary 1.1.11 in [58] it follows that due to $\mathcal{L}^n(\Omega - D) > 0$,

$$\|\nabla^m u\|_{L^p(\Omega)} + \|u\|_{L^1(\Omega - D)}$$

is a norm on the Sobolev space $W^{m,p}(\Omega)$, which is equivalent to $\|\cdot\|_{m,p;\Omega}$. Part b) now immediately follows from a) via approximation with smooth functions in the area strict topology (cf. Lemma 1.1.1). □

1.2 A Density Result in $BV^m(\Omega) \cap L^q(\Omega - D)$

In the context of the inpainting problem, intersection spaces of type $BV^m(\Omega) \cap L^q(\Omega - D)$ play an important role. As the embedding theorem from the preceding section shows, these are proper subspaces of $BV^m(\Omega)$ if $m < n$ and $q > n/(n - m)$. Here we will be concerned with an adaption of Lemma 1.1.1, where now the approximating smooth function is additionally required to respect the q-integrability of $u \in W^{m,p}(\Omega) \cap L^q(\Omega - D)$ and $u \in BV^m(\Omega) \cap L^q(\Omega - D)$, respectively. In the first-order case ($m = 1$), this was achieved by using a Lipschitz-truncation argument, the details can be found in [41], Lemma 2.2.4 and Lemma 2.2.7 (cf. also Lemma 2.1 in [59]). However, in the higher-order setting, this procedure fails due to the fact that for $m \geq 2$ there exists no nontrivial function $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi(u) \in W^{m,p}(\Omega)$ holds for all $u \in W^{m,p}(\Omega)$ (see [60]), in general. Therefore, we pursue a different approach which is based on "local translations" of the function u. Roughly speaking, this means that, via a partition of unity, u is split into a sum of functions with compact support each of which is then slightly shifted across the boundary of $\Omega - D$ with the result that the domain of q-integrability of the reassembled function is extended outside of $\Omega - D$. The approximating function is then obtained from mollification. In contrast to the first-order case, this technique requires the additional assumption of Lipschitz regularity of the boundary $\partial(\Omega - D)$ (or, equivalently, of ∂D).

Theorem 1.2.1
Let $\Omega \subset \mathbb{R}^n$ be open and bounded with Lipschitz boundary, and let $D \subset \Omega$ be an open subset with $0 \leq \mathcal{L}^n(D) < \mathcal{L}^n(\Omega)$ and such that $\Omega - D$ has Lipschitz boundary, as well. Then, for arbitrary values $1 \leq p, q < \infty$, the following assertions hold:

a) Any function $u \in W^{m,p}(\Omega) \cap L^q(\Omega - D)$ can be approximated by a sequence of smooth functions $(\varphi_k)_{k=1}^{\infty} \subset C^{\infty}(\overline{\Omega})$ such that

$$\|u - \varphi_k\|_{m,p;\Omega} + \|u - \varphi_k\|_{q;\Omega - D} \to 0 \text{ for } k \to \infty.$$

b) Any function $u \in BV^m(\Omega) \cap L^q(\Omega - D)$ can be approximated by a sequence *of smooth functions* $(\varphi_k)_{k=1}^{\infty} \subset C^{\infty}(\overline{\Omega})$ *such that*

$$\|u-\varphi_k\|_{m-1,1;\Omega} + \|u-\varphi_k\|_{q;\Omega-D} + \left| |\nabla^m u|(\Omega) - \int\limits_{\Omega} |\nabla^m \varphi_k| \, \mathrm{d}x \right|$$
$$+ \left| \int\limits_{\Omega} \sqrt{1+|\nabla^m u|^2} - \int\limits_{\Omega} \sqrt{1+|\nabla^m \varphi_k|^2} \, \mathrm{d}x \right| \to 0 \quad \textit{for} \quad k \to \infty.$$

Remark 1.2.1 *a) For $p > 1$, part a) can be deduced from an approximation Lemma by Hedberg (see [61], Theorem 3.4.1) even without the assumption of $\Omega - D$ having Lipschitz boundary. The proof avoids truncation and uses arguments from Potential Theory instead. Therefore it fails for $p = 1$ and a fortiori in the BV-case.*

b) It will be evident from the proof that it makes no difference whether we require $\Omega - D$ or D to have Lipschitz boundary in the statement of Theorem 1.2.1.

c) Theorem 1.2.1 was first published in [50] under more restrictive assumptions on the geometry of the set D, which could later be relaxed to those given above (cf. [51]).

Proof. We begin with part a) and first look at the special case in which Ω is a cuboid in $\mathbb{R}^n$,

$$\Omega = (a_1, b_1) \times ... \times (a_n, b_n)$$

and $\Omega - D$ is bounded by the graph of a Lipschitz continuous function ϕ with Lipschitz constant $L := \mathrm{Lip}(\phi)$, i.e.

$$\Omega - D = \Big\{ (x_1, ..., x_n) \in \Omega \, \Big| \, x_n < \phi(x_1, ..., x_{n-1}) \Big\}.$$

We define

$$\Omega_{-1} := \emptyset,$$
$$\Omega_i := \left\{ x \in \Omega \, \Big| \, \mathrm{dist}(x, \partial\Omega) > \frac{1}{i+1} \right\}, \; i \in \mathbb{N}_0.$$

Then, for $j \in \mathbb{N}_0$, the sets $A_j := \Omega_{j+1} - \overline{\Omega_{j-1}}$ cover Ω. Let $(\eta_j)_{j=0}^{\infty} \subset C_0^{\infty}(\Omega)$ denote a partition of unity with respect to this covering, i.e. $\mathrm{spt}\, \eta_j \Subset A_j$ and $\sum\limits_{j=0}^{\infty} \eta_j \equiv 1$. Let further C denote the cone

$$C := \Big\{ (x_1, ..., x_n) \in \mathbb{R}^n \, \Big| \, x_n < -L \cdot \big| (x_1, ..., x_{n-1}, 0) \big| \Big\}$$

and let ρ_ε be a symmetric mollifier supported in the ball $B_\varepsilon(0)$ for some $\varepsilon > 0$. Note that for any $x \in \overline{\Omega - D}$ we then have

$$(C + x) \cap \Omega \subset \Omega - D.$$

For a given $\delta > 0$, we will first construct a function $\varphi_\delta \in C^\infty(\Omega)$ such that

$$\|u - \varphi_\delta\|_{m,p;\Omega} + \|u - \varphi_\delta\|_{q;\Omega - D} < \delta. \tag{1.2}$$

To this purpose, consider $u_j := \eta_j u$ and the shifted functions

$$u_j^{h_j} := u_j(x_1, ..., x_n - h_j)$$

for some fixed $h_j > 0$, which is small enough such that $\operatorname{spt}(\eta_j) + (0, ..., 0, h_j) \Subset \Omega$. Since translations act continuously on $L^p(\mathbb{R}^n)$ (and hence on Sobolev spaces),

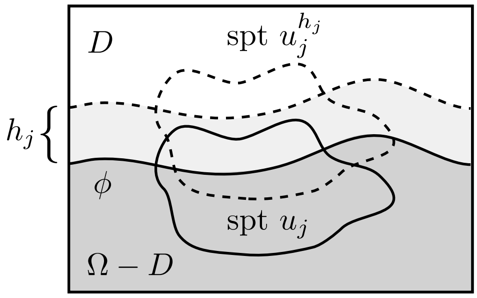

Fig. 1.1

we can choose a decreasing sequence of positive numbers h_j such that $\operatorname{spt} u_j + (0, ..., 0, h_j) \Subset A_j$ for each j and

$$\|u_j - u_j^{h_j}\|_{m,p;\Omega} + \|u_j - u_j^{h_j}\|_{q;\Omega - D} < \delta 2^{-(j+2)}. \tag{1.3}$$

Further, we can select a decreasing sequence of positive numbers ε_j which satisfy

$$B_{\varepsilon_j}(0) - (0, ..., 0, h_j) \Subset C, \tag{1.4}$$

$$\big(\operatorname{spt}(\eta_j) + (0, ..., 0, h_j)\big)^{\varepsilon_j} \Subset A_j, \tag{1.5}$$

where for a subset $A \subset \mathbb{R}^n$ we denote by A^{ε_j} the outer parallel set in distance ε_j. Moreover, we may assume that it holds

$$\|\rho_{\varepsilon_j} * u_j^{h_j} - u_j^{h_j}\|_{m,p;\Omega} + \|\rho_{\varepsilon_j} * u_j^{h_j} - u_j^{h_j}\|_{q;\Omega - D} < \delta 2^{-(j+2)}. \tag{1.6}$$

Note that, due to (1.4), we have $\rho_{\varepsilon_j} * u_j^{h_j} \in L^q(\Omega - D)$ since $u_j^{h_j}$ is q-integrable on $\big\{x \in \Omega \,|\, x_n < \phi(x_1, ..., x_{n-1}) + h_j\big\}$ (cf. Figure 1.1). By (1.3) and (1.6) we further have

$$\|u_j - \rho_{\varepsilon_j} * u_j^{h_j}\|_{m,p;\Omega} + \|u_j - \rho_{\varepsilon_j} * u_j^{h_j}\|_{q;\Omega - D} < \delta 2^{-(j+1)}. \tag{1.7}$$

Consequently, $\varphi_\delta := \sum\limits_{j=0}^{\infty} \rho_{\varepsilon_j} * u_j^{h_j}$ is a smooth function that satisfies (1.2). Furthermore, from the construction of φ_δ it is clear that

$$\mathcal{T}_k u = \mathcal{T}_k \varphi_\delta \quad \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega \quad \text{for } k = 1, ..., m-1, \tag{1.8}$$

where $\mathcal{T}_k$ denotes the k-th boundary trace operator from Lemma 1.1.2.

Now we consider the general case. Let $\delta > 0$ be given. Since u can be extended outside of Ω we may, w.l.o.g., assume $u \in W^{m,p}(\mathbb{R}^n)$. Cover $\partial(\Omega - D)$ by a finite number of cuboids $Q_1, ..., Q_N$ such that $(\Omega - D) \cap Q_i$ lies beneath the graph of a Lipschitz function (with respect to a suitable local coordinate system), and such that $\Omega - D$ is a compact subset of $(\Omega - D) \cup Q_1 \cup ... \cup Q_N$. This means that, on each of the cuboids, we are in the situation of our special case. Starting with Q_1, we therefore find a smooth function $\varphi_1 \in C^\infty(Q_1)$ such that

$$\|u - \varphi_1\|_{m,p;Q_1} + \|u - \varphi_1\|_{q;(\Omega-D)\cap Q_1} < \frac{\delta}{2N}$$

and by (1.8), the function u_1 defined through

$$u_1(x) := \begin{cases} u(x), & x \in \mathbb{R}^n - Q_1, \\ \varphi_1(x), & x \in Q_1 \end{cases}$$

is an element of $W^{m,p}(\mathbb{R}^n) \cap L^q(\Omega - D)$ for which it holds that

$$\|u - u_1\|_{m,p;\mathbb{R}^n} + \|u - u_1\|_{q;\Omega-D} < \frac{\delta}{2N}.$$

Continuing this process on Q_2 with u replaced by u_1 and so on, we finally end up with a function u_N which satisfies

$$\|u - u_N\|_{m,p;\mathbb{R}^n} + \|u - u_N\|_{q;\Omega-D} < \frac{\delta}{2}$$

and which is locally q-integrable on

$$U := (\Omega - D) \cup \bigcup_{i=1}^{N} Q_i \Supset (\Omega - D).$$

Hence we can choose $\varepsilon < \operatorname{dist}(\Omega - D, \partial U)$ small enough, such that

$$\|\rho_\varepsilon * u_N - u_N\|_{m,p,\mathbb{R}^n} + \|\rho_\varepsilon * u_N - u_N\|_{q;\Omega-D} < \frac{\delta}{2}$$

and observe that $\varphi_\delta := \rho_\varepsilon * u_N \in C^\infty(\mathbb{R}^n) \subset C^\infty(\overline{\Omega})$ approximates u as desired.

We continue with the proof of part b), keeping the notation from above. Let $u \in BV^m(\Omega) \cap L^q(\Omega - D)$. Again it will essentially suffice to prove the claim in the special case of Ω being an n-dimensional cuboid and $\partial(\Omega - D)$ being the graph of a Lipschitz continuous function. For a given $\delta > 0$, we choose a sequence $(h_j)_{j=0}^\infty$ of positive numbers such that $\operatorname{spt}(\eta_j) + (0, ..., 0, h_j) \Subset A_j$ and such that the following conditions are satisfied:

$$\|(\eta_j u)^{h_j} - \eta_j u\|_{m-1,1;\Omega} + \|(\eta_j u)^{h_j} - \eta_j u\|_{q;\Omega-D} < \delta 2^{-(j+2)}, \tag{1.9}$$

$$\left\| \big(\nabla^m(\eta_j u) - \eta_j \nabla^m u\big)^{h_j} - \big(\nabla^m(\eta_j u) - \eta_j \nabla^m u\big) \right\|_{1;\Omega} < \delta 2^{-(j+2)}. \tag{1.10}$$

Note that $\nabla^m(\eta_j u) - \eta_j \nabla^m u \in L^1(\Omega)$ so that we can require h_j to satisfy (1.10). Furthermore, since $\sum_{j=0}^{\infty} \eta_j \equiv 1$ on Ω, it holds $\sum_{j=0}^{\infty} \big(\nabla^m(\eta_j u) - \eta_j \nabla^m u\big) \equiv 0$. Consider now

$$u_\delta := \sum_{j=0}^{\infty} (\eta_j u)^{h_j}.$$

We claim that u_δ approximates u in $BV^m(\Omega) \cap L^q(\Omega - D)$ with respect to the metric $\rho(.,.)$ that defines the area-strict topology on $BV^m(\Omega)$, see (1.1). First, from (1.9) it follows that

$$\|u - u_\delta\|_{m-1,1;\Omega} + \|u - u_\delta\|_{q;\Omega - D} < \delta/2.$$

Let us compute the total variation of $\nabla^m u_\delta$. For a measure $\mu \in \mathcal{M}(\Omega)$, let μ^{h_j} denote the image measure under translation by h_j in the n-th coordinate direction. By Proposition 3.18 in [1] it then holds $\nabla^m(\eta_j u)^{h_j} = \big(\nabla^m(\eta_j u)\big)^{h_j}$ and therefore (recall $\sum_{j=0}^{\infty} \eta_j \equiv 1$ and note that the occurring sums are locally finite):

$$\begin{aligned}
\nabla^m u_\delta &= \sum_{j=0}^{\infty} \nabla^m(\eta_j u)^{h_j} = \sum_{j=0}^{\infty} \big(\nabla^m(\eta_j u)\big)^{h_j} \\
&= \sum_{j=0}^{\infty} \Big[(\eta_j \nabla^m u)^{h_j} + (\nabla^m(\eta_j u) - \eta_j \nabla^m u)^{h_j}\Big] \\
&= \sum_{j=0}^{\infty} \Big[(\eta_j \nabla^m u)^{h_j} + \Big\{(\nabla^m(\eta_j u) - \eta_j \nabla^m u)^{h_j} - (\nabla^m(\eta_j u) - \eta_j \nabla^m u)\Big\}\Big],
\end{aligned}$$

so that

$$\begin{aligned}
&\int_\Omega |\nabla^m u_\delta| \le \sum_{j=0}^{\infty} \int_\Omega |(\eta_j \nabla^m u)^{h_j}| \\
&+ \sum_{j=0}^{\infty} \int_\Omega \Big|(\nabla^m(\eta_j u) - \eta_j \nabla^m u)^{h_j} - (\nabla^m(\eta_j u) - \eta_j \nabla^m u)\Big| \, \mathrm{d}x \\
&\le \left(\sum_{j=0}^{\infty} \int_\Omega \Big|(\eta_j \nabla^m u)^{h_j}\Big|\right) + \frac{\delta}{2} = \left(\sum_{j=0}^{\infty} \int_\Omega \eta_j \, \mathrm{d}|\nabla^m u|\right) + \frac{\delta}{2} = \int_\Omega |\nabla^m u| + \frac{\delta}{2}.
\end{aligned}$$

Letting $\delta \downarrow 0$, it follows $\limsup_{\delta \to 0} |\nabla^m u_\delta|(\Omega) \le |\nabla^m u|(\Omega)$ and the lower semicontinuity of the total variation together with $u_\delta \to u$ in $W^{m-1,1}(\Omega)$ thus implies

$$|\nabla^m u_\delta|(\Omega) \to |\nabla^m u|(\Omega).$$

Moreover, we claim that it even holds

$$\nabla_a^m u_\delta \to \nabla_a^m u \text{ in } L^1\big(\Omega, \mathbb{R}^{n^m}\big)$$

as $\delta \downarrow 0$ ($\nabla^m_a u$ denoting the Lebesgue density of the measure $\nabla^m u$). To justify this, we observe that on Ω_i it holds

$$\nabla^m u_\delta|_{\Omega_i} = \left(\sum_{j=0}^{i} \left(\nabla^m (\eta_j u)\right)^{h_j} \right) \Bigg|_{\Omega_i}$$

and since for two measures μ and ν it holds $(\mu + \nu)^a = \mu^a + \nu^a$ as well as $(\mu^{h_j})^a = (\mu^a)^{h_j}$, it follows

$$\chi_{\Omega_i} \cdot \nabla^m_a u_\delta = \chi_{\Omega_i} \cdot \left(\sum_{j=0}^{i} \left(\nabla^m_a (\eta_j u)\right)^{h_j} \right) = \chi_{\Omega_i} \cdot \left(\sum_{j=0}^{\infty} \left(\nabla^m_a (\eta_j u)\right)^{h_j} \right).$$

Due to $\chi_{\Omega_i} \cdot \nabla^m_a u_\delta \xrightarrow{i\to\infty} \nabla^m_a u_\delta$ in $L^1(\Omega, \mathbb{R}^{n^m})$, we infer

$$\nabla^m_a u_\delta = \sum_{j=0}^{\infty} \left(\nabla^m_a (\eta_j u)\right)^{h_j}.$$

Since $\nabla^m_a (\eta_j u) \in L^1(\Omega, \mathbb{R}^{n^m})$, we can choose h_j small enough such that

$$\left\| \left(\nabla^m_a (\eta_j u)\right)^{h_j} - \nabla^m_a (\eta_j u) \right\|_{1;\Omega} < \delta 2^{-(j+1)}$$

and therefore $\|\nabla^m_a u_\delta - \nabla^m_a u\|_{1;\Omega} < \delta$. Hence

$$\nabla^m_a u_\delta \to \nabla^m_a u \text{ in } L^1(\Omega) \quad \text{as } \delta \downarrow 0$$

and since the functional $v \mapsto \int_\Omega \sqrt{1 + v(x)^2}\, \mathrm{d}x$ is continuous on $L^1(\Omega)$, we infer that

$$\int_\Omega \sqrt{1 + |\nabla^m_a u_\delta|^2}\, \mathrm{d}x \xrightarrow{\delta\downarrow 0} \int_\Omega \sqrt{1 + |\nabla^m_a u|^2}\, \mathrm{d}x. \tag{1.11}$$

Now $|\nabla^m u_\delta|(\Omega) \to |\nabla^m u|(\Omega)$ together with $|\nabla^m_a u_\delta|(\Omega) \to |\nabla^m_a u|(\Omega)$ implies

$$|\nabla^m_s u_\delta|(\Omega) \to |\nabla^m_s u|(\Omega), \tag{1.12}$$

which together with (1.11) proves

$$\int_\Omega \sqrt{1 + |\nabla^m u_\delta|^2} \to \int_\Omega \sqrt{1 + |\nabla^m u|^2} \quad \text{as } \delta \downarrow 0.$$

Hence, for $\delta > 0$ arbitrarily small, we can choose $0 < \widetilde{\delta} \leq \delta$ such that

$$\rho(u, u_{\widetilde{\delta}}) + \|u - u_{\widetilde{\delta}}\|_{q;\Omega - D} < \delta/2.$$

Then, we may follow the arguments from the proof of Theorem 2.2 in [54] and choose a sequence $\varepsilon_j \downarrow 0$ such that the smooth function

$$v_{\widetilde{\delta}} := \sum_{j=0}^{\infty} \rho_{\varepsilon_j} * (\eta_j u_{\widetilde{\delta}})$$

satisfies

$$\rho(v_{\widetilde{\delta}}, u_{\widetilde{\delta}}) + \|v_{\widetilde{\delta}} - u_{\widetilde{\delta}}\|_{q;\Omega - D} < \delta/2,$$

where we tacitly assume that $\varepsilon_j < \min\{h_{j-1}, h_j, h_{j+1}\}$, so that $\rho_{\varepsilon_j} * (\eta_j u_{\widetilde{\delta}}) \in L^q(\Omega - D)$. Altogether, we find that $v_{\widetilde{\delta}}$ approximates u as required. The general case now follows via the same covering argument as in the proof of part a). $\square$

Chapter 2

Relaxation and Dual Formulation

Following the ideas in [36] and [37], this chapter will treat the relaxation and dual formulation of the variational Problem (V). To this purpose we choose a suitable extension of the functional I to the class $BV^m(\Omega) \supset W^{m,1}(\Omega)$, where we can prove the existence of a generalized minimizer due to the BV^m-compactness property (see Lemma 1.1.5). We will see that this function coincides with the solution of (V) in the Sobolev class whenever such a minimizer exists. In the second Section, the dual of our minimization problem $I \to \min$ in the sense of convex optimization (cf. e.g. [62]) will be introduced and shown to possess a unique solution in the class $L^\infty\big(\Omega, \mathbb{R}^{n^m}\big)$. Eventually, the connection between the generalized BV^m-minimizers and the dual solution will be established in form of a so called duality formula. The results of this chapter are published in [51] with the sole difference that in this reference, for simplicity, only the second order case ($m = 2$) was treated.

2.1 Generalized Solutions in $BV^m(\Omega)$

The concept of convex functions of a measure (see Appendix B) enables us to give a natural extension of the quantity $\int_\Omega F(\nabla^m u)\,\mathrm{d}x$, defined on the Sobolev space $W^{m,1}(\Omega)$, to the class of $BV^m(\Omega)$-functions through the formula

$$\int\limits_\Omega F(\nabla^m u) := \int\limits_\Omega F\left(\nabla^m_a u\right)\,\mathrm{d}x + \int\limits_\Omega F^\infty\left(\frac{\nabla^m_s u}{|\nabla^m_s u|}\right)\mathrm{d}|\nabla^m_s u|.$$

Here, $\nabla^m u = \nabla^m_a u \cdot \mathcal{L}^n + \nabla^m_s u$ is the Lebesgue decomposition of the measure $\nabla^m u$ and

$$F^\infty(Z) := \lim_{t\to\infty} \frac{F(tZ)}{t} \quad \text{for all } Z \in \mathbb{R}^{n^m}$$

is the so called *recession function*, which is well defined due to our assumption (F2) on F. Now we define the functional $K : BV^m(\Omega) \to [0, \infty]$ by

$$K[u] := \int_\Omega F(\nabla^m u) + \lambda \int_{\Omega - D} \phi\big(|u - f|\big)\,\mathrm{d}x. \tag{2.1}$$

Note that for $u \in W^{m,1}(\Omega)$ it holds $\nabla_s^m u \equiv 0$ and therefore $K \equiv I$ on $W^{m,1}(\Omega)$, which means that K is a reasonable extension of our primal functional.

The following lemma is an immediate consequence from Reshetnyak's continuity Theorem (see Theorem B.1 in Appendix B) and will be very useful in the study of the functional K:

Lemma 2.1.1
Let $F : \mathbb{R}^{n^m} \to \mathbb{R}$ satisfy (F1)-(F3), $(w_k)_{k\in\mathbb{N}} \in BV^m(\Omega)$ be a bounded sequence and $w \in BV^m(\Omega)$.

(a) If $w_k \to w$ in $W^{m-1,1}(\Omega)$ for $k \to \infty$, then

$$\int_\Omega F(\nabla^m w) \le \liminf_{k\to\infty} \int_\Omega F(\nabla^m w_k). \tag{2.2}$$

(b) If we know in addition

$$\int_\Omega \sqrt{1 + |\nabla^m w_k|^2} \to \int_\Omega \sqrt{1 + |\nabla^m w|^2} \quad \textit{for } k \to \infty,$$

then

$$\int_\Omega F(\nabla^m w) = \lim_{k\to\infty} \int_\Omega F(\nabla^m w_k).$$

In contrast to the primal problem (V), the relaxed problem

$$K[u] \to \min \ \text{ in } BV^m(\Omega) \tag{$\widetilde{V}$}$$

always possesses a (not necessarily unique!) minimizer even under our general assumptions on the regularizer F and the penalty function ϕ. The next theorem collects some properties of these minimizers:

Theorem 2.1.1
Let Ω and D be as in Theorem 1.2.1, assume F satisfies (F1)-(F3) and that ϕ fulfills (ϕ1), (ϕ2) for some $q \ge 1$. Then we have:

a) The problem $(\widetilde{V})$ admits at least one solution $u \in BV^m(\Omega)$.

b) Suppose that u and $\widetilde{u}$ are two distinct solutions of $(\widetilde{V})$. Then $u = \widetilde{u}$ almost everywhere on $\Omega - D$ and $\nabla_a^m u = \nabla_a^m \widetilde{u}$ a.e. on Ω. In particular, the solution u is unique if $\mathcal{L}^n(D) = 0$ (i.e. in the case of pure denoising).

c) We have

$$\inf_{W^{m,1}(\Omega)} I = \inf_{BV^m(\Omega)} K.$$

d) Let $\mathcal{M}$ denote the set of all $W^{m-1,1}$-cluster points of I-minimizing sequences from the space $W^{m,1}(\Omega)$. Then $\mathcal{M}$ coincides with the set of all K-minimizers in $BV^m(\Omega)$.

e) If $\mathcal{M}$ contains a function $u \in W^{m,1}(\Omega)$, then it already holds $\mathcal{M} = \{u\}$.

Proof. By part b) of Lemma 1.1.6, it follows that any K-minimizing sequence $(u_k)_{k=1}^{\infty}$ is uniformly bounded in $BV^m(\Omega) \cap L^q(\Omega - D)$ and hence, by the BV^m-compactness property (and possibly after passing to a suitable subsequence) there is a function $u \in BV^m(\Omega)$ such that $u_k \to u$ in $W^{m-1,1}(\Omega)$ and a.e. on Ω. Furthermore, Fatou's lemma implies

$$\int\limits_{\Omega - D} \phi\big(|u - f|\big)\, \mathrm{d}x \leq \liminf_{k \to \infty} \int\limits_{\Omega - D} \phi\big(|u_k - f|\big)\, \mathrm{d}x$$

and an application of Lemma 2.1.1 a) yields that the limit function u is in fact a minimizer of the functional K. Part b) is an immediate consequence of the strict convexity of F and ϕ. For part c), we note that $\inf_{BV^m} K \leq \inf_{W^{m,1}} I$ is clear since $K \equiv I$ on $W^{m,1}(\Omega)$. For the other inequality, we choose a sequence of smooth functions

$$(\varphi_l)_{l=1}^{\infty} \subset C^{\infty}(\Omega) \cap L^q(\Omega - D)$$

that approximates u in the area-strict topology of BV^m as in Theorem 1.2.1. Since by (ϕ2) the convex functional

$$L^q(\Omega - D) \ni u \mapsto \int\limits_{\Omega - D} \phi\big(|u - f|\big)\, \mathrm{d}x$$

is locally bounded and therefore continuous, it follows that

$$\int\limits_{\Omega - D} \phi\big(|\varphi_l - f|\big)\, \mathrm{d}x \xrightarrow{l \to \infty} \int\limits_{\Omega - D} \phi\big(|u - f|\big)\, \mathrm{d}x$$

and together with part b) of Lemma 2.1.1 this gives

$$\inf_{BV^m} K = \lim_{l \to \infty} K[\varphi_l] = \lim_{l \to \infty} I[\varphi_l] \geq \inf_{W^{m,1}} I.$$

Now to d). Again, Theorem 1.2.1 in combination with Lemma 2.1.1 implies that any K-minimizer occurs as the $W^{m,1}$-limit of an I-minimizing sequence. Moreover, let $w \in BV^m(\Omega)$ be any $W^{m,1}$-limit of an I-minimizing sequence $(w_k)_{k=1}^{\infty}$. Then by part a) of Lemma 2.1.1 it holds

$$K[w] \leq \liminf_{k \to \infty} K[u_k] = \liminf_{k \to \infty} I[u_k] = \inf_{W^{m,1}(\Omega)} I \stackrel{c)}{=} \inf_{BV^m(\Omega)} K$$

so that w is in fact K-minimal. It remains to give a proof of the statement in e). Assume that there exist two distinct elements $u \in \mathcal{M} \cap W^{m,1}(\Omega)$ and $\widetilde{u} \in \mathcal{M}$. Then $K[u] = K[\widetilde{u}]$ together with b) implies

$$\int_\Omega F^\infty \left(\frac{\nabla_s^m \widetilde{u}}{|\nabla_s^m \widetilde{u}|} \right) \mathrm{d}|\nabla_s^m \widetilde{u}| = 0.$$

Since $F(Z) > 0$ if $Z \neq 0 \in \mathbb{R}^{n^m}$, this implies

$$\frac{\nabla_s^m \widetilde{u}}{|\nabla_s^m \widetilde{u}|} = 0 \quad |\nabla_s^m \widetilde{u}| \text{ -a.e.},$$

hence $\nabla_s^m \widetilde{u} \equiv 0$, i.e. $\widetilde{u} \in W^{m,1}(\Omega)$. Then b) implies

$$\nabla^m \widetilde{u} = \nabla_a^m \widetilde{u} = \nabla_a^m u = \nabla^m u \qquad \text{a.e. on } \Omega,$$

so that $\widetilde{u} - u = p$ is a polynomial of degree at most $m-1$. From $\widetilde{u} = u$ a.e. on $\Omega - D$ together with $\mathcal{L}^n(\Omega - D) > 0$ we finally infer $p \equiv 0$ and thus $\widetilde{u} = u$ a.e. on Ω. □

2.2 The Dual Problem

In this section, we are going to describe the convex dual of problem (V). This approach has already been successfully pursued for $m = 1$ in [37], [63] and [59], whose results are now transfered to the m-th order setting. Following these ideas, we pass to the dual formulation via Lagrangians as it is described in detail in Chapter III, section 4 of the monograph [62]. This method is based on the notion of the *convex conjugate* of a functional, which is defined as follows (cf. [62], p. 16): let V denote some Banach space with dual V^* and $G : V \to \mathbb{R} \cup \{\infty\}$ a functional on V. Then, for every $u^* \in V^*$, we define the convex dual functional of G through the formula

$$G^*(u^*) := \sup_{u \in V} \Big[u^*(u) - G(u) \Big].$$

It is shown in Proposition 2.1 from p. 271 in [62] that for a convex integral functional on $L^p(\Omega, \mathbb{R}^N)$ of the form $G(u) = \int_\Omega F(u)\,\mathrm{d}x$ with $F : \mathbb{R}^N \to \mathbb{R}$ it holds

$$G^*(v) = \int_\Omega F^*(v)\,\mathrm{d}x \quad \text{for all } v \in L^{p^*}(\Omega, \mathbb{R}^N),$$

where p^* is the conjugate exponent of p and

$$F^*(y) := \sup_{x \in \mathbb{R}^N} \Big[x \cdot y - F(x) \Big], \quad y \in \mathbb{R}^N. \tag{2.3}$$

Moreover, Proposition 4.1 on p. 18 of [62] shows that for a convex function $F : \mathbb{R}^N \to \mathbb{R}$ it always holds $F^{**} = F$, which gives us the following representation of our functional I: for all $w \in W^{m,1}(\Omega) \cap L^q(\Omega - D)$ it holds

$$I[w] = \sup_{\kappa \in L^\infty(\Omega, \mathbb{R}^{n^m})} \int_\Omega \Big[\kappa : \nabla^m w - F^*(\kappa)\Big] \,\mathrm{d}x + \lambda \int_{\Omega - D} \phi\big(|w - f|\big) \,\mathrm{d}x. \tag{2.4}$$

This motivates the following definition: for all $w \in W^{m,1}(\Omega) \cap L^q(\Omega - D)$ and $\kappa \in L^\infty\big(\Omega, \mathbb{R}^{n^m}\big)$, we define the *Lagrangian* of our variational problem (V) as the bivariate functional

$$l(w, \kappa) := \int_\Omega \Big[\kappa : \nabla^m w - F^*(\kappa)\Big] \,\mathrm{d}x + \lambda \int_{\Omega - D} \phi\big(|w - f|\big) \,\mathrm{d}x.$$

The *dual functional* $R : L^\infty(\Omega, \mathbb{R}^{n^m}) \to [-\infty, \infty]$ is then obtained by exchanging "sup" with "inf" in the above formula for I, i.e.

$$R[\kappa] := \inf_{w \in W^{m,1}(\Omega)} l(w, \kappa), \tag{2.5}$$

and the *dual problem* consists in maximizing R:

$$R[\kappa] \to \max \quad \text{in } L^\infty(\Omega, \mathbb{R}^{n^m}). \tag{V^*}$$

Theorem 2.2.1
Let Ω and D be as in Theorem 1.2.1, assume F satisfies (F1)-(F3) and that ϕ fulfills (ϕ1), (ϕ2) for some $q \in [1, 2]$. Then we have:

(a) The dual problem (V^*) admits a unique solution $\sigma \in L^\infty\big(\Omega, \mathbb{R}^{n^m}\big)$.

(b) The solution from a) is unique and if $u \in \mathcal{M} \subset BV^m(\Omega)$ is any minimizer of ($\widetilde{V}$), then the duality formula holds:

$$\sigma = DF\big(\nabla^m_a u\big)$$

(note that by Theorem 2.1.1 b) $\nabla^m_a u$ is independent of the choice of $u \in \mathcal{M}$).

(c) The functionals I and R obey the so called "inf-sup" *relation:*

$$\inf_{W^{m,1}(\Omega)} I = \sup_{L^\infty(\Omega)} R.$$

Proof. As in the first-order case (see, e.g. [37], [59] or [41]), we will construct a maximizer of the dual functional R as the limit of a sequence of solutions to a family of regularized problems. For $0 < \delta < 1$ we define

$$\begin{aligned}
J_\delta[u] &:= \frac{\delta}{2} \int_\Omega |\nabla^m u|^2 \,\mathrm{d}x + I[u] \\
&= \frac{\delta}{2} \int_\Omega |\nabla^m u|^2 + F(\nabla^m u) \,\mathrm{d}x + \lambda \int_{\Omega - D} \phi\big(|u - f|\big) \,\mathrm{d}x \\
&=: \int_\Omega F_\delta(\nabla^m u) \,\mathrm{d}x + \lambda \int_{\Omega - D} \phi\big(|u - f|\big) \,\mathrm{d}x,
\end{aligned}$$

where we have set

$$F_\delta(Z) := \frac{\delta}{2}|Z|^2 + F(Z) \quad \text{for all } Z \in \mathbb{R}^{n^m}.$$

Lemma 2.2.1
Under the assumptions of Theorem 2.2.1 we have:

a) For fixed $\delta \in (0,1)$, the problem

$$J_\delta \to \min \ \text{ in } W^{m,2}(\Omega) \tag{V_δ}$$

admits a unique solution $u_\delta \in W^{m,2}(\Omega) \cap L^q(\Omega - D)$.

b) We have

$$\sup_{\delta>0} \|u_\delta\|_{m,1;\Omega} < \infty. \tag{2.6}$$

c) It holds (not necessarily uniform in δ)

$$u_\delta \in W^{m+1,2}_{\mathrm{loc}}(\Omega).$$

Proof of the Lemma. Ad a). For $\delta \in (0,1)$ fixed, consider a J_δ-minimizing sequence $v_k \in W^{m,2}(\Omega)$. By the structure of J we clearly have

$$\sup_k \|\nabla^m v_k\|_{L^2(\Omega)} < \infty, \tag{2.7}$$

$$\sup_k \|v_k\|_{L^q(\Omega-D)} < \infty. \tag{2.8}$$

Quoting Lemma 1.1.6 it therefore follows

$$\sup_k \|v_k\|_{m,2;\Omega} < \infty. \tag{2.9}$$

Then there exists a function $u_\delta \in W^{m,2}(\Omega)$ such that, after passing to a suitable subsequence, $v_k \rightharpoonup u_\delta$ in $W^{m,2}(\Omega)$ and classical results on lower semicontinuity of (quasi-)convex functionals of power growth (see, e.g. [64]) together with Fatou's lemma directly imply the J_δ-minimality of u_δ. Clearly $u_\delta \in L^q(\Omega - D)$ due to $\int_{\Omega-D} \phi\big(|u_\delta - f|\big)\,\mathrm{d}x < \infty$ and (ϕ2). Assume v is a second J_δ-minimizer. The strict convexity of F_δ would imply $\nabla^m v = \nabla^m u_\delta$ a.e. on Ω, so that u_δ and v only differ by a polynomial p of degree at most $m-1$. Furthermore, the strict convexity of the penalizer ϕ yields $v = u_\delta$ a.e. on $\Omega - D$ and since we assume $\mathcal{L}^n(\Omega - D) > 0$ it must hold $p \equiv 0$, hence $v = u_\delta$ a.e. on Ω. For part b), we note that

$$\sup_{\delta>0} \|\nabla^m u_\delta\|_{L^1(\Omega)} < \infty, \tag{2.10}$$

$$\sup_{\delta>0} \|u_\delta - f\|_{L^q(\Omega-D)} < \infty, \tag{2.11}$$

$$\sup_{\delta>0} \delta \int_\Omega |\nabla^m u_\delta|^2\,\mathrm{d}x < \infty, \tag{2.12}$$

is immediate from $J_\delta[u_\delta] \le J_\delta[0] = \text{const}$. Lemma 1.1.6 a) then implies (2.6). Finally, part c) is obtained via an application of the well-known difference quotient technique to the quadratic minimization problem $J_\delta \to \min$ for fixed δ: for the first-order case $m = 1$, we may quote [41], Lemma 7.1.1 b). If $m \ge 2$, we note that u_δ solves the following Euler equation:

$$\int\limits_{\Omega} DF_\delta(\nabla^m u_\delta) : \nabla^m \varphi \,\mathrm{d}x = -\lambda \int\limits_{\Omega-D} \phi'\big(|u_\delta - f|\big) \operatorname{sgn}(u_\delta - f)\varphi \,\mathrm{d}x \tag{2.13}$$

for all $\varphi \in W^{m,2}(\Omega) \cap L^q(\Omega - D)$. We may choose $\varphi := \Delta_\gamma^{-h}(\eta^6 \Delta_\gamma^h u_\delta)$, where $\gamma \in \mathbb{R}^n$ is an arbitrary vector, $\eta \in C_0^\infty(\Omega)$ and

$$\Delta_\gamma^h u_\delta(x) := \frac{u_\delta(x + h\gamma) - u_\delta(x)}{h}.$$

Due to (ϕ3), the right-hand side of (2.13) can now be estimated by

$$\begin{aligned}
&-\lambda \int\limits_{\Omega-D} \phi'\big(|u_\delta - f|\big) \operatorname{sgn}(u_\delta - f) \Delta_\gamma^{-h}(\eta^6 \Delta_\gamma^h u_\delta) \,\mathrm{d}x \\
&\le c\lambda \int\limits_{\Omega-D} \big(1 + |u_\delta - f|^{q-1}\big) \big|\Delta_\gamma^{-h}(\eta^6 \Delta_\gamma^h u_\delta)\big| \,\mathrm{d}x \\
&\le c\lambda \int\limits_{\Omega-D} 1 + |u_\delta - f|^{2(q-1)} \,\mathrm{d}x + c \int\limits_{\Omega-D} \big|\Delta_\gamma^{-h}(\eta^6 \Delta_\gamma^h u_\delta)\big|^2 \,\mathrm{d}x \\
&\le c\lambda \int\limits_{\Omega-D} 1 + |u_\delta - f|^{2(q-1)} \,\mathrm{d}x + c \int\limits_{\Omega-D} |\nabla u_\delta|^2 + |\nabla^2 u_\delta|^2 \,\mathrm{d}x,
\end{aligned}$$

where we have applied Lemma 7.23 from [65]. Due to our assumption $q \le 2$ and (2.11), the terms in the last line are bounded by a constant and from this point on, we can argue just like in [66], "*Step 2*" on p. 353 to finish the reasoning. □

We proceed with the proof of Theorem 2.2.1. By the compactness property of $BV^m(\Omega)$ (cf. Lemma 1.1.5) and (2.6) there exists $u \in BV^m(\Omega)$ such that (for a suitable sequence $\delta \downarrow 0$) it holds $u_\delta \to u$ in $W^{m-1,1}(\Omega)$. Furthermore, an application of Fatou's Lemma gives

$$\int\limits_{\Omega-D} \phi\big(|u - f|\big) \,\mathrm{d}x \le \liminf_{\delta \downarrow 0} \int\limits_{\Omega-D} \phi\big(|u_\delta - f|\big) \,\mathrm{d}x \le J_\delta[0] = \text{const}.$$

which by (ϕ2) implies $u \in L^q(\Omega - D)$. Together with Lemma 2.1.1 a), we infer that

$$K[u] \le \liminf_{\delta \downarrow 0} K[u_\delta] = \liminf_{\delta \downarrow 0} I[u_\delta] \le \liminf_{\delta \downarrow 0} I_\delta[u_\delta] \le \liminf_{\delta \downarrow 0} I_\delta[w] = I[w]$$

for all $w \in W^{m,2}(\Omega)$, which, by an approximation argument, holds even for all $w \in W^{m,1}(\Omega)$. This shows that u is a solution of $(\widetilde{V})$. In view of the duality relation that is stated in Theorem 2.2.1 b), it is reasonable to consider the functions

$$\sigma_\delta := DF_\delta(\nabla^m u_\delta) \quad \text{as well as} \quad \tau_\delta := DF(\nabla^m u_\delta) \tag{2.14}$$

and to investigate their behavior as $\delta \downarrow 0$. Due to the boundedness of $|\nabla F|$ (cf. (F2)), the family of the τ_δ is uniformly bounded in $L^\infty(\Omega, \mathbb{R}^{n^m})$ and there exists $\tau \in L^\infty(\Omega, \mathbb{R}^{n^m})$ such that (possibly for another subsequence $\delta \downarrow 0$)

$$\tau_\delta \overset{*}{\rightharpoonup} \tau \text{ in } L^\infty(\Omega, \mathbb{R}^{n^m}) \text{ as } \delta \downarrow 0. \tag{2.15}$$

Furthermore, (2.12) implies that $\delta \nabla^m u_\delta \to 0$ in $L^2(\Omega, \mathbb{R}^{n^m})$ so that there exists $\sigma \in L^2(\Omega, \mathbb{R}^{n^m}))$ and another subsequence for which

$$\sigma_\delta \rightharpoonup \sigma \text{ in } L^2(\Omega, \mathbb{R}^{n^m}) \text{ as } \delta \downarrow 0. \tag{2.16}$$

From the formula $\sigma_\delta = \tau_\delta + \delta \nabla^m u_\delta$ it then follows that $\sigma = \tau$ a.e. on Ω. We are going to prove that τ in fact maximizes R. Therefore, we firstly note that due to their J_δ-minimality the functions u_δ satisfy the Euler equation (2.13) from above:

$$\begin{aligned} \delta \int_\Omega \nabla^m u_\delta : \nabla^m \varphi \,\mathrm{d}x + \int_\Omega \tau_\delta : \nabla^m \varphi \,\mathrm{d}x & \\ + \lambda \int_{\Omega - D} \phi'\big(|u_\delta - f|\big) \operatorname{sgn}(u_\delta - f) \varphi \,\mathrm{d}x &= 0 \end{aligned} \tag{2.17}$$

for all $\varphi \in W^{m,2}(\Omega) \cap L^q(\Omega - D)$. We further observe that thanks to ($\phi 3$) the sequence $\phi'\big(|u_\delta - f|\big) \operatorname{sgn}(u_\delta - f)$ is bounded in $L^{\frac{q}{q-1}}(\Omega)$, so that we may assume

$$\phi'\big(|u_\delta - f|\big) \operatorname{sgn}(u_\delta - f) \rightharpoonup \phi'\big(|u - f|\big) \operatorname{sgn}(u - f) \quad \text{in } L^{\frac{q}{q-1}}(\Omega).$$

By $\delta \nabla^m u_\delta \to 0$ in L^2 and $\tau_\delta \overset{*}{\rightharpoonup} \tau$, we can pass to the limit $\delta \downarrow 0$ in (2.17) and obtain

$$\int_\Omega \tau : \nabla^m \varphi \,\mathrm{d}x + \lambda \int_{\Omega - D} \phi'\big(|u - f|\big) \operatorname{sgn}(u - f) \varphi \,\mathrm{d}x = 0. \tag{2.18}$$

We actually need the validity of this equation for all φ in the domain of I, i.e. $\varphi \in W^{m,1}(\Omega) \cap L^q(\Omega - D)$. Therefore, we employ Theorem 1.2.1 to approximate $\varphi \in W^{m,1}(\Omega) \cap L^q(\Omega - D)$ with a sequence of smooth functions $(\varphi_k) \subset C^\infty(\overline{\Omega})$ such that $\varphi_k \to \varphi$ in $W^{m,1}(\Omega)$ and $\varphi_k \to \varphi$ in $L^q(\Omega - D)$. The equality in (2.18) is then preserved in the limit $k \to \infty$. We continue with the observation that τ_δ and $DF(\nabla^m u_\delta)$ are connected through the following relation (cf. [62], Proposition 5.1 on p. 21):

$$F(\nabla^m u_\delta) = \tau_\delta : \nabla^m u_\delta - F^*(\tau_\delta).$$

In $J_\delta[u_\delta]$, we replace $F(\nabla^m u_\delta)$ by the above expression and subtract (2.17) with the admissible choice $\varphi = u_\delta$ to obtain

$$\begin{aligned} J_\delta[u_\delta] =& \frac{\delta}{2}\int_\Omega |\nabla^m u_\delta|^2 \,\mathrm{d}x + \int_\Omega \tau_\delta : \nabla^m u_\delta - F^*(\tau_\delta)\,\mathrm{d}x + \lambda \int_{\Omega - D} \phi\big(|u_\delta - f|\big)\,\mathrm{d}x \\ =& -\frac{\delta}{2}\int_\Omega |\nabla^m u_\delta|^2 \,\mathrm{d}x - \int_\Omega F^*(\tau_\delta)\,\mathrm{d}x \\ &+ \lambda \int_{\Omega - D} \phi\big(|u_\delta - f|\big) - \phi'\big(|u_\delta - f|\big)\,\mathrm{sgn}(u_\delta - f) u_\delta \,\mathrm{d}x \end{aligned}$$

and therefore

$$\begin{aligned} J_\delta[u_\delta] \le & -\int_\Omega F^*(\nabla^m u_\delta)\,\mathrm{d}x \\ & + \lambda \int_{\Omega - D} \phi\big(|u_\delta - f|\big) - \phi'\big(|u_\delta - f|\big)\,\mathrm{sgn}(u_\delta - f) u_\delta \,\mathrm{d}x. \end{aligned} \tag{2.19}$$

From the definition of R and (2.4) it is clear that $\sup\limits_{L^\infty(\Omega)} R \le \inf\limits_{W^{m,1}(\Omega)} I$. Moreover, $\inf\limits_{W^{m,1}(\Omega)} I \le I[u_\delta] \le J_\delta[u_\delta]$, and we infer that

$$\begin{aligned} \sup_{L^\infty(\Omega)} R[\kappa] \le & -\int_\Omega F^*(\tau_\delta)\,\mathrm{d}x \\ & + \lambda \int_{\Omega - D} \phi\big(|u_\delta - f|\big) - \phi'\big(|u_\delta - f|\big)\,\mathrm{sgn}(u_\delta - f) u_\delta \,\mathrm{d}x. \end{aligned} \tag{2.20}$$

Now we pass to the limit $\delta \downarrow 0$ in the above inequality, noting that as a convex function, F^* is weak-$*$ lower semicontinuous (cf. [33] Theorem 3.23 and Remark 3.25 (ii)), so that

$$\limsup_{\delta \downarrow 0} \int_\Omega -F^*(\tau_\delta)\,\mathrm{d}x \le \int_\Omega -F^*(\tau)\,\mathrm{d}x.$$

Furthermore, using $u_\delta = u_\delta - f + f$, we may write

$$\begin{aligned} &\int_{\Omega - D} \phi\big(|u_\delta - f|\big) - \phi'\big(|u_\delta - f|\big)\,\mathrm{sgn}(u_\delta - f) u_\delta \,\mathrm{d}x \\ &= \int_{\Omega - D} \phi\big(|u_\delta - f|\big) - \phi'\big(|u_\delta - f|\big)|u_\delta - f|\,\mathrm{d}x + \int_{\Omega - D} \phi'\big(|u_\delta - f|\big)\,\mathrm{sgn}(u_\delta - f) f \,\mathrm{d}x. \end{aligned}$$

By the strict convexity of ϕ, the term $\phi(|x|) - \phi'(|x|)|x|$ is nonpositive and since

$u_\delta \to u$ pointwise a.e., we can apply Fatou's Lemma to obtain

$$\limsup_{\delta\downarrow 0} \int\limits_{\Omega - D} \phi\big(|u_\delta - f|\big) - \phi'\big(|u_\delta - f|\big)|u_\delta - f| \,\mathrm{d}x \\ \leq \int\limits_{\Omega - D} \phi\big(|u - f|\big) - \phi'\big(|u - f|\big)|u - f| \,\mathrm{d}x.$$

Due to

$$\phi'\big(|u_\delta - f|\big)\operatorname{sgn}(u_\delta - f) \rightharpoonup \phi'\big(|u - f|\big)\operatorname{sgn}(u - f) \text{ in } L^{\frac{q}{q-1}}(\Omega)$$

and $f \in L^\infty(\Omega)$ it further holds

$$\int\limits_{\Omega - D} \phi'\big(|u_\delta - f|\big)\operatorname{sgn}(u_\delta - f) f \,\mathrm{d}x \xrightarrow{\delta\downarrow 0} \int\limits_{\Omega - D} \phi'\big(|u - f|\big)\operatorname{sgn}(u - f) f \,\mathrm{d}x,$$

and letting $\delta \downarrow 0$ in (2.20) yields

$$\sup_{\kappa \in L^\infty(\Omega)} R[\kappa] \leq -\int\limits_{\Omega} F^*(\tau)\,\mathrm{d}x \\ + \lambda \int\limits_{\Omega - D} \phi\big(|u - f|\big) - \phi'\big(|u - f|\big)\operatorname{sgn}(u - f) u \,\mathrm{d}x. \tag{2.21}$$

We are going to show that the right-hand side of this inequality equals $R[\tau]$. Observing

$$R[\tau] = \inf_{w \in W^{m,1}(\Omega)} \int\limits_{\Omega} \tau : \nabla^m w - F^*(\tau)\,\mathrm{d}x + \lambda \int\limits_{\Omega - D} \phi\big(|w - f|\big)\,\mathrm{d}x$$

and using the identity (2.18) it follows

$$\begin{aligned} R[\tau] &= -\int\limits_{\Omega} F^*(\tau)\,\mathrm{d}x \\ &\quad + \inf_{w \in W^{m,1}} \left[\lambda \int\limits_{\Omega - D} \phi\big(|w - f|\big) - \phi'\big(|u - f|\big)\operatorname{sgn}(u - f) w \,\mathrm{d}x\right] \\ &= -\int\limits_{\Omega} F^*(\tau)\,\mathrm{d}x \\ &\quad + \inf_{w \in W^{m,1}} \Bigg[\lambda \int\limits_{\Omega - D} \phi\big(|w - f|\big) - \phi'\big(|u - f|\big)\operatorname{sgn}(u - f)(w - f) \\ &\quad + \phi'\big(|u - f|\big)\operatorname{sgn}(u - f) f \,\mathrm{d}x\Bigg]. \end{aligned}$$

For any $a \in \mathbb{R}$, the term $\phi(|t|) - at$ is minimal for $t_0 \in \mathbb{R}$ such that $\phi'(|t_0|)\,\mathrm{sgn}(t_0) = a$. Consequently,

$$\phi\big(|u-f|\big) - \phi'\big(|u-f|\big)\,\mathrm{sgn}(u-f)u \le \phi\big(|w-f|\big) - \phi'\big(|u-f|\big)\,\mathrm{sgn}(u-f)w$$

holds for all $w \in W^{m,1}(\Omega)$, and

$$R[\tau] = \int_{\Omega} F^*(\tau)\,\mathrm{d}x + \lambda \int_{\Omega - D} \phi\big(|u-f|\big) - \phi'\big(|u-f|\big)\,\mathrm{sgn}(u-f)u.$$

(2.21) thus implies

$$\sup_{\kappa \in L^\infty(\Omega)} R[\kappa] \le R[\tau],$$

i.e. τ is R-maximal and

$$R[\tau] \le \inf_{w \in W^{m,1}(\Omega)} I[w] \le \limsup_{\delta \downarrow 0} I[u_\delta] \le \limsup_{\delta \downarrow 0} J_\delta[u_\delta] \overset{(2.19)}{\le} R[\tau]$$

yields

$$\inf_{w \in W^{m,1}(\Omega)} I[w] = R[\tau] = \sup_{\kappa \in L^\infty(\Omega)} R[\kappa].$$

We have thereby proved part a) of Theorem 2.2.1 as well as the "inf-sup" relation from part c), and it remains to establish the duality formula from part b). To this purpose, we adopt the ideas from [63] which are based on the following observation:

Lemma 2.2.2
Let $u \in \mathcal{M} \subset BV^m(\Omega)$ be any minimizer of the functional K from (2.1). Then the tensor

$$\sigma_0 := \nabla F(\nabla^m_a u)$$

is a maximizer of the dual functional R.

Proof of Lemma 2.2.2. Note at first that since $|\nabla F|$ is bounded, $\sigma_0(x)$ lies in $L^\infty(\Omega, \mathbb{R}^{n^m})$, i.e., σ_0 is admissible. For $v \in W^{m,1}(\Omega) \cap L^q(\Omega - D)$ and $\kappa = \sigma_0$ we thus obtain

$$l(v, \sigma_0) = \int_{\Omega} DF(\nabla^m_a u) : \nabla^m v - F^*\big(DF(\nabla^m_a u)\big)\,\mathrm{d}x + \lambda \int_{\Omega - D} \phi\big(|v-f|\big)\,\mathrm{d}x,$$

which, by means of the duality relation

$$F(Z) + F^*(DF(Z)) = Z : DF(Z), \text{ valid for all } Z \in \mathbb{R}^{n^m},$$

reads as

$$\begin{aligned} l(v, \sigma_0) = &\int_{\Omega} F\big(DF(\nabla^m_a u)\big)\,\mathrm{d}x + \int_{\Omega} \big(\nabla^m v - \nabla^m_a u\big) : DF(\nabla^m_a u)\,\mathrm{d}x \\ &+ \lambda \int_{\Omega - D} \phi\big(|v-f|\big)\,\mathrm{d}x. \end{aligned} \tag{2.22}$$

Since u minimizes K in $BV^m(\Omega)$, we have that

$$
\begin{aligned}
0 &= \frac{d}{dt}\bigg|_{t=0} K[u+tv] \\
&= \int_\Omega DF\big(\nabla_a^m u\big) : \nabla^m v \,\mathrm{d}x + \lambda \int_{\Omega - D} \phi'\big(|u-f|\big)\operatorname{sgn}(u-f) v \,\mathrm{d}x.
\end{aligned}
\tag{2.23}
$$

Here we have used that for $v \in W^{m,1}(\Omega)$ the singular part

$$
\nabla_s^m(u+tv) = \nabla_s^m u
$$

is independent of the variable t. The K-minimality of u further implies the identity

$$
\begin{aligned}
0 = \frac{d}{dt}\bigg|_{t=0} K[u+tu] &= \int_\Omega DF\big(\nabla_a^m u\big) : \nabla_a^m u \,\mathrm{d}x \\
&+ \int_\Omega F^\infty\left(\frac{\nabla_s^m u}{|\nabla_s^m u|}\right) d|\nabla^s u| + \lambda \int_{\Omega - D} \phi'\big(|u-f|\big)\operatorname{sgn}(u-f) u \,\mathrm{d}x.
\end{aligned}
\tag{2.24}
$$

Inserting (2.23) and (2.24) in (2.22) now yields

$$
\begin{aligned}
l(v,\sigma_0) = &\int_\Omega F\big(DF(\nabla_a^m u)\big)\,\mathrm{d}x + \int_\Omega F^\infty\left(\frac{\nabla_s^m u}{|\nabla_s^m u|}\right) d|\nabla^s u| \\
&+ \lambda \int_{\Omega - D} \phi\big(|v-f|\big) - \phi'\big(|u-f|\big)\operatorname{sgn}(u-f)(u-v)\,\mathrm{d}x.
\end{aligned}
\tag{2.25}
$$

Observing that

$$
\begin{aligned}
T &:= \int_{\Omega - D} \phi\big(|v-f|\big) - \phi'\big(|u-f|\big)\operatorname{sgn}(u-f)(u-v)\,\mathrm{d}x \\
&= \int_{\Omega - D} \phi\big(|v-f|\big) - \phi'\big(|u-f|\big)\operatorname{sgn}(u-f)(u-f) \\
&\quad + \phi'\big(|u-f|\big)\operatorname{sgn}(u-f)(v-f)\,\mathrm{d}x,
\end{aligned}
$$

and using

$$
\begin{aligned}
&\phi\big(|v-f|\big) - \phi'\big(|u-f|\big)\operatorname{sgn}(u-f)(v-f) \\
&\geq \phi\big(|u-f|\big) - \phi'\big(|u-f|\big)\operatorname{sgn}(u-f)(u-f),
\end{aligned}
$$

we obtain

$$
T \geq \lambda \int_{\Omega - D} \phi\big(|u-f|\big)\,\mathrm{d}x. \tag{2.26}
$$

Combining (2.26) with (2.25), it follows that

$$
l(v,\sigma_0) \geq K[u].
$$

From the definition of the dual functional R and the K-minimality of u we thus infer

$$R[\sigma_0] \geq K[u] = \inf_{BV^m(\Omega)} K = \inf_{W^{m,1}(\Omega)} I = \sup_{L^\infty(\Omega)} R,$$

i.e. σ_0 is R-maximal. This completes the proof of Lemma 2.2.2. $\square$

Part b) of Theorem 2.2.1 will now follow once we have shown that σ_0 is the *unique* R-maximizer. Assume that the dual problem admits a second solution $\widetilde{\sigma}$, satisfying $\widetilde{\sigma} \neq \sigma_0$ on a set of positive measure. Arguing as in the proof of Theorem 2.15 in [67], the strict convexity of F suffices to deduce the inequality

$$\int_\Omega (-F^*)\left(\frac{\widetilde{\sigma}+\sigma_0}{2}\right) \mathrm{d}x > \frac{1}{2}\int_\Omega (-F^*)(\widetilde{\sigma}) \,\mathrm{d}x + \frac{1}{2}\int_\Omega (-F^*)(\sigma_0) \,\mathrm{d}x.$$

Moreover, the functional

$$L^\infty\big(\Omega, \mathbb{R}^{m^n}\big) \ni \kappa \mapsto \inf_{v \in W^{1,1}(\Omega)} \int_\Omega \kappa : \nabla^m v - \chi_{\Omega - D}\phi\big(|v-f|\big) \,\mathrm{d}x$$

is concave and, together with the above inequality, this yields

$$R\left[\frac{\widetilde{\sigma}+\sigma_0}{2}\right] > \frac{1}{2}R[\widetilde{\sigma}] + \frac{1}{2}R[\sigma_0],$$

which is a contradiction to the R-maximality of σ_0 and $\widetilde{\sigma}$. This completes the proof of Theorem 2.2.1.

Remark 2.2.1
Note that the identity

$$DF(\nabla^m_a u) = \tau = \sigma \quad \textit{a.e. on } \Omega,$$

with τ, σ as defined in (2.15) and (2.16), respectively, is an immediate consequence of the proof.

$\square$

Part II

Fine Regularity Properties in Lower Dimensions

Chapter 3

The One-Dimensional Case

In the second part of this thesis, we will have a closer look at two specific cases of our general variational problem (V), starting with the simplest one where we choose $m = n = 1$. The one-dimensional setting comes with the advantage of being accessible to a variety of methods that do not have an equivalent in higher dimensions. On the other hand, studying the behavior of solutions in one dimension permits a better understanding of the general case. A particular example for this will be given in Section 3.6, where examining the dependence of the regularity behavior of the minimizer u on the ellipticity parameter μ (cf. (F_μ) in the introduction) for a certain choice of the data function f, will show that, for $\mu > 2$, problem (V) does in general not admit a solution in the Sobolev class. Besides these theoretical aspects, one-dimensional TV-type denoising models seem to have some interesting applications as well, mainly in connection with the recovery of piecewise constant data as it is frequently encountered in practical sciences such as geophysics or biophysics (cf. [68] and the introduction of [69]), whereas in [70], TV-models have been applied to the filtering of gravitational wave signals. In this context we also mention the works [9], [71] and [72] which, like us, treat the classical TV-model as well as higher-order variants in one dimension from a more theoretical point of view.

In contrast to the foregoing part, we will in the following not be concerned with proving things in utmost generality. On the contrary: for the sake of simplicity, we restrict ourselves to the minimization problem

$$J[u] := \int_0^1 F(\dot{u})\,dt + \frac{\lambda}{2}\int_0^1 (u-f)^2\,dt \to \min \text{ in } W^{1,1}(0,1) \tag{3.1}$$

for a density function $F : \mathbb{R} \to \mathbb{R}$, which in addition to (F1)–(F3) satisfies the strict inequality

$$F''(t) > 0 \quad \text{for all } t \in \mathbb{R} \tag{F4}$$

and with a fixed quadratic fidelity term. Note that we do *not* consider inpainting.

Here we use the notation $\dot{u} := \frac{d}{dt}u$ to denote the (weak) derivative of a function $u : (0,1) \to \mathbb{R}$ and $\lambda > 0$ is a regularization parameter which controls the balance between the smoothing and the data-fitting effect that results from the minimization of the first and the second integral, respectively. $f \in L^\infty([0,1])$ represents the given data for which, without loss of generality, we may assume

$$0 \le f(t) \le 1 \quad \text{for almost all } t \in [0,1]. \tag{3.2}$$

Since, in view of applications, F should be an approximation to the TV-density, reasonable choices of F are e.g. given by the regularized TV-density, $F_\varepsilon(p) := \sqrt{\varepsilon^2 + p^2} - \varepsilon$ for some $\varepsilon > 0$ or $F(p) := \Phi_\mu(|p|)$ as defined in (6). In regard to regularity properties we will work with densities that satisfy the condition of μ-ellipticity (see (F_μ)) in the simplified form

$$F''(t) \ge \frac{c_1}{(1+|t|)^\mu}, \tag{F5}$$

with a constant $c_1 > 0$ and parameter $\mu \in (1,\infty)$. We further define the *critical value* of the parameter λ to be

$$\lambda_\infty = \lambda_\infty(F) := \lim_{t\to\infty} F'(t). \tag{3.3}$$

This value will play a crucial role in the study of regularity properties of J-minimizers.

Example 3.0.1. For F_ε it is immediate that $\lambda_\infty(F_\varepsilon) = 1$ independently of ε, whereas for $F = \Phi_\mu$ we have

$$\lambda_\infty(\Phi_\mu) = \frac{1}{\mu - 1}.$$

The following table gives a first overview of the various regularity results of this chapter, which are published in [52]:

Data f	Density F	Bound on λ	Regularity of u	Reference
$L^\infty(0,1)$	(F1)-(F4)	$0 < \lambda < \lambda_\infty$	$C^{1,1}([0,1])$	Theorem 3.2.1 a)
$L^\infty(0,1)$	(F1)-(F4)	$\lambda > 0$	$W^{2,\infty}_{\mathrm{loc}}(\mathrm{Reg}(u))$	Theorem 3.3.1 b)
$W^{1,2}_{\mathrm{loc}}(a,b)$	(F1)-(F4), (F5)	$\lambda > 0$	$W^{1,1}(a,b)$ $\cap W^{1,2}_{\mathrm{loc}}(a,b)$	Theorem 3.3.1 c)
continuous at t_0	(F1)-(F4)	$\lambda > 0$	continuous at t_0	Theorem 3.4.1 a)
$W^{1,1}(0,1)$	(F1)-(F4)	$\lambda(\frac{1}{2} + \|\dot{f}\|_1)$ $< \omega_\infty$	$C^{1,1}([0,1])$	Theorem 3.4.1 c)
$L^\infty(0,1)$	(F1)-(F5) $\mu \in (1,2]$	$\lambda > 0$	$C^{1,1}([0,1])$	Theorem3.6.1
$L^\infty(0,1)$	(F1)-(F4) $\omega_\infty = \infty$	$\lambda > 0$	$C^{1,1}([0,1])$	Corollary 3.6.2 a)
$L^\infty(0,1)$	(F1)-(F4)	$0 < \lambda < \lambda_\mu$	$W^{2,1}(0,1)$	Theorem 3.6.3

3.1 Sobolev and BV-functions of One Real Variable

We start with some basic facts about functions on $\mathbb{R}$, referring to the relevant sections of [1] or the classical monograph [73] for proofs and details. One main advantage of the one-dimensional setting in compare to the general case consists in the fact that here we have a more classical characterization of Sobolev- and BV functions. We begin with the space $W^{1,1}(\Omega)$, where now $\Omega = (a, b) \subset \mathbb{R}$ is an open interval. In one dimension, the functions f having a weak derivative in the space $L^1(\Omega)$ are exactly the "absolutely continuous" functions (denoted by $f \in AC(\Omega)$), which classically means that f can be represented by an integral of a suitable L^1-density $g = \dot{f}$,

$$f(t) = \int_a^t g(s)\,\mathrm{d}s + f(a).$$

In particular, every $f \in W^{1,1}(\Omega)$ is even uniformly continuous which implies that it has a continuous extension to the boundary points of Ω.

For BV functions in one dimension, we simply write

$$Du := D_1 u$$

to denote the distributional derivative of u, which is a real-valued Radon measure. Defining the so called *pointwise variation* by

$$\mathrm{pV}(u, \Omega) := \sup \left\{ \sum_{i=1}^{n-1} |u(t_{i+1}) - u(t_i)| \;:\; n \geq 2,\; a < t_1 < \dots < t_n < b \right\},$$

as well as the *essential variation* through

$$\mathrm{eV}(u, \Omega) := \inf \Big\{ \mathrm{pV}(v, \Omega) \;:\; v = u \; \mathcal{L}^1\text{-a.e. in } \Omega \Big\},$$

the following is proved in [1] (see Theorem 3.27, p. 135 and Theorem 3.28, p. 136):

Lemma 3.1.1
Let $u \in L^1(a, b)$ satisfy $\mathrm{eV}(u, (a, b)) < \infty$. Then it holds:

a) $u \in BV(a, b)$ and $\mathrm{eV}(u, (a, b)) = |Du|(a, b)$.

b) There always exists a so called "good" L^1-representative $\overline{u} = u$ a.e. for which it holds

$$|Du|(a, b) = \mathrm{pV}(\overline{u}, (a, b)).$$

Moreover, $\overline{u}$ is continuous up to a countable set of jump-type discontinuities. In particular, the left- and the right limit of u exist at all points of Ω, i.e. $\overline{u}$ is a regulated function.

In what follows, we will tacitly identify any BV-function with its good representative. We further note that this representative has a classical derivative at $\mathcal{L}^1$-almost all points of Ω which is an L^1-representative of the absolutely continuous (with respect to $\mathcal{L}^1$) part of the measure Du, and which, by an abuse of notation, will be denoted by $\dot{u}$, also. Hence Du reads as

$$Du = \overbrace{\dot{u} \cdot \mathcal{L}^1}^{=:\, D_a u} + \overbrace{\sum_{k \in \mathbb{N}} h(x_k) \delta_{x_k} + D_c u}^{=:\, D_s u}, \tag{3.4}$$

and it holds (compare [1], Corollary 3.33)

$$|Du|(a,b) = \int_a^b |\dot{u}| dt + \sum_{k \in \mathbb{N}} |h(x_k)| + |D_c u|(a,b).$$

Here, for $k \in \mathbb{N}$, x_k denotes the discontinuity points of the good representative of u,

$$h(x_k) := \lim_{x \downarrow x_k} u(x) - \lim_{x \uparrow x_k} u(x)$$

is the "jump-heigh" and δ_{x_k} is Dirac's measure of mass 1 concentrated at x_k. The sum $\sum_{k \in \mathbb{N}} h(x_k) \delta_{x_k}$ is the so called *jump part* $D_j u$ of Du which, together with the residual part $D_c u := D_s u - D_j u$ (the so called *Cantor part*), forms the *singular part* $D_s u$ in the Lebesgue decomposition of Du.

3.2 Sobolev Regularity

Of course, even in the one-dimensional case, the space $W^{1,1}(\Omega)$ is not reflexive and the existence of a solution of problem (3.1) is the exception rather than the rule. Interestingly, the following theorem shows that it suffices to put a constraint on the parameter λ to guarantee the existence of a J-minimizing function in the Sobolev class:

Theorem 3.2.1
Suppose (3.2) holds for the data f and let F satisfy (F1)–(F4). Assume further that the parameter λ is bounded by

$$\lambda < \lambda_\infty(F) \tag{3.5}$$

with $\lambda_\infty(F)$ defined in (3.3). Then it holds:

a) Problem (3.1) admits a unique solution $u \in W^{1,1}(0,1) = AC\big([0,1]\big)$ and this solution satisfies $0 \le u(x) \le 1$ for all $x \in [0,1]$.

b) The minimizer u belongs to the class $W^{2,\infty}(0,1) = C^{1,1}\big([0,1]\big)$ and solves the following Neumann-type boundary value problem

$$\begin{cases} \ddot{u} = \lambda \dfrac{u - f}{F''(\dot{u})} \ \text{a.e. on } (0,1), \\ \dot{u}(0) = \dot{u}(1) = 0. \end{cases} \tag{BVP}$$

Remark 3.2.1
The bound $\lambda < \lambda_\infty$ is the most general condition under which we can establish existence of a Sobolev solution of problem (3.1) and is far from being optimal. In Section 3.4 we will point out the dependence of the regularity behavior of u on the qualities of the data function f.

Remark 3.2.2
Note that, in contrast to the n-dimensional setting, Theorem 3.2.1 goes without additional constraints on the density F (e.g. μ-ellipticity condition).

Proof. We start with the statement of part a). Let us assume the validity of the hypotheses of Theorem 3.2.1. We first note that problem (3.1) has at most one solution thanks to the strict convexity of the data fitting quantity $\int_0^1 (w-f)^2 \,\mathrm{d}t$ with respect to w. Next we show that there exists at least one solution. To this purpose we once more employ the δ-regularization from Lemma 2.2.1, i.e. for fixed $\delta \in (0,1]$ we consider the problem

$$J_\delta[w] := \int_0^1 F_\delta(\dot{w}) \,\mathrm{d}t + \frac{\lambda}{2} \int_0^1 (w-f)^2 \,\mathrm{d}t \to \min \text{ in } W^{1,2}(0,1), \tag{3.6}$$

where

$$F_\delta(t) := \frac{\delta}{2}|t|^2 + F(t), \quad t \in \mathbb{R}. \tag{3.7}$$

Quoting Lemma 2.2.1, it holds that problem (3.6) admits a unique solution u_δ which lies in the local class $W^{2,2}_{\mathrm{loc}}(\Omega)$ and satisfies

$$\sup_{0\le\delta<1} \|u_\delta\|_{1,1;\Omega} < \infty \tag{3.8}$$

$$\text{as well as} \qquad \sup_{0\le\delta<1} \delta \int_0^1 |\dot{u}_\delta|^2 \,\mathrm{d}t < \infty. \tag{3.9}$$

Moreover, it holds

$$0 \le u_\delta(t) \le 1 \text{ for all } t \in [0,1], \tag{3.10}$$

which can easily be proved by contradiction: if this were not the case, then

$$v_\delta(t) := \min\{u_\delta(t), 1\} - \min\{u_\delta(t), 0\}$$

would be a distinct $W^{1,2}(0,1)$-function, satisfying $|v_\delta(t)| \le |u_\delta(t)|$ as well as $|\dot{v}_\delta(t)| \le |\dot{u}_\delta(t)|$ for $\mathcal{L}^1$-a.a. $t \in [0,1]$, hence $J_\delta[v_\delta] \le J_\delta[u_\delta]$. This, however, contradicts the uniqueness of the minimizer u_δ.

Remark 3.2.3
We emphasize that in the one-dimensional cased $\dot{u}_\delta(t)$ exists for all $t \in (0,1)$ in the classical sense as a continuous function due to the embedding $W^{2,2}_{\mathrm{loc}}(0,1) \hookrightarrow C^1(0,1)$.

We continue with the observation that from the assumptions (F1)–(F4) imposed on the density F and the definition of λ_∞ (compare (3.3)) we can infer

$$\operatorname{Im}(F') = (-\lambda_\infty, \lambda_\infty). \tag{3.11}$$

Next, we fix $\lambda \in (0, \lambda_\infty)$ and observe the validity of the following lemma which, despite its elementary nature, will be important during the further proof:

Lemma 3.2.1
The inverse function of $F_\delta' : \mathbb{R} \to \mathbb{R}$ is uniformly (in δ) bounded on the interval $[-\lambda, \lambda]$.

Proof of Lemma 3.2.1. Since F' is an odd, strictly increasing function that induces a diffeomorphism between $\mathbb{R}$ and the open interval $(-\lambda_\infty, \lambda_\infty)$. Let us write $(F')^{-1}\big([-\lambda, \lambda]\big) = [-\alpha, \alpha]$, where $F'(\alpha) = \lambda$. Next we choose $t \in [-\lambda, \lambda]$ and assume that $(F_\delta')^{-1}(t) > \alpha$. Then it follows (note that F_δ' is strictly increasing)

$$t > F_\delta'(\alpha) = \delta\alpha + F'(\alpha) = \delta\alpha + \lambda > \lambda,$$

which is clearly a contradiction. The case $(F_\delta')^{-1}(t) < -\alpha$ is treated in the same manner and the lemma is proved. $\square$

After these preparations, we proceed with the proof of Theorem 3.2.1 a). First, we introduce (compare (2.14))

$$\sigma_\delta := F_\delta'(\dot{u}_\delta) \in C^0(0,1). \tag{3.12}$$

Using (F2) together with (3.9), we obtain

$$\sigma_\delta \in L^2(0,1) \quad \text{uniformly in } \delta. \tag{3.13}$$

Owing to their J_δ-minimality, the u_δ solve the Euler equation

$$0 = \int_0^1 F_\delta'(\dot{u}_\delta)\dot{\varphi}\,\mathrm{d}t + \lambda \int_0^1 (u_\delta - f)\varphi\,\mathrm{d}t \tag{3.14}$$

for all $\varphi \in W^{1,2}(0,1)$. Note that this means exactly that σ_δ is weakly differentiable with derivative

$$\dot{\sigma}_\delta = \lambda(u_\delta - f) \quad \text{a.e. on } (0,1). \tag{3.15}$$

Combining (3.10) with (3.13) and (3.15) it follows (recall our assumption $0 \le f \le 1$ a.e. on $(0,1)$)

$$\sigma_\delta \in W^{1,\infty}(0,1) = C^{0,1}\big([0,1]\big) \quad \text{uniformly in } \delta \text{ and } \|\dot{\sigma}_\delta\|_\infty \le \lambda. \tag{3.16}$$

Choosing $\varphi \in C^1([0,1])$ in (3.14) and using (3.15) together with the fundamental theorem of calculus (which holds for the class of AC-functions; see e.g. [74] (18.16) Theorem on p. 285 or [73] Chapter 2), we infer that

$$0 = \int_0^1 (\dot{\sigma}_\delta \varphi + \sigma_\delta \dot{\varphi})\,\mathrm{d}t = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(\sigma_\delta \varphi)\,\mathrm{d}t = \sigma_\delta(1)\varphi(1) - \sigma_\delta(0)\varphi(0),$$

and since $\varphi \in C^1([0,1])$ is arbitrary it must hold

$$\sigma_\delta(0) = \sigma_\delta(1) = 0. \tag{3.17}$$

Note that (3.16) and (3.17) imply

$$\|\sigma_\delta\|_\infty \leq \lambda. \tag{3.18}$$

At this point, the definition of σ_δ, (3.16), (3.17), (3.18) and Lemma 3.2.1 yield the existence of a constant $M > 0$, independent of δ, such that

$$\|\dot{u}_\delta\|_\infty \leq M. \tag{3.19}$$

Here we have made essential use of the restriction $\lambda < \lambda_\infty$. As a consequence, there exists a function $u \in W^{1,\infty}(0,1)$ such that $u_\delta \rightrightarrows u$ as $\delta \downarrow 0$ and $\dot{u}_\delta \rightharpoonup \dot{u}$ in $L^p(0,1)$ for all finite $p > 1$ as $\delta \downarrow 0$, at least for a subsequence. Our goal is to show that u is J-minimal: thanks to the J_δ-minimality of u_δ it follows that for all $v \in W^{1,2}(0,1)$ it holds

$$J[u_\delta] \leq J_\delta[u_\delta] \leq J_\delta[v] \overset{\delta\downarrow 0}{\to} J[v].$$

By lower semicontinuity, we further have

$$J[u] \leq \liminf_{\delta \to 0} J[u_\delta]$$

and thus $J[u] \leq J[v]$ for all $v \in W^{1,2}(0,1)$. This yields $J[u] \leq J[w]$ for all $w \in W^{1,1}(0,1)$ by approximating w with functions $v_k \in W^{1,2}(0,1)$ in the $W^{1,1}$-norm, which finally proves u to be a solution of problem (3.1) and thereby part a) of Theorem 3.2.1. We continue with the proof of part b). By (3.14) it holds

$$\ddot{u}_\delta = \lambda \frac{(u_\delta - f)}{F''_\delta(\dot{u}_\delta)} \quad \text{a.e. on } (0,1),$$

so that $\dot{u}_\delta \in W^{1,\infty}(0,1)$ uniformly (in δ) on account of (3.19). Consequently, the functions $\dot{u}_\delta$ have a unique Lipschitz extension to the boundary points 0 and 1, which particularly implies the differentiability of u_δ at 0 and 1 , i.e. there is a clear meaning of $\dot{u}_\delta(0)$ and $\dot{u}_\delta(1)$. For continuity reasons, the defining equation (3.12) for σ_δ extends to the boundary points of $(0,1)$ as well and since F'_δ vanishes exactly in the origin, it follows from (3.17) that $\dot{u}_\delta(0) = \dot{u}_\delta(1) = 0$. Combining this with the uniform boundedness of u_δ in $C^{1,1}([0,1])$, we see that

$u \in C^{1,1}\big([0,1]\big)$ holds together with the boundary condition $\dot{u}(0) = \dot{u}(1) = 0$. Furthermore, u solves the Euler equation

$$0 = \int_0^1 F'(\dot{u})\dot{\varphi}\,\mathrm{d}t + \lambda \int_0^1 (u-f)\varphi\,\mathrm{d}t$$

for all $\varphi \in C_0^1(0,1)$ and we infer the validity of the relation

$$\frac{d}{dt}F'(\dot{u}) = \lambda(u-f) \quad \text{a.e. on } (0,1).$$

Consequently, it holds

$$\ddot{u} = \lambda \frac{u-f}{F''(\dot{u})} \quad \text{a.e. on } (0,1),$$

together with $\dot{u}(0) = \dot{u}(1) = 0$, i.e. u solves the boundary value problem (BVP), as it was claimed in part b). $\square$

3.3 The Relaxed Problem

For arbitrarily large values of the parameter λ, problem (3.1) does not necessarily possess a solution. For that reason, we have to study the one-dimensional version of the relaxed problem $(\widetilde{V})$ from Section 2.1 in the space $BV(0,1)$. Here, the functional K from (2.1) takes a particularly simple form: under our assumptions imposed on F, the recession function $F^\infty(t) := \lim\limits_{s\to\infty} F(st)/s$ simplifies to $F^\infty(t) = \lambda_\infty |t|$ with λ_∞ as defined in (3.3), and the relaxed problem therefore reads as

$$K[w] = \int_0^1 F(\dot{w})\,\mathrm{d}t + \lambda_\infty \int_0^1 |D_s w| + \frac{\lambda}{2}\int_0^1 (w-f)^2\,\mathrm{d}t \to \min \text{ in } BV(0,1). \tag{3.20}$$

Quoting Theorem 2.1.1, (3.20) has a solution $u \in BV(0,1)$ which is even unique due to our assumption $D = \emptyset$ during this chapter. But as the following theorem shows, we now obtain much finer properties of u:

Theorem 3.3.1
Suppose (3.2) for the data function f and let the density F satisfy (F1)-(F4) Moreover, let $\lambda > 0$ denote any number. Then it holds:

a) Problem (3.20) admits a unique solution $u \in BV(0,1)$, satisfying $0 \le u \le 1$ a.e.

b) There is an open subset $\mathrm{Reg}(u)$ *of* $(0,1)$ *such that* $u \in W^{2,\infty}_{\mathrm{loc}}(\mathrm{Reg}(u))$ *and* $\mathcal{L}^1\big((0,1) - \mathrm{Reg}(u)\big) = 0$*. We have*

$$\mathrm{Reg}(u) := \big\{ s \in (0,1) : s \text{ is a Lebesgue point of } \dot{u} \big\},$$

where $\dot{u}$ *is defined in (3.4). Moreover, there are numbers* $0 < t_1 \le t_2 < 1$ *such that* $u \in C^{1,1}\big([0,t_1] \cup [t_2,1]\big)$*.*

c) If F*, in addition to (F1)-(F4), satisfies*

$$F''(t) \le c_2 \frac{1}{1+|t|} \tag{F6}$$

for all $t \in \mathbb{R}$ *and a constant* $c_2 > 0$*, and if there is a subinterval* $(a,b) \subset (0,1)$ *such that* $f \in W^{1,2}_{\mathrm{loc}}(a,b)$*, then we also have* $u \in W^{1,1}(a,b) \cap W^{1,2}_{\mathrm{loc}}(a,b)$*. In case* $(a,b) = (0,1)$ *we get that* $u \in W^{1,1}(0,1) \cap W^{1,2}_{\mathrm{loc}}(0,1)$ *is* J*-minimizing in* $W^{1,1}(0,1)$*.*

Remark 3.3.1
The requirement (F6) is not as restrictive as it may appear at first sight. In particular, it is easy to confirm that for a given $\varepsilon > 0$ *and* $\mu > 1$ *our examples from the introduction,* $F(p) := F_\varepsilon(p)$ *and* $F(p) := \Phi_\mu(|p|)$ *satisfy condition (F6).*

Proof. Let us assume the validity of the hypotheses of Theorem 3.3.1. As already mentioned above, part a) follows from Theorem 2.1.1 where it was also shown that a BV-minimal function u can be obtained as the L^1-limit of the regularizing sequence u_δ from Lemma 2.2.1. That u is indeed unique with this property follows from part b) of the aforesaid theorem. Due to (3.10) and $u_\delta \to u$ $\mathcal{L}^1$-a.e. it further holds $0 \le u \le 1$ a.e.

Now to part b). With σ_δ as defined in the proof of Theorem 2.1.1 (see (3.12)), we recall that we have (3.15)-(3.17) at hand. Note that at this stage no bound on λ was necessary. Thus, there exists $\sigma \in W^{1,\infty}(0,1)$ with $\sigma_\delta \rightrightarrows \sigma$ as $\delta \downarrow 0$ (at least for a subsequence). Moreover

$$\begin{cases} \dot{\sigma} = \lambda(u - f) \text{ and } |\dot{\sigma}(t)| \le \lambda \quad \text{a.e.}, \\ |\sigma(t)| \le \lambda \quad \text{on } [0,1], \\ \sigma(0) = \sigma(1) = 0. \end{cases} \tag{3.21}$$

By Theorem 2.2.1 from Section 2.2, σ is the unique solution of the dual problem associated to (3.1) and it holds

$$\sigma = F'(\dot{u}) \quad \text{a.e.} \tag{3.22}$$

Due to the continuity of σ (recall (3.21)), (3.22) particularly holds for all *Lebesgue points* of $\dot{u}$, i.e. all $t \in (0,1)$ such that

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} |\dot{u}(s) - \dot{u}(t)| \, \mathrm{d}s = 0.$$

Thus, identifying $\dot{u}$ in the following with its Lebesgue point representative, we have the formula

$$\sigma(t) = F'(\dot{u}(t)), \text{ for all } t \in (0,1) - A. \tag{3.23}$$

where A is the set

$$A := \left\{ t \in (0,1) \,:\, \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} |\dot{u}(s) - \dot{u}(t)| \,\mathrm{d}s \neq 0 \right\}.$$

Note that by Lebesgue's differentiation theorem (see Corollary 1, p. 44 in [75]) A is a nullset, i.e. $\mathcal{L}^1(A) = 0$. Let us fix some point $t_0 \in (0,1) - A$. Then, due to (3.22) it holds $|\sigma(t_0)| < \lambda_\infty$ and since σ is continuous, there exists $\varepsilon > 0$ with

$$|\sigma(t)| \leq \lambda_\infty - \alpha \quad \text{for all } t \in [t_0 - \varepsilon, t_0 + \varepsilon], \tag{3.24}$$

if $\alpha > 0$ is chosen appropriately small. Recalling $\sigma_\delta \rightrightarrows \sigma$, (3.24) yields for δ small enough

$$|\sigma_\delta(t)| \leq \lambda_\infty - \frac{\alpha}{2} \quad \text{for all } t \in [t_0 - \varepsilon, t_0 + \varepsilon]. \tag{3.25}$$

Quoting Lemma 3.2.1, $(F_\delta')^{-1}$ is uniformly (with respect to δ) bounded on the interval $\big[-\lambda_\infty + \frac{\alpha}{2}, \lambda_\infty - \frac{\alpha}{2}\big]$. Hence, there exists a number $L > 0$, independent of δ, such that (compare (3.19))

$$\|\dot{u}_\delta\|_{L^\infty(t_0-\varepsilon, t_0+\varepsilon)} \leq L \text{ for all small enough } \delta. \tag{3.26}$$

Since u is the L^1-limit of the sequence (u_δ), (3.26) implies

$$u \in C^{0,1}\big([t_0 - \varepsilon, t_0 + \varepsilon]\big)$$

and using the Euler equation (3.14) for u_δ on $(t_0 - \varepsilon, t_0 + \varepsilon)$, we deduce

$$\ddot{u}_\delta = \lambda \frac{(u_\delta - f)}{F_\delta''(\dot{u}_\delta)} \quad \text{a.e. on } (t_0 - \varepsilon, t_0 + \varepsilon).$$

This yields the existence of a number $L' > 0$ such that, independently of δ, it holds

$$\|\ddot{u}_\delta\|_{L^\infty(t_0-\varepsilon, t_0+\varepsilon)} \leq L'. \tag{3.27}$$

From (3.27) it finally follows that

$$u \in C^{1,1}\big([t_0 - \varepsilon, t_0 + \varepsilon]\big)$$

which proves u to be of class $C^{1,1}$ in a neighborhood of every point $t \in (0,1) - A$. Recalling (3.21), we then infer that (3.24) must hold on a suitable interval $[0, t_1]$ (choose, for instance, $t_1 < \sup\{s \in [0,1] \,:\, |\sigma(s)| < \lambda_\infty\}$). Hence, $u \in C^{1,1}\big([0, t_1]\big)$ follows and, using analogous arguments, we can show the existence of a number t_2 with $0 < t_1 \le t_2 < 1$ and such that $u \in C^{1,1}([t_2, 1])$. This proves part b) of the theorem.

For proving part c), our strategy is to show $u_\delta \in W^{1,2}_{\mathrm{loc}}(a,b)$ uniformly with respect to δ. Along with the fact that the K-minimizing function $u \in BV(0,1)$ is obtained as the limit of the sequence u_δ, we then infer that $u \in BV(a,b) \cap W^{1,2}_{\mathrm{loc}}(a,b)$, hence $u \in W^{1,1}(a,b)$. We first recall $u_\delta \in W^{2,2}_{\mathrm{loc}}(0,1)$ (compare Lemma 2.2.1 c)) so that $F'_\delta(\dot u_\delta)$ is of class $W^{1,2}_{\mathrm{loc}}(0,1)$, satisfying

$$(F'_\delta(\dot u_\delta))' = F''_\delta(\dot u_\delta)\ddot u_\delta \quad \text{a.e. on } (0,1).$$

From (3.14) we therefore get

$$\int_0^1 F''_\delta(\dot u_\delta)\ddot u_\delta \dot\varphi \,\mathrm{d}t = \lambda \int_0^1 (u_\delta - f)\dot\varphi \,\mathrm{d}t \tag{3.28}$$

for all $\varphi \in C^\infty_0(0,1)$ and by approximation, (3.28) remains valid for functions $\varphi \in W^{1,2}(0,1)$ whose support is a compact subset of $(0,1)$. Next, we fix a point $x_0 \in (a,b)$, a number $R > 0$ such that $(x_0 - 2R, x_0 + 2R) \Subset (a,b)$ and $\eta \in C^\infty_0(x_0 - 2R, x_0 + 2R)$ with $\eta \equiv 1$ on $(x_0 - R, x_0 + R)$, $0 \le \eta \le 1$ as well as $|\dot\eta| \le \frac{c}{R}$. Choosing $\varphi := \eta^2 \dot u_\delta$ in (3.28) we obtain

$$\begin{aligned} I_0 :&= \int_{x_0-2R}^{x_0+2R} F''_\delta(\dot u_\delta)(\ddot u_\delta)^2 \eta^2 \,\mathrm{d}t \\ &= -2 \int_{x_0-2R}^{x_0+2R} F''_\delta(\dot u_\delta)\ddot u_\delta \dot u_\delta \dot\eta\eta \,\mathrm{d}t + \lambda \int_{x_0-2R}^{x_0+2R} (u_\delta - f)\dot\varphi \,\mathrm{d}t \\ &=: I_1 + \lambda I_2. \end{aligned} \tag{3.29}$$

We continue with I_1 for which Young's inequality yields ($\varepsilon > 0$ can be chosen arbitrarily small)

$$|I_1| \le \varepsilon I_0 + c\varepsilon^{-1} \int_{x_0-2R}^{x_0+2R} F''_\delta(\dot u_\delta)\dot u_\delta^2 \dot\eta^2 \,\mathrm{d}t. \tag{3.30}$$

An integration by parts (recall $f \in W^{1,2}_{\mathrm{loc}}(a,b)$) further gives for I_2

$$I_2 = -\int_{x_0-2R}^{x_0+2R} (\dot u_\delta - \dot f)\dot u_\delta \eta^2 \,\mathrm{d}t = -\int_{x_0-2R}^{x_0+2R} \dot u_\delta^2 \eta^2 \,\mathrm{d}t + \int_{x_0-2R}^{x_0+2R} \dot f \dot u_\delta \eta^2 \,\mathrm{d}t. \tag{3.31}$$

Putting together (3.30) and (3.31) and absorbing terms (choose $\varepsilon > 0$ sufficiently small), (3.29) implies

$$\begin{aligned}
&\int\limits_{x_0-2R}^{x_0+2R} F''_\delta(\dot u_\delta)(\ddot u_\delta)^2\eta^2\,\mathrm{d}t + \lambda\int\limits_{x_0-2R}^{x_0+2R} \dot u_\delta^2\eta^2\,\mathrm{d}t\\
&\le c\int\limits_{x_0-2R}^{x_0+2R} F''_\delta(\dot u_\delta)\dot u_\delta^2\dot\eta^2\,\mathrm{d}t + c\int\limits_{x_0-2R}^{x_0+2R} |\dot f||\dot u_\delta|\eta^2\,\mathrm{d}t.
\end{aligned}\tag{3.32}$$

The first integral on the right-hand side of (3.32) can be handled by the uniform estimate $J_\delta[u_\delta]\le J[0]$, the linear growth of F and condition (F6). More precisely, we have

$$\int\limits_{x_0-2R}^{x_0+2R} F''_\delta(\dot u_\delta)\dot u_\delta^2\dot\eta^2\,\mathrm{d}t \le c(R)\int\limits_{x_0-2R}^{x_0+2R}\big(\delta + (1+\dot u_\delta^2)^{-\frac12}\big)\dot u_\delta^2\,\mathrm{d}t \le c(R),$$

where $c(R)$ denotes a local constant being independent of δ. To the second integral we apply Young's inequality which yields

$$\int\limits_{x_0-2R}^{x_0+2R} |\dot f||\dot u_\delta|\eta^2\,\mathrm{d}t \le \varepsilon\int\limits_{x_0-2R}^{x_0+2R}\dot u_\delta^2\eta^2\,\mathrm{d}t + c\varepsilon^{-1}\int\limits_{x_0-2R}^{x_0+2R}\dot f^2\eta^2\,\mathrm{d}t.$$

Absorbing terms by choosing $\varepsilon>0$ sufficiently small, (3.32) implies (recall $\eta\equiv 1$ on (x_0-R,x_0+R) and $f\in W^{1,2}_{\mathrm{loc}}(a,b)$ once again)

$$\int\limits_{x_0-R}^{x_0+R} F''_\delta(\dot u_\delta)(\ddot u_\delta)^2\,\mathrm{d}t + \lambda\int\limits_{x_0-R}^{x_0+R}\dot u_\delta^2\,\mathrm{d}t \le c(f,R),\tag{3.33}$$

where $c(f,R)$ is a local constant, independent of δ. This proves

$$u_\delta\in W^{1,2}(x_0-R,x_0+R)\text{ uniformly with respect to }\delta$$

and part c) of the theorem now follows from a covering argument. □

Remark 3.3.2
From the proof of part b) we see how the set of possible singularities,

$$\mathrm{Sing}(u) := [0,1]-\mathrm{Reg}(u),$$

can be given in terms of the function σ: due to (3.22), we have $|\sigma(t)|<\lambda_\infty$ at almost all points $t\in[0,1]$ and since σ is continuous it therefore holds

$$-\lambda_\infty\le\sigma(t)\le\lambda_\infty\ \text{ for all } t\in[0,1].$$

We claim that $\mathrm{Sing}(u)$ *is exactly the set of points where $|\sigma|$ attains the maximal value λ_∞, i.e.*

$$\mathrm{Sing}(u) = \big\{t\in[0,1]\,:\,|\sigma(t)|=\lambda_\infty\big\}.$$

Indeed, let $t_0 \in [0,1]$ be a regular point of u, i.e. there is a small neighborhood $(t_0-\varepsilon, t_0+\varepsilon)$ of t_0 such that u is of class $C^{1,1}(t_0-\varepsilon, t_0+\varepsilon)$. Hence $|\dot{u}|$ is bounded on $(t_0-\varepsilon, t_0+\varepsilon)$ and (3.23) along with the continuity of σ implies $|\sigma(t_0)| < \lambda_\infty$. Conversely, if $s_0 \in [0,1]$ is a point where $|\sigma(s_0)| < \lambda_\infty$ the arguments given after (3.23) show that s_0 is a regular point.

3.4 The Dependence on the Regularity of the Data

Part c) of Theorem 3.3.1 already indicated the connection between the regularity of the solution u of problem (3.20) and the regularity of the data function f. In this section, we are going to study some typical model cases of data functions as they might appear in practice, such as piecewise constant or triangular signals (see Figure 3.1), and how their regularity behavior influences the minimizer u.

Theorem 3.4.1
Suppose (3.2) holds for the data function f and let the density F satisfy (F1)-(F4). Let u denote the unique K-minimizer from Theorem 2.2.1. Then, the following assertions hold:

a) If $t_0 \in (0,1)$ is a point, where some representative of the data function f is continuous. Then the good representative of u introduced in front of (3.4) is continuous at t_0, as well.

b) Assume that there is an interval $[a,b] \subset (0,1)$ such that $f \in W^{1,\infty}(a,b)$. Then we have $u \in C^2(a,b)$.

c) Suppose $f \in W^{1,1}(0,1)$ and define

$$\omega_\infty := \lim_{t\to\infty} tF'(t) - F(t) \in [0,\infty). \tag{3.34}$$

If $\lambda\left(\frac{1}{2} + \|\dot{f}\|_1\right) < \omega_\infty$, then $u \in C^{1,1}\big([0,1]\big)$.

Corollary 3.4.1
If the data function f is globally Lipschitz-continuous on $[0,1]$, then $u \in C^2\big([0,1]\big)$.

Proof of Corollary 3.4.1. Applying Theorem 3.4.1 b) with a and b arbitrarily close to 0 and 1, respectively, yields $u \in C^2(0,1)$. Hence u satisfies the differential equation from Theorem 2.1.1 c)

$$\ddot{u} = \lambda \frac{u-f}{F''(\dot{u})} \tag{3.35}$$

in the classical sense at all points of $(0,1)$. Due to Theorem 2.2.1 b) we have $t_1, t_2 > 0$ for which $u \in C^1\big([0,t_1] \cup [t_2,1]\big)$ and therefore $\dot{u}$ is uniformly continuous on $[0,1]$, which means that the right-hand side of equation (3.35) belongs to the space $C^0\big([0,1]\big)$. Hence $\ddot{u}$ exists even in 0 and 1 as a continuous function on $[0,1]$. $\square$

Remark 3.4.1 *(i) From part a) we infer that if f is continuous on an interval $(a,b) \subset [0,1]$, then also $u \in C^0(a,b)$.*

(ii) We would like to remark that part b) in particular applies to piecewise affine data functions such as triangular or rectangular signals as shown in Figure 3.1. For these types of data functions, we obtain K-minimizers which are differentiable away from the discontinuity points of the data.

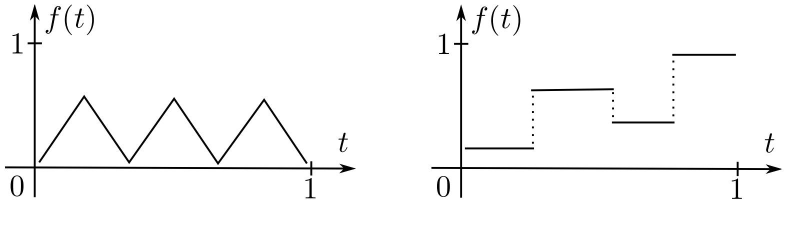

Fig. 3.1

(iii) The point of the assertion in c) is that even though full $C^{1,1}$-regularity may fail to hold in general if the parameter λ exceeds λ_∞, it can still hold up to $2\lambda_\infty$ provided the oscillation of the data f is sufficiently small. If e.g. F is the regularized minimal surface integrand $F(t) := F_\varepsilon(t) = \sqrt{\varepsilon^2 + t^2} - \varepsilon$, it is easily seen that

$$\omega_\infty(F_\varepsilon) = \varepsilon.$$

Consequently, we get full $C^{1,1}$-regularity for all parameters λ up to the bound

$$\frac{\varepsilon}{\frac{1}{2} + \|\dot{f}\|_1},$$

which can be larger than $\lambda_\infty(F_\varepsilon) = 1$. For $F(t) = \Phi_\mu(|t|)$ it holds $\lambda_\infty = \frac{1}{\mu - 1}$, whereas

$$\lim_{t\to\infty} t\Phi'_\mu(t) - \Phi_\mu(t) = \begin{cases} \dfrac{1}{\mu-1}\dfrac{1}{\mu-2}, & \mu > 2, \\ \infty & 1 < \mu \le 2, \end{cases}$$

so that ω_∞ is even unbounded if we let μ approach 2 from above.

Proof. We begin with part a). Without loss of generality we will in the following identify f with the representative that is continuous in t_0. Moreover, we recall that we always consider the "good" representative of u as specified in Section 3.1. Assume that the statement is false, i.e. the left- and the right limit of u at t_0,

$$u^-(t_0) := \lim_{t_k \uparrow t_0} u(t_k) \quad \text{and} \quad u^+(t_0) := \lim_{t_k \downarrow t_0} u(t_k)$$

do not coincide. W.l.o.g. we may assume

$$u^-(t_0) < f(t_0) \quad \text{and} \quad u^+(t_0) \ge f(t_0), \tag{3.36}$$

and it will be clear from the proof that the other possible cases can be treated analogously. Let $h_0 := u^+(t_0) - u^-(t_0)$ denote the jump-height at t_0. Then (3.36) implies the existence of $\varepsilon > 0$ and $0 < d < h_0$ such that

$$u(t) < f(t) - d \text{ for all } t \in [t_0 - \varepsilon, t_0].$$

We may further assume that u is continuous at $t_0 - \varepsilon$. Now define $\tilde{u}$ by

$$\tilde{u}(t) := u(t) + d\chi_{[t_0-\varepsilon,t_0]}(t).$$

This means that on $[t_0 - \varepsilon, t_0]$ we shift the graph of u a little closer to the graph of f so that in particular

$$\int_0^1 (\tilde{u} - f)^2 \,\mathrm{d}t < \int_0^1 (u - f)^2 \,\mathrm{d}t. \tag{3.37}$$

Let us write (compare (3.4)) $Du = \dot{u}\mathcal{L}^1 + \sum_{k=0}^{\infty} h_k \delta_{x_k} + D^c u$, where x_k are the discontinuity points of u. Clearly, $\tilde{u} \in BV(0,1)$ and it holds

$$D\tilde{u} = \dot{u}\mathcal{L}^1 + (h_0 - d)\delta_{t_0} + d\delta_{t_0-\varepsilon} + \sum_{k=1}^{\infty} h_k \delta_{x_k} + D^c u$$

and in conclusion

$$\begin{aligned} K[\tilde{u}] = \int_0^1 F(\dot{u}) \,\mathrm{d}t + \lambda_\infty \left(|h_0 - d| + d + \sum_{k=1}^{\infty} |h_k| \right) \\ + \lambda_\infty |D^c u|(0,1) + \frac{\lambda}{2} \int_0^1 (\tilde{u} - f)^2 \,\mathrm{d}t. \end{aligned}$$

Since $d < h_0$ and due to (3.37) this implies

$$K[\tilde{u}] < K[u],$$

which is a contradiction to the minimality of u.

Now to part b). First, we note that due to part b) of Theorem 3.3.1 there are points s_1 and s_2 in (a,b), arbitrarily close to a and b, respectively, such that u is $C^{1,1}$-regular in a small neighborhood of s_1 and s_2. Hence, the singular set

$$S := \mathrm{Sing}(u) \cap [s_1, s_2]$$

is a compact subset of the open interval (s_1, s_2). Moreover, by part a) of Theorem 3.4.1 we have $u \in C^0(a,b)$. Assume $S \neq \emptyset$. Then there exists $\overline{s} := \inf S > a$ (which is an element of S itself since $\mathrm{Sing}(u)$ is closed). In particular, $\sigma(\overline{s}) = \pm\lambda_\infty$ (cf. Remark 3.3.2), i.e. σ has a maximum respectively minimum in $\overline{s}$ and since $\dot{\sigma} = \lambda(u - f) \in C^0(a,b)$ it follows that

$$\dot{\sigma}(\overline{s}) = 0,$$

which means that

$$u(\overline{s}) = f(\overline{s}). \tag{3.38}$$

W.l.o.g. we may assume $\sigma(\overline{s}) = \lambda_\infty$. Since σ is continuous in $\overline{s}$ it must hold that, for any sequence $t_k \uparrow \overline{s}$ approaching $\overline{s}$ from the left, $\sigma(t_k) \to \lambda_\infty$ and because of $\dot{u} = DF^{-1}(\sigma)$,

$$\dot{u}(t_k) \to \infty \text{ for any sequence } t_k \uparrow \overline{s}. \tag{3.39}$$

In particular, for arbitrary $M > 0$ there exists $\varepsilon > 0$ such that

$$\dot{u}(t) > M \text{ for } t \in [\overline{s} - \varepsilon, \overline{s}). \tag{3.40}$$

Now choose $M := \|\dot{f}\|_{\infty;[s_1,s_2]}$ in (3.40). Then

$$\dot{u} - \dot{f} > 0 \text{ on } [\overline{s} - \varepsilon, \overline{s}),$$

which is not compatible with (3.38) unless $u - f < 0$ on $[\overline{s} - \varepsilon, \overline{s})$. But in this case, the differential equation

$$\ddot{u} = \lambda \frac{u - f}{F''(\dot{u})} \text{ a.e. on } [\overline{s} - \varepsilon, \overline{s}) \tag{3.41}$$

implies that $\dot{u}$ is strictly decreasing on $[\overline{s} - \varepsilon, \overline{s})$ and thereby $\dot{u}(\overline{s} - \varepsilon) \geq \dot{u}(s)$ for all $s \in [\overline{s} - \varepsilon, \overline{s})$. This clearly contradicts (3.39). Since s_1, s_2 can be chosen arbitrarily near to boundary points of (a, b), we infer $\mathrm{Sing}(u) \cap (a, b) = \emptyset$, i.e. $u \in W^{2,\infty}_{\mathrm{loc}}(a, b)$. Consequently, (3.41) holds at almost all points of (a, b) and by the continuity of $\dot{u}$, the right-hand side of (3.41) is continuous and thus $u \in C^2(a, b)$.

It remains prove part c). In Section 2.2 we have seen that the function σ maximizes the dual functional R in $L^\infty(0, 1)$ which, with the Lagrangian

$$l(v, \kappa) = \int_0^1 \kappa \dot{v} \, \mathrm{d}t - \int_0^1 F^*(\kappa) \, \mathrm{d}t + \underbrace{\frac{\lambda}{2} \int_0^1 (v - f)^2 \, \mathrm{d}t}_{=: \Psi(v)},$$

is defined by

$$R[\kappa] = \inf_{v \in W^{1,1}(0,1)} l(v, \kappa), \ \kappa \in L^\infty(0, 1).$$

Since $\sigma \in W^{1,\infty}(0, 1)$ along with $\sigma(0) = F'(\dot{u}(0)) = 0 = F'(\dot{u}(1)) = \sigma(1)$ (cf. (3.21)), an integration by parts yields the following representation of $R[\sigma]$ (cf.

also Theorem 9.8.1 on p. 366 in [76]):

$$
\begin{aligned}
R[\sigma] &= \inf_{v\in W^{1,1}(0,1)} \int_0^1 \sigma \dot{v}\,\mathrm{d}t - \int_0^1 F^*(\sigma)\,\mathrm{d}t + \Psi(v) \\
&= -\int_0^1 F^*(\sigma)\,\mathrm{d}t - \sup_{v\in W^{1,1}(0,1)} \left(-\int_0^1 \sigma \dot{v}\,\mathrm{d}t - \Psi(v) \right) \\
&= -\int_0^1 F^*(\sigma)\,\mathrm{d}t - \sup_{v\in W^{1,1}(0,1)} \left(\int_0^1 \dot{\sigma} v\,\mathrm{d}t - \Psi(v) \right) \\
&= -\int_0^1 F^*(\sigma)\,\mathrm{d}t - \Psi^*(\dot{\sigma}),
\end{aligned}
$$

where Ψ^* denotes the convex dual of Ψ, see Section 2.2. We want to determine $\Psi^*(\dot{\sigma})$ explicitely. By definition we have

$$
\begin{aligned}
\Psi^*(\dot{\sigma}) &= \sup_{v\in W^{1,1}(0,1)} \left(\int_\Omega v\dot{\sigma}\,\mathrm{d}x - \frac{\lambda}{2} \int_\Omega (v-f)^2\,\mathrm{d}x \right) \\
&= \sup_{v\in W^{1,1}(0,1)} \int_\Omega v\Big(\dot{\sigma} - \frac{\lambda}{2} v + \lambda f\Big)\,\mathrm{d}x - \frac{\lambda}{2} \int_\Omega f^2\,\mathrm{d}x.
\end{aligned}
$$

Applying Hölder's inequality, we find that

$$
\int_\Omega v\Big(\dot{\sigma} - \frac{\lambda}{2} v + \lambda f\Big)\,\mathrm{d}x \le -\frac{\lambda}{2}\|v\|_2^2 + \|\dot{\sigma} + \lambda f\|_2 \|v\|_2 \tag{3.42}
$$

and the term $-\frac{\lambda}{2} \cdot t^2 + \|\dot{\sigma} + \lambda f\|_2 \cdot t$ is maximal for $t = \|\frac{\dot{\sigma}}{\lambda} + f\|_2$. Indeed, the left-hand side of (3.42) attains this maximal value with the choice $v = \frac{\dot{\sigma}}{\lambda} + f$ and it follows

$$
\begin{aligned}
\Psi^*(\dot{\sigma}) &= \int_0^1 \left(\frac{\dot{\sigma}}{\lambda} + f\right) \dot{\sigma}\,\mathrm{d}t - \frac{\lambda}{2} \int_0^1 \left(\frac{\dot{\sigma}}{\lambda} + f\right)^2 \mathrm{d}t \\
&= \int_0^1 \frac{\dot{\sigma}^2}{2\lambda} + \dot{\sigma} f\,\mathrm{d}t.
\end{aligned}
$$

For $R[\sigma]$ we thus obtain

$$
R[\sigma] = -\int_0^1 \frac{\dot{\sigma}^2}{2\lambda} + \dot{\sigma} f\,\mathrm{d}t - \int_0^1 F^*(\sigma)\,\mathrm{d}t. \tag{3.43}
$$

Now assume that $\mathrm{Sing}(u) \neq 0$. By Remark 3.3.2, this means that there exists at least one point $t \in [0,1]$ where $\sigma(t) = \pm\lambda_\infty$. Let $\hat{t} := \inf \mathrm{Sing}(u)$. Since

$\sigma(0)=0$ it follows that $\hat{t}>0$ and, without loss of generality, we may assume $\sigma(\hat{t})=\lambda_\infty$. Let $\varphi\in C_0^\infty\big([0,\hat{t})\big)$ be an arbitrary test function. On $[0,\hat{t})$ it holds $|\sigma|<\lambda_\infty$ and since $\operatorname{spt}\varphi$ is a compact subset of $[0,\hat{t})$ (and σ is continuous) there exists $\varepsilon_0>0$ such that $|\sigma(t)+\varepsilon\varphi(t)|\le\lambda_\infty-\delta$ for some $\delta>0$ and for all $0\le\varepsilon<\varepsilon_0$. By Theorem 26.4 and Corollary 26.4.1 in [77], F^* is finite and continuously differentiable on $(-\lambda_\infty,\lambda_\infty)$ (with derivative $(F^*)'=(F')^{-1}$) so that

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} F^*(\sigma(t)+\varepsilon\varphi(t))=(F^*)'\big(\sigma(t)\big)\varphi(t)$$

exists. We may therefore use the representation (3.43) and the maximality of σ to derive the following Euler equation:

$$-\int_0^1\frac{\dot\sigma}{\lambda}\dot\varphi+f\dot\varphi\,\mathrm{d}t-\int_0^1(F^*)'(\sigma)\varphi\,\mathrm{d}t=0. \tag{3.44}$$

Since $[0,\hat{t})\subset\mathrm{Reg}(u)$ and $f\in W^{1,1}(0,1)$ by assumption, we have (see (3.21))

$$\dot\sigma=\lambda(u-f)\in W^{1,1}(0,\hat{t}) \tag{3.45}$$

and therefore $\sigma\in W^{2,1}(0,1)$. Hence (3.44) implies the following differential equation:

$$\frac{\ddot\sigma}{\lambda}+\dot f-(F^*)'(\sigma)=0\ \text{ a.e. on }(0,\hat{t}). \tag{3.46}$$

Let $s_k\subset[0,\hat{t})$ $(k\in\mathbb{N})$ denote a sequence with $s_k\uparrow\hat{t}$ as $k\to\infty$. Multiplying (3.46) with $\dot\sigma$ and integrating by parts (recall $\dot\sigma\in W^{1,1}(0,\hat{t})$) then yields

$$\frac{\dot\sigma(s_k)^2}{2\lambda}-\frac{\dot\sigma(0)^2}{2\lambda}+\int_0^{s_k}\dot f\dot\sigma\,\mathrm{d}t-F^*(\sigma_{s_k})=0.$$

Since $\dot\sigma$ is bounded by λ, this implies the estimate

$$F^*(\sigma(s_k))\le\lambda\left(\frac{1}{2}+\|\dot f\|_1\right) \tag{3.47}$$

and thereby

$$\lim_{k\to\infty}F^*(\sigma(s_k))\le\lambda\left(\frac{1}{2}+\|\dot f\|_1\right). \tag{3.48}$$

But the following calculation (cf. also Figure 3.2) shows that the limit on the left-hand side is just the quantity ω_∞ from the assumptions of part c):

$$\lim_{q\uparrow\lambda_\infty}F^*(q)=\lim_{q\uparrow\lambda_\infty}\int_0^q(F^*)'(t)\,\mathrm{d}t=\lim_{q\uparrow\lambda_\infty}\int_0^q(F')^{-1}(t)\,\mathrm{d}t$$

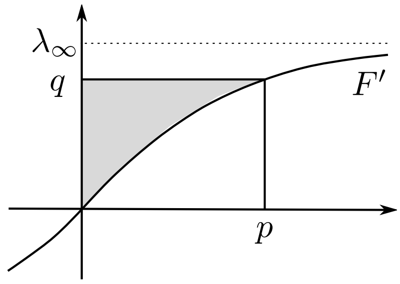

Fig. 3.2

$$= \lim_{q \uparrow \lambda_\infty} q(F')^{-1}(q) - \int\limits_0^{(F')^{-1}(q)} F'(t)\,\mathrm{d}t \overset{p:=(F')^{-1}(q)}{=} \lim_{p\uparrow\infty} pF'(p) - F(p).$$

Hence (3.48) is in contradiction to our assumption and $\mathrm{Sing}(u) = \emptyset$ must hold. $\square$

3.5 Discontinuous Minimizers

The regularity results from the previous sections raise the question, whether our model (3.1) is able to produce a discontinuous output at all. Keeping in mind that it is the distinguishing feature of the TV-model to maintain sharp discontinuities of the data, it is desirable to know under which conditions the generalized minimizer u from Theorem 3.3.1 is strictly contained in the class $BV(0,1) - W^{1,1}(0,1)$. The one dimensional setting enables us to give an explicit example where, for large enough values of the parameter λ, the functional J does not attain its minimum in the Sobolev class. This example can easily be transported to higher dimensions, thereby showing that (V) is indeed a practically useful alternative to the TV-model. In the following, we denote by u_λ for $\lambda \in [0, \infty)$ the unique minimizer of

$$K_\lambda[w] := \int\limits_0^1 F(\dot{w})\,\mathrm{d}t + \lambda_\infty |D_s w|(0,1) + \frac{\lambda}{2} \int\limits_0^1 (w - f)^2\,\mathrm{d}t$$

for a fixed density F satisfying (F1)-(F4) and data f as in (3.2). We make the following (trivial) observation on the convergence behavior of u_λ as $\lambda \to \infty$:

Lemma 3.5.1
Let F, f and u_λ be defined as above. Then it holds

$$u_\lambda \to f \text{ in } L^2(0,1) \text{ as } \lambda \to \infty.$$

Proof. This is immediate from $\frac{\lambda}{2}\|u_\lambda - f\|_{L^2(0,1)} \leq \sqrt{K_\lambda[u_\lambda]} \leq \sqrt{K_\lambda[0]}$. $\square$

We will in the following fix a certain data function f, which is designed to be sort of a "model case" of a discontinuous data function. Namely we set $f = f_0$ with

$$f_0 : [0,1] \to [0,1],\ f_0(t) = \begin{cases} 0, \ 0 \le t \le \dfrac{1}{2}, \\ 1, \ \dfrac{1}{2} < t \le 1. \end{cases} \tag{3.49}$$

By a slight abuse of notation, we still denote by u_λ the BV-minimizer of K_λ, even if $f = f_0$. Based upon our previous results, we can gather the following properties of u_λ:

Lemma 3.5.2
Let F, $f = f_0$ and u_λ be as above. Then we have:

a) $u_\lambda \in C^2\big([0,1] - \{\frac{1}{2}\}\big)$, $0 \le u_\lambda \le 1$ a.e. and u_λ satisfies

$$\begin{cases} \ddot{u}_\lambda = \lambda \dfrac{u_\lambda}{F''(\dot{u}_\lambda)},\ \dot{u}_\lambda(0) = 0,\ \textit{on } [0, 1/2) \quad (1), \\ \ddot{u}_\lambda = \lambda \dfrac{1-u_\lambda}{F''(\dot{u}_\lambda)},\ \dot{u}_\lambda(1) = 0,\ \textit{on } (1/2, 1] \quad (2), \end{cases}$$

b) $\dot{u}_\lambda$ is increasing on $[0, \frac{1}{2})$ and $\dot{u}_\lambda$ is decreasing on $[0, \frac{1}{2})$,

c) u_λ is increasing on $[0,1]$.

d) The two continuous branches, $u_\lambda|_{[0,\frac{1}{2})}$ and $u_\lambda|_{(\frac{1}{2},1]}$, of u_λ are symmetric with respect to the point $(\frac{1}{2}, \frac{1}{2})$, i.e.

$$u_\lambda(t) = \underbrace{1 - u_\lambda(1-t)}_{=:\, \tilde{u}_\lambda(t)},\ \textit{for } t \in [0,1] - \{1/2\}\,.$$

e) $0 \le u_\lambda < \frac{1}{2}$ on $[0, \frac{1}{2})$ and $\frac{1}{2} < u_\lambda \le 1$ on $(\frac{1}{2}, 1]$.

f) u_λ converges locally uniformly to f_0 on $[0, \frac{1}{2})$ and on $(\frac{1}{2}, 1]$ as $\lambda \to \infty$.

Proof. a) $0 \le u_\lambda \le 1$ is clear since f_0 fulfills (3.2) and $u_\lambda \in C^2\big([0,1] - \{\frac{1}{2}\}\big)$ is immediate from part b) of Theorem 3.4.1. That u_λ satisfies the system (1), (2) at all points where it is differentiable follows as in Theorem 3.2.1 b) from the K_λ-minimality.

b) By part a) we have $u_\lambda - f_0 \ge 0$ on $[0, \frac{1}{2})$ and $u_\lambda - f_0 \le 0$ on $(\frac{1}{2}, 1]$ so that (1), (2) imply $\ddot{u}_\lambda \ge 0$ on $[0, \frac{1}{2})$ and $\ddot{u}_\lambda \le 0$ on $(\frac{1}{2}, 1]$.

c) Due to $\dot{u}_\lambda(0) = 0$ and b) we have $\dot{u}_\lambda(0) \ge 0$ on $[0, \frac{1}{2})$. Similarly, $\dot{u}_\lambda(1) = 0$ together with b) implies $\dot{u}_\lambda(t) \ge 0$ on $(\frac{1}{2}, 1]$.

d) We show $K[\tilde{u}_\lambda] = K[u_\lambda]$. The result then follows from the uniqueness of the K-minimizer in $BV(0,1)$ in the case of pure denoising (Theorem 2.2.1 a)). Let

$$h := \lim_{t \downarrow \frac{1}{2}} u_\lambda(u) - \lim_{t \uparrow \frac{1}{2}} u_\lambda(u)$$

denote the height of the (possible) jump of u_λ at $t = 1/2$. Then the distributional derivative of u_λ is given by

$$Du_\lambda = \dot{u}_\lambda + h\delta_{1/2},$$

where $\delta_{1/2}$ is Dirac's measure of mass 1 concentrated in $\frac{1}{2}$. Hence

$$\begin{aligned} K[u_\lambda] &= \int_0^{\frac{1}{2}} F(\dot{u}_\lambda)\,\mathrm{d}t + \int_{\frac{1}{2}}^1 F(\dot{u}_\lambda)\,\mathrm{d}t + \lambda_\infty |h\delta_{1/2}|(0,1) + \frac{\lambda}{2}\int_0^1 (u_\lambda - f_0)^2\,\mathrm{d}t \\ &= \int_0^{\frac{1}{2}} F(\dot{u}_\lambda)\,\mathrm{d}t + \int_{\frac{1}{2}}^1 F(\dot{u}_\lambda)\,\mathrm{d}t + \lambda_\infty h + \frac{\lambda}{2}\int_0^1 (u_\lambda - f_0)^2\,\mathrm{d}t. \end{aligned}$$

For $\tilde{u}_\lambda$ we obtain

$$\begin{aligned} &K[\tilde{u}_\lambda] \\ &= \int_0^{\frac{1}{2}} F(\dot{u}_\lambda(1-t))\,\mathrm{d}t + \int_{\frac{1}{2}}^1 F(\dot{u}_\lambda(1-t))\,\mathrm{d}t + \lambda_\infty |h\delta_{1/2}|(0,1) + \frac{\lambda}{2}\int_0^1 (\tilde{u}_\lambda - f_0)^2\,\mathrm{d}t \\ &= \int_0^{\frac{1}{2}} F(\dot{u}_\lambda)\,\mathrm{d}t + \int_{\frac{1}{2}}^1 F(\dot{u}_\lambda)\,\mathrm{d}t + \lambda_\infty h + \frac{\lambda}{2}\int_0^1 (\tilde{u}_\lambda - f_0)^2\,\mathrm{d}t, \end{aligned}$$

but clearly $\int_0^1 (\tilde{u}_\lambda - f_0)^2\,\mathrm{d}t = \int_0^1 (u_\lambda - f_0)^2\,\mathrm{d}t$ and therefore $K[\tilde{u}_\lambda] = K[u_\lambda]$.

e) Consider the function

$$v_\lambda(t) := \begin{cases} \min\left\{\frac{1}{2}, u_\lambda(t)\right\}, & \text{if } t < \frac{1}{2}, \\ \max\left\{\frac{1}{2}, u_\lambda(t)\right\}, & \text{if } t > \frac{1}{2}, \\ \frac{1}{2}, \quad \text{if } t = \frac{1}{2}. \end{cases}$$

Now assume that there exists $t_0 \in [0, \frac{1}{2})$ with $u_\lambda(t_0) \geq \frac{1}{2}$. Then due to part c) and d) it holds $u_\lambda > \frac{1}{2}$ on $(t_0, \frac{1}{2})$ and $u_\lambda < \frac{1}{2}$ on $(\frac{1}{2}, 1-t_0)$, hence $v_\lambda \in W^{1,1}(0,1)$. Clearly, $|\dot{v}_\lambda| \leq |\dot{u}_\lambda|$ and $\int_0^1 (v_\lambda - f_0)^2\,\mathrm{d}t < \int_0^1 (u_\lambda - f_0)^2\,\mathrm{d}t$, so that $K[v_\lambda] < K[u_\lambda]$, in contradiction to the minimality of u_λ. This means that $u_\lambda < \frac{1}{2}$ on $[0, 1/2)$ and the symmetry from part d) implies $u_\lambda > \frac{1}{2}$ on $(1/2, 1]$.

f) By Lemma 3.5.1,

$$\|u_\lambda\|_{L^2(0,\frac{1}{2})} \to 0 \text{ as } \lambda \to \infty. \tag{3.50}$$

Fix $s \in (0, \frac{1}{2})$. Due to part c) it holds

$$\int_0^{\frac{1}{2}} u_\lambda^2 \,\mathrm{d}t \geq \int_s^{\frac{1}{2}} u_\lambda^2 \,\mathrm{d}t \geq \int_s^{\frac{1}{2}} u_\lambda(s)^2 \,\mathrm{d}t = \left(\frac{1}{2} - s\right) \sup_{t\in[0,s]} |u_\lambda(t)|,$$

so that (3.50) implies $\sup_{t\in[0,s]} |u_\lambda(t)| \to 0$ on every interval $[0, s]$ with $s < \frac{1}{2}$. The corresponding statement on $(\frac{1}{2}, 1]$ follows via the symmetry from d). □

Corollary 3.5.1
Under the assumptions of Lemma 3.5.2 the BV-minimum u_λ is of class $C^{1,1}\big([0,1]\big)$ if $\lambda \leq 4\lambda_\infty$.

Proof. By Remark 3.3.2, u_λ is $C^{1,1}$ if the corresponding function $\sigma_\lambda := F'(\dot{u})$ satisfies $|\sigma_\lambda| < \lambda_\infty$ on $[0,1]$. Moreover, due to (3.21) we have $\dot{\sigma}_\lambda(t) = \lambda(u_\lambda - f_0) \geq 0$ on $[0, \frac{1}{2})$ and $\dot{\sigma}_\lambda(t) \leq 0$ on $(\frac{1}{2}, 1]$, so that $\sigma_\lambda(t)$ reaches its maximal value in $t = \frac{1}{2}$. Together with part e) of the above Lemma we thus obtain

$$\sigma_\lambda(t) \leq \sigma_\lambda\Big(\frac{1}{2}\Big) = \lambda \int_0^{\frac{1}{2}} u_\lambda - f \,\mathrm{d}t < \lambda \int_0^{\frac{1}{2}} \frac{1}{2} - 0 \,\mathrm{d}t = \frac{\lambda}{4},$$

so that $|\sigma_\lambda| < \lambda_\infty$ if $\lambda \leq 4\lambda_\infty$. □

In what follows, we are going to show that the bound $\lambda \leq 4\lambda_\infty$ is optimal in the context of Corollary 3.5.1, which is to say that whenever $\lambda > 4\lambda_\infty$, there exists a density function F (satisfying (F1)-(F4)) such that the corresponding K-minimizer (where $f = f_0$) u_λ is discontinuous (exactly at $t = \frac{1}{2}$). The proof of this fact relies on a comparison of u_λ with the minimizer of the standard TV-model:

$$K_{TV}[w] := \int_0^1 |Dw| + \frac{\lambda}{2} \int_0^1 (w - f)^2 \,\mathrm{d}t \to \min \text{ for } w \in BV(0,1).$$

Lemma 3.5.3
Suppose f is some arbitrary data function satisfying (3.2). Then it holds:

a) The problem $K_{TV} \to \min$ admits a unique solution u_{TV} in the class $BV(0,1)$. It holds $0 \leq u_{TV} \leq 1$ a.e. on $(0,1)$.

b) If we choose $f = f_0$ from (3.49), then the K_{TV}-minimum u_{TV} is constant on each of the intervals $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1)$.

c) For $f = f_0$ we actually have: $u_{TV}\big|_{[0,\frac{1}{2})} \equiv \frac{1}{2}$, if $\lambda \le 4$ and $u_{TV}\big|_{[0,\frac{1}{2})} \equiv \frac{2}{\lambda}$, if $\lambda > 4$. In particular, u_{TV} is continuous on $[0,1]$ if $\lambda \le 4$ and discontinuous with a jump of height $\frac{\lambda-4}{\lambda}$ at $t = \frac{1}{2}$ if $\lambda > 4$.

Remark 3.5.1*a) As in Theorem 3.4.1 a), we can show that any point of continuity of the data function f is a point of continuity of the TV-minimum u_{TV}.*

b) Part b) of the above lemma could be stated in wider generality: if f is constant on some open interval $(a,b) \subset [0,1]$, then so is u_{TV}. The proof works analogously to that of part b).

Proof. a) This follows along the lines of Theorem 2.1.1 and 3.3.1 a).

b) We identify u_{TV} with its good representative (cf. Lemma 3.1.1), such that

$$\int_0^1 |Du_{TV}| = \mathrm{pV}(u,(0,1)).$$

Now consider the function

$$\widetilde{u}_{TV}(t) := \begin{cases} \inf\limits_{0 \le s \le \frac{1}{2}} u_{TV}(s), & \text{if } t \in \left[0, \dfrac{1}{2}\right], \\ \sup\limits_{\frac{1}{2} < s \le 1} u_{TV}(s), & \text{if } t \in \left(\dfrac{1}{2}, 1\right]. \end{cases}$$

Then it holds

$$\int_0^1 |D\widetilde{u}_{TV}| = \left| \sup_{t \in (\frac{1}{2},1]} u_{TV}(t) - \inf_{t \in [0,\frac{1}{2})} u_{TV}(t) \right| \le \mathrm{pV}(u_{TV},(0,1)) = \int_0^1 |Du_{TV}|$$

as well as

$$\int_0^1 (\widetilde{u}_{TV} - f_0)^2 \,\mathrm{d}t \le \int_0^1 (u_{TV} - f_0)^2 \,\mathrm{d}t.$$

Hence, $K_{TV}[\widetilde{u}_{TV}] \le K_{TV}[u_{TV}]$, so that the uniqueness statement of part a) implies $u_{TV} = \widetilde{u}_{TV}$ a.e.

c) By part b), there are numbers $x, y \in [0,1]$ such that $u_{TV} \equiv x$ on $[0,\frac{1}{2}]$ and $u_{TV} \equiv y$ on $(\frac{1}{2},1]$. Thus

$$K_{TV}[u_{TV}] = \frac{\lambda}{4}(x^2 + (1-y)^2) + |x-y|.$$

It is easily confirmed that this expression is minimal for $x = y = \frac{1}{2}$, if $\lambda \le 4$ and $x = \frac{2}{\lambda}$, $y = 1 - \frac{2}{\lambda}$ if $\lambda > 4$. Hence, for $\lambda \le 4$, the minimizer $u_{TV} \equiv \frac{1}{2}$ is continuous on $[0,1]$ and discontinuous with a jump of height $1 - \frac{4}{\lambda}$ for values $\lambda > 4$. □

In the following calculations, we specify the density F, setting $F = F_\varepsilon$ with

$$F_\varepsilon(t) := \sqrt{\varepsilon^2 + t^2} - \varepsilon$$

for some $\varepsilon > 0$. We then consider the problem (note that $\lambda_\infty(F_\varepsilon) = 1$)

$$K_\varepsilon[w] := \int_0^1 F_\varepsilon(\dot{w})\,\mathrm{d}t + |D_s w|(0,1) + \frac{\lambda}{2}\int_0^1 (w-f)^2\,\mathrm{d}t \to \min \text{ in } BV(0,1), \tag{3.51}$$

where the parameter λ is fixed, and ε varies in $(0,\infty)$.

Theorem 3.5.1
Suppose f is as in (3.2) and denote by $u_\varepsilon \in BV(0,1)$ the unique K_ε-minimizer. Let u_{TV} be as in Lemma 3.5.3. Then there exists a subsequence $\varepsilon \downarrow 0$ such that

$$u_\varepsilon \to u_{TV}\ \mathcal{L}^1\text{-a.e. on } (0,1) \quad \text{and} \quad u_\varepsilon \rightharpoonup u_{TV} \text{ in } L^2(0,1).$$

Moreover, u_ε converges locally uniformly to u_{TV} on the open set

$$S := \left\{ s \in [0,1] \,:\, \underbrace{\left|\lambda \int_0^s u_{TV} - f\,\mathrm{d}t\right|}_{=:\,\sigma_{TV}(s)} < 1 \right\}.$$

Remark 3.5.2 *(i) In Theorem 3.5.1 we can replace F_ε with $\mu \cdot \Phi_\mu$ or any other one-parameter family of density functions which approximates $|\cdot|$ uniformly.*

(ii) For $f = f_0$, part c) of Lemma 3.5.3 implies that $S = [0,1]$ if $\lambda < 4$ and $S = [0,1] - \{\frac{1}{2}\}$ for $\lambda \geq 4$.

(iii) Note that always $S \neq \emptyset$: since $\sigma_{TV}(0) = 0$ and σ_{TV} is continuous by definition, S contains a small interval $[0,\delta)$ for some $\delta > 0$. In particular, we always have $u_\varepsilon(0) \to u_{TV}(0)$ as $\varepsilon \downarrow 0$.

Proof. By $K_\varepsilon[u_\varepsilon] \leq K_\varepsilon[0]$, the family u_ε is uniformly bounded in $BV(0,1)$ as well as in $L^2(0,1)$. Hence, by the BV-compactness theorem and the weak compactness of L^2, there exists $\overline{u} \in BV(0,1)$ such that $u_\varepsilon \to \overline{u}$ in $L^1(0,1)$ as well as $u_\varepsilon \rightharpoonup \overline{u}$ in $L^2(0,1)$. Since $|t| - \varepsilon \leq F_\varepsilon(t)$ holds for all $t \in \mathbb{R}$ and $\varepsilon > 0$, we have

$$K_{TV}[u_\varepsilon] - \varepsilon \leq K_\varepsilon[u_\varepsilon] \leq K_\varepsilon[u_{TV}]$$

which yields

$$K_{TV}[\overline{u}] \leq \liminf_{\varepsilon\downarrow 0} K_{TV}[u_\varepsilon] - \varepsilon \leq \lim_{\varepsilon\downarrow 0} K_\varepsilon[u_{TV}] = K_{TV}[u_{TV}],$$

and hence $\overline{u} = u_{TV}$ a.e. by the uniqueness of u_{TV}.

Now let $s \in [0,1]$ be some point with $\sigma_{TV}(s) < 1$. Since $u_\varepsilon \to u_{TV}$ in $L^1(0,1)$, it follows that

$$\sigma_\varepsilon(s) := \lambda \int_0^s u_\varepsilon - f \,\mathrm{d}t \xrightarrow{\varepsilon \downarrow 0} \lambda \int_0^s u_{TV} - f \,\mathrm{d}t < 1$$

so that there exists $\varepsilon_0 > 0$ with $|\sigma_\varepsilon(s)| < 1$ for all $\varepsilon < \varepsilon_0$. From (3.21) it follows that σ_ε is Lipschitz continuous with Lipschitz constant

$$\|\dot{\sigma}_\varepsilon\|_\infty \leq \lambda \|u_\varepsilon - f\|_\infty \leq \lambda,$$

being uniformly bounded with respect to the parameter ε. Hence we can choose $\delta > 0$ small enough such that on the interval $(s-\delta, s+\delta)$ it holds $|\sigma_\varepsilon| < 1$ for all $\varepsilon < \varepsilon_0$. With the same arguments as in the proof of Theorem 3.3.1 b) we then see that u_ε is $C^{1,1}$ on $(s-\delta, s+\delta)$ together with

$$\dot{u}_\varepsilon = (F'_\varepsilon)^{-1}(\sigma_\varepsilon) \text{ for all } \varepsilon < \varepsilon_0.$$

Since $|\sigma_\varepsilon|$ is uniformly bounded away from 1 and

$$(F'_\varepsilon)^{-1}(r) = \frac{\varepsilon r}{\sqrt{1-r^2}} \rightrightarrows 0 \quad \text{locally on } (-1,1) \text{ as } \varepsilon \downarrow 0,$$

it follows that $\dot{u}_\varepsilon$ is uniformly bounded on $(s-\delta, s+\delta)$. Consequently, the Arzelà-Ascoli Theorem implies the existence of a subsequence $\varepsilon \downarrow 0$ for which $u_\varepsilon \rightrightarrows u_{TV}$ on the interval $(s-\delta, s+\delta)$. □

After these preparations we have now everything at hand to prove the optimality of the bound $\lambda \leq 4\lambda_\infty$ for our example data f_0 in Corollary 3.5.1:

Lemma 3.5.4
Choose $f = f_0$ as in (3.49) and let u_ε denote the unique BV-minimum of K_ε from (3.51). If $\lambda > 4$, then there exists a value $\varepsilon > 0$ for which u_ε is discontinuous at $t = \frac{1}{2}$. In particular, since $\lambda_\infty(F_\varepsilon) = 1$, the bound $\lambda \leq 4\lambda_\infty$ given in Corollary 3.5.1 cannot be improved.

Proof. By part d) of Lemma 3.5.2, u_ε is not continuous unless $u_\varepsilon(1/2) = 1/2$. This means that it suffices to prove that u_ε is bounded away from $1/2$ on $[0, 1/2)$ for small enough values of ε to infer that the minimizer has a jump at $t = \frac{1}{2}$. To this purpose, consider equation (1) from part a) of Lemma 3.5.2:

$$\begin{aligned} &\ddot{u}_\varepsilon(t) = \lambda \frac{u_\varepsilon(t)}{F''_\varepsilon(\dot{u}_\varepsilon(t))} \\ \Leftrightarrow\ &F''_\varepsilon(\dot{u}_\varepsilon(t))\ddot{u}_\varepsilon(t) = \lambda u_\varepsilon(t) \\ \Leftrightarrow\ &\frac{d}{dt} F'_\varepsilon(\dot{u}_\varepsilon(t))\dot{u}_\varepsilon(t) = \lambda u_\varepsilon(t)\dot{u}_\varepsilon(t). \end{aligned}$$

An integration from 0 to s (where s is some value in the interval $[0, \frac{1}{2})$) then yields:

$$\begin{aligned} \int_0^s \frac{d}{dt} F_\varepsilon'(\dot u_\varepsilon(t)) \dot u_\varepsilon(t)\,\mathrm{d}t &= \int_0^s \lambda u_\varepsilon(t)\dot u_\varepsilon(t)\,\mathrm{d}t \\ \Leftrightarrow \Big[F_\varepsilon'(\dot u_\varepsilon(t))\dot u_\varepsilon(t)\Big]_0^s - \int_0^s \underbrace{F_\varepsilon'(\dot u_\varepsilon(t))\ddot u_\varepsilon(t)}_{=\frac{d}{dt}F_\varepsilon'(\dot u_\varepsilon(t))}\,\mathrm{d}t &= \Big[\frac{\lambda}{2} u_\varepsilon(t)^2\Big]_0^s, \end{aligned}$$

and with $\dot u_\varepsilon(0) = 0$ and $F_\varepsilon'(0) = 0$, we arrive at

$$\dot u_\varepsilon(s) F_\varepsilon'(\dot u_\varepsilon(s)) - F_\varepsilon(\dot u_\varepsilon(s)) = \frac{\lambda}{2}\big(u_\varepsilon(s)^2 - u_\varepsilon(0)^2\big). \tag{3.52}$$

The left-hand side of (3.52) is nonnegative due to $F_\varepsilon'' > 0$ and we therefore obtain:

$$u_\varepsilon(s) = \sqrt{u_\varepsilon(0)^2 + \frac{2}{\lambda}\big(\dot u_\varepsilon(s)F_\varepsilon'(\dot u_\varepsilon(s)) - F_\varepsilon(\dot u_\varepsilon(s))\big)} \quad \text{for } s \in [0, 1/2)\,. \tag{3.53}$$

By the convexity of the function F_ε, the term $pF_\varepsilon'(p) - F_\varepsilon(p)$ is increasing on $[0, \infty)$ and can therefore be bounded by $\lim\limits_{t\to\infty} tF_\varepsilon'(t) - F_\varepsilon(t) = \omega_\infty(F_\varepsilon) = \varepsilon$, so that

$$u_\varepsilon(s) \le \sqrt{u_\varepsilon(0)^2 + \frac{2\varepsilon}{\lambda}} \quad \text{for all } s \in \Big[0, \frac{1}{2}\Big).$$

If now $\lambda > 4$, Remark 3.5.2 (iii) together with Lemma 3.5.3 implies

$$\sqrt{u_\varepsilon(0)^2 + \frac{2\varepsilon}{\lambda}} \xrightarrow{\varepsilon\downarrow 0} \frac{2}{\lambda} < \frac{1}{2}.$$

Hence there exist $\varepsilon_0 > 0$ and $\delta > 0$ such that $u_{\varepsilon_0} \le \frac{1}{2} - \delta$ on $[0, \frac{1}{2})$, which means that u_ε must have a jump discontinuity at $t = \frac{1}{2}$. □

3.6 μ-elliptic Densities

So far, we have seen that we can prove the existence of a solution to problem (3.1) despite the nonreflexivity of the function space $W^{1,1}(\Omega)$, provided the value of the parameter λ does not exceed a certain threshold. However, the results from [35], [36], [37] and [38] indicate that such a solution exists even for arbitrary values of λ, if we choose the density function F in a certain class. These are the so called "μ-elliptic" functions for which, in addition to (F1)-(F3), also (F5) is satisfied for some $\mu \in (1, \infty)$. In the first-order case, it was shown in [36], [38] and [78] that the assumption $\mu < 2$ suffices to show that the functional I attains its minimum in the Sobolev class. In the one dimensional setting here, we will now study this dependence of the regularity behavior of the function u on the parameter μ more closely. Namely we will show that for any $\mu > 2$ there exists a density F, satisfying (F1)-(F5), and data f such that the solution u of (3.20) is discontinuous and consequently is not an element of $W^{1,1}(0,1)$. This counterexample can be transported easily to the higher-dimensional case $n \geq 2$, thus confirming the optimality of the bound $\mu \leq 2$.

Theorem 3.6.1
Suppose f is as in (3.2) and consider a density F satisfying (F1)-(F5). Moreover, fix any number $\lambda > 0$. Then, if

$$\mu \in (1,2] \tag{3.54}$$

holds, the unique solution $u \in BV(0,1)$ of problem (3.20) is of class $C^{1,1}\big([0,1]\big)$.

Proof. We recall the definition of the set $\mathrm{Reg}(u)$ from Theorem 3.3.1:

$$\mathrm{Reg}(u) := \big\{t \in [0,1] \,:\, u \text{ is } C^{1,1} \text{ in a neighborhood of } t\big\}.$$

From Theorem 3.3.1 b) we deduce that $\mathrm{Sing}(u) := [0,1] - \mathrm{Reg}(u)$ is a compact subset of $(0,1)$. Assume that $\mathrm{Sing}(u) \neq \emptyset$ and let $s := \inf \mathrm{Sing}(u) > 0$. Then $u \in C^{1,1}\big([0,s)\big)$ and therefore it holds

$$\ddot{u}F''(\dot{u}) = \lambda(u - f) \text{ a.e. on } [0,s). \tag{3.55}$$

From (3.55) we deduce (compare the derivation of (3.52)) the validity of

$$\dot{u}(t)F'(\dot{u}(t)) - F(\dot{u}(t)) = \frac{\lambda}{2}\big(u(t)^2 - u(0)^2\big) - \int_0^t f(\tau)\dot{u}(\tau)\,d\tau \tag{3.56}$$

for $t \in [0,s)$. Setting $\omega(p) := pF'(p) - F(p)$ for $p \in \mathbb{R}$, (3.56) implies (recall $0 \leq u, f \leq 1$ a.e. on $(0,1)$)

$$|\omega(\dot{u}(t))| \leq \frac{\lambda}{2} + |Du|(0,1) < \infty,\ t \in [0,s). \tag{3.57}$$

By the convexity of F (together with $F(0) = 0$) we see that $\omega(p) \geq 0$, $\omega(0) = 0$. Moreover,

$$\omega(p) = \int_0^p \omega'(q)\,dq = \int_0^q qF''(q)\,dq$$

and hence (F5) together with assumption (3.54) implies

$$\lim_{p\to\infty} \omega(p) = \infty, \qquad \lim_{p\to-\infty} \omega(p) = \infty. \tag{3.58}$$

Since we assume that u is singular in s, it must hold

$$\lim_{k\to\infty} |\dot{u}(t_k)| = \infty$$

for a suitable sequence $t_k \uparrow s$, since otherwise

$$|\sigma(s)| = \lim_{t\to s} \big|(F')^{-1}(\dot{u}(t))\big| < \lambda_\infty,$$

yielding $s \in \mathrm{Reg}(u)$ (cf. Remark (3.3.2)). But this contradicts (3.57) on account of (3.58). $\square$

Next, we look at the case $\mu > 2$. We fix $f = f_0$ from (3.49) and choose $F = \Phi_\mu\big(|\cdot|\big)$ (see (6)). We will see that if $\mu > 2$ and $\lambda > \lambda_\infty$ is appropriately large, the BV-minimum of the functional

$$K_\mu[w] := \int_0^1 \Phi_\mu(\dot{w})\,\mathrm{d}t + \lambda_\infty |D_s w|(0,1) + \frac{\lambda}{2}\int (w - f_0)^2\,\mathrm{d}t$$

is discontinuous at $t = \frac{1}{2}$. Actually, this is a consequence of a more general statement: going through the arguments of the proof of Theorem 3.6.1 the reader will notice that the critical factor is the behavior of the function $\omega(p)$ as $|p| \to \infty$:

Theorem 3.6.2
Assume F satisfies (F1)-(F4) and let $u \in BV(0,1)$ denote the unique solution of (3.20). Define ω_∞ as in (3.34). Then it holds:

a) In case $\omega_\infty = \infty$, u is of class $C^{1,1}\big([0,1]\big)$ for every value of λ and for arbitrary data f as in (3.2).

b) If $\omega_\infty < \infty$ and $f = f_0$ from (3.49), then there is a critical value λ_{crit} of the parameter λ such that u is discontinuous (exactly at $t = 1/2$) provided we choose $\lambda > \lambda_{\mathrm{crit}}$.

Remark 3.6.1
It is easily seen that for $F = \Phi_\mu\big(|\cdot|\big)$ it holds

$$\omega_\infty = \begin{cases} +\infty, & \text{if } \mu \le 2, \\ \dfrac{1}{\mu - 1}\dfrac{1}{\mu - 2}, & \text{if } \mu > 2. \end{cases}$$

Hence, for $\mu > 2$ (and $f = f_0$) problem (V) does in general not admit a solution in the Sobolev class.

Proof. a) $\mathrm{Sing}(u) = \emptyset$ follows from the same arguments that were used to prove Theorem 3.6.1.

b) As in the proof of Lemma 3.5.4, we deduce the identity

$$u(s) = \sqrt{u(0)^2 + \frac{2}{\lambda}\omega(\dot{u}(s))} \text{ for } s \in \Big[0, \frac{1}{2}\Big).$$

If now the function ω is bounded, then by Lemma 3.5.2 f) the right-hand side converges to 0 as $\lambda \to \infty$. In particular, there is a value $\lambda_{\mathrm{crit}} > 0$ such that

$$\sqrt{u(0)^2 + \frac{2}{\lambda}\omega_\infty} < \frac{1}{2} \text{ for } \lambda > \lambda_{\mathrm{crit}}. \tag{3.59}$$

Lemma 3.5.2 d) then implies that u is discontinuous at $t = \frac{1}{2}$ if $\lambda > \lambda_{\mathrm{crit}}$. $\square$

Remark 3.6.2
The relation (3.59) provides a lower bound for the critical parameter value:

$$\lambda_{\text{crit}} > 2\omega_\infty\Big(\frac{1}{4} - u(0)^2\Big).$$

Remark 3.6.3
Setting

$$\Omega := (0,1) \times \ldots \times (0,1),\ F(x_1, \ldots, x_n) := \Phi_\mu\Big(\sqrt{x_1^2 + \ldots + x_n^2}\Big) \text{ as well as}$$
$$f(x_1, \ldots, x_n) := f_0(x_1)$$

with f_0 from (3.49), we can transfer this example for a discontinuous solution of (V) to the n-dimensional case.

Example 3.6.1. The following simulation confirms the statement of Theorem 3.6.2 b) numerically: Figure 3.3 depicts the BV-minimizer of K_μ for $\mu = 3$, on the left with $\lambda = 4$, and on the right with $\lambda = 5$. The solution develops a discontinuity at $t = 1/2$ for $\lambda > \lambda_{\text{crit}} \approx 4.16$.

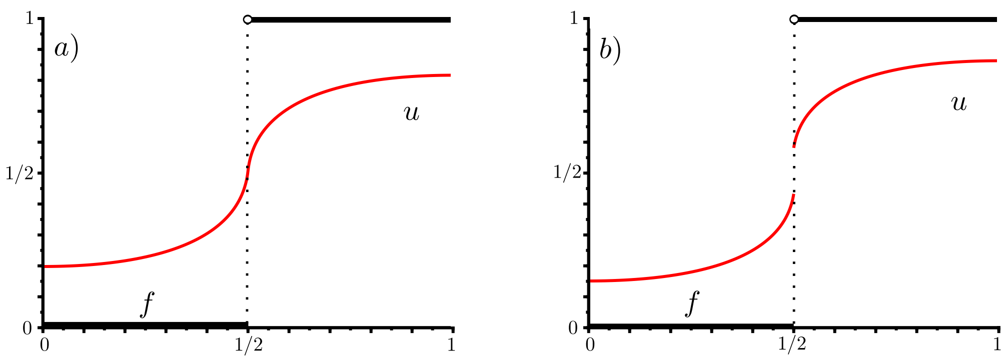

Fig. 3.3: Example plots of the K-minimizer u for $\mu = 3$ and $a)$ $\lambda = 4$, $b)$ $\lambda = 5$.

So far, our approach to problem (V) was by variational means. However, we could just as well take the boundary value problem (BVP) as our starting point and study it with methods from the theory of differential equations. In the one dimensional setting, this strategy is especially promising since we are then dealing with an ordinary rather than a partial differential equation. In the articles [79] and [80], Thompson has worked out an extensive theory for a large class of two-point boundary value problems with both continuous and measurable right-hand sides, which applies to our situation in the following way:

Theorem 3.6.3
Suppose (3.2) holds for f and let F fulfill (F1)-(F5). If the parameter λ satisfies

$$0 < \lambda < c_1 \int_1^\infty \frac{s\,ds}{(1+s)^\mu} =: \lambda_\mu,$$

where c_1 is as in (F5) on p. 39, then there exists a function $v \in W^{2,1}(0,1)$, satisfying $0 \le v(t) \le 1$ for almost all $t \in [0,1]$ and which solves the Neumann problem (BVP) a.e. on $[0,1]$. Furthermore, v solves (3.1).

Remark 3.6.4 *a) The reader familiar with the theory of lower and upper solutions will recognize in the above bound λ_μ a "Nagumo-condition" (see, e.g. [81]), which guarantees a priori bounds on the first derivative of the solution v.*

b) If f is continuous, it follows from the differential equation that $v \in C^2\big([0,1]\big)$.

c) At the example of $F(p) = \Phi_\mu(|p|)$, we would like to demonstrate how λ_μ might actually improve the bound $\lambda < \lambda_\infty$ from Theorem 3.2.1: obviously, the integral defining λ_μ diverges for $1 < \mu \le 2$. If $\mu > 2$, it is not difficult to show that the optimal constant c_1 in (F5) is given by $\frac{2}{(\mu-1)(\mu-2)}$. This yields

$$\lambda_\mu = \frac{2^{2-\mu}\mu}{(\mu-1)^2(\mu-2)^2}.$$

It holds $\lambda_\mu > \lambda_\infty(\Phi_\mu) = \frac{1}{\mu-1}$ for $\mu < 2.9$.

Proof. We essentially have to show that, for $\lambda < \lambda_\mu$, the conditions of Theorem 6, p. 295, in [80] are fulfilled. Without further explanation we adopt the notation of this work. First of all, we notice that, due to our restriction $0 \le f(t) \le 1$, the constant functions $\alpha(t) \equiv 0$ and $\beta(t) \equiv 1$ are a trivial lower and upper solution of (BVP), respectively: it holds

$$0 \ge \lambda \frac{0-f}{F''(0)} \quad \text{as well as} \quad 0 \le \lambda \frac{1-f}{F''(0)}.$$

Secondly, the right-hand side of the equation (BVP) may be considered as a tri-variate function

$$\Phi(t, v, \dot v) = \lambda \frac{v - f(t)}{F''(\dot v)},$$

with

$$\Phi : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R},\ \Phi(t,y,p) := \lambda \frac{y - f(t)}{F''(p)}$$

being a *Carathéodory function* (even if f is merely measurable). Using (F5), we can estimate Φ by

$$|\Phi(t,y,p)| \le \frac{\lambda}{c_1}\big(1 + |p|\big)^\mu.$$

In accordance with the notation of [80], we now define

$$\begin{aligned} h(p) &:= \frac{\lambda}{c_1}(1+p)^{\mu}, \quad \overline{h}(p) \equiv 1 \quad \text{and} \\ r(t) &:= \varepsilon \quad \text{for some arbitrarily small } \varepsilon > 0. \end{aligned}$$

Furthermore, we set

$$K := \sup \left\{ \frac{s}{h(s)} \,\middle|\, s \in [1, \infty) \right\} < \infty.$$

If we now choose λ small enough such that

$$\lambda < \frac{c_1}{(1+K\varepsilon)} \int_1^{\infty} \frac{s\,ds}{(1+s)^{\mu}}, \tag{3.60}$$

we find that for some large enough $L > 0$, Φ satisfies the following *Bernstein-Nagumo-Zwirner* condition (compare [80], Definition 4):

$$\begin{cases} |\Phi(t, y, p)| \leq h(|p|)\overline{h}(p) + r(t) \text{ for all } (t, y) \in [0, 1] \times [0, 1] \text{ and} \\ \displaystyle\int_1^{L} \frac{s\,ds}{h(s)} > 1 + K\varepsilon. \end{cases}$$

In the next step, we reformulate the boundary conditions of (BVP) as a "set condition" (as it is required in the context of [80]):

$$(v(0), \dot{v}(0)) \in \mathcal{J}(0) \quad \text{and} \quad (v(1), \dot{v}(1)) \in \mathcal{J}(1)$$

where

$$\mathcal{J}(0) = \mathcal{J}(1) := [0, 1] \times \{0\}.$$

To verify that the sets $\mathcal{J}(0) = \mathcal{J}(1) := [0, 1] \times \{0\}$ are of "compatible type 1" in the sense of Definition 14 in [80] is straightforward. Let us further define the sets S_0, S_1, S_2 and S_3 according to Definition 15 in [80] (see the Figure 3.4). Then we have

$$\mathcal{J}(0) \cap \{S_0 \cup S_2\} = \mathcal{J}(1) \cap \{S_1 \cup S_3\} = \{(0, 0), (0, 1)\} \neq \emptyset,$$

which means that all conditions of Theorem 6 are fulfilled and we can infer that there exists a solution $v \in W^{2,1}(0, 1)$ of (BVP). Moreover, $0 \leq v(t) \leq 1$ for almost all $t \in [0, 1]$ follows since $\alpha(t) \equiv 0$ and $\beta(t) \equiv 1$ is an upper and lower solution, respectively. Note that letting ε tend to zero in (3.60) yields the claimed bound λ_μ for λ.

Let now $v \in W^{2,1}(0, 1)$ be a solution of (BVP). We want to show that v coincides with the K-minimizer u from Theorem 3.3.1 a.e. on $[0, 1]$. Using the convexity of the functional J, we see that for any $w \in C^{1,1}\big([0, 1]\big)$ it holds

$$J[w] \geq J[v] + \langle DJ[v], w - v \rangle \tag{3.61}$$

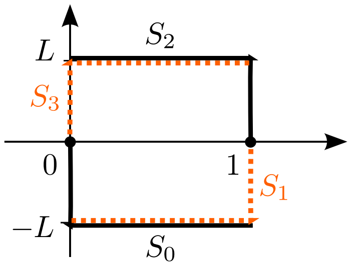

Fig. 3.4

where we use the shorthand notation

$$\langle DJ[v], w - v\rangle := \int_0^1 F'(\dot v)(\dot w - \dot v)\,\mathrm{d}t + \lambda \int_0^1 (v - f)(w - v)\,\mathrm{d}t.$$

On account of $F'(0) = 0$ we have

$$\begin{aligned}\int_0^1 F'(\dot v)(\dot w - \dot v)\,\mathrm{d}t &= \int_0^1 \frac{d}{\mathrm{d}t}\big[F'(\dot v)(w - v)\big]\,\mathrm{d}t - \int_0^1 F''(\dot v)\ddot v(w - v)\,\mathrm{d}t \\ &= -\int_0^1 F''(\dot v)\ddot v(w - v)\,\mathrm{d}t.\end{aligned}$$

By assumption, v solves (BVP) a.e. on $(0, 1)$, which implies

$$\langle DJ[v], w - v\rangle = -\int_0^1 (w - v)\big[F''(\dot v)\ddot v - \lambda(v - f)\big]\,\mathrm{d}t = 0 \tag{3.62}$$

for all $w \in C^{1,1}\big([0, 1]\big)$. Therefore, (3.61) yields

$$J[v] \le J[w] \quad \text{for all } w \in C^{1,1}\big([0, 1]\big).$$

Now let u denote the unique minimizer of K in $BV(0, 1)$. By Theorem 1.2.1, there exists a sequence $u_k \in C^\infty\big([0, 1]\big)$ such that

$$|Du_k|(0, 1) \xrightarrow{k\to\infty} |Du|(0, 1), \quad u_k \to u \text{ in } L^1(0, 1),$$

as well as

$$\int_0^1 \sqrt{1 + |Du_k|^2} \xrightarrow{k\to\infty} \int_0^1 \sqrt{1 + |Du|^2}$$

and from Reshetnyak's continuity theorem (see Lemma 2.1.1) we thus infer

$$J[v] \le J[u_k] = K[u_k] \xrightarrow{k\to\infty} K[u].$$

Since u is K-minimal, it follows that

$$K[u] \leq K[v] = J[v] \leq K[u],$$

which means $K[u] = K[v]$ and therefore $u = v$ a.e. by the uniqueness of the K-minimizer. $\square$

Chapter 4

The Two-Dimensional Case

As we have already outlined in the introduction of this thesis, a major field of application for variational models of linear growth is the mathematics of image processing. In this context, $f : \mathbb{R}^2 \supset \Omega - D \to [0,1]$ models the data of a black and white picture that might be afflicted with a noise, or where parts of the data (namely on the deficiency set D) are even entirely missing. Solving the problems (V) and $(\widetilde{V})$ then corresponds to a restoration of the original image from the data f, and yields particularly good results if the density F is of "TV-type" (i.e. approximates $|\cdot|$). However, the solutions of the first-order model are typically downgraded by the staircasing effect, which serves as the main motivation for the study of the higher-order model with $m \geq 2$. Against this backdrop, it is plausible to take a closer look at the two-dimensional case, which is the matter of this chapter. In view of existence and regularity question, the case $n = 2$ will be seen to stand out against the general, higher-dimensional setting. The results of this chapter are published in [51].

4.1 Sobolev Regularity

As it was shown in [82] for $m = 1$, the assumption (F_μ) together with $\mu < 2$ suffices to prove the existence of a classical solution of (V) in the Sobolev class even in arbitrary dimensions $n \in \mathbb{N}$. Furthermore, the results from Section 3.6 prove that the bound $\mu \leq 2$ is optimal in this respect. For systems, Tietz has proved (see [78] or [41]) that this still holds if the regularizing density F is rotationally invariant, i.e. if it only depends on the modulus $|\nabla u|$. These results mainly rely on the "maximum principle" from [40], which does not have an analog at higher orders. However, if we restrict ourselves to the planar setting $\Omega \subset \mathbb{R}^2$, then the required degree of integrability is granted by Sobolev's embedding Theorem and this allows us to prove the following:

Theorem 4.1.1
Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and $D \subset \Omega$ a measurable subset

with $\mathcal{L}^2(\Omega - D) > 0$. Assume that F satisfies (F1)-(F3),(F_μ) and that ϕ fulfills (ϕ1) and (ϕ2) for some $q \in [1,2]$. Let further $f \in L^\infty(\Omega - D)$ hold for the data function. Then, for any $1 \leq s < 2$ we have:

a) If $m = 1$ and $\mu \in (1,2)$, then (V) has a unique solution in the Sobolev space $W^{1,1}(\Omega) \cap W^{2,s}_{\mathrm{loc}}(\Omega)$.

b) If $m = 2$ and $\mu \in \big(1, \frac{3}{2}\big)$, then (V) has a unique solution in the Sobolev space $W^{2,1}(\Omega) \cap W^{3,s}_{\mathrm{loc}}(\Omega)$.

c) If $m \geq 3$ and $\mu \in \big(1,2\big)$, then (V) has a unique solution in the Sobolev space $W^{m,1}(\Omega) \cap W^{m+1,1}_{\mathrm{loc}}(\Omega)$.

Remark 4.1.1
The uniqueness part follows directly from part e) of Theorem 2.1.1 in each case.

Proof. Throughout the following calculations, we apply the summation convention that the sum is taken with respect to the index "i" whenever it appears twice within the same formula. We do not give the proof of part a), which was treated in [41] (even for arbitrary dimensions $n \geq 2$). Let us therefore start with b), i.e. $m = 2$. The strategy of the proof will be the same at any order so that we start our computations with general $m \in \mathbb{N}$ and later switch to $m = 2$ where this becomes relevant. Our arguments rely on the K-minimizing sequence u_δ, whose properties were gathered in Lemma 2.2.1. In particular, we have equation (2.17) serving as our starting point: for all $\varphi \in W^{m,2}(\Omega)$ it holds

$$\int\limits_\Omega DF_\delta(\nabla^m u_\delta) : \nabla^m \varphi \,\mathrm{d}x + \lambda \int\limits_{\Omega - D} \phi'\big(|u_\delta - f|\big)\,\mathrm{sgn}(u_\delta - f)\varphi \,\mathrm{d}x = 0. \tag{4.1}$$

Replacing φ with $D_i\varphi$ ($i \in \{1, ..., n\}$) for some $\varphi \in W^{m+1,2}(\Omega)$, an integration by parts leads to

$$\int\limits_\Omega D^2F_\delta(\nabla^m u_\delta)\big(D_i\nabla^m u_\delta, \nabla^m \varphi\big) \,\mathrm{d}x = \lambda \int\limits_{\Omega - D} \phi'\big(|u_\delta - f|\big)\,\mathrm{sgn}(u_\delta - f) D_i\varphi \,\mathrm{d}x, \tag{4.2}$$

which, by an approximation argument (cf. Theorem 1.2.1), holds even for $\varphi \in W^{m,2}(\Omega)$. Let now $x_0 \in \Omega$ be an arbitrary point and $R > 0$ such that the ball $B_{2R}(x_0)$ is compactly contained in Ω. Choose a test function $\eta \in C_0^\infty(\Omega)$ that fulfills

$$\begin{cases} \mathrm{supp}(\eta) \subset B_{2R}(x_0), \\ 0 \leq \eta \leq 1, \\ |\nabla^k \eta| \leq \dfrac{c}{R^k} \text{ for } k = 1, ..., m \text{ and} \\ \eta \equiv 1 \text{ on } B_R(x_0), \end{cases} \tag{4.3}$$

where c is some positive constant. Due to Lemma 2.2.1 c) we have that $\varphi := \eta^{2m} D_i u_\delta \in W^{m,2}(\Omega)$ is an admissible choice, for which (4.2) reads as

$$\begin{aligned}
&\int_\Omega D^2 F_\delta(\nabla^m u_\delta)\big(D_i \nabla^m u_\delta, D_i \nabla^m u_\delta\big)\eta^{2m}\,\mathrm{d}x \\
&= -\int_\Omega D^2 F_\delta(\nabla^m u_\delta)\big(D_i \nabla^m u_\delta, \nabla^m(\eta^{2m} D_i u_\delta) - \eta^{2m} D_i \nabla^m u_\delta\big)\,\mathrm{d}x \\
&\underbrace{+\lambda \int_{\Omega - D} \phi'\big(|u_\delta - f|\big)\operatorname{sgn}(u_\delta - f) D_i(\eta^{2m} D_i u_\delta)\,\mathrm{d}x.}_{=:\,T}
\end{aligned}$$

To the first summand on the right-hand side, we apply the Cauchy-Schwarz inequality followed by Young's inequality, giving

$$\begin{aligned}
&\int_\Omega D^2 F_\delta(\nabla^m u_\delta)\big(D_i \nabla^m u_\delta, D_i \nabla^m u_\delta\big)\eta^{2m}\,\mathrm{d}x \\
&\leq \varepsilon \int_\Omega D^2 F_\delta(\nabla^m u_\delta)\big(D_i \nabla^m u_\delta, D_i \nabla^m u_\delta\big)\eta^{2m}\,\mathrm{d}x \\
&+ c(\varepsilon, \eta) \int_{B_{2R}(x_0)} \sum_{k=0}^{m} \big|D^2 F_\delta(\nabla^m u_\delta)\big| |\nabla^k u_\delta|^2\,\mathrm{d}x + |T|,
\end{aligned}$$

where $\varepsilon > 0$ can be chosen arbitrarily small. After absorbing terms on the left-hand side, it remains

$$\begin{aligned}
&\int_\Omega D^2 F_\delta(\nabla^m u_\delta)\big(D_i \nabla^m u_\delta, D_i \nabla^m u_\delta\big)\eta^{2m}\,\mathrm{d}x \\
&\leq c\Bigg[\underbrace{\int_{B_{2R}(x_0)} \sum_{k=0}^{m} \big|D^2 F_\delta(\nabla^m u_\delta)\big| |\nabla^k u_\delta|^2\,\mathrm{d}x}_{=:\,S} + |T|\Bigg].
\end{aligned} \tag{4.4}$$

Introducing the quantity

$$\varphi_\delta := \big(1 + |\nabla^m u_\delta|\big)^{1-\frac{\mu}{2}}, \tag{4.5}$$

we observe that, in combination with (F_μ), inequality (4.4) now implies

$$\int_{B_{2R}(x_0)} |\nabla \varphi_\delta|^2\,\mathrm{d}x \leq c\big(|S| + |T|\big). \tag{4.6}$$

Note that if the right-hand side of (4.6) is bounded uniformly in δ, Sobolev's embedding Theorem yields

$$\nabla^m u_\delta \in L^p_{\mathrm{loc}}\big(\Omega, \mathbb{R}^{n^m}\big) \text{ uniformly with respect to } \delta \tag{4.7}$$

for all exponents $p < \infty$. We may therefore choose a subsequence $\delta \downarrow 0$ such that

$$u_\delta \to u \text{ in } W^{m-1,1}\big(\Omega\big) \quad \text{as well as} \quad u_\delta \rightharpoondown u \text{ in } W^{m,p}_{\mathrm{loc}}(\Omega)$$

for some $u \in BV^m(\Omega)$, which minimizes the relaxed functional K from (2.1). In particular, $u \in BV^m(\Omega) \cap W^{m,1}_{\mathrm{loc}}(\Omega) \subset W^{m,1}(\Omega)$ and since u is K-minimal (and $K \equiv I$ on $W^{m,1}(\Omega)$), it follows that u is I-minimal in the Sobolev class. Consequently, we have to examine the behavior of the quantities $|S|$ and $|T|$ for $m = 2$ and $m \geq 3$, respectively. Starting with $|S|$, we note that

$$S \leq \int\limits_\Omega \big|D^2 F_\delta(\nabla^m u_\delta)\big| |\nabla^m u_\delta|^2 \, \mathrm{d}x + c\|u_\delta\|^2_{m-1,2;\Omega}.$$

Since $n = 2$ and $u_\delta \in W^{m,1}(\Omega)$ uniformly, Sobolev's embedding theorem guarantees the boundedness of the second summand. To the first one, we apply the second inequality in (F_μ) together with (2.12) from the proof of Lemma 2.2.1, yielding

$$\int\limits_\Omega \big|D^2 F_\delta(\nabla^m u_\delta)\big| |\nabla^m u_\delta| \, \mathrm{d}x \leq c \left(1 + \int\limits_\Omega \frac{|\nabla^m u_\delta|^2}{1 + |\nabla^m u_\delta|} \, \mathrm{d}x\right) < \infty.$$

We proceed with the quantity T. Let $m = 2$. Using the growth estimate $(\phi 3)$, we have

$$\begin{aligned}
|T| &\leq c \int\limits_{\Omega - D} \Big(1 + |u_\delta - f|^{q-1}\Big) |D_i(\eta^4 D_i u_\delta)| \, \mathrm{d}x \\
&\leq c \left[\int\limits_{\Omega - D} \Big(1 + |u_\delta - f|^{q-1}\Big) |\nabla u_\delta| \, \mathrm{d}x + \int\limits_{\Omega - D} \eta^4 \Big(1 + |u_\delta - f|^{q-1}\Big) |\nabla^2 u_\delta| \, \mathrm{d}x\right] \\
&\leq c \left[1 + \int\limits_{\Omega - D} \eta^4 |u_\delta - f|^{q-1} |\nabla^2 u_\delta| \, \mathrm{d}x\right],
\end{aligned}$$

since, by Sobolev's embedding Theorem, $|u_\delta - f| \in L^p(\Omega - D)$ uniformly for any $p \in [1, \infty)$. An application of Young's inequality further yields

$$\begin{aligned}
|T| &\leq c \left[1 + \int\limits_{\Omega - D} \eta^4 |u_\delta - f|^{(q-1)\frac{\mu}{\mu - 1}} \, \mathrm{d}x + \int\limits_\Omega \eta^4 |\nabla^2 u_\delta|^\mu \, \mathrm{d}x\right] \\
&\leq c \left[1 + \int\limits_\Omega \eta^4 |\nabla^2 u_\delta|^\mu \, \mathrm{d}x\right]
\end{aligned} \tag{4.8}$$

and it remains to give a bound on the integral

$$T' := \int\limits_\Omega \eta^4 |\nabla^2 u_\delta|^\mu \, \mathrm{d}x.$$

Here, we benefit from the μ-ellipticity condition (F_μ), once again. With

$$\psi_\delta := \big(1 + |\nabla^2 u_\delta|\big)^{\frac{\mu}{2}},$$

we may estimate

$$T' \le c \int\limits_\Omega (\eta^2 \psi_\delta)^2 \,\mathrm{d}x. \tag{4.9}$$

Sobolev's inequality yields

$$\begin{aligned} \int\limits_\Omega (\eta^2\psi_\delta)^2 \,\mathrm{d}x &\le c \left(\int\limits_\Omega \big|\nabla(\eta^2\psi_\delta)\big| \,\mathrm{d}x \right)^2 \overset{\mu<2}{\le} c \left(1 + \int\limits_\Omega \eta^2 |\nabla\psi_\delta| \,\mathrm{d}x \right)^2 \\ &\le c \left[1 + \left(\int\limits_\Omega \eta^2 |\nabla\psi_\delta| \,\mathrm{d}x \right)^2 \right]. \end{aligned} \tag{4.10}$$

Observing (cf. (4.5)) the relation

$$\psi_\delta = \varphi_\delta^\alpha \qquad \text{with } \alpha = \frac{\mu}{2-\mu},$$

we obtain (recall our choice of η)

$$\begin{aligned} \int\limits_\Omega \eta^2 |\nabla\psi_\delta| \,\mathrm{d}x &\le c \int\limits_{B_{2R}(x_0)} \eta^2 |\nabla\varphi_\delta| \varphi_\delta^{\alpha-1} \,\mathrm{d}x \\ &\le c \left(\int\limits_{B_{2R}(x_0)} \eta^4 |\nabla\varphi_\delta|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \cdot \left(\int\limits_{B_{2R}(x_0)} \varphi_\delta^{2\alpha-2} \,\mathrm{d}x \right)^{\frac{1}{2}}. \end{aligned} \tag{4.11}$$

Inserting (4.11) into (4.10) and going back to (4.9) we infer

$$T' \le c \left[1 + \left(\int\limits_{B_{2R}(x_0)} \varphi_\delta^{2\alpha-2} \,\mathrm{d}x \right) \cdot \left(\int\limits_{B_{2R}(x_0)} \eta^4 |\nabla\varphi_\delta|^2 \,\mathrm{d}x \right) \right] \tag{4.12}$$

and therefore it follows from (4.8) that

$$|T| \le c \left[1 + \left(\int\limits_{B_{2R}(x_0)} \varphi_\delta^{2\alpha-2} \,\mathrm{d}x \right) \cdot \left(\int\limits_{B_{2R}(x_0)} \eta^2 |\nabla\varphi_\delta|^2 \,\mathrm{d}x \right) \right]. \tag{4.13}$$

With (4.13) at hand, we return to (4.6), observing first

$$|\nabla\varphi_\delta|^2 \le c\, D^2F_\delta(\nabla^2 u_\delta)\big(D_i\nabla^2 u_\delta, D_i\nabla^2 u_\delta\big).$$

Second we note

$$\varphi_\delta^{2\alpha-2} = \big(1 + |\nabla^2 u_\delta|\big)^{(\alpha-1)(2-\mu)} \sim |\nabla^2 u_\delta|^{(\alpha-1)(2-\mu)}$$

with exponent

$$(\alpha - 1)(2 - \mu) = 2\mu - 2 < 1 \Leftrightarrow \mu < \frac{3}{2}.$$

Thus, by Lemma 2.2.1 b) and Hölder's inequality we obtain

$$\int\limits_{B_{2R}(x_0)} \varphi_\delta^{(\alpha-1)(2-\mu)} \, \mathrm{d}x \le c\mathcal{L}^n \left(B_{2R}(x_0)\right)^\gamma$$

for some positive exponent γ and (4.6) in combination with our previous results yields

$$\left(1 - c\mathcal{L}^n \left(B_{2R}(x_0)\right)^\gamma\right) \int\limits_{B_{2R}(x_0)} \eta^4 D^2 F_\delta(\nabla^2 u_\delta)\big(D_i \nabla^2 u_\delta, D_i \nabla^2 u_\delta\big) \le c. \tag{4.14}$$

From this inequality we deduce: if we restrict ourselves to radii $R \le R_0$ for some $R_0 > 0$ independent of δ, then

$$\int\limits_{B_{2R}(x_0)} |\nabla \varphi_\delta|^2 \, \mathrm{d}x \le c(R) < \infty$$

and hence

$$|\nabla^2 u_\delta| \in L^p_{\mathrm{loc}}(\Omega) \text{ for all } p < \infty. \tag{4.15}$$

Let now $s \in [1, 2)$ be given and let $\Omega^* \Subset \Omega$ denote some compact subset. Using Hölder's inequality together with (4.14) and (4.15), we obtain

$$\begin{aligned}
\int\limits_{\Omega^*} |\nabla^3 u_\delta|^s \, \mathrm{d}x &= \int\limits_{\Omega^*} \big(1 + |\nabla^2 u_\delta|\big)^{-\mu\frac{s}{2}} |\nabla^3 u_\delta|^s \big(1 + |\nabla^2 u_\delta|\big)^{\mu\frac{s}{2}} \, \mathrm{d}x \\
&\le \left(\int\limits_{\Omega^*} \big(1 + |\nabla^2 u_\delta|\big)^{-\mu} |\nabla^3 u_\delta|^2 \, \mathrm{d}x \right)^{\frac{s}{2}} \left(\int\limits_{\Omega^*} \big(1 + |\nabla^2 u_\delta|\big)^{\frac{\mu s}{2-s}} \, \mathrm{d}x \right)^{1-\frac{s}{2}} < c,
\end{aligned}$$

with a constant $c = c(\Omega^*)$ independent of δ due to (4.15). This proves part b) of Theorem 4.1.1.

We proceed with a brief discussion of the case $m \ge 3$. Looking at the relevant quantity T from (4.6), we find that

$$|T| \le c \left[1 + \int\limits_{\Omega - D} |u_\delta - f|^{q-1} |\nabla u_\delta| \, \mathrm{d}x + \int\limits_{\Omega - D} \eta^2 |u_\delta - f|^{q-1} |\nabla^2 u_\delta| \, \mathrm{d}x \right].$$

Note that by Sobolev's embedding Theorem we even have $|u_\delta - f| \in L^\infty(\Omega)$ and $|\nabla^2 u_\delta| \in L^2(\Omega)$ (uniformly in δ) if $m \ge 3$, so that T is obviously bounded. We may thus adopt the arguments for $m = 2$ to deduce

$$\varphi_\delta \in W^{1,2}_{\mathrm{loc}}(\Omega) \quad \text{uniformly in } \delta, \tag{4.16}$$

i.e. $|\nabla^m u_\delta| \in L^p(\Omega)$ for all $p < \infty$ and the claim $u_\delta \in W^{m+1,s}_{\mathrm{loc}}(\Omega)$ for $s \in \big[1, 2\big)$ follows the same way. □

4.2 The Case of Pure Denoising

In this short addendum we show that, under the condition $\mathcal{L}^n(D) = 0$ (i.e. in the case of "pure denoising"), the bound $\mu < \frac{3}{2}$ from part b) of Theorem 4.1.1 can be improved to the more natural assumption $\mu \in (1,2)$. In fact, if we choose the quadratic fidelity function $\phi(t) = \frac{1}{2}t^2$, then we have:

Theorem 4.2.1
Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain, assume $f \in L^\infty(\Omega)$, let F satisfy (F1)-(F3) and (F_μ) together with $\mu \in (1,2)$ and choose $\phi(t) := \frac{1}{2}t^2$. Then, for $m = 2$, problem (V) admits a unique solution u in the space $W^{2,1}(\Omega) \cap W^{3,s}_{\mathrm{loc}}(\Omega)$ for any $s \in [1,2)$.

Proof. Going back to the proof of Theorem 4.1.1 a), we see that the critical term T now takes the form

$$T = \lambda \int_\Omega (u_\delta - f) D_i(\eta^2 u_\delta)\,\mathrm{d}x$$

which expands to

$$T = \lambda \int_\Omega u_\delta D_i(\eta^2 D_i u_\delta)\,\mathrm{d}x - \lambda \int_\Omega f D_i(\eta^2 D_i u_\delta)\,\mathrm{d}x.$$

An integration by parts then yields

$$T = -\lambda \int_\Omega \eta^2 |\nabla u_\delta|^2\,\mathrm{d}x - \lambda \int_\Omega f D_i(\eta^2 D_i u_\delta)\,\mathrm{d}x,$$

and, together with the boundedness of f, (4.6) implies

$$\begin{aligned}\int_\Omega D^2 F_\delta(\nabla^m u_\delta)(D_i\nabla^m u_\delta, D_i \nabla^m u_\delta)\eta^2\,\mathrm{d}x + \lambda \int_\Omega \eta^2 |\nabla u_\delta|^2\,\mathrm{d}x \\ \leq c(\eta) + \lambda \|f\|_{\infty;\Omega}\left\{\int_\Omega |\nabla \eta^2||\nabla u_\delta|\,\mathrm{d}x + \int_\Omega \eta^2 |\nabla^2 u_\delta|\,\mathrm{d}x\right\}.\end{aligned} \tag{4.17}$$

By Lemma 2.2.1 all terms on the right-hand side of (4.17) are bounded independently of δ and thus

$$\sup_{\delta>0} \int_{\Omega^*} D^2 F_\delta(\nabla^2 u_\delta)(D_i \nabla^2 u_\delta, D_i \nabla^2 u_\delta)\,\mathrm{d}x < \infty \tag{4.18}$$

on all compact subsets $\Omega^* \Subset \Omega$. The assertion of Theorem 4.2.1 now follows from the same arguments as in the proof of Theorem 4.1.1. $\square$

4.3 Partial Hölder Regularity

In this section, we use the well-known blow-up technique by Evans and Gariepy (see [47]) to prove that the Sobolev solution from Theorem 4.1.1 is actually partially Hölder continuous:

Theorem 4.3.1
Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and let $D \subset \Omega$ be a measurable subset with $\mathcal{L}^2(\Omega - D) > 0$. Assume further $f \in L^\infty(\Omega - D)$ holds for the data function, that the density F is of class C^2, satisfying (F1)-(F3) as well as (F_μ) together with

$$\begin{aligned} \mu < 2 \quad & \textit{if } D = \emptyset, \\ \textit{or } \; \mu < \frac{3}{2} \quad & \textit{in case of general } D, \end{aligned}$$

and set $\phi(t) := \frac{1}{2}t^2$. Then for the $W^{m,1}$-solution u of problem (V) from Theorem 4.1.1, the following assertions hold:

a) There is an open subset Ω_0 of Ω with full $\mathcal{L}^2$-measure such that the minimizer u is of class $C^{m,\alpha}(\Omega_0)$ for any $\alpha \in (0,1)$.

b) The set $\Omega - \Omega_0$ of possible singularities has Hausdorff dimension zero, i.e. $\mathcal{H}^\varepsilon(\Omega - \Omega_0) = 0$ for any $\varepsilon > 0$.

c) Let $\sigma \in L^\infty(\Omega, \mathbb{R}^{n^m})$ denote the solution of the dual problem from Theorem 2.2.1 a). Then it holds $\sigma \in W^{1,2}_{loc}(\Omega, \mathbb{R}^{n^m})$.

Remark 4.3.1
Our restriction to the quadratic fidelity term $\phi(t) := \frac{1}{2}t^2$ is merely for notational simplicity. We could just as well do the following calculations with general ϕ, satisfying (ϕ1) and (ϕ2) for some $q \in [1,2]$.

Proof. As already mentioned, the basis for the proof of part a) is the so called blow-up technique whose idea together with appropriate references is explained in the monograph [46]; we also suggest to consult the paper [47]. Our arguments follow these ideas and their higher-order generalizations developed in [83], section 3. However, for the sake of of simplicity, we restrict ourselves in the following calculations to $m = 2$, i.e.

$$I[u] = \int\limits_{\Omega} F(\nabla^2 u)\,\mathrm{d}x + \frac{\lambda}{2} \int\limits_{\Omega - D} (u - f)^2\,\mathrm{d}x.$$

We emphasize that the general case follows from the same arguments.

For disks $B_\rho(x) \Subset \Omega$ we define the so called *excess function* by

$$E(x,\rho) := \fint_{B_\rho(x)} \left|\nabla^2 u(y) - (\nabla^2 u)_{x,\rho}\right|^2\,\mathrm{d}y,$$

where $u \in W^{2,1}(\Omega)$ is the Sobolev minimum of the functional I, which exists due to Theorem 4.1.1 and satisfies $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ for any $p \in [1,\infty)$ (hence E sis well defined). Note that by

$$(\nabla^2 u)_{x,\rho} := \fint_{B_\rho(x)} \nabla^2 u(y)\,\mathrm{d}y$$

we denote the mean value of $\nabla^2 u$ on the disk $B_\rho(x)$. The essential step is to show the following *excess decay*, or "*blow-up" lemma*:

Lemma 4.3.1
Given $L > 0$, define the constant $C^*(L)$ according to (4.32) below and set $C_* = C_*(L) := 2C^*(L)$. Then, for any $\tau \in (0, \frac{1}{2})$ there is $\varepsilon = \varepsilon(L,\tau)$ such that whenever

$$|(\nabla^2 u)_{x,r}| \le L, \quad E(x,r) + r \le \varepsilon, \tag{4.19}$$

then also

$$E(x,\tau r) \le \tau^2 C_*(L) \cdot \big(E(x,r) + r\big) \tag{4.20}$$

for disks $B_r(x) \Subset \Omega$.

Remark 4.3.2 *a) Due to Lebesgue's differentiation theorem, the condition (4.19) is valid for $\mathcal{L}^2$-almost all points $x \in \Omega$, i.e. the set*

$$\Omega_0 := \Big\{x \in \Omega \,:\, \limsup_{r\to 0} |(\nabla^2 u)_{x,r}| < \infty\Big\} \cap \Big\{x \in \Omega \,:\, \liminf_{r\to 0} E(x,r) = 0\Big\}$$

has full Lebesgue measure.

b) That in fact $\nabla^2 u$ is Hölder continuous on Ω_0 and that Ω_0 is an open subset of Ω follows from Lemma 4.3.1 in a standard way, as e.g. outlined in detail on p. 95 ff. in the monograph [46]: by iteration, inequality (4.20) yields ($0 < \alpha < 1$)

$$E(x,\tau r) \le \tau^{2\alpha}(E(x,r) + r),$$

where τ is such that $C_*(L)\tau^{2-2\alpha} = 1$. This implies

$$E(x,\rho) \le c\left(\frac{\rho}{r}\right)^{2\alpha}(E(x,r) + r)$$

for all $\rho \le r$ and Morrey's integral characterization of Hölder continuity (cf. [46], chapter III, Theorem 1.3) then implies the assertion of Theorem 4.3.1 a).

We continue with an indirect proof of the blow-up Lemma. Fix $L > 0$. If the statement of the lemma is false, then there is $\tau \in (0, \frac{1}{2})$ and a sequence $B_{r_k}(x_k) \Subset \Omega$ ($k \in \mathbb{N}$) of disks with

$$\big|(\nabla^2 u)_{x_k,r_k}\big| \le L, \quad E(x_k,r_k) + r_k =: \lambda_k^2 \to 0, \tag{4.21}$$

but at the same time

$$E(x_k, \tau r_k) > \tau^2 C_*(L)\lambda_k^2. \tag{4.22}$$

Now we rescale the function u and subtract a suitable second-degree polynomial, setting

$$a_k := (u)_{x_k,r_k},\ A_k := (\nabla u)_{x_k,r_k},\ H_k = (\nabla^2 u)_{x_k,r_k} \tag{4.23}$$

$$u_k(z) := \frac{1}{\lambda_k r_k^2}\left[u(x_k + r_k z) - r_k A_k \cdot z - a_k - \frac{r_k^2}{2} H_k(z,z) \right.$$
$$\left. + \frac{r_k^2}{2} \fint_{B_1(0)} H_k(y,y)dy\right],$$

where $z \in B_1(0)$. These scalings are chosen in such a way that $(u_k)_{0,1} = 0$, $(\nabla u_k)_{0,1} = 0$, $(\nabla^2 u_k)_{0,1} = 0$ and we further have

$$\nabla u_k(z) = \frac{1}{\lambda_k r_k}\left[\nabla u(x_k + r_k z) - A_k - \frac{1}{2} r_k \nabla (H_k^{\alpha\beta} z_\alpha z_\beta)\right], \tag{4.24}$$

$$\nabla^2 u_k(z) = \frac{1}{\lambda_k}\left[\nabla^2 u(x_k + r_k z) - H_k\right], \tag{4.25}$$

(Note that we apply summation convention with respect to the Greek indices) as well as

$$\fint_{B_1(0)} |\nabla^2 u_k(z)|^2 \, \mathrm{d}z = \lambda_k^{-2} E(x_k, r_k) \underset{(4.21)}{\le} 1. \tag{4.26}$$

Hence we may assume that for a suitable subsequence $k \to \infty$ there exists a function $\hat{u}$ in $W^{2,2}(B_1(0))$ such that

$$u_k \rightharpoonup \hat{u} \text{ in } W^{2,2}(B_1(0)) \tag{4.27}$$

and consequently

$$\lambda_k \nabla^2 u_k \to 0 \text{ in } L^2(B_1(0)) \text{ and a.e.} \tag{4.28}$$

According to (4.21) we may further assume that

$$H_k \to: H \tag{4.29}$$

for a 2×2-matrix H as $k \to \infty$. We claim that the function $\hat{u}$ fulfills the following constant coefficient elliptic system (implying the smoothness of $\hat{u}$):

$$\int_{B_1(0)} D^2 F(H)(\nabla^2 \hat{u}, \nabla^2 \psi) \, \mathrm{d}z = 0 \ \text{ for all } \psi \in C_0^\infty(B_1(0)). \tag{4.30}$$

Fix some function ψ and set

$$\varphi(x) := \psi\left(\frac{x - x_k}{r_k}\right), \quad x \in B_{r_k}(x_k).$$

By the minimality of u, it holds

$$0 = \underbrace{\int\limits_{B_{r_k}(x_k)} DF(\nabla^2 u) : \nabla^2 \varphi \, \mathrm{d}x}_{=: S_1} + \underbrace{\lambda \int\limits_{B_{r_k}(x_k)-D} (u-f)\varphi \, \mathrm{d}x}_{=: S_2}$$

On any open subset $\Omega^* \Subset \Omega$, both u and f are bounded (recall $u \in W^{3,s}_{\mathrm{loc}}(\Omega)$, $s < 2$ by Theorem 4.1.1). Thus we can estimate S_2 by

$$\begin{aligned}|S_2| &\le c \int\limits_{B_{r_k}(x_k)} |\varphi| \, \mathrm{d}x = c \int\limits_{B_{r_k}(x_k)} \left| \psi\left(\frac{x-x_k}{r_k}\right)\right| \mathrm{d}x = c r_k^2 \int\limits_{B_1(0)} |\psi(z)| \, \mathrm{d}z \\ &\le C(\psi) r_k^2.\end{aligned}$$

After the coordinate transformation $z = \frac{x-x_k}{r_k}$, the integral S_1 reads as

$$\begin{aligned}S_1 &= \int\limits_{B_1(0)} DF(H_k + \lambda_k \nabla^2 u_k) : \nabla^2 \psi \, \mathrm{d}z \\ &= \int\limits_{B_1(0)} \int\limits_0^1 \frac{\mathrm{d}}{\mathrm{d}s} DF(H_k + s\lambda_k \nabla^2 u_k) : \nabla^2 \psi \, \mathrm{d}s \, \mathrm{d}z + \underbrace{\int\limits_{B_1(0)} DF(H_k) : \nabla^2 \psi \, \mathrm{d}z}_{= 0 \text{ due to } \psi \in C_0^\infty(B_1(0))} \\ &= \int\limits_{B_1(0)} \int\limits_0^1 D^2F(H_k + s\lambda_k \nabla^2 u_k)(\nabla^2 u_k, \nabla^2 \psi) \lambda_k \, \mathrm{d}s \, \mathrm{d}z,\end{aligned}$$

and together with our estimate for S_2 this yields

$$\left| \int\limits_{B_1(0)} \int\limits_0^1 D^2F(H_k + s\lambda_k \nabla^2 u_k)(\nabla^2 u_k, \nabla^2 \psi) \, \mathrm{d}s \, \mathrm{d}z \right| \le C(\psi) \lambda_k^{-1} r_k^2. \tag{4.31}$$

Because of (4.21), $r_k \lambda_k^{-1} \le \lambda_k \to 0$ for $k \to \infty$ and thus (note that r_k is bounded) also $r_k^2 \lambda_k^{-1} \to 0$. Now we turn to the left-hand side of (4.31). Let $\delta > 0$ be given. By (4.28) and Egorov's Theorem (see e.g. Theorem 1.34 in [1]) there is a set $S \subset B_1(0)$ with $\mathcal{L}^2(B_1(0) - S) < \delta$ and $\lambda_k \nabla^2 u_k \rightrightarrows 0$ a.e. on S. With (4.27) and (4.28) it follows:

$$\int\limits_S \int\limits_0^1 D^2F(H_k + \lambda_k s \nabla^2 u_k)(\nabla^2 u_k, \nabla^2 \psi) \, \mathrm{d}s \, \mathrm{d}z \to \int\limits_S D^2F(H)(\nabla^2 \hat{u}, \nabla^2 \psi) \, \mathrm{d}z.$$

At the same time, due to the boundedness of D^2F and by Hölder's inequality we infer that

$$
\begin{aligned}
&\left| \int\limits_{B_1(0)-S} \int\limits_0^1 D^2F(H_k+\lambda_k s\nabla^2 u_k)(\nabla^2 u_k,\nabla^2\psi)\,\mathrm{d}s\,\mathrm{d}z \right| \\
&\le c\|\nabla^2 u_k\|_{2;B_1(0)}\|\nabla^2\psi\|_{2;B_1(0)-S} \\
&\underset{(4.27)}{\le} C\|\nabla^2\psi\|_{2;B_1(0)-S} \le C(\psi)\delta^{\frac{1}{2}},
\end{aligned}
$$

and since δ can be chosen arbitrarily small, this proves

$$
\int\limits_{B_1(0)} \int\limits_0^1 D^2F(H_k+s\lambda_k\nabla^2 u_k)(\nabla^2 u_k,\nabla^2\psi)\,\mathrm{d}s\,\mathrm{d}z \to \int\limits_{B_1(0)} D^2F(H)(\nabla^2\hat{u},\nabla^2\psi)\,\mathrm{d}z
$$

so that (4.30) follows. We can therefore draw on the results in [84] and [85] on higher-order elliptic systems (see also the comments subsequent to (3.10) in [83]) and find a constant $C^*(L)$ such that

$$
\fint_{B_\tau(0)} |\nabla^2\hat{u}-(\nabla^2\hat{u})_{0,\tau}|^2\,\mathrm{d}z \le C^*(L)\tau^2, \tag{4.32}
$$

which together with the definition of $C_*(L)$ contradicts (4.22) once that (4.27) is improved to

$$
\nabla^2 u_k \to \nabla^2\hat{u} \text{ in } L^2_{\mathrm{loc}}(B_1(0)). \tag{4.33}
$$

In fact, after scaling (4.22) reads as

$$
\fint_{B_\tau(0)} \left|\nabla^2 u_k-(\nabla^2 u_k)_{0,\tau}\right|^2\mathrm{d}z = \lambda_k^{-2}E(x_k,\tau r_k) > \tau^2 C_*(L),
$$

and hence, along with (4.33) we obtain

$$
\fint_{B_\tau(0)} |\nabla^2\hat{u}-(\nabla^2\hat{u})_{0,\tau}|^2\,\mathrm{d}z \ge \tau^2 C_*(L),
$$

which contradicts (4.32). In order to complete the proof of the blow-up lemma we therefore need to verify (4.33). To do this, we proceed just like in [83] and notice that we have (cf. (3.14) therein) the relation

$$
\lim_{k\to\infty} \int\limits_{B_\rho(0)} \Big(1+|H_k|+\lambda_k|\nabla^2\hat{u}|+\lambda_k|\nabla^2 w_k|\Big)^{-\mu}|\nabla^2 w_k|^2\,\mathrm{d}z = 0, \tag{4.34}
$$

where $\rho\in(0,1)$ and

$$
w_k := u_k-\hat{u}. \tag{4.35}
$$

Since the derivation of (4.34) is somewhat lengthy, we postpone its proof to the end and continue to establish (4.33) with its help. Fix $\rho \in (0,1)$ and choose $M \geq 1$: it holds

$$\int\limits_{B_\rho(0)} |\nabla^2 w_k|^2 \,\mathrm{d}z = \int\limits_{\substack{B_\rho(0)\cap \\ \left[\lambda_k|\nabla^2 u_k|\leq M\right]}} |\nabla^2 w_k|^2 \,\mathrm{d}z + \varepsilon_k,$$

$$\varepsilon_k := \int\limits_{\substack{B_\rho(0)\cap \\ \left[\lambda_k|\nabla^2 u_k|> M\right]}} |\nabla^2 w_k|^2 \,\mathrm{d}z.$$

Due to $\nabla^2 \hat{u} \in L^\infty_{\mathrm{loc}}(B_1(0))$ and the boundedness of the sequence H_k, equation (4.34) implies

$$\lim_{k\to\infty} \int\limits_{\substack{B_\rho(0)\cap \\ \left[\lambda_k|\nabla^2 u_k|\leq M\right]}} |\nabla^2 w_k|^2 \,\mathrm{d}z = 0. \tag{4.36}$$

Let

$$\varphi_k := \lambda_k^{-1} \left\{ \big(1 + |H_k + \lambda_k \nabla^2 u_k|\big)^{1-\frac{\mu}{2}} - \big(1 + |H_k|\big)^{1-\frac{\mu}{2}} \right\}.$$

We claim the validity of

$$\sup_k \int\limits_{B_\rho(0)} |\nabla \varphi_k|^2 \,\mathrm{d}z \leq c(\rho) < \infty. \tag{4.37}$$

Accepting this inequality for the moment, we further observe

$$|\varphi_k| = \lambda_k^{-1} \left| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} \big(1 + |H_k + s\lambda_k \nabla^2 u_k|\big)^{1-\frac{\mu}{2}} \,\mathrm{d}s \right| \leq c|\nabla^2 u_k|,$$

so that (4.26) implies the $L^2(B_1(0))$-boundedness of the φ_k and thereby

$$\sup_{k\in\mathbb{N}} \|\varphi_k\|_{W^{1,2}(B_\rho(0))} < \infty. \tag{4.38}$$

By the definition of the φ_k, for $M \geq M_0$ sufficiently large, we have

$$\varphi_k \geq \frac{1}{2}\lambda_k^{-1}\big(\lambda_k|\nabla^2 u_k|\big)^{1-\frac{\mu}{2}} = \frac{1}{2}\lambda_k^{-\frac{\mu}{2}}|\nabla^2 u_k|^{1-\frac{\mu}{2}}$$

on the set $B_\rho(0) \cap \big[\lambda_k|\nabla^2 u_k| > M\big]$, and therefore

$$|\nabla^2 u_k|^2 \leq (2\varphi_k)^{\frac{4}{2-\mu}} \lambda_k^{\frac{2\mu}{2-\mu}} \text{ on } B_\rho(0) \cap \big[\lambda_k|\nabla^2 u_k| > M\big].$$

According to (4.38), $\varphi_k^{\frac{4}{2-\mu}}$ is uniformly integrable and thus

$$\int\limits_{\substack{B_\rho(0)\cap\\ \left[\lambda_k|\nabla^2 u_k|>M\right]}} |\nabla^2 u_k|^2\,\mathrm{d}z \to 0 \quad \text{for } k\to\infty. \tag{4.39}$$

From $\lambda_k|\nabla^2 u_k| \to 0$ a.e. (cf. (4.28)) it further follows that

$$\lim_{k\to\infty} \int\limits_{\substack{B_\rho(0)\cap\\ \left[\lambda_k|\nabla^2 u_k|>M\right]}} |\nabla^2 \hat{u}|\,\mathrm{d}z = 0, \tag{4.40}$$

and (4.39), (4.40) together with (4.36) imply $\nabla^2 w_k \to 0$ in $L^2(B_\rho(0))$, i.e. (4.33) holds, which proves part a) of Theorem 4.3.1 once that the technicalities (4.34) and (4.37) have been established. This is about to follow:

ad (4.37). We start with the Euler equation (4.2), choosing

$$\varphi := \eta^6 D_i(u_\delta - P),$$

where P denotes a polynomial of degree at most two and η is as in (4.3). It holds:

$$\begin{aligned}
&\int\limits_{B_{2R}(x_0)} \eta^6 D^2 F_\delta(\nabla^2 u_\delta)\big(D_i\nabla^2 u_\delta, D_i\nabla^2 u_\delta\big)\,\mathrm{d}x\\
&= -\int\limits_{B_{2R}(x_0)} D^2F_\delta(\nabla^2 u_\delta)\big(D_i\nabla^2 u_\delta, \nabla^2\eta^6 D_i[u_\delta - P] + 2\nabla\eta^6\otimes\nabla D_i(u_\delta - P)\big)\,\mathrm{d}x\\
&\quad + \lambda \int\limits_{B_{2R}(x_0)-D} (u_\delta - f) D_i\big(\eta^6 D_i(u_\delta - P)\big)\,\mathrm{d}x.
\end{aligned}$$

From (4.7) we infer that $u_\delta \in L^\infty_{\mathrm{loc}}(\Omega)$ uniformly. Applying the Cauchy-Schwarz inequality to the first integral on the right-hand side, Young's inequality and using the boundedness of D^2F we get the following estimate, which corresponds to inequality (2.11) in [83]:

$$\begin{aligned}
&\int\limits_{B_{2R}(x_0)} \eta^6 D^2 F_\delta(\nabla^2 u_\delta)\big(D_i\nabla^2 u_\delta, D_i\nabla^2 u_\delta\big)\,\mathrm{d}x\\
&\le c\Bigg\{R^{-4}\int\limits_{B_{2R}(x_0)} |\nabla(u_\delta - P)|^2\,\mathrm{d}x + R^{-2}\int\limits_{B_{2R}(x_0)} |\nabla^2(u_\delta - P)|^2\,\mathrm{d}x\\
&+ \int\limits_{B_{2R}(x_0)} |\nabla^2(u_\delta - P)|\,\mathrm{d}x + R^{-1}\int\limits_{B_{2R}(x_0)} \big|\nabla(u_\delta - P)\big|\,\mathrm{d}x\Bigg\}\\
&\le c\Bigg\{R^{-4}\int\limits_{B_{2R}(x_0)} |\nabla(u_\delta - P)|^2\,\mathrm{d}x + R^{-2}\int\limits_{B_{2R}(x_0)} |\nabla^2(u_\delta - P)|^2\,\mathrm{d}x + R^2\Bigg\}.
\end{aligned} \tag{4.41}$$

Let φ_δ be as in (4.5). Due to (4.16), we may choose a sequence $\delta \downarrow 0$ such that

$$\varphi_\delta \rightharpoonup \hat{\varphi} \text{ in } W^{1,2}_{\mathrm{loc}}(\Omega).$$

for some function $\hat{\varphi} \in W^{1,2}_{\mathrm{loc}}(\Omega)$. Moreover, since $\nabla^2 u_\delta \to \nabla^2 u$ a.e. on Ω we find that

$$\hat{\varphi} = \big(1 + |\nabla^2 u|\big)^{1-\frac{\mu}{2}} \quad \text{a.e. on } \Omega. \tag{4.42}$$

After passing to the limit $\delta \downarrow 0$ in (4.41), the first inequality in the ellipticity estimate (F_μ) implies

$$\int\limits_{B_R(x_0)} |\nabla \hat{\varphi}|^2 \,\mathrm{d}x$$
$$\leq c \left\{ R^{-4} \int\limits_{B_{2R}(x_0)} |\nabla u - \nabla P|^2 \,\mathrm{d}x + R^{-2} \int\limits_{B_{2R}(x_0)} |\nabla^2 u - \nabla^2 P|^2 \,\mathrm{d}x + R^2 \right\}.$$

Now let $\rho \in (0,1)$ be arbitrary. If we choose in (4.3) $\eta \equiv 1$ on $B_{\rho R}(x_0)$, $\operatorname{spt}\eta \subset B_R(x_0)$, etc., then it is clear that, in place of the latter inequality, we obtain

$$\int\limits_{B_{\rho R}(x_0)} |\nabla \hat{\varphi}|^2 \,\mathrm{d}x$$
$$\leq c(\rho) \left\{ R^{-4} \int\limits_{B_R(x_0)} |\nabla u - \nabla P|^2 \,\mathrm{d}x + R^{-2} \int\limits_{B_R(x_0)} |\nabla^2 u - \nabla^2 P|^2 \,\mathrm{d}x + R^2 \right\}. \tag{4.43}$$

By (4.42) together with (4.25) and $\nabla^3 u_k(z) = r_k^2 \lambda_k^{-2} \nabla^3 u(x_k + r_k z)$ it is not difficult to verify the relation

$$\int\limits_{B_\rho(0)} |\nabla \varphi_k|^2 \,\mathrm{d}z \leq c\lambda_k^{-2} \int\limits_{B_{\rho r_k}(x_k)} |\nabla \hat{\varphi}|^2 \,\mathrm{d}x.$$

Hence (4.43) yields

$$\int\limits_{B_\rho(0)} |\nabla \varphi_k|^2 \,\mathrm{d}z \leq c(\rho)\lambda_k^{-2} \Bigg\{ r_k^{-2} \int\limits_{B_1(0)} |\nabla u(x_k + r_k z) - \nabla P(x_k + r_k z)|^2 \,\mathrm{d}x$$
$$+ \int\limits_{B_1(0)} |\nabla^2 u(x_k + r_k z) - \nabla^2 P(x_k + r_k z)|^2 \,\mathrm{d}x + r_k^2 \Bigg\},$$

which holds for all $k \in \mathbb{N}$. If we now choose (for $k \in \mathbb{N}$ fixed) $P(x) = P_k(x)$, with

$$P_k(x) = r_k A_k \cdot \frac{x - x_k}{r_k} - a_k - \frac{1}{2} r_k^2 H_k\Big(\frac{x - x_k}{r_k}, \frac{x - x_k}{r_k}\Big) + \frac{1}{2} r_k^2 \fint_{B_1(0)} H_k(y,y)dy \tag{4.44}$$

where A_k, a_k and H_k are as in (4.23), we obtain the estimate

$$\int\limits_{B_\rho(0)} |\nabla\varphi_k|^2 \,\mathrm{d}z \le c(\rho)\lambda_k^{-2}\Bigg\{\lambda_k^2 \int\limits_{B_1(0)} |\nabla u_k(z)|^2 \,\mathrm{d}z + \lambda_k^2 \int\limits_{B_1(0)} |\nabla^2 u_k(z)|^2 \,\mathrm{d}z + r_k^2\Bigg\},$$

and (4.26) together with $\lambda_k^{-2} r_k^2 \to 0$ now implies that the right-hand side is bounded uniformly in k, which proves (4.37).

ad (4.34). Fix a cut-off function $\eta \in C_0^\infty(B_1(0))$, $0 \le \eta \le 1$. Noting that the function

$$\mathbb{R} \ni s \mapsto \int\limits_{B_1(0)} \eta F\big(H_k + \lambda_k \nabla^2 \hat{u} + s\lambda_k \nabla^2 w_k\big) \,\mathrm{d}x$$

is twice continuously differentiable on $\mathbb{R}$, an application of the Taylor formula with expansion point $s = 0$ yields:

$$\begin{aligned}
\lambda_k^{-2} \int\limits_{B_1(0)} & \eta \Big[F(H_k + \lambda_k \nabla^2 u_k) - F(H_k + \lambda_k \nabla^2 \hat{u})\Big] \,\mathrm{d}z \\
&\qquad - \lambda_k^{-1} \int\limits_{B_1(0)} \eta DF(H_k + \lambda_k \nabla^2 \hat{u}) : \nabla^2 w_k \,\mathrm{d}z \\
= & \underbrace{\int\limits_{B_1(0)} \int\limits_0^1 \eta D^2 F\big(H_k + \lambda_k \nabla^2 \hat{u} + s\lambda_k \nabla^2 w_k\big)(\nabla^2 w_k, \nabla^2 w_k)(1-s) \,\mathrm{d}s \,\mathrm{d}z}_{=: R(1)}. \qquad (4.45)
\end{aligned}$$

Due to (F_μ), we can estimate the remainder $R(1)$ by

$$R(1) \ge c \int\limits_{B_1(0)} \int\limits_0^1 \eta\big(1 + |H_k + \lambda_k D^2 \hat{u} + s\lambda_k D^2 w_k|\big)^{-\mu} |D^2 w_k|^2 (1-s) \,\mathrm{d}s \,\mathrm{d}z.$$

Consequently,

$$R(1) \ge \frac{c}{2} \int\limits_{B_1(0)} \eta\big(1 + |H_k| + \lambda_k |\nabla^2 \hat{u}| + \lambda_k |\nabla^2 w_k|^2\big)^{-\mu} |\nabla^2 w_k|^2 \,\mathrm{d}z. \qquad (4.46)$$

and we see that, for proving (4.34), it will suffice to show that the left-hand side of (4.45) converges to zero as $k \to \infty$ (note that $R(1)$ is obviously nonnegative). We make use of the convexity of F to obtain

$$\begin{aligned}
\eta F\Big(H_k + \lambda_k \nabla^2 u_k\Big) - \eta F\Big(H_k + \lambda_k \nabla^2 \hat{u}\Big) &= \eta F\Big(H_k + \lambda_k \nabla^2 u_k\Big) \\
-\Big\{\eta F\Big(H_k + \lambda_k \nabla^2 \hat{u}\Big) + (1-\eta) F\Big(H_k + \lambda_k \nabla^2 u_k\Big)\Big\} &+ (1-\eta) F\Big(H_k + \lambda_k \nabla^2 u_k\Big) \\
&\le \eta F\Big(H_k + \lambda_k \nabla^2 u_k\Big) + (1-\eta) F\Big(H_k + \lambda_k \nabla^2 u_k\Big) \\
&\quad - F\Big(H_k + \eta\lambda_k \nabla^2 \hat{u} + (1-\eta)\lambda_k \nabla^2 u_k\Big) \\
&= F\Big(H_k + \lambda_k \nabla^2 u_k\Big) - F\Big(H_k + \lambda_k\Big[\eta \nabla^2 \hat{u} + (1-\eta)\nabla^2 u_k\Big]\Big),
\end{aligned}$$

hence:

$$\left.\begin{aligned}
&\text{l.h.s. of (4.45)}\\
&\leq \lambda_k^{-2} \int\limits_{B_1(0)} F(H_k + \lambda_k \nabla^2 u_k)\,\mathrm{d}z\\
&- \lambda_k^{-2} \int\limits_{B_1(0)} F\left(H_k + \lambda_k \left[\eta \nabla^2 \hat{u} + (1-\eta)\nabla^2 u_k\right]\right)\,\mathrm{d}z\\
&- \lambda_k^{-1} \int\limits_{B_1(0)} \eta DF(H_k + \lambda_k \nabla^2 \hat{u}) : \nabla^2 w_k \,\mathrm{d}z =: I_1 - I_2 - I_3.
\end{aligned}\right\} \tag{4.47}$$

The I-minimality of u implies that for all $v \in W^{2,1}(\Omega)$, which satisfy $\operatorname{spt}(u - v) \Subset B_{r_k}(x_k)$, it holds

$$\begin{aligned}
I_1 =&\lambda_k^{-2} \int\limits_{B_1(0)} F\big(\nabla^2 u(r_k z + x_k)\big)\,\mathrm{d}z = \lambda_k^{-2} r_k^{-2} \int\limits_{B_{r_k}(x_k)} F(\nabla^2 u)\,\mathrm{d}x\\
\leq&\lambda_k^{-2} r_k^{-2} \Bigg\{ \int\limits_{B_{r_k}(x_k)} F(\nabla^2 v)\,\mathrm{d}x + \frac{\lambda}{2} \int\limits_{B_{r_k}(x_k)-D} |v-f|^2\,\mathrm{d}x\\
&- \frac{\lambda}{2} \int\limits_{B_{r_k}(x_k)-D} |u-f|^2\,\mathrm{d}x \Bigg\}.
\end{aligned} \tag{4.48}$$

Now we set

$$v_k(z) := u_k(z) + \eta(z)(\hat{u} - u_k)(z),\ z \in B_1(0),$$

and define

$$\tilde{v}_k(z) := \lambda_k r_k^2 v_k(z) + P_k(z),$$

where $P_k(z)$ is as in (4.44). Finally, we define

$$\hat{v}_k(x) := \tilde{v}_k\left(\frac{x - x_k}{r_k}\right).$$

Then $\operatorname{spt}\big(u - \hat{v}_k\big) \Subset B_{r_k}(x_k)$, and

$$\begin{aligned}
\nabla^2 \hat{v}_k(x) &= r_k^{-2} \nabla^2 \tilde{v}_k \left(\frac{x - x_k}{r_k}\right)\\
&= \lambda_k \nabla^2 v_k \left(\frac{x - x_k}{r_k}\right) + \nabla^2 P_k \left(\frac{x - x_k}{r_k}\right) r_k^{-2}\\
&= \lambda_k \nabla^2 \Big\{ u_k + \eta(\hat{u} - u_k) \Big\} \left(\frac{x - x_k}{r_k}\right) + H_k,
\end{aligned}$$

which means that

$$
\begin{aligned}
&\lambda_k^{-2} r_k^{-2} \int\limits_{B_{r_k}(x_k)} F(\nabla^2 \hat{v}_k)\,\mathrm{d}x \\
&= \lambda_k^{-2} \int\limits_{B_1(0)} F\big(H_k + \lambda_k \nabla^2 \{u_k + \eta(\hat{u} - u_k)\}\big)\,\mathrm{d}z.
\end{aligned}
$$

Going back to (4.48), we infer that

$$
\begin{aligned}
I_1 \le \overbrace{\lambda_k^{-2} \int\limits_{B_1(0)} F\big(H_k + \lambda_k \nabla^2 \{u_k + \eta(\hat{u} - u_k)\}\big)\,\mathrm{d}z}^{=:\, \widetilde{I}_1} \\
+ \frac{\lambda}{2} \lambda_k^{-2} r_k^{-2} \int\limits_{B_{r_k}(x_k) - D} \big(|f - \hat{v}_k|^2 - |f - u|^2\big)\,\mathrm{d}x.
\end{aligned} \tag{4.49}
$$

Using that f is in $L^\infty(\Omega - D)$, we can estimate the second term on the right-hand side of (4.49) by

$$
\begin{aligned}
&\lambda_k^{-2} r_k^{-2} \int\limits_{B_{r_k}(x_k) - D} \big(|f - \hat{v}_k|^2 - |f - u|^2\big)\,\mathrm{d}x \\
&\le c \lambda_k^{-2} r_k^{-2} \int\limits_{B_{r_k}(x_k)} |\hat{v}_k - u| \underbrace{(1 + |u| + |\hat{v}_k|)}_{\text{locally bounded on } \Omega}\,\mathrm{d}x \\
&\le c \lambda_k^{-2} r_k^{-2} \int\limits_{B_{r_k}(x_k)} |\hat{v}_k - u|\,\mathrm{d}x \\
&= c \lambda_k^{-2} \int\limits_{B_1(0)} |\tilde{v}_k(z) - u(x_k + r_k z)|\,\mathrm{d}z \\
&= c \lambda_k^{-2} \int\limits_{B_1(0)} |\lambda_k r_k^2 v_k(z) + P_k(z) - u(x_k + r_k z)|\,\mathrm{d}z \\
&= c \lambda_k^{-1} r_k^2 \int\limits_{B_1(0)} \left| v_k(z) - r_k^{-2} \lambda_k^{-1} \Big(\big(u(x_k + r_k z) - P_k(z)\big) \Big) \right| \mathrm{d}z \\
&= c \lambda_k^{-1} r_k^2 \int\limits_{B_1(0)} |v_k - u_k|\,\mathrm{d}z \\
&= c \lambda_k^{-1} r_k^2 \int\limits_{B_1(0)} \eta |\hat{u} - u_k|\,\mathrm{d}z \\
&\le c \lambda_k^{-1} r_k^2 =: \varepsilon_k \to 0 \text{ as } k \to \infty,
\end{aligned}
$$

where the last estimate uses (4.27). The above estimate combined with (4.47) and (4.49) now implies

$$\text{l.h.s. of (4.45)} \le \widetilde{I}_1 - I_2 - I_3 + \frac{\lambda}{2}\varepsilon_k. \tag{4.50}$$

Starting from (4.50), we can basically follow the arguments in [83], p. 209: let $X_k := H_k + \lambda_k\big[(1-\eta)\nabla^2 u_k + \eta\nabla^2\hat{u}\big]$, $Z_k := 2\nabla\eta\otimes\nabla(\hat{u}-u_k) + \nabla^2\eta(\hat{u}-u_k)$. Observing that

$$X_k + \lambda_k Z_k = H_k + \lambda_k\nabla^2\big\{u_k + \eta(\hat{u}-u_k)\big\},$$

an application of Taylor's formula yields

$$\begin{aligned}\widetilde{I}_1 - I_2 = \lambda_k^{-1} &\int\limits_{B_1(0)} DF(X_k) : Z_k \,\mathrm{d}z \\ &+ \int\limits_{B_1(0)}\int\limits_0^1 D^2F\big(X_k + s\lambda_k Z_k\big)(Z_k, Z_k)(1-s)\,\mathrm{d}s\,\mathrm{d}z.\end{aligned}$$

Using the boundedness of D^2F, this implies

$$\widetilde{I}_1 - I_2 \le \lambda_k^{-1}\int\limits_{B_1(0)} DF(X_k) : Z_k\,\mathrm{d}z + c\int\limits_{B_1(0)} |Z_k|^2\,\mathrm{d}z.$$

By (4.27) and the Rellich-Kondrachov embedding theorem, we may assume that $u_k \to \hat{u}$ in $W^{1,2}(B_1(0))$ which implies $c\int_{B_1(0)}|Z_k|^2\,\mathrm{d}z \to 0$. Hence (4.50) yields

$$\left.\begin{aligned}&\text{l.h.s. of (4.45)}\\ &\le \lambda_k^{-1}\int\limits_{B_1(0)} DF(X_k) : Z_k\,\mathrm{d}z\\ &-\lambda_k^{-1}\int\limits_{B_1(0)} \eta DF(H_k + \lambda_k\nabla^2\hat{u}) : \nabla^2 w_k\,\mathrm{d}z + \varepsilon_k\end{aligned}\right\} \tag{4.51}$$

$$\begin{aligned}&= \lambda_k^{-1}\int\limits_{B_1(0)} \Big(DF(X_k) - DF\big(H_k + \lambda_k\nabla^2\hat{u}\big)\Big) : Z_k\,\mathrm{d}z\\ &+ \lambda_k^{-1}\int\limits_{B_1(0)} DF(H_k + \lambda_k\nabla^2\hat{u}) : \nabla^2(\eta w_k)\,\mathrm{d}z + \varepsilon_k\\ &=: \lambda_k^{-1}I_4 + \lambda_k^{-1}I_5 + \varepsilon_k,\end{aligned}$$

with $\varepsilon_k \to 0$ as $k\to\infty$. Using the ideas from [83], p. 211, the integral I_4 can be estimated by

$$\begin{aligned}\lambda_k^{-1}I_4 &= \int\limits_{B_1(0)}\int\limits_0^1 D^2F(H_k + \lambda_k\nabla^2\hat{u} + s\lambda_k(1-\eta)\nabla^2 w_k)(\nabla^2 w_k, Z_k)(1-\eta)\,\mathrm{d}s\,\mathrm{d}z\\ &\underset{\substack{\text{Hölder}\\ +(\mathrm{F}_\mu)}}{\le} c\ \underbrace{\|\nabla^2 w_k\|_{2;B_1(0)}}_{\text{bounded by (4.27)}} \cdot \underbrace{\|Z_k\|_{2;B_1(0)}}_{\to 0 \text{ by (4.27)}} < \infty.\end{aligned}$$

Moreover, we have

$$
\begin{aligned}
\lambda_k^{-1} I_5 &= \lambda_k^{-1} \int\limits_{B_1(0)} \int\limits_0^1 D^2F\big(H_k + s\lambda_k \nabla^2 \hat{u}\big)\big(\lambda_k \nabla^2 \hat{u}, \nabla^2(\eta w_k)\big)\, \mathrm{d}s\, \mathrm{d}z \\
&= \int\limits_{B_1(0)} \int\limits_0^1 D^2F\big(H_k + s\lambda_k \nabla^2 \hat{u}\big)\big(\nabla^2 \hat{u}, \nabla^2(\eta w_k)\big)\, \mathrm{d}s\, \mathrm{d}z \\
&\longrightarrow 0 \text{ for } k \to \infty,
\end{aligned}
$$

since $\nabla^2(\eta w_k) \rightharpoondown 0$ in L^2 by (4.27), and consequently

$$
\text{r.h.s. of (4.51)} \longrightarrow 0 \quad \text{for } k \to \infty,
$$

which finally proves (4.34).

The next step in the proof of Theorem 4.3.1 is to verify the assertion on the Hausdorff-dimension of the singular set $\Omega - \Omega_0$ given in part b). We therefore notice that by Remark 4.3.2 a) we have

$$
\begin{aligned}
&\Omega - \Omega_0 \\
&= \underbrace{\Big\{x \in \Omega \,:\, \limsup_{r \to 0} |(\nabla^2 u)_{x,r}| = \infty\Big\}}_{=:\,\Omega_1} \cup \underbrace{\Big\{x \in \Omega \,:\, \liminf_{r \to 0} E(x,r) > 0\Big\}}_{=:\,\Omega_2}.
\end{aligned}
$$

Since $\nabla^2 u \in W^{1,s}_{\mathrm{loc}}(\Omega)$ for all $s < 2$, it follows from [46], Theorem 2.1 on p. 100 that

$$
\mathcal{H}^{2-s+\kappa}(\Omega_1) = 0 \quad \forall \kappa > 0
$$

and hence $\mathcal{H}^{\varepsilon}(\Omega_1) = 0$ for $\varepsilon > 0$ arbitrarily small. Further, by the Sobolev-Poincaré inequality it holds

$$
E(x,r)^{\frac{1}{2}} \le c r \fint_{B_r(x)} |\nabla^3 u| dy \le c \left(r^{s-2} \int\limits_{B_r(x)} |\nabla^3 u|^s dy \right)^{\frac{1}{s}},
$$

and Theorem 2.2 on p. 101 [46] now states that $|\nabla^3 u|^s \in L^1_{\mathrm{loc}}(\Omega)$ in combination with the above estimate yields $\mathcal{H}^{2-s}(\Omega_2) = 0$, and $\mathcal{H}^{\varepsilon}(\Omega_2) = 0$ for any $\varepsilon > 0$ which concludes the proof of Theorem 4.3.1 b). It remains to give the proof of Theorem 4.3.1 c): Let $\sigma_\delta := DF_\delta(\nabla^2 u_\delta)$. By an application of the Cauchy-Schwarz inequality to the bilinear form $D^2F_\delta(\cdot,\cdot)$, we have

$$
\begin{aligned}
D_i\sigma_\delta : D_i\sigma_\delta &= D^2F_\delta(\nabla^2 u_\delta)\big(D_i \nabla^2 u_\delta, D_i \sigma_\delta\big) \\
&\le D^2F_\delta(\nabla^2 u_\delta)\big(D_i \nabla^2 u_\delta, D_i \nabla^2 u_\delta\big)^{\frac{1}{2}} D^2F_\delta(\nabla^2 u_\delta)\big(D_i\sigma_\delta, D_i\sigma_\delta\big)^{\frac{1}{2}},
\end{aligned}
$$

so that $|\nabla\sigma_\delta|^2 \le \Theta_\delta \big|D^2F_\delta(\nabla^2 u_\delta)\big|^{\frac{1}{2}} |\nabla\sigma_\delta|$ where we use the shorthand notation

$$\Theta_\delta := D^2F_\delta(\nabla^2 u_\delta)\big(D_i\nabla^2 u_\delta, D_i\nabla^2 u_\delta\big)^{\frac{1}{2}}.$$

According to inequality (4.14), we know that

$$\Theta_\delta \in L^2_{\mathrm{loc}}(\Omega) \quad \text{uniformly in } \delta.$$

By the condition of μ-ellipticity (F_μ) it further holds $\big|D^2F_\delta(Z)\big| \le c(1+\delta)$ and thus

$$|\nabla\sigma_\delta| \le c\sqrt{1+\delta}\,\Theta_\delta,$$

which implies $\sigma_\delta \in W^{1,2}_{\mathrm{loc}}(\Omega)$ uniformly (in δ) and there exists a function $\overline{\sigma} \in W^{1,2}_{\mathrm{loc}}(\Omega)$ such that

$$\sigma_\delta \rightharpoonup \overline{\sigma} \quad \text{in } W^{1,2}_{\mathrm{loc}}(\Omega)$$

for a suitable subsequence $\delta \downarrow 0$. But then it must hold $\overline{\sigma} = \sigma = DF(\nabla^2 u)$, since (due to $u_\delta \in W^{3,s}_{\mathrm{loc}}(\Omega)$ uniformly for $s < 2$) $\nabla^2 u_\delta \to \nabla^2 u$ a.e. and DF is continuous. This verifies part c) and thereby finishes the proof of Theorem 4.3.1. $\square$

Part III

An Alternative Approach to the Higher-Order Case

Chapter 5

Coupling Models: Relaxed and Dual Solutions

In the third and last part of this thesis we will have a look at a certain class of coupled variational problems which represent an alternative approach to the higher-order case. In this respect, the work [23] serves as a model and we are concerned with transferring the theory developed therein for superlinear densities to our linear-growth case. Thereby, we make use of the techniques from [36] and [37] which will allow us to prove existence as well as (partial) Hölder regularity of both generalized and dual solutions.

More precisely, the underlying idea is the following: for a reduction of the differentiability order of the functional

$$I[u] := \int_\Omega F(\nabla^m u)\,\mathrm{d}x + \lambda \int_{\Omega - D} \phi\big(|u-f|\big)\,\mathrm{d}x,\ u \in W^{m,1}(\Omega)$$

from (V), we introduce a vector-valued variable $v \in \mathbb{R}^n$ along with the additional term

$$\int_\Omega |\nabla u - v|\,\mathrm{d}x$$

and then minimize the bivariate functional

$$I_1(u,v) := \int_\Omega F(\nabla^{m-1} v)\,\mathrm{d}x + \int_\Omega |\nabla u - v|\,\mathrm{d}x + \lambda \int_{\Omega - D} \phi\big(|u-f|\big)\,\mathrm{d}x$$

in the class $W^{1,1}(\Omega) \times W^{m-1,1}(\Omega, \mathbb{R}^n)$. Minimizing the middle term, the so called "*coupling term*", entails "$v \approx \nabla u$", so that "$\nabla^{m-1} v \approx \nabla^m u$". We have thereby reduced the order of our model by one and it is clear that an iteration of this procedure, introducing $m-1$ further coupling terms, leads to a functional which involves only first derivatives:

$$I_m(u, v_1, ..., v_{m-1}) := \int\limits_\Omega F(\nabla v_{m-1})\,\mathrm{d}x + \int\limits_\Omega |\nabla v_{m-2} - v_{m-1}|\,\mathrm{d}x$$
$$+ \int\limits_\Omega |\nabla v_{m-3} - v_{m-2}|\,\mathrm{d}x + ... + \int\limits_\Omega |\nabla u - v_1|\,\mathrm{d}x + \lambda \int\limits_{\Omega - D} \phi\big(|u - f|\big)\,\mathrm{d}x,$$

which is to be minimized in the product space

$$(u, v_1, ..., v_{m-1}) \in W^{1,1}(\Omega) \times W^{1,1}(\Omega, \mathbb{R}^n) \times ... \times W^{1,1}(\Omega, \mathbb{R}^{n^{m-1}}).$$

Of course, every further iteration of the coupling procedure may possibly lead to a degradation of the quality in which u approximates the solution of problem (V). However, the reduction of the order is extremely advantageous when it comes to numerical computations.

Let us continue with a few words on the general assumptions of this part. As before, $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, $D \subset \Omega$ is a (possibly empty) measurable subset, where now we do not only require $\mathcal{L}^n(\Omega - D) > 0$ but that even $\Omega - \overline{D} \neq \emptyset$. The data function $f : \Omega - D \to \mathbb{R}$ is supposed to be a bounded and measurable real-valued function, i.e.

$$f \in L^\infty(\Omega - D).$$

For simplicity, we will assume $n = 2$ and choose the quadratic data term $\phi(t) = t^2$. However, we would like to emphasize that the results of this chapter which concern the existence of generalized and dual solutions remain true in arbitrary dimensions $n \geq 2$ and even for vector valued data $f \in L^\infty(\Omega - D, \mathbb{R}^N)$ in combination with general penalty terms $\int_{\Omega - D} \phi\big(|u - f|\big)\,\mathrm{d}x$ ($\phi : [0, \infty) \to \mathbb{R}$ satisfying $(\phi 1)$ and $(\phi 2)$ for some $q \in [1, 2]$). Note, however, that this is *not* true for the regularity results of Chapter 6!

Let $F : \mathbb{R}^{2\times 2} \to \mathbb{R}$ and $G : \mathbb{R}^2 \to \mathbb{R}$ be strictly convex functions, satisfying the following set of conditions (where c denotes a generic positive constant):

(F1) $F \in C^2(\mathbb{R}^{2\times 2})$, $F(-p) = F(p)$, $F(0) = 0$, $|\nabla F(p)| \leq c \;\forall p \in \mathbb{R}^{2\times 2}$,

(F2) $0 < D^2F(p)(q, q) \leq c\dfrac{1}{1 + |p|}|q|^2$ for all $p, q \in \mathbb{R}^{2\times 2}$,

(F3) $c_1|p| - c_2 \leq F(p)$ for some $c_1 > 0$, $c_2 \in \mathbb{R}$ and all $p \in \mathbb{R}^{2\times 2}$.

(G1) $G \in C^2(\mathbb{R}^2)$, $G(-y) = G(y)$, $G(0) = 0$, $|\nabla G(y)| \leq c \;\forall y \in \mathbb{R}^2$,

(G2) $0 < D^2G(y)(x, x) \leq c\dfrac{1}{1 + |y|}|x|^2$ for all $x, y \in \mathbb{R}^2$,

(G3) $c_1|y| - c_2 \leq G(y)$ for some $c_1 > 0$, $c_2 \in \mathbb{R}$ and all $y \in \mathbb{R}^2$.

We further use the short-hand notation

$$\mathcal{V} := W^{1,1}(\Omega) \times W^{1,1}(\Omega, \mathbb{R}^2). \tag{5.1}$$

The underlying problem then reads as

$$E(u,v) := \alpha \int_\Omega F(\nabla v)\,\mathrm{d}x + \beta \int_\Omega G(\nabla u - v)\,\mathrm{d}x + \int_{\Omega - D} (u-f)^2\,\mathrm{d}x \\ \to \min \text{ in } \mathcal{V}, \tag{$\mathcal{P}$}$$

where α and β are positive parameters which control the balance between the "leading term" $\int_\Omega F(\nabla v)\,\mathrm{d}x$ and the "coupling term" $\int_\Omega G(\nabla u - v)\,\mathrm{d}x$.

Of course, as for the problem (V), we can in general not expect to find a solution of $(\mathcal{P})$ in the nonreflexive function class $\mathcal{V}$ and therefore have to consider suitably relaxed variants of the above problem.

5.1 Relaxation in $BV(\Omega) \times BV(\Omega, \mathbb{R}^2)$

The first method is the relaxation of $(\mathcal{P})$ in the space $BV(\Omega) \times BV(\Omega, \mathbb{R}^2)$ using the concept of convex functions of a measure (see Appendix B). This means that we replace E with the functional

$$\widetilde{E}(u,v) = \alpha \int_\Omega F(\nabla v) + \beta \int_\Omega G(\nabla u - v \cdot \mathcal{L}^2) + \int_{\Omega - D} (u-f)^2\,\mathrm{d}x$$

and look for solutions of

$$\widetilde{E}(u,v) \to \min \text{ in } \widetilde{\mathcal{V}} := BV(\Omega) \times BV(\Omega, \mathbb{R}^2), \tag{$\widetilde{\mathcal{P}}$}$$

where

$$\int_\Omega F(\nabla v) := \int_\Omega F(\nabla_a v)\,\mathrm{d}x + \int_\Omega F^\infty\left(\frac{\nabla_s v}{|\nabla_s v|}\right)\mathrm{d}|\nabla_s v|,$$

and

$$\int_\Omega G(\nabla u - v) := \int_\Omega G(\nabla_a u - v)\,\mathrm{d}x + \int_\Omega G^\infty\left(\frac{\nabla_s u}{|\nabla_s u|}\right)\mathrm{d}|\nabla_s u|.$$

The next theorem gathers our results concerning the existence of solutions and their connection to minimizing sequences of the primal problem $(\mathcal{P})$ (cf. Theorem 2.1.1):

Theorem 5.1.1
Under our general assumptions regarding Ω, D, f, F and G from the introduction of Part III it holds:

a) Problem $(\widetilde{\mathcal{P}})$ has at least one solution $(u,v) \in \widetilde{\mathcal{V}} = BV(\Omega) \times BV(\Omega, \mathbb{R}^2)$.

b) If (u,v) and $(\widetilde{u}, \widetilde{v})$ are two distinct solutions of $(\widetilde{\mathcal{P}})$, then

$$u = \widetilde{u} \text{ a.e. on } \Omega - D,\ \nabla_a u - v = \nabla_a \widetilde{u} - \widetilde{v} \text{ a.e. on } \Omega$$
$$\text{and } \nabla_a v = \nabla_a \widetilde{v} \text{ a.e. on } \Omega.$$

In particular, if $D = \emptyset$, i.e. in the case of pure denoising, the solution of $(\widetilde{\mathcal{P}})$ is unique.

c) It holds $\inf\limits_{\widetilde{\mathcal{V}}} \widetilde{E} = \inf\limits_{\mathcal{V}} E$.

d) The set $\mathcal{M} \subset BV(\Omega) \times BV(\Omega, \mathbb{R}^2)$ of all solutions of $(\widetilde{\mathcal{P}})$ coincides with the set of all $L^1(\Omega) \times L^1(\Omega, \mathbb{R}^2)$-cluster points of E-minimizing sequences in $W^{1,1}(\Omega) \times W^{1,1}(\Omega, \mathbb{R}^2)$. If E admits a minimizer (u,v) in the Sobolev class $\mathcal{V}$, then $\mathcal{M} = \big\{(u,v)\big\}$.

Proof of Theorem 5.1.1. A key tool is the following adaption of the density result from Section 1.2 (cf. also Lemma 2.2 in [63]):

Lemma 5.1.1
Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Given $(u,v) \in BV(\Omega) \times BV(\Omega, \mathbb{R}^2)$, there is a sequence $(\varphi_k, \psi_k) \subset C^\infty(\overline{\Omega}) \times C^\infty(\overline{\Omega}, \mathbb{R}^2)$ such that

$$\varphi_k \to u \text{ in } L^2(\Omega), \tag{5.2}$$
$$\psi_k \to v \text{ in } L^2(\Omega, \mathbb{R}^2), \tag{5.3}$$
$$\int_\Omega \sqrt{1 + |\nabla \psi_k|^2}\, \mathrm{d}x \to \int_\Omega \sqrt{1 + |\nabla v|^2}, \tag{5.4}$$
$$\int_\Omega \sqrt{1 + |\nabla \varphi_k - \psi_k|^2}\, \mathrm{d}x \to \int_\Omega \sqrt{1 + |\nabla u - v \cdot \mathcal{L}^2|^2}. \tag{5.5}$$

Proof of the Lemma. First, we note that the existence of a sequence ψ_k with the properties (5.3) and (5.4) follows directly from Lemma 1.1.1 (note that due to $\Omega \subset \mathbb{R}^2$ it holds $u \in L^2(\Omega)$, $v \in L^2(\Omega, \mathbb{R}^2)$ thanks to embedding theorems, and since the approximating sequence ψ_k is obtained via mollification, it particularly follows $\psi_k \to v$ in $L^2(\Omega, \mathbb{R}^2)$). Let $S : C_0^\infty(\Omega) \times C_0^\infty(\Omega, \mathbb{R}^2) \to C_0^\infty(\Omega, \mathbb{R}^2)$ be defined by

$$S(\eta, \vartheta) := \nabla \eta - \vartheta.$$

Then S is a first-order differential operator with constant coefficients. Moreover, it is easy to verify that S satisfies the conditions of Theorem 2.2 in [54] which yields the existence of a sequence $(\varphi_k, \widetilde{\psi}_k) \subset C^\infty(\overline{\Omega}) \times C^\infty(\overline{\Omega}, \mathbb{R}^2)$ such that

$$\begin{aligned}
&(\varphi_k, \widetilde{\psi}_k) \to (u, v) \text{ in } L^2(\Omega) \times L^2(\Omega, \mathbb{R}^2) \quad \text{ as well as} \\
&\int\limits_\Omega \sqrt{1 + |S(\varphi_k, \widetilde{\psi}_k)|^2} \,\mathrm{d}x \to \int\limits_\Omega \sqrt{1 + |S(u, v)|^2} = \int\limits_\Omega \sqrt{1 + |\nabla u - v \cdot \mathcal{L}^2|^2}.
\end{aligned} \tag{5.6}$$

Furthermore,

$$\left| \int\limits_\Omega \sqrt{1 + |S(\varphi_k, \widetilde{\psi}_k)|^2} \,\mathrm{d}x - \int\limits_\Omega \sqrt{1 + |S(\varphi_k, \psi_k)|^2} \,\mathrm{d}x \right| \leq \int\limits_\Omega |\widetilde{\psi}_n - \psi_n| \,\mathrm{d}x \to 0,$$

since $\widetilde{\psi}_k$, ψ_k both converge to v in $L^1(\Omega, \mathbb{R}^2)$ which, together with (5.6), proves that (φ_k, ψ_k) approximates (u, v) as claimed. □

We continue with the proof of Theorem 5.1.1. Ad a). Let $(u_k, v_k) \in BV(\Omega) \times BV(\Omega, \mathbb{R}^2)$ denote an $\widetilde{E}$ minimizing sequence. By Lemma 5.1.1 in combination with Reshetnyak's continuity theorem (see Theorem B.1 in Appendix B) we may assume that $(u_k, v_k) \in W^{1,1}(\Omega) \times W^{1,1}(\Omega, \mathbb{R}^2)$. Thanks to the linear growth of F and G, it is clear that there are constants $M_1, M_2, M_3 > 0$ such that

$$\sup_{k \in \mathbb{N}} \int\limits_\Omega |\nabla v_k| \,\mathrm{d}x \leq M_1, \tag{5.7}$$

$$\sup_{k \in \mathbb{N}} \int\limits_\Omega |\nabla u_k - v_k| \,\mathrm{d}x \leq M_2 \tag{5.8}$$

$$\text{and } \sup_{k \in \mathbb{N}} \int\limits_{\Omega - D} |u_k| \,\mathrm{d}x \leq M_3. \tag{5.9}$$

Now we need the following variant of Poincaré's inequality (cf. Lemma 4.2 in [23]):

Lemma 5.1.2
Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary and $\rho \in C_0^1(\Omega)$ with $\int_\Omega \rho \,\mathrm{d}x = 1$. Then there is a constant $c > 0$ which only depends on Ω and ρ, such that for any function $u \in W^{1,1}(\Omega)$ it holds

$$\left\| u - \int_\Omega \rho u \,\mathrm{d}x \right\|_{1;\Omega} \leq c \|\nabla u\|_{1;\Omega}.$$

Continuing with the proof of Theorem 5.1.1, we choose ρ as in the Lemma and such that $\operatorname{spt} \rho \subset \Omega - D$ (note that this is possible due to our assumption $\Omega - \overline{D} \neq \emptyset$). Then we have:

$$\sup_{k \in \mathbb{N}} \left| \int\limits_\Omega \rho \nabla u_k \,\mathrm{d}x - \int\limits_\Omega \rho v_k \,\mathrm{d}x \right| \leq \|\rho\|_\infty \sup_{k \in \mathbb{N}} \int\limits_\Omega |\nabla u_k - v_k| \,\mathrm{d}x \leq \|\rho\|_\infty M_2. \tag{5.10}$$

Furthermore:

$$\left|\int\limits_{\Omega} \rho \nabla u_k \, \mathrm{d}x\right| = \left|\int\limits_{\Omega} \nabla \rho u_k \, \mathrm{d}x\right| \leq \|\nabla \rho\|_\infty \int\limits_{\Omega - D} |u_k| \, \mathrm{d}x \leq \|\nabla \rho\|_\infty M_3. \qquad (5.11)$$

Thus, from (5.10) and (5.11) we infer

$$\sup_{k \in \mathbb{N}} \left|\int\limits_{\Omega} \rho v_k \, \mathrm{d}x\right| < \infty$$

and (5.7) together with Lemma 5.1.2 therefore implies

$$\sup_{k \in \mathbb{N}} \|v_k\|_{1,1;\Omega} < \infty.$$

But then the boundedness of v_k in $L^1(\Omega, \mathbb{R}^2)$ along with (5.8) and (5.9) gives (by another application of Poincaré's inequality):

$$\sup_{k \in \mathbb{N}} \|u_k\|_{1,1;\Omega} < \infty,$$

so that (u_k, v_k) is indeed bounded in $BV(\Omega) \times BV(\Omega, \mathbb{R}^2)$. By the BV-compactness theorem, there exists $(u, v) \in BV(\Omega) \times BV(\Omega, \mathbb{R}^2)$ such that

$$(u_k, v_k) \to (u, v) \text{ in } L^1(\Omega) \times L^1(\Omega, \mathbb{R}^2) \text{ and } a.e..$$

That (u, v) is indeed $\widetilde{E}$-minimal is now immediate since the relaxation $\widetilde{E}$ is lower semicontinuous with respect to weak-$*$ convergence of measures by Reshetnyak's theorem (see Appendix B, Theorem B.1 a)).

The statements of part b) are a mere consequence of the strict convexity of the functions F, G and the quadratic fidelity term.

For the equality of the infima of E and $\widetilde{E}$ that is stated in c), we first note that $\inf \widetilde{E} \leq \inf E$ is clear from $E = \widetilde{E}$ on $\mathcal{V}$. For the opposite inequality, let $(u_k, v_k) \in W^{1,1}(\Omega) \times W^{1,1}(\Omega, \mathbb{R}^2)$ be a sequence which approximates an $\widetilde{E}$-minimum (u, v) in the sense of Lemma 5.1.1. Thus, Reshetnyak's theorem gives

$$\inf_{\mathcal{V}} E \leq E(u_k, v_k) = \widetilde{E}(u_k, v_k) \xrightarrow{k \to \infty} \inf_{\widetilde{\mathcal{V}}} \widetilde{E}.$$

Now to part d). That every $\widetilde{E}$-minimizer is indeed the L^1-limit of an E-minimizing sequence in the Sobolev class $\mathcal{V}$ follows from Lemma 5.1.1 together with $\widetilde{E}_{|\mathcal{V}} = E$ and part c). That every such limit indeed minimizes $\widetilde{E}$ is a consequence of the above mentioned lower semicontinuity property of the relaxation. It remains to prove that $(\widetilde{\mathcal{P}})$ has a unique solution if $\mathcal{M} \cap \mathcal{V} \neq \emptyset$. Assume therefore that $(u, v) \in \mathcal{V}$ minimizes $\widetilde{E}$ and let $(\widetilde{u}, \widetilde{v})$ be another element of $\mathcal{M}$. From $\widetilde{E}(u, v) = E(u, v) = \widetilde{E}(\widetilde{u}, \widetilde{v})$ and part b) we infer

$$\int\limits_{\Omega} F^\infty\left(\frac{\nabla_s \widetilde{v}}{|\nabla_s \widetilde{v}|}\right) \mathrm{d}|\nabla_s \widetilde{v}| + \int\limits_{\Omega} G^\infty\left(\frac{\nabla_s \widetilde{u}}{|\nabla_s \widetilde{u}|}\right) \mathrm{d}|\nabla_s \widetilde{u}| = 0$$

and therefore

$$\frac{\nabla_s \widetilde{v}}{|\nabla_s \widetilde{v}|} = 0 \quad |\nabla_s \widetilde{v}|\text{-a.e.} \quad \text{and} \quad \frac{\nabla_s \widetilde{u}}{|\nabla_s \widetilde{u}|} = 0 \quad |\nabla_s \widetilde{u}|\text{-a.e.},$$

i.e. $\nabla_s \widetilde{v} \equiv 0$ and $\nabla_s \widetilde{v} \equiv 0$, i.e. $(\widetilde{u}, \widetilde{v}) \in \mathcal{V}$. But then b) implies $\nabla \widetilde{v} = \nabla v$ and thereby $v = \widetilde{v} + c$. Further it follows from $\nabla u - v = \nabla \widetilde{u} - \widetilde{v}$ that $\nabla u = \nabla \widetilde{u} + c$, i.e. $u(x) = \widetilde{u}(x) + c \cdot x + b$, for some $b \in \mathbb{R}$, $c \in \mathbb{R}^2$. Finally, due to $u = \widetilde{u}$ on $\Omega - D$ along with $\mathcal{L}^2(\Omega - D) > 0$, we infer $b = c = 0$. Hence $u = \widetilde{u}$ and $v = \widetilde{v}$ a.e. on Ω. □

5.2 The Dual Problem

As done in Section 2.2 for the problem (V), we are now going to study the convex dual of $(\mathcal{P})$. In order to simplify our notation, we define the linear operator

$$\Lambda : \mathcal{V} \to \mathcal{Y},\ \mathbf{u} = (u, v) \mapsto (\nabla u - v, \nabla v), \tag{5.12}$$

where

$$\mathcal{Y} := L^1(\Omega, \mathbb{R}^2) \times L^1(\Omega, \mathbb{R}^{2\times 2}),$$

as well as the function

$$\mathcal{F} : \mathbb{R}^2 \times \mathbb{R}^{2\times 2} \to \mathbb{R},\ (y, p) \mapsto \alpha F(p) + \beta G(y).$$

Then problem $(\mathcal{P})$ can be written in short-hand notation as

$$E(\mathbf{u}) = \int_\Omega \mathcal{F}(\Lambda \mathbf{u})\, \mathrm{d}x + \int_{\Omega - D} (u - f)^2\, \mathrm{d}x \to \min, \tag{5.13}$$

where $\mathbf{u} = (u, v) \in W^{1,1}(\Omega) \times W^{1,1}(\Omega, \mathbb{R}^2)$. By means of this representation, it is easy to see how to apply the results from [62], Remark 4.1 and 4.2 on pp. 60-61 in order to obtain the problem in duality to $(\mathcal{P})$: first, for $\boldsymbol{w} = (u, v) \in \mathcal{V}$ and $\mathbf{y} = (\kappa, \lambda) \in \mathcal{Y}^* = L^\infty(\Omega, \mathbb{R}^2) \times L^\infty(\Omega, \mathbb{R}^{2\times 2})$, we define the associated Lagrangian through

$$\ell(\boldsymbol{w}, \mathbf{y}) := \int_\Omega \Lambda(\boldsymbol{w}) \odot \mathbf{y}\, \mathrm{d}x - \int_\Omega \mathcal{F}^*(\mathbf{y})\, \mathrm{d}x + \int_{\Omega - D} (u - f)^2\, \mathrm{d}x, \tag{5.14}$$

where for $(x, p), (y, q) \in \mathbb{R}^2 \times \mathbb{R}^{2\times 2}$ we set

$$(x, p) \odot (y, q) := x \cdot y + p : q,$$

"$\cdot$" and "$:$" denoting the canonical dot products of $\mathbb{R}^2$ and $\mathbb{R}^{2\times 2}$, respectively. Furthermore, $\mathcal{F}^*$ is the convex dual to the function $\mathcal{F}$ which, by Remark 4.3 in [62] on p. 61 is given by

$$\mathcal{F}^*(\kappa, \lambda) = \alpha F^*(\lambda) + \beta G^*(\kappa),$$

with F^*, G^* as defined by the formula (2.3) from Section 2.2. Hence, we may write (5.14) as

$$\ell(\boldsymbol{w}, \mathbf{y}) \\ = \int_\Omega \nabla v : \kappa + (\nabla u - v) \cdot \lambda \,\mathrm{d}x - \int_\Omega \alpha F^*(\lambda) + \beta G^*(\kappa) \,\mathrm{d}x + \int_{\Omega - D} (u - f)^2 \,\mathrm{d}x$$

and it holds (see [62], p. 56)

$$E(\boldsymbol{w}) = \sup_{\mathbf{y} \in \mathcal{Y}^*} \ell(\boldsymbol{w}, \mathbf{y}).$$

Just like in Section 2.2 from the first part, the dual functional $R : Y^* \to [0, \infty]$ is now defined as

$$R(\mathbf{y}) := \inf_{\boldsymbol{w} \in \mathcal{V}} \ell(\boldsymbol{w}, \mathbf{y}), \quad \mathbf{y} \in \mathcal{Y}^*, \tag{5.15}$$

and the dual problem consists in maximizing R:

$$R \to \max \text{ in } L^\infty(\Omega, \mathbb{R}^2) \times L^\infty(\Omega, \mathbb{R}^{2\times 2}). \tag{$\mathcal{P}^*$}$$

Theorem 5.2.1
Under our general assumptions regarding Ω, D, f, F and G it holds:

a) Problem $(\mathcal{P}^*)$ has at least one solution $(\rho, \sigma) \in L^\infty(\Omega, \mathbb{R}^2) \times L^\infty(\Omega, \mathbb{R}^{2\times 2})$.

b) The problems $(\mathcal{P})$ and $(\mathcal{P}^*)$ are related via the "inf-sup" relation:

$$\inf_{\boldsymbol{w} \in \mathcal{V}} E(\boldsymbol{w}) = \sup_{\mathbf{y} \in \mathcal{Y}^*} R(\mathbf{y}).$$

c) Let $(u, v) \in BV(\Omega) \times BV(\Omega, \mathbb{R}^2)$ be a solution of the relaxed problem $(\widetilde{\mathcal{P}})$. Then the following formula holds:

$$(\rho, \sigma) = D\mathcal{F}\big(\Lambda^a(u, v)\big) = \beta DG(\nabla_a u - v) \oplus \alpha DF(\nabla_a v) \text{ a.e. on } \Omega, \tag{5.16}$$

where we declare

$$\Lambda^a(u, v) := (\nabla_a u - v, \nabla_a v).$$

In particular, the solution of the dual problem is unique by Theorem 5.1.1 b).

Proof of Theorem 5.2.1. Since the proof relies more or less on the same arguments as used in the proof of Theorem 2.2.1 in Section 2.2, we will merely give a rather condensed sketch of it.

As in the uncoupled case (see Lemma 2.2.1), we employ a suitable δ-approximation of problem $(\mathcal{P})$. This means that for $\delta \in (0, 1)$ we consider the problem

$$\begin{aligned} E_\delta(u, v) := \frac{\delta}{2} \int_\Omega |\nabla u|^2 + |\nabla v|^2 \,\mathrm{d}x + E(u, v) \\ \to \min \text{ in } W^{1,2}(\Omega) \times W^{1,2}(\Omega, \mathbb{R}^2). \end{aligned} \tag{$\mathcal{P}_\delta$}$$

Lemma 5.2.1
Under our general assumptions regarding Ω, f, F and G it holds:

a) For any $\delta \in (0,1)$, problem $(\mathcal{P}_\delta)$ admits a unique solution $\mathbf{u}_\delta = (u_\delta, v_\delta)$ in the space $W^{1,2}(\Omega) \times W^{1,2}(\Omega, \mathbb{R}^2)$.

b) The family of the $\mathbf{u}_\delta$'s fulfills:

$$\sup_{\delta \in (0,1)} \delta \int_\Omega |\nabla u_\delta|^2 + |\nabla v_\delta|^2 \, \mathrm{d}x < \infty, \tag{5.17}$$

$$\textit{as well as} \quad \sup_{\delta \in (0,1)} \int_{\Omega - D} |u_\delta|^2 \, \mathrm{d}x < \infty. \tag{5.18}$$

c) It holds (not necessarily uniformly with respect to δ)

$$\mathbf{u}_\delta \in W^{2,2}_{\mathrm{loc}}(\Omega) \times W^{2,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^2). \tag{5.19}$$

Proof of Lemma 5.2.1. Ad a). Let $\delta \in (0,1)$ be fixed. Quoting standard results concerning the weak lower semicontinuity of convex functionals on Sobolev spaces (see, e.g. [64]), the existence of a minimizer follows via the direct method once we have shown that an E_δ minimizing sequence is bounded in $W^{1,2}(\Omega) \times W^{1,2}(\Omega, \mathbb{R}^2)$. So let us fix $\delta \in (0,1)$ and denote by $\mathbf{u}_k = (u_k, v_k)$ such a minimizing sequence. From $E_\delta(\mathbf{u}_k) \le E_\delta(0,0) = E(0,0)$, it is clear that $|\nabla v_k|$ and $|\nabla u_k|$ are bounded in $L^2(\Omega)$. Furthermore, $f - u_k$ is bounded in $L^2(\Omega - D)$. By Poincaré's inequality we therefore have

$$\int_\Omega \Big| u_k(x) - \big(u_k(x)\big)_{\Omega - D} \Big|^2 \, \mathrm{d}x \le c \int_\Omega |\nabla u_k|^2 \, \mathrm{d}x,$$

where $\big(u_k(x)\big)_{\Omega - D} := \fint_{\Omega - D} u_k(t) \, \mathrm{d}t$. It thus follows that u_k is bounded in $W^{1,2}(\Omega)$. But then,

$$\int_\Omega G(\nabla u_k - v_k) \, \mathrm{d}x \le E(0,0)$$

implies that also $|v_k|$ is bounded in $L^1(\Omega)$ and another application of Poincaré's inequality yields the boundedness of v_k in $W^{1,2}(\Omega, \mathbb{R}^2)$.

Ad b): this follows immediately from $E_\delta(\mathbf{u}_\delta) \le E_\delta(0,0) = E(0,0)$.

Ad c): let $\delta \in (0,1)$ be fixed. The proof of this statement is a standard application of the difference quotient technique to the quadratic variational problems

$$E_\delta(u, v_\delta) \to \min \ \text{in} \ W^{1,2}(\Omega), \quad \text{with } v_\delta \text{ fixed}$$

and

$$E_\delta(u_\delta, v) \to \min \ \text{in} \ W^{1,2}(\Omega, \mathbb{R}^2), \quad \text{with } u_\delta \text{ fixed},$$

respectively. We refer to Appendix C for the details of the calculation. $\square$

The core of the proof of Theorem 5.2.1 now consists in a careful analysis of the convergence behavior of $\mathbf{u}_\delta$ as δ approaches zero. Our claim is that (at least for a subsequence) $\mathbf{u}_\delta$ converges in $L^1(\Omega) \times L^1(\Omega, \mathbb{R}^2)$ towards a solution of the relaxed problem $(\widetilde{\mathcal{P}})$, and that

$$\begin{aligned} \boldsymbol{\sigma}_\delta = (\rho_\delta, \sigma_\delta) &:= \delta \Lambda \mathbf{u}_\delta + D\mathcal{F}(\Lambda \mathbf{u}_\delta) \\ &= \Big[\delta(\nabla u_\delta - v_\delta) + \beta DG(\nabla u_\delta - v_\delta)\Big] \oplus \Big[\delta \nabla v_\delta + \alpha DF(\nabla v_\delta)\Big] \end{aligned} \tag{5.20}$$

converges weakly in $L^2(\Omega, \mathbb{R}^2) \times L^2(\Omega, \mathbb{R}^{2\times 2})$ towards a solution of the dual problem $(\mathcal{P}^*)$.

Lemma 5.2.2
The family $\mathbf{u}_\delta$ is uniformly bounded in the space $\mathcal{V} = W^{1,1}(\Omega) \times W^{1,1}(\Omega, \mathbb{R}^2)$. In particular, there exists a function $\overline{\mathbf{u}} = (\overline{u}, \overline{v}) \in BV(\Omega) \times BV(\Omega, \mathbb{R}^2)$ such that $\mathbf{u}_\delta \to \overline{\mathbf{u}}$ in $L^1(\Omega) \times L^1(\Omega, \mathbb{R}^2)$ and a.e. for a suitable sequence $\delta \downarrow 0$.

Proof of Lemma 5.2.2. We start with the observation that due to $E_\delta(\mathbf{u}_\delta) \le E_\delta(0,0) = E(0,0)$ and the linear growth of F and G we have the following bounds:

$$\sup_{\delta\in(0,1)} \int_\Omega |\nabla v_\delta| \,\mathrm{d}x \le M_1', \tag{5.21}$$

$$\sup_{\delta\in(0,1)} \int_\Omega |\nabla u_\delta - v_\delta| \,\mathrm{d}x \le M_2', \text{ as well as} \tag{5.22}$$

$$\sup_{\delta\in(0,1)} \int_{\Omega - D} |u_\delta|^2 \,\mathrm{d}x \le M_3' \tag{5.23}$$

for constants $M_1', M_2', M_3' > 0$. From here on, we may repeat the arguments from p. 98 to prove the boundedness of (u_δ, v_δ) in $W^{1,1}(\Omega) \times W^{1,1}(\Omega, \mathbb{R}^2)$ via Lemma 5.1.2, and the existence of a function $\overline{\mathbf{u}}$ as claimed follows from the BV-compactness property. $\square$

We will see later during the proof of Theorem 5.2.1, that the function $\overline{\mathbf{u}}$ from Lemma 5.2.2 in fact minimizes $\widetilde{E}$. Let us fix a null-sequence $\delta \downarrow 0$ as in Lemma 5.2.2. By (5.17), it holds $\delta u_\delta \to 0$ in $W^{1,2}(\Omega, \mathbb{R}^2)$ as well as $\delta v_\delta \to 0$ in $W^{1,2}(\Omega, \mathbb{R}^{2\times 2})$, so that

$$\delta \Lambda \mathbf{u}_\delta = \big(\delta(\nabla u_\delta - v_\delta), \delta \nabla v_\delta\big) \to 0 \text{ in } L^2(\Omega, \mathbb{R}^2) \times L^2(\Omega, \mathbb{R}^{2\times 2}). \tag{5.24}$$

Since further $|\nabla \mathcal{F}(y,p)|$ is bounded for all $(y,p) \in \mathbb{R}^2 \times \mathbb{R}^{2\times 2}$ by our assumptions on F and G, it follows that

$$\sup_{\delta\in(0,1)} \int_\Omega |\boldsymbol{\sigma}_\delta|^2 \,\mathrm{d}x < \infty. \tag{5.25}$$

Thus there exists $\boldsymbol{\sigma} \in L^2(\Omega, \mathbb{R}^2) \times L^2(\Omega, \mathbb{R}^{2\times 2})$ such that

$$\boldsymbol{\sigma}_\delta \rightharpoonup \boldsymbol{\sigma} \text{ in } L^2(\Omega, \mathbb{R}^2) \times L^2(\Omega, \mathbb{R}^{2\times 2}) \text{ as } \delta \downarrow 0,$$

at least for a suitable subsequence. Furthermore, setting $\boldsymbol{\tau}_\delta := D\mathcal{F}(\Lambda \mathbf{u}_\delta)$, we may assume that there exists $\boldsymbol{\tau} \in L^\infty(\Omega, \mathbb{R}^2) \times L^\infty(\Omega, \mathbb{R}^{2\times 2})$ such that

$$\boldsymbol{\tau}_\delta \overset{*}{\rightharpoonup} \boldsymbol{\tau}. \tag{5.26}$$

Due to $\boldsymbol{\sigma}_\delta = \delta \Lambda \mathbf{u}_\delta + \boldsymbol{\tau}_\delta$ and (5.24) it must further hold

$$\boldsymbol{\sigma} = \boldsymbol{\tau} \text{ a.e. on } \Omega. \tag{5.27}$$

Next we observe that, thanks to its E_δ-minimality, $\mathbf{u}_\delta$ satisfies for all pairs of functions $\boldsymbol{\phi} = (\varphi, \psi) \in W^{1,2}(\Omega, \mathbb{R}) \times W^{1,2}(\Omega, \mathbb{R}^2)$ the following Euler-Lagrange equation:

$$\begin{aligned} \delta \int_\Omega \nabla u_\delta \cdot \nabla \varphi + \nabla v_\delta : \nabla \psi \, \mathrm{d}x + \int_\Omega D\mathcal{F}(\Lambda \mathbf{u}_\delta) \odot \Lambda \boldsymbol{\phi} \, \mathrm{d}x \\ +2 \int_{\Omega - D} (u_\delta - f) \varphi \, \mathrm{d}x = 0. \end{aligned} \tag{EL}$$

This can be decoupled into the two equations

$$\begin{aligned} 0 = \int_\Omega DF_\delta(\nabla v_\delta) : \nabla \psi \, \mathrm{d}x - \beta \int_\Omega DG(\nabla u_\delta - v_\delta) \cdot \psi \, \mathrm{d}x \\ \text{for all } \psi \in W^{1,2}(\Omega, \mathbb{R}^2), \end{aligned} \tag{EL1}$$

where $F_\delta(p) := \frac{\delta}{2}|p|^2 + \alpha F(p)$ for all $p \in \mathbb{R}^{2\times 2}$, and

$$\begin{aligned} 0 = \delta \int_\Omega \nabla u_\delta \cdot \nabla \varphi \, \mathrm{d}x + \beta \int_\Omega DG(\nabla u_\delta - v_\delta) \cdot \nabla \varphi \, \mathrm{d}x \\ + 2 \int_{\Omega - D} (u_\delta - f) \varphi \, \mathrm{d}x \text{ for all } \varphi \in W^{1,2}(\Omega). \end{aligned} \tag{EL2}$$

We note that $\boldsymbol{\phi} = \mathbf{u}_\delta$ is admissible in (EL) and, using the duality relation (see [62], Proposition 5.1 on p. 21)

$$\mathcal{F}(\Lambda \mathbf{u}_\delta) = \boldsymbol{\tau}_\delta \odot \Lambda \mathbf{u}_\delta - \mathcal{F}^*(\boldsymbol{\tau}_\delta), \tag{5.28}$$

we obtain the formula

$$\begin{aligned} E_\delta(\mathbf{u}_\delta) &= -\frac{\delta}{2} \int_\Omega |\nabla u_\delta|^2 + |\nabla v_\delta|^2 \, \mathrm{d}x - \int_\Omega \mathcal{F}^*(\boldsymbol{\tau}_\delta) \, \mathrm{d}x - \int_{\Omega - D} u_\delta^2 \, \mathrm{d}x + \int_{\Omega - D} f^2 \, \mathrm{d}x \\ &\leq - \int_\Omega \mathcal{F}^*(\boldsymbol{\tau}_\delta) \, \mathrm{d}x - \int_{\Omega - D} u_\delta^2 \, \mathrm{d}x + \int_{\Omega - D} f^2 \, \mathrm{d}x. \end{aligned} \tag{5.29}$$

Note that from the definition of R (see (5.15)) it is clear that

$$\sup_{\mathbf{y}\in\mathcal{Y}^*} R(\mathbf{y}) \leq \inf_{\boldsymbol{w}\in\mathcal{V}} E(\boldsymbol{w})$$

and, observing that $\inf_{\boldsymbol{w}\in\mathcal{V}} E(\boldsymbol{w}) \leq E(\mathbf{u}_\delta) \leq E_\delta(\mathbf{u}_\delta)$, we obtain after passing to $\limsup_{\delta\downarrow 0}$ on both sides of (5.29) the inequality

$$\sup_{\mathbf{y}\in\mathcal{Y}^*} R(\mathbf{y}) \leq \inf_{\boldsymbol{w}\in\mathcal{V}} E(\boldsymbol{w}) \leq -\int_\Omega \mathcal{F}^*(\boldsymbol{\tau})\,\mathrm{d}x - \int_{\Omega-D} \overline{u}^2\,\mathrm{d}x + \int_{\Omega-D} f^2\,\mathrm{d}x. \tag{5.30}$$

Here we have used that $\mathcal{F}^*$ is convex and therefore weakly-$*$ lower semicontinuous, as well as Fatou's Lemma. Passing to the limit $\delta\downarrow 0$ in (EL) (using $\delta\nabla u_\delta \to 0$ in $L^2(\Omega,\mathbb{R}^2)$ and $\delta\nabla v_\delta \to 0$ in $L^2(\Omega,\mathbb{R}^{2\times 2})$ as well as $u_\delta \rightharpoondown \overline{u}$ in $L^2(\Omega - D)$) further yields the relation

$$\int_\Omega \boldsymbol{\tau}\odot\Lambda\boldsymbol{w}\,\mathrm{d}x + 2\int_{\Omega-D}(\overline{u}-f)w_1\,\mathrm{d}x = 0, \tag{5.31}$$

which (by a standard approximation argument) holds for all $\boldsymbol{w} = (w_1, w_2) \in \mathcal{V}$. From

$$R[\tau] = \inf_{\boldsymbol{w}\in\mathcal{V}} \ell(\boldsymbol{w},\boldsymbol{\tau})$$

and (5.31) we can therefore deduce the formula

$$R(\boldsymbol{\tau}) = -\int_\Omega \mathcal{F}^*(\boldsymbol{\tau})\,\mathrm{d}x + \inf_{\boldsymbol{w}=(w_1,w_2)\in V}\left[\int_{\Omega-D}(\overline{u}-w_1)^2 + f^2 - \overline{u}^2\,\mathrm{d}x\right],$$

and since the term in brackets is obviously minimal for $w_1 = \overline{u}$, (5.30) allows us to infer the equation

$$\sup_{\mathbf{y}\in\mathcal{Y}^*} R(\mathbf{y}) = R(\boldsymbol{\tau}) = \inf_{\boldsymbol{w}\in\mathcal{V}} E(\boldsymbol{w}),$$

i.e. the inf-sup relation. Furthermore, we see that $\boldsymbol{\tau} = \boldsymbol{\sigma}$ maximizes the dual functional and in particular, (5.29) implies

$$\inf_{\boldsymbol{w}\in\mathcal{V}} E(\boldsymbol{w}) \leq \limsup_{\delta\downarrow 0} E(\mathbf{u}_\delta) \leq R(\boldsymbol{\tau}),$$

which shows that $\mathbf{u}_\delta$ is indeed an E-minimizing sequence. Part a) and b) of Theorem 5.2.1 are thereby proved.

It remains to establish the duality formula that is claimed in part c) of Theorem 5.2.1. Arguing as in the proof of Theorem 2.15 in [67] (cf. also Theorem 2.2.1 in Section 2.2 of this thesis), it is essentially enough to revise the single steps with F replaced by $\mathcal{F}$ and ∇_a replaced by Λ^a, but for the reader's convenience, we give a sketch of the main arguments. We first claim that the tensor

$$\mathbf{t} := D\mathcal{F}(\Lambda^a(u,v))$$

is in fact a maximizer of the dual functional R. Note that due to our assumptions (F1) and (G1) it holds $\mathbf{t} \in L^\infty(\Omega, \mathbb{R}^2) \times L^\infty(\Omega, \mathbb{R}^{2\times 2})$, i.e. $\mathbf{t}$ is admissible in R. Let $\boldsymbol{w} = (w_1, w_2) \in \mathcal{V}$ be arbitrary and let $\ell(\cdot,\cdot)$ be the Lagrangian defined in (5.14). Using the duality relation

$$\mathcal{F}^*(D\mathcal{F}(\mathbf{p})) = \mathbf{p} \odot D\mathcal{F}(\mathbf{p}) - \mathcal{F}(\mathbf{p}),\ \mathbf{p} \in \mathbb{R}^2 \times \mathbb{R}^{2\times 2}$$

from Proposition 5.1 on p. 21 of [62] with the choice $\mathbf{p} = \Lambda^a(u, v)$, we obtain

$$\begin{aligned} \ell(\boldsymbol{w}, \mathbf{t}) &= \int_\Omega D\mathcal{F}(\Lambda^a(u,v)) \odot \Lambda\boldsymbol{w} - \mathcal{F}^*\big(D\mathcal{F}(\Lambda^a(u,v))\big)\,\mathrm{d}x + \int_{\Omega - D} (w_1 - f)^2\,\mathrm{d}x \\ &= \int_\Omega \mathcal{F}(\Lambda^a(u,v))\,\mathrm{d}x + \int_\Omega (\Lambda\boldsymbol{w} - \Lambda^a(u,v)) \odot \mathbf{t}\,\mathrm{d}x + \int_{\Omega - D} (w_1 - f)^2\,\mathrm{d}x. \end{aligned} \tag{5.32}$$

The $\widetilde{E}$-minimality of (u, v) implies the equation

$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \widetilde{E}(u + tw_1, v + tw_2) &= \int_\Omega D\mathcal{F}(\Lambda^a(u,v)) \odot \Lambda\boldsymbol{w}\,\mathrm{d}x \\ + 2 \int_{\Omega - D} (u - f) w_1\,\mathrm{d}x, \end{aligned} \tag{5.33}$$

where we used $(\nabla_s w_1, \nabla_s w_2) = (0, 0) \in \mathbb{R}^2 \times \mathbb{R}^{2\times 2}$ and, likewise, we obtain

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \widetilde{E}(u + tu, v + tv) \\ &= \int_\Omega D\mathcal{F}(\Lambda^a(u,v)) \odot \Lambda^a(u,v)\,\mathrm{d}x + \int_\Omega \mathcal{F}^\infty(\Lambda^s(u,v))\,\mathrm{d}x \\ &\quad + 2 \int_{\Omega - D} (u - f) u\,\mathrm{d}x, \end{aligned} \tag{5.34}$$

which is due to

$$\Lambda^a(u + tu, v + tv) = (1 + t)\Lambda^a(u, v)$$

and

$$\Lambda^s(u + tu, v + tv) = (1 + t)\Lambda^s(u, v) = (1 + t)(\nabla_s u, \nabla_s v).$$

Note that here we have tacitly set

$$\Lambda^s(u, v) := (\nabla_s u, \nabla_s v)$$

as well as

$$\mathcal{F}^\infty(\Lambda^s(u,v)) := \alpha \int_\Omega F^\infty\left(\frac{\mathrm{d}\nabla_s u}{\mathrm{d}|\nabla_s u|}\right)\mathrm{d}|\nabla_s u| + \beta \int_\Omega G^\infty\left(\frac{\mathrm{d}\nabla_s v}{\mathrm{d}|\nabla_s v|}\right)\mathrm{d}|\nabla_s v|.$$

Inserting (5.33) and (5.34) into (5.32), it now follows that

$$\begin{aligned}\ell(\boldsymbol{w},\mathbf{t}) &= \int\limits_{\Omega} \mathcal{F}(\Lambda^a(u,v))\,\mathrm{d}x + \int\limits_{\Omega} \mathcal{F}^\infty(\Lambda^s(u,v)) \\ &+ \int\limits_{\Omega-D} \underbrace{(w_1-f)^2 + 2(u-f)(u-w_1)}_{=(u-f)^2+(u-w_1)^2}\,\mathrm{d}x \\ &= \widetilde{E}(u,v) + \int\limits_{\Omega-D} (u-w_1)^2\,\mathrm{d}x \geq \widetilde{E}(u,v),\end{aligned} \tag{5.35}$$

which implies

$$R[\mathbf{t}] = \inf_{\boldsymbol{w}\in\mathcal{V}} \ell(\boldsymbol{w},\mathbf{t}) \geq \widetilde{E}(u,v) = \inf_{\widetilde{\mathcal{V}}} \widetilde{E} = \inf_{\mathcal{V}} E \overset{\text{Thm.5.2.1}a)}{=} \sup_{\mathcal{Y}^*} R,$$

i.e. $\mathbf{t}$ is indeed a maximizer of the dual functional. Now assume that there exists another maximizer $\widetilde{\mathbf{t}}$ of $\mathbb{R}$. Arguing as in the proof of Theorem 2.15 in [67], we will see how this assumption leads to a contradiction. Let therefore

$$U := \mathrm{Im}(D\mathcal{F})$$

which, by our assumptions (F2) and (G2) on the second derivatives of F and G and the inverse function theorem, is an open set. Furthermore, Theorem 26.5 on p. 258 in [77] proves that U is convex. Consider now the partition of U into the subsets

$$\begin{aligned}\Sigma_1 &:= \{x \in \Omega \,:\, \mathbf{t}(x) \neq \widetilde{\mathbf{t}}(x) \text{ and } \widetilde{\mathbf{t}}(x) \in U\}, \\ \Sigma_2 &:= \{x \in \Omega \,:\, \mathbf{t}(x) \neq \widetilde{\mathbf{t}}(x) \text{ and } \widetilde{\mathbf{t}}(x) \in \partial U\}, \\ \text{and } \Sigma_3 &:= \{x \in \Omega \,:\, \widetilde{\mathbf{t}}(x) \notin \overline{U}\}.\end{aligned}$$

Since $\widetilde{\mathbf{t}}$ is a maximizer of the functional R and $\mathcal{F}^* = +\infty$ on $(\mathbb{R}^2 \times \mathbb{R}^{2\times 2}) - \overline{U}$ (cf. Theorem 26.5 in [77]), it directly follows that $\mathcal{L}^2(\Sigma_3) = 0$. Making use of the strict convexity of the conjugate function $\mathcal{F}^*$ on U (which also follows from Theorem 26.5 in [77]) as well as the concavity of the mapping

$$L^\infty(\Omega,\mathbb{R}^2) \times L^\infty(\Omega,\mathbb{R}^{2\times 2}) \ni \mathbf{y} \mapsto \inf_{\boldsymbol{w}=(w_1,w_2)\in\mathcal{V}} \int\limits_{\Omega} \Lambda\boldsymbol{w} \odot \mathbf{y}\,\mathrm{d}x + \int\limits_{\Omega-D} (w_1-f)^2\,\mathrm{d}x,$$

we can furthermore verify $\mathcal{L}^2(\Sigma_1) = \mathcal{L}^2(\Sigma_2) = 0$ along the lines of the proof of Theorem 2.15 in [67]. Hence the set $\{x \in \Omega \,:\, \mathbf{t} \neq \widetilde{\mathbf{t}}\} = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ is a null set, which thereby proves part c) of Theorem 5.2.1. □

Chapter 6

Regularity Properties

In accordance with the general scheme of this thesis, after having established the existence of generalized and dual solutions, we now proceed with a discussion of Sobolev and Hölder regularity. Again we will need to restrict our considerations to the two-dimensional setting, where we have all the required embeddings at hand. For proving the existence of a classical solution of problem $(\mathcal{P})$ in the Sobolev class, we will once more employ the concept of μ-ellipticity where, in the context of our coupling model, we will have to require the main as well as the coupling term to be μ- and ν-elliptic, respectively, with parameters (μ, ν) varying in some bounded subregion of $[1, \infty) \times [1, \infty)$. Additionally, since our coupling model involves the vector variable v, it will become necessary to assume that the density F is rotationally invariant. In the second section, by adapting the technique that was developed in [36] for the first-order case, we will show that these classical solutions are actually partially Hölder continuous on a dense open subset of Ω.

6.1 Sobolev Regularity

After having discussed the relaxed and dual formulation of problem $(\mathcal{P})$, we now consider the question whether or not there exists a "classical" solution in the Sobolev class $W^{1,1}(\Omega) \times W^{1,1}(\Omega, \mathbb{R}^2)$. To that purpose, as in our previous considerations in Chapter 4, we impose an additional μ-ellipticity condition (cf. (F_μ) in the introduction) on our densities F and G. This means that we replace the rather general ellipticity condition (F2) with the stronger assumption

$$\begin{aligned} &c_1 \frac{1}{(1+|p|)^\mu} |q|^2 \leq D^2F(p)(q,q) \leq c_2 \frac{1}{1+|p|} |q|^2, \text{ for some} \\ &c_1, c_2 > 0, \text{ parameter } \mu \in (1, \infty) \text{ and for all } p, q \in \mathbb{R}^{2\times 2} \end{aligned} \tag{F2$'$}$$

and, likewise, (G2) is replaced with

$$\begin{aligned}&c_1\frac{1}{(1+|y|)^\nu}|x|^2 \le D^2G(y)(x,x) \le c_2\frac{1}{1+|y|}|x|^2 \text{ for some}\\ &c_1,c_2>0,\ \text{ parameter } \nu\in(1,\infty) \text{ and for all } x,y\in\mathbb{R}^2.\end{aligned} \tag{G2$'$}$$

We want to emphasize that, unlike to the statements of Theorems 5.1.1 and 5.2.1, the planarity assumption $\Omega \subset \mathbb{R}^2$ is essential for the regularity results to follow. We will also have to make a distinction between the case of pure denoising $D = \emptyset$ and the general case. Precisely we have:

Theorem 6.1.1
Together with our general assumptions regarding Ω and f, let F satisfy (F1), (F2)', (F3) and let G satisfy (G1), (G2)', (G3) with parameters $\mu,\nu > 1$. Then it holds:

a) If $D = \emptyset$ (pure denoising) and

$$(\mu,\nu) \in (1, 3/2) \times (1,2), \tag{6.1}$$

then problem $(\mathcal{P})$ admits a unique solution (u,v) in the Sobolev class

$$\mathcal{V} = W^{1,1}(\Omega) \times W^{1,1}(\Omega,\mathbb{R}^2).$$

It even holds $(u,v) \in W^{1,p}_{\mathrm{loc}}(\Omega) \times W^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^2)$ for every $p \in [1,\infty)$.

b) If $D \neq \emptyset$, and if we replace the quadratic fidelity term $\int_{\Omega-D}(u-f)^2\,\mathrm{d}x$ in $(\mathcal{P})$ e.g. with

$$\int_\Omega \sqrt{1+|u-f|^2}dx,$$

and (6.1) with the stronger condition

$$(\mu,\nu) \in (1, 3/2) \times (1, 3/2), \tag{6.2}$$

then the statement of Theorem 6.1.1 a) holds for the problem

$$\begin{aligned}\widehat{E}(u,v) := \alpha\int_\Omega F(\nabla v)\,\mathrm{d}x + \beta\int_\Omega G(\nabla u - v)\,\mathrm{d}x\\ + \int_{\Omega-D}\sqrt{1+|u-f|^2}\,\mathrm{d}x \to \min \ \textit{in } \mathcal{V}.\end{aligned} \tag{$\widehat{\mathcal{P}}$}$$

Remark 6.1.1 *a) The uniqueness of a possible Sobolev-minimizer follows directly from Theorem 5.1.1 part d).*

b) The results from [38] indicate that the bound $\mu < 3/2$ is not optimal, whereas the considerations from Section 3.6 have proved that that $\mu,\nu < 2$ is necessary for making a general statement on the existence of a solution in the Sobolev class.

c) In the case $D \neq \emptyset$, we may just as well choose any fidelity term of the more general form

$$\int_\Omega \phi(|u - f|)dx$$

with a convex, differentiable and increasing function $\phi : [0, \infty) \to [0, \infty)$ of linear growth.

Proof of Theorem 6.1.1. We begin with part a), i.e. $D = \emptyset$ in the following calculations. Let $\mathbf{u}_\delta = (u_\delta, v_\delta)$ be the $\widetilde{E}$-minimizing sequence as constructed in the previous section. Our proof mainly relies on the following lemma:

Lemma 6.1.1
Under the assumptions of Theorem 6.1.1 we have that

$$\varphi_\delta := \big(1 + |\nabla v_\delta|\big)^{1-\frac{\mu}{2}}, \tag{6.3}$$

$$\widetilde{\varphi}_\delta := \big(1 + |\nabla u_\delta|\big)^{1-\frac{\nu}{2}} \tag{6.4}$$

are uniformly bounded in $W^{1,2}_{\mathrm{loc}}(\Omega)$.

Proof of Lemma 6.1.1: Ad (6.3). Throughout the following, we use summation convention with respect to the index $i \in \{1, 2\}$ and denote by c a generic positive constant. We start with the discussion of the quantity φ_δ. First, we note that the uniform boundedness of φ_δ in $L^2_{\mathrm{loc}}(\Omega)$ is clear since we assume $\mu > 1$ and v_δ is uniformly bounded in $W^{1,1}(\Omega, \mathbb{R}^2)$ by Lemma 5.2.2. Choosing $\partial_i\psi$ for some $\psi \in C^\infty_0(\Omega, \mathbb{R}^2)$ in the Euler equation (EL1) and performing an integration by parts, we obtain

$$0 = \int_\Omega D^2F_\delta(\nabla v_\delta)(D_i\nabla v_\delta, \nabla\psi)\,\mathrm{d}x + \beta\int_\Omega DG(\nabla u - v_\delta)\cdot\partial_i\psi\,\mathrm{d}x, \tag{EL1'}$$

which, by a standard approximation argument, holds even for all $\psi \in W^{1,2}(\Omega, \mathbb{R}^2)$ which are compactly supported in Ω. Let $x_0 \in \Omega$ denote some point and let $R > 0$ be such that $B_{2R}(x_0) \Subset \Omega$. We choose $\psi = \eta^2 D_i v_\delta$, where $\eta \in C^\infty_0(\Omega)$ satisfies the conditions listed in (4.3). With this choice of ψ, equation (EL1)' reads as

$$\begin{aligned} 0 = &\int_\Omega D^2F_\delta(\nabla v_\delta)\big(D_i\nabla v_\delta, \nabla(\eta^2 D_i v_\delta)\big)\,\mathrm{d}x \\ &+ \beta\int_\Omega DG(\nabla u_\delta - v_\delta)\cdot D_i(\eta^2 D_i v_\delta)\,\mathrm{d}x \end{aligned}$$

which can be expanded to

$$
\begin{aligned}
0 = \int_\Omega & D^2F_\delta(\nabla v_\delta)\big(D_i\nabla v_\delta, D_i\nabla v_\delta\big)\eta^2\,\mathrm{d}x \\
& \underbrace{+\,2\int_\Omega D^2F_\delta(\nabla v_\delta)\big(\eta D_i\nabla v_\delta, D_iv_\delta\otimes\nabla\eta\big)\,\mathrm{d}x}_{=:\,T_1} \\
& \qquad\qquad \underbrace{+\,\beta\int_\Omega DG(\nabla u_\delta - v_\delta)\cdot D_i(\eta^2 D_iv_\delta)\,\mathrm{d}x.}_{=:\,T_2}
\end{aligned}
\tag{6.5}
$$

We define

$$
\Theta_\delta := D^2F_\delta(\nabla v_\delta)\big(D_i\nabla v_\delta, D_i\nabla v_\delta\big)^{\frac{1}{2}}
$$

and may then write (6.5) for short as

$$
\int_\Omega \Theta_\delta^2\eta^2\,\mathrm{d}x = -T_1 - T_2. \tag{6.6}
$$

Recalling (F2)$'$, we see that the first claim of Lemma 6.1.1 follows via a uniform estimate of the integral $\int_\Omega \Theta_\delta^2\eta^2\,\mathrm{d}x$ on the left-hand side of (6.6). So let us have a look at the quantity T_1 first. Applying the Cauchy-Schwarz inequality to the bilinear form $D^2F_\delta(\nabla v_\delta)(\,\cdot\,,\,\cdot\,)$ and then Young's inequality, we infer that (for $\varepsilon > 0$ arbitrarily small)

$$
|T_1| \le \varepsilon\int_\Omega \Theta_\delta^2\eta^2\,\mathrm{d}x + \varepsilon^{-1}\int_\Omega D^2F_\delta(\nabla v_\delta)\big(D_iv_\delta\otimes\nabla\eta, D_iv_\delta\otimes\nabla\eta\big)\,\mathrm{d}x. \tag{6.7}
$$

Choosing $\varepsilon = \frac{1}{2}$, the first summand can be absorbed in the right-hand side of (6.6) whereas to the second summand we apply the estimate (F2)$'$ as well as Lemma 5.2.1 b) and Lemma 5.2.2 with the result

$$
\begin{aligned}
&\int_\Omega D^2F_\delta(\nabla v_\delta)\big(D_iv_\delta\otimes\nabla\eta, D_iv_\delta\otimes\nabla\eta\big)\,\mathrm{d}x \\
&\le \frac{c}{R^2} + \frac{c}{R^2}\int_\Omega \frac{1}{1+|\nabla v_\delta|}|\nabla v_\delta|^2\,\mathrm{d}x = c(R).
\end{aligned}
\tag{6.8}
$$

Hence, we have shown

$$
\int_\Omega \Theta_\delta^2\eta^2\,\mathrm{d}x \le c(R) - T_2,
$$

and it remains to estimate T_2. Therefore, we notice that due to our assumption (G1) on the function G, we have $DG(\nabla u_\delta - v_\delta) \in L^\infty(\Omega,\mathbb{R}^2)$ uniformly and thus

$$
\begin{aligned}
|T_2| &\le c\int_\Omega |D_i(\eta^2 D_i v_\delta)|\,\mathrm{d}x \le c\int_\Omega |\nabla\eta||\nabla v_\delta|\,\mathrm{d}x + c\int_\Omega \eta^2|\nabla^2 v_\delta|\,\mathrm{d}x\\
&\overset{\text{Lemma 5.2.2}}{=} c(R) + c\int_\Omega \eta^2|\nabla^2 v_\delta|\,\mathrm{d}x\\
&=: c(R) + T_3.
\end{aligned}
$$

For the quantity T_3 we observe

$$
T_3 = \int_\Omega \eta^2\big(1+|\nabla v_\delta|\big)^{\frac{\mu}{2}}\frac{|\nabla^2 v_\delta|}{\big(1+|\nabla v_\delta|\big)^{\frac{\mu}{2}}}\,\mathrm{d}x,
$$

which, using Young's inequality, can be estimated through

$$
T_3 \le \varepsilon\int_\Omega \eta^2\frac{|\nabla^2 v_\delta|^2}{\big(1+|\nabla v_\delta|\big)^{\mu}}\,\mathrm{d}x + \varepsilon^{-1}\int_\Omega \eta^2\big(1+|\nabla v_\delta|\big)^{\mu}\,\mathrm{d}x, \tag{6.9}
$$

where $\varepsilon > 0$ is arbitrary. Therefore (F2)$'$ implies

$$
T_3 \le c\varepsilon\int_\Omega \Theta_\delta^2\eta^2\,\mathrm{d}x + \varepsilon^{-1}\int_\Omega \eta^2\big(1+|\nabla v_\delta|\big)^{\mu}\,\mathrm{d}x =: c\varepsilon\int_\Omega \eta^2\Theta_\delta^2\,\mathrm{d}x + T_4.
$$

Choosing ε small enough, we may absorb the first term in the left-hand side of (6.6). Furthermore, with the notation

$$
\omega_\delta := \big(1+|\nabla v_\delta|\big)^{\frac{\mu}{2}} \tag{6.10}
$$

we may write

$$
T_4 = \int_\Omega (\eta\omega_\delta)^2\,\mathrm{d}x.
$$

This integral can be treated exactly as the corresponding quantity T' from (4.9) in Chapter 4: First, we apply the Sobolev inequality to get

$$
\begin{aligned}
|T_4| &\le c\left(\int_\Omega |\nabla(\eta\omega_\delta)|\,\mathrm{d}x\right)^2\\
&\le c(R) + c\left(\int_\Omega \eta|\nabla\omega_\delta|\,\mathrm{d}x\right)^2.
\end{aligned}
$$

Observing the relation $\omega_\delta = \varphi_\delta^{\frac{\mu}{2-\mu}}$, it follows that

$$
|T_4| \le c(R) + c\left(\int_\Omega \eta|\varphi_\delta|^{\frac{2\mu-2}{2-\mu}}|\nabla\varphi_\delta|\,\mathrm{d}x\right)^2,
$$

and an application of the Hölder inequality then yields

$$\begin{aligned} T_4 &\le c(R) + c \int\limits_{B_{2R}(x_0)} \varphi_\delta^{\frac{4\mu-4}{2-\mu}} \,\mathrm{d}x \int\limits_\Omega \eta^2 |\nabla \varphi_\delta|^2 \,\mathrm{d}x \\ &\le c(R) + c \int\limits_{B_{2R}(x_0)} \varphi_\delta^{\frac{4\mu-4}{2-\mu}} \,\mathrm{d}x \int\limits_\Omega \eta^2 \Theta_\delta^2 \,\mathrm{d}x. \end{aligned}$$

Here, our assumption (6.1) is indispensable for $\frac{4\mu-4}{2-\mu}(1-\frac{\mu}{2}) = 2\mu - 2 < 1$, which enables us to apply Hölder's inequality once more, giving:

$$\int\limits_{B_{2R}(x_0)} \varphi_\delta^{\frac{4\mu-4}{2-\mu}} \,\mathrm{d}x \le \pi^s R^{2s} \left(\int\limits_\Omega 1 + |\nabla v_\delta| \,\mathrm{d}x \right)^{2\mu-2} \le cR^{2s}, \tag{6.11}$$

where $s = 3 - 2\mu > 0$ and the constant c is independent of the Radius R. Combining our estimates of T_1 and T_2 with (6.6), we arrive at

$$(1 - cR^{2s}) \int\limits_\Omega \eta^2 \Theta_\delta^2 \,\mathrm{d}x \le c(R). \tag{6.12}$$

Thus, for radii $R < R_0$ (with R_0 suitably small such that $cR_0^{2s} < 1$), we end up with the uniform estimate

$$\int\limits_\Omega \eta^2 \Theta_\delta^2 \,\mathrm{d}x \le c(R_0). \tag{6.13}$$

Claim (6.3) of Lemma 6.1.1 now follows from a covering argument.

Note that as a consequence of (6.3) and Sobolev's embedding Theorem (recall $n = 2$), we obtain

$$\nabla v_\delta \in L^p_{\mathrm{loc}}(\Omega, \mathbb{R}^{2\times 2}) \text{ for any } p \in [1, \infty), \text{ uniformly with respect to } \delta. \tag{6.14}$$

In particular (due to our assumption $n = 2$) it follows that

$$v_\delta \in L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^{2\times 2}) \text{ uniformly with respect to } \delta. \tag{6.15}$$

Therefore, after passing to a suitable subsequence $\delta \downarrow 0$, we infer that ∇v_δ has a weak $L^p_{\mathrm{loc}}(\Omega, \mathbb{R}^{2\times 2})$-limit (for some $p > 1$) and since $v_\delta \to v$ in $L^1(\Omega, \mathbb{R}^2)$ and a.e., it holds $v \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$. Hence $v \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2) \cap BV(\Omega, \mathbb{R}^2)$ and thereby

$$v \in W^{1,1}(\Omega, \mathbb{R}^2).$$

Let us now turn to the corresponding quantity $\widetilde{\varphi}_\delta$ involving the scalar function u_δ. We start with the Euler equation (EL2) (keep in mind that $D = \emptyset$ in the setting of Theorem 6.1.1 a)), where we choose $\varphi = D_i(\eta^2 D_i u_\delta)$ for some

$\eta \in C_0^1(\Omega_0)$ satisfying the set of conditions from (4.3). Writing $G_\delta(x) := \delta|x|^2 + \beta G(x)$ for $x \in \mathbb{R}^2$, (EL2) (after an integration by parts) reads as:

$$\begin{aligned}\int_\Omega D^2G_\delta(\nabla u_\delta - v_\delta)\big(D_i\nabla u_\delta, \nabla(\eta^2 D_i u_\delta)\big)\,\mathrm{d}x&\\ - \beta\int_\Omega D^2G(\nabla u_\delta - v_\delta)\big(D_i v_\delta, \nabla(\eta^2 D_i u_\delta)\big)\,\mathrm{d}x&\\ - 2\int_\Omega (u_\delta - f)D_i(\eta^2 D_i u_\delta)\,\mathrm{d}x = 0.&\end{aligned} \tag{6.16}$$

Now we define

$$\widetilde{\Theta}_\delta := D^2G_\delta(\nabla u_\delta - v_\delta)\big(D_i\nabla u_\delta, D_i\nabla u_\delta\big)^{1/2}$$

due to which we may write (6.16) as

$$\begin{aligned}\int_\Omega \eta^2\widetilde{\Theta}_\delta^2\,\mathrm{d}x =& - 2\int_\Omega D^2G_\delta(\nabla u_\delta - v_\delta)\big(\eta D_i\nabla u_\delta, \nabla\eta D_i u_\delta\big)\,\mathrm{d}x\\ &+ \beta\int_\Omega D^2G(\nabla u_\delta - v_\delta)\big(\eta D_i v_\delta, \eta D_i\nabla u_\delta\big)\,\mathrm{d}x\\ &+ 2\beta\int_\Omega D^2G(\nabla u_\delta - v_\delta)\big(\eta D_i v_\delta, \nabla\eta D_i u_\delta\big)\,\mathrm{d}x\\ &+ 2\int_\Omega (u_\delta - f)D_i(\eta^2 D_i u_\delta)\,\mathrm{d}x\\ &=: -T_1' + T_2' + T_3' + T_4'.\end{aligned} \tag{6.17}$$

First, we note that due to (G2)$'$ it holds

$$\begin{aligned}\eta^2\widetilde{\Theta}_\delta^2 &\geq c_1\frac{1}{\big(1 + |\nabla u_\delta - v_\delta|\big)^\nu}|\eta\nabla^2 u_\delta|^2\\ &\overset{(6.15)}{\geq} c_1\frac{1}{\big(1 + |\nabla u_\delta| + \|v_\delta\|_{L^\infty(B_{2R}(x_0))}\big)^\nu}|\eta\nabla^2 u_\delta|^2\\ &\geq c\eta^2|\nabla\widetilde{\varphi}_\delta|^2.\end{aligned} \tag{6.18}$$

Hence, by (6.17) and our choice of η we have

$$\int_{B_R(x_0)} |\nabla\widetilde{\varphi}_\delta|^2\,\mathrm{d}x \leq c\big(|T_1'| + |T_2'| + |T_3'| + |T_4'|\big).$$

To the Integral T_1', we first apply the Cauchy-Schwarz inequality to the bilinear form $D^2G_\delta(\nabla u_\delta - v_\delta)(\cdot,\cdot)$, followed by Young's inequality to obtain

$$|T_1'| \leq \varepsilon\int_\Omega \eta^2\widetilde{\Theta}_\delta^2\,\mathrm{d}x + c(\varepsilon)\int_\Omega D^2G_\delta(\nabla u - v_\delta)(\nabla\eta D_i u_\delta, \nabla\eta D_i u_\delta)\,\mathrm{d}x,$$

where $\varepsilon > 0$ is arbitrarily small. The first summand can be absorbed in the left-hand side of (6.17). For the second term, consider the set

$$\Sigma := \Big\{ x \in B_{2R}(x_0) \,:\, |\nabla u_\delta(x)| \le \|v_\delta\|_{L^\infty(B_{2R}(x_0))} + 1 \Big\}.$$

We observe that

$$\begin{aligned}
&\int_\Omega D^2G(\nabla u_\delta - v_\delta)(\nabla\eta D_i u_\delta, \nabla\eta D_i u_\delta)\,\mathrm{d}x \\
&\le \int_\Sigma |D^2G(\nabla u_\delta - v_\delta)||\nabla\eta|^2|\nabla u_\delta|^2\,\mathrm{d}x \\
&\quad + \int_{B_{2R}(x_0)-\Sigma} D^2G(\nabla u_\delta - v_\delta)(\nabla\eta D_i u_\delta, \nabla\eta D_i u_\delta)\,\mathrm{d}x \\
&\overset{\text{(G2)}'\,\&\,(6.15)}{\le} c\left(\frac{1}{R^2} + \frac{1}{R^2}\int_{B_{2R}(x_0)-\Sigma} \frac{1}{1+|\nabla u_\delta - v_\delta|}|\nabla u_\delta|^2\,\mathrm{d}x\right) \\
&\le c\left(\frac{1}{R^2} + \frac{1}{R^2}\int_{B_{2R}(x_0)-\Sigma} \frac{1}{1+|\nabla u_\delta| - \|v_\delta\|_{L^\infty(B_{2R}(x_0))}}|\nabla u_\delta|^2\,\mathrm{d}x\right) \le c(R).
\end{aligned}$$

Together with Lemma 5.2.1 b), this implies the boundedness of $|T_1'|$. To the quantity T_2' we apply the Cauchy-Schwarz inequality and then Young's inequality with the following result:

$$|T_2'| \le \varepsilon \int_\Omega \eta^2\widetilde{\Theta}_\delta^2\,\mathrm{d}x + c(\varepsilon)\int_\Omega D^2G(\nabla u_\delta - v_\delta)(\eta D_i v_\delta, \eta D_i v_\delta)\,\mathrm{d}x.$$

Again, we absorb the first term in the left-hand side of (6.17) and the second term is bounded by (6.14). Combining the arguments for T_1' and T_2', we can estimate T_3' by

$$|T_3'| \le \frac{c}{R^2}$$

and (6.17) consequently reads as

$$\int_\Omega \eta^2\widetilde{\Theta}_\delta^2\,\mathrm{d}x \le c\left(1 + \frac{1}{R^2}\right) + |T_4'|. \tag{6.19}$$

It remains to give a bound on $|T_4'|$. An integration by parts yields (here we need the assumption $D = \emptyset$!)

$$T_4' = -\int_\Omega \eta^2|\nabla u_\delta|^2\,\mathrm{d}x - \int_\Omega f D_i(\eta^2 D_i\nabla u_\delta)\,\mathrm{d}x.$$

The Dirichlet integral can be moved to the left-hand side of (6.19), so that

$$\int_\Omega \eta^2 \widetilde{\Theta}_\delta^2 \,\mathrm{d}x + \int_\Omega \eta^2 |\nabla u_\delta|^2 \,\mathrm{d}x \le c(R) + \int_\Omega |f||D_i(\eta^2 D_i \nabla u_\delta)| \,\mathrm{d}x. \tag{6.20}$$

Note that by our assumptions we have $f \in L^\infty(\Omega)$ and thus (6.20) together with Lemma 5.2.2 implies

$$\int_\Omega \eta^2 \widetilde{\Theta}_\delta^2 \,\mathrm{d}x + \int_\Omega \eta^2 |\nabla u_\delta|^2 \,\mathrm{d}x \le c(R) + c \int_\Omega \eta^2 |\nabla^2 u_\delta| \,\mathrm{d}x.$$

The non-constant term on the right-hand side can now be estimated just like the corresponding term T_3 in (6.9), which yields

$$\int_\Omega \eta^2 |\nabla^2 u_\delta| \,\mathrm{d}x \le c\varepsilon \int_\Omega \eta^2 \widetilde{\Theta}_\delta^2 \,\mathrm{d}x + \varepsilon^{-1} \int_\Omega \eta^2 \big(1 + |\nabla u_\delta - v_\delta|\big)^\nu \,\mathrm{d}x.$$

For ε small enough, the first term can be absorbed in the left-hand side of (6.20) and to the second term we apply Young's inequality once again (making use of $\nu < 2$ and (6.15)), which results in

$$\int_\Omega \eta^2 \big(1 + |\nabla u_\delta - v_\delta|\big)^\nu \,\mathrm{d}x \le \varepsilon \int_\Omega \eta^2 |\nabla u|^2 \,\mathrm{d}x + c(R).$$

The Dirichlet integral on the right-hand side can be absorbed in the left-hand side of (6.20) if we choose ε small enough. Then claim (6.4) now follows from (6.19) and (6.18). Via Sobolev's embedding Theorem, (6.4) yields

$$\nabla u_\delta \in L^p_{\mathrm{loc}}(\Omega) \text{ for any } p \in [1, \infty) \text{ and uniform with respect to } \delta, \tag{6.21}$$

which allows us to infer $u \in W^{1,1}(\Omega)$ and $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ as for v_δ. From (6.21) it even follows

$$u_\delta \in L^\infty_{\mathrm{loc}}(\Omega) \text{ uniformly with respect to } \delta. \tag{6.22}$$

Let us now briefly comment on part b) of Theorem 6.1.1. If $D \neq \emptyset$, we cannot readily perform an integration by parts to estimate the crucial quantity

$$T_4' := \int_{\Omega - D} (u_\delta - f) D_i(\eta^2 D_i u_\delta) \,\mathrm{d}x.$$

However, switching to an error term of linear growth, T_4' turns into

$$T_4'' := \int_{\Omega - D} \frac{|u_\delta - f|}{\sqrt{1 + |u_\delta - f|^2}} \operatorname{sgn}(u_\delta - f) D_i(\eta^2 D_i u_\delta) \,\mathrm{d}x,$$

so that

$$|T_4''| \le \int_{\Omega - D} |D_i(\eta^2 D_i u_\delta)| \,\mathrm{d}x \le c(R) + c \int_\Omega \eta^2 |\nabla^2 u_\delta| \,\mathrm{d}x,$$

and we can employ the same arguments that have been used for the term T_3 to infer $u \in W^{1,1}(\Omega)$ even for $D \neq \emptyset$. $\square$

6.2 Partial $C^{1,\alpha}$-regularity

In this final section we want to prove that the Sobolev solution (u, v), whose existence was established in the preceding section under the assumption that the densities F and G are μ-elliptic, is actually a pair of locally Hölder continuous functions if we additionally assume that F is rotationally invariant. This means that, in addition to the fulfillment of (F1), (F2)' and (F3), we require F to be of the particular form

$$F(p) = g\big(|p|^2\big) \tag{F4}$$

with a convex, increasing function $g : [0, \infty) \to [0, \infty)$, which is at least of class C^2. We restrict ourselves to the case of pure denoising ($D = \emptyset$) (see Remark 6.2.1). Then we can show:

Theorem 6.2.1
Together with our general assumptions regarding Ω and f, assume $D = \emptyset$ and let F satisfy (F1), (F2)', (F3), (F4), and let G satisfy (G1), (G2)' and (G3) with parameter μ and ν satisfying (6.1). Let (u, v) be the $W^{1,1}(\Omega) \times W^{1,1}(\Omega, \mathbb{R}^2)$-minimizer of problem $(\mathcal{P})$ from Theorem 6.1.1. Then there is an open set $\Omega_0 \subset \Omega$ of full measure, i.e.

$$\mathcal{L}^2(\Omega - \Omega_0) = 0, \tag{6.23}$$

such that ∇v and ∇u satisfy a local Hölder-condition on Ω_0. Precisely:

$$(u, v) \in C^{1,\alpha_1}(\Omega) \times C^{1,\alpha_2}(\Omega, \mathbb{R}^2) \quad \textit{for all pairs} \quad (\alpha_1, \alpha_2) \in (0, 1) \times (0, 1).$$

Moreover,

$$(u, v) \in W^{2,2}_{\mathrm{loc}}(\Omega) \times W^{2,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^2).$$

For the set $\Omega - \Omega_0$ of possible singularities it further holds

$$\mathcal{H}\text{-dim}(\Omega - \Omega_0) = 0, \tag{6.24}$$

which means that the ε-dimensional Hausdorff measure $\mathcal{H}^\varepsilon(\Omega_0)$ is zero for every $\varepsilon > 0$.

Remark 6.2.1
For $D \neq \emptyset$ the statement of Theorem 6.2.1 still holds if the modified problem $(\widehat{\mathcal{P}})$ is considered, where (6.1) is replaced with (6.2) and F satisfies the structure condition (F4).

Remark 6.2.2
Let us mention once more that, in contrast to Theorems 5.1.1 and 5.2.1, the statements of Theorems 1.2.1 and 6.2.1 crucially depend on the planarity assumption $\Omega \subset \mathbb{R}^2$.

Proof of Theorem 6.2.1. Throughout the following, we use summation convention with respect to the index $i \in \{1,2\}$ and denote by c a generic positive constant. We follow the ideas in [36], where, based on results by Frehse and Seregin (see [48] and [49]), partial $C^{1,\alpha}$-regularity of the minimizer of uncoupled first-order problems was established. The proof is split into two parts, where we discuss the regularity of v and u separately.

Part 1. We begin with the following observation concerning the sequence (u_δ, v_δ) as introduced in Lemma 5.2.1:

Lemma 6.2.1
Under the assumptions of Lemma 6.1.1 it holds for any $s \in [1,2)$:

$$(u_\delta, v_\delta) \text{ is uniformly bounded in } W^{2,s}_{\mathrm{loc}}(\Omega) \times W^{2,s}_{\mathrm{loc}}(\Omega, \mathbb{R}^2). \tag{6.25}$$

Moreover,

$$\omega_\delta := \big(1 + |\nabla v_\delta|\big)^{\frac{\mu}{2}} \text{ is uniformly bounded in } W^{1,2}_{\mathrm{loc}}(\Omega). \tag{6.26}$$

Proof of the lemma. Ad (6.25). Recalling the definition of φ_δ from Lemma 6.1.1 as well as inequality (F2)$'$, we see that the uniform boundedness of $\nabla\varphi_\delta$ in $L^2_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$, which is obtained from (6.14), implies that for any compact subset $\Omega^* \Subset \Omega$, there is a constant $c(\Omega^*) > 0$ (independent of δ!) such that

$$\int_{\Omega^*} \frac{|\nabla^2 v_\delta|^2}{\big(1 + |\nabla v_\delta|\big)^\mu}\,\mathrm{d}x \le c(\Omega^*).$$

Let now $s \in (1,2)$ be arbitrary. We may write

$$\int_{\Omega^*} |\nabla^2 v_\delta|^s\,\mathrm{d}x = \int_{\Omega^*} \left(\frac{|\nabla^2 v_\delta|^2}{\big(1 + |\nabla v_\delta|\big)^\mu}\right)^{\frac{s}{2}} \big(1 + |\nabla v_\delta|\big)^{\mu\frac{s}{2}}\,\mathrm{d}x$$

and an application of Hölder's inequality yields

$$\int_{\Omega^*} |\nabla^2 v_\delta|^s\,\mathrm{d}x \le \left(\int_{\Omega^*} \frac{|\nabla^2 v_\delta|^2}{\big(1 + |\nabla v_\delta|\big)^\mu}\,\mathrm{d}x\right)^{\frac{s}{2}} \left(\int_{\Omega^*} \big(1 + |\nabla v_\delta|\big)^{\frac{2-s}{2}}\,\mathrm{d}x\right)^{\frac{2}{2-s}},$$

so that $v_\delta \in W^{2,s}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ follows from (6.3) and (6.14). The same argument works for u_δ if we replace φ_δ with $\widetilde{\varphi}_\delta$ above and use (6.21) instead of (6.14) (when μ is replaced with ν, of course).

We continue with (6.26). Let $B_{2R}(x_0) \subset \Omega$ and η be as in (4.3). Setting

$$\Gamma_\delta := 1 + |\nabla v_\delta|^2,$$

we observe

$$\begin{aligned}\int\limits_{B_R(x_0)} |\nabla\omega_\delta|^2\,\mathrm{d}x \le c \int\limits_{B_R(x_0)} &\big(1+|\nabla v_\delta|\big)^{\mu-2}|\nabla^2 v_\delta|^2\,\mathrm{d}x\\ = c\int\limits_{B_R(x_0)} &\big(1+|\nabla v_\delta|\big)^{-\mu}|\nabla^2 v_\delta|^2\big(1+|\nabla v_\delta|\big)^{2\mu-2}\,\mathrm{d}x\\ \overset{(\mathrm{F2})'}{\le} c \int\limits_{B_{2R}(x_0)} &\eta^2 D^2F_\delta(\nabla v_\delta)\big(D_i\nabla v_\delta, D_i\nabla v_\delta\big)\Gamma_\delta^{\mu-1}\,\mathrm{d}x.\end{aligned} \tag{6.27}$$

Choosing $\psi = D_i(\eta^2 D_i v_\delta \Gamma_\delta^{\mu-1})$ in (EL1) yields

$$\begin{aligned}&\int\limits_{B_{2R}(x_0)} \eta^2 D^2F_\delta(\nabla v_\delta)\big(D_i\nabla v_\delta, D_i\nabla v_\delta\big)\Gamma_\delta^{\mu-1}\,\mathrm{d}x\\ &=-\beta\int\limits_{B_{2R}(x_0)} DG(\nabla u_\delta - v_\delta)\cdot D_i(\eta^2 D_i v_\delta\Gamma_\delta^{\mu-1})\,\mathrm{d}x\\ &\quad-2\eta\int\limits_{B_{2R}(x_0)} D^2F_\delta(\nabla v_\delta)\big(D_i\nabla v_\delta, D_i v_\delta\otimes\nabla\eta\big)\Gamma_\delta^{\mu-1}\,\mathrm{d}x\\ &\quad-\int\limits_{B_{2R}(x_0)} D^2F_\delta(\nabla v_\delta)\big(D_i\nabla v_\delta, D_i v_\delta\otimes\nabla\Gamma_\delta^{\mu-1}\big)\eta^2\,\mathrm{d}x\\ &=:-I_1-I_2-I_3.\end{aligned}$$

We start with the term I_2. Applying the Cauchy-Schwarz inequality to the bilinear form $D^2F_\delta(\cdot,\cdot)$ followed by Young's inequality yields

$$\begin{aligned}|I_2| \le& c\Bigg[\int\limits_{B_{2R}(x_0)} D^2F_\delta(\nabla v_\delta)(D_i\nabla v_\delta, D_i\nabla v_\delta)\eta^2\,\mathrm{d}x\\ &+\int\limits_{B_{2R}(x_0)} D^2F_\delta(\nabla v_\delta)(D_i v_\delta\otimes\nabla\eta, D_i v_\delta\otimes\nabla\eta)\Gamma_\delta^{2\mu-2}\,\mathrm{d}x\Bigg]\\ \le& \frac{c}{R^2}\Bigg[\int\limits_{B_{2R}(x_0)}\Theta_\delta^2\eta^2\,\mathrm{d}x+\int\limits_{B_{2R}(x_0)}\delta|\nabla v_\delta|^2\Gamma_\delta^{2\mu-2}+\frac{1}{1+|\nabla v_\delta|}|\nabla v_\delta|^2\Gamma_\delta^{2\mu-2}\,\mathrm{d}x\Bigg],\end{aligned}$$

which is bounded due to (6.13) and (6.14). Let us continue with I_3. At this point, we make use of the structure condition (F4) which enables us to write (cf. the calculation on the bottom of page 62 in [67]):

$$\begin{aligned}&D^2F_\delta(\nabla v_\delta)\big(D_i\nabla v_\delta, D_i v_\delta\otimes\nabla\Gamma_\delta^{\mu-1}\big)\eta^2\\ &\qquad=(\mu-1)\frac{1}{2}D^2F_\delta(\nabla v_\delta)\big(e_i\otimes\nabla|\nabla v_\delta|^2, e_i\otimes\nabla|\nabla v_\delta|^2\big)\Gamma_\delta^{\mu-2}>0,\end{aligned}$$

where e_i denotes the canonical basis of $\mathbb{R}^2$. Therefore, we may just neglect the term I_3 and it remains to give a bound on the quantity I_1. We note that due to the boundedness of $DG(\nabla u_\delta - v_\delta)$ it holds

$$
\begin{aligned}
|I_1| &\le c \int\limits_{B_{2R}(x_0)} \big|D_i(\eta^2 D_i v_\delta \Gamma_\delta^{\mu-1})\big| \,\mathrm{d}x \\
&\le \frac{c}{R}\left[\int\limits_{B_{2R}(x_0)} |\nabla v_\delta|\Gamma_\delta^{\mu-1}\,\mathrm{d}x + \int\limits_{B_{2R}(x_0)} \eta^2|\nabla^2 v_\delta|\Gamma_\delta^{\mu-1}\,\mathrm{d}x\right].
\end{aligned}
\tag{6.28}
$$

The first term in the bracket is bounded by (6.14). For the second one, we note that an application of Young's inequality yields

$$
\begin{aligned}
\int\limits_{B_{2R}(x_0)} \eta^2|\nabla^2 v_\delta|\Gamma_\delta^{\mu-1}\,\mathrm{d}x &\le c\left[\int\limits_{B_{2R}(x_0)} \eta^2\frac{|\nabla^2 v_\delta|^2}{\big(1+|\nabla v_\delta|\big)^\mu}\,\mathrm{d}x + \int\limits_{B_{2R}(x_0)} \eta^2\Gamma_\delta^{\frac{5}{2}\mu-2}\,\mathrm{d}x\right] \\
&\overset{(\mathrm{F2})'}{\le} c\left[\int\limits_{\Omega} \Theta_\delta^2\eta^2\,\mathrm{d}x + \int\limits_{B_{2R}(x_0)} \Gamma_\delta^{\frac{5}{2}\mu-2}\,\mathrm{d}x\right],
\end{aligned}
$$

which is bounded by (6.13) and (6.14). Consequently, (6.27) implies (6.26) of Lemma 6.2.1. □

We continue with the proof of Theorem 6.2.1. In the differentiated Euler-Lagrange equation (EL1)' (see p. 110), we now choose $\psi = \eta^2\big(D_i v_\delta - \overline{D_i v_\delta}\big)$, where we set

$$\overline{D_i v_\delta} := \fint_\Omega D_i v_\delta\,\mathrm{d}x$$

and $\eta \in C_0^\infty(\Omega)$ satisfies the set of conditions from (4.3). Denoting by T the annulus $B_{2R}(x_0) - B_R(x_0)$, (EL1)' reads as (remember $\eta \equiv$ const. outside T)

$$
\begin{aligned}
0 = \int\limits_\Omega \Theta_\delta^2\eta^2\,\mathrm{d}x &+ 2\int\limits_T D^2F_\delta(\nabla v_\delta)\big(D_i\nabla v_\delta, (D_i v_\delta - \overline{D_i v_\delta})\otimes\nabla\eta\big)\eta\,\mathrm{d}x \\
&+ \beta\int\limits_\Omega \underbrace{DG(\nabla u_\delta - v_\delta)}_{\in L^\infty(\Omega,\mathbb{R}^2)}\cdot D_i\big(\eta^2(D_i v_\delta - \overline{D_i v_\delta})\big)\,\mathrm{d}x
\end{aligned}
$$

and we infer that for some constant $c > 0$, independent of η, it holds

$$
\begin{aligned}
\int\limits_\Omega \Theta_\delta^2\eta^2\,\mathrm{d}x \le c\Bigg[&\int\limits_T \big|D^2F_\delta(\nabla v_\delta)\big(D_i\nabla v_\delta, (D_i v_\delta - \overline{D_i v_\delta})\otimes\nabla\eta\big)\eta\big|\,\mathrm{d}x \\
&+\beta\int\limits_\Omega \Big|D_i\big(\eta^2(D_i v_\delta - \overline{D_i v_\delta})\big)\Big|\,\mathrm{d}x\Bigg] =: c\Big[S_1 + S_2\Big].
\end{aligned}
\tag{6.29}
$$

In S_1, an application the Cauchy-Schwarz inequality to the bilinear form $D^2F_\delta(\cdot,\cdot)$ yields

$$S_1 \le \int_T \overbrace{D^2F_\delta(v_\delta)\big(D_i\nabla v_\delta, D_i\nabla v_\delta\big)^{\frac{1}{2}}}^{=\Theta_\delta}$$
$$\cdot D^2F_\delta(\nabla v_\delta)\big((D_iv_\delta - \overline{D_iv_\delta})\otimes\nabla\eta, (D_iv_\delta - \overline{D_iv_\delta})\nabla\eta\big)^{\frac{1}{2}}\,\mathrm{d}x.$$

Now, by Hölder's inequality, we infer

$$\begin{aligned} S_1 \le & \left[\int_T \Theta_\delta^2\,\mathrm{d}x\right]^{\frac{1}{2}} \\ & \cdot\left[\int_T D^2F_\delta(\nabla v_\delta)\big(\nabla\eta\otimes(D_iv_\delta - \overline{D_iv_\delta}), \nabla\eta\otimes(D_iv_\delta - \overline{D_iv_\delta})\big)\,\mathrm{d}x\right]^{\frac{1}{2}} \\ \le & c\left[\int_T \Theta_\delta^2\,\mathrm{d}x\right]^{\frac{1}{2}}\left[\int_T |\nabla\eta|^2|D_iv_\delta - \overline{D_iv_\delta}|^2\,\mathrm{d}x\right]^{\frac{1}{2}} \end{aligned}$$

and hence

$$\begin{aligned} S_1 &\le \frac{c}{R}\left[\int_T \Theta_\delta^2\,\mathrm{d}x\right]^{\frac{1}{2}}\left[\int_T |D_iv_\delta - \overline{D_iv_\delta}|^2\,\mathrm{d}x\right]^{\frac{1}{2}} \\ &\le \frac{c}{R}\left[\int_T \Theta_\delta^2\,\mathrm{d}x\right]^{\frac{1}{2}}\int_T |\nabla^2 v_\delta|\,\mathrm{d}x, \end{aligned} \tag{6.30}$$

where the Sobolev-Poincaré inequality has been applied in the last step. Next, we note that by (F2)$'$ we have

$$\big(1+|\nabla v_\delta|\big)^{-\frac{\mu}{2}}|\nabla^2 v_\delta| \le c\Theta_\delta$$

and therefore

$$|\nabla^2 v_\delta| \le c\Theta_\delta\big(1+|\nabla v_\delta|\big)^{\frac{\mu}{2}} = c\Theta_\delta\omega_\delta, \tag{6.31}$$

with ω_δ as in Lemma 6.2.1. Consequently, (6.30) reads as

$$S_1 \le \frac{c}{R}\left[\int_T \Theta_\delta^2\,\mathrm{d}x\right]^{1/2}\int_T |\nabla^2 v_\delta|\,\mathrm{d}x \overset{(6.31)}{\le} \frac{c}{R}\left[\int_T \Theta_\delta^2\,\mathrm{d}x\right]^{1/2}\int_T \Theta_\delta\omega_\delta\,\mathrm{d}x. \tag{6.32}$$

The term S_2 can be treated as follows:

$$S_2 \le c\left[\frac{1}{R}\int_T |D_iv_\delta - \overline{D_iv_\delta}|\,\mathrm{d}x + \int_\Omega \eta^2|\nabla^2 v_\delta|\,\mathrm{d}x\right]$$

which, after an application of Poincaré's inequality and using (6.31), becomes

$$S_2 \le c \left[\int_T \Theta_\delta \omega_\delta \,\mathrm{d}x + \int_\Omega \eta^2 \Theta_\delta \omega_\delta \,\mathrm{d}x \right].$$

The estimates of S_1 and S_2 together with (6.29) now establish the crucial inequality

$$\int_{B_{2R}(x_0)} \eta^2 \Theta_\delta^2 \,\mathrm{d}x \le \frac{c}{R} \left[\int_T \Theta_\delta^2 \,\mathrm{d}x + R^2 \right]^{\frac{1}{2}} \int_T \Theta_\delta \omega_\delta \,\mathrm{d}x + c \int_{B_{2R}(x_0)} \eta^2 \Theta_\delta \omega_\delta \,\mathrm{d}x, \tag{6.33}$$

which holds for all radii $0 < R < R_0$ and all points $x_0 \in \Omega$ for which $B_{2R_0}(x_0) \Subset \Omega$, with a constant c only depending on R_0. To the last term, we apply Young's and Hölder's inequality to get

$$\int_{B_{2R}(x_0)} \eta^2 \Theta_\delta \omega_\delta \,\mathrm{d}x \le c \int_{B_{2R}(x_0)} \Theta_\delta^2 \eta^2 \,\mathrm{d}x + cR^{\frac{2}{q}}, \tag{6.34}$$

where, due to (6.14), the exponent q can be chosen arbitrarily in $(1, \infty)$. We fix exponents $\gamma < \widetilde{\gamma} < 2$ and observe that from (6.34) and (6.33) (note that we may assume $R < 1$) we obtain the estimate:

$$\int_{B_R(x_0)} \Theta_\delta^2 \,\mathrm{d}x \le \frac{c}{R} \left[\int_T \Theta_\delta^2 \,\mathrm{d}x + R^\gamma \right]^{\frac{1}{2}} \int_T \Theta_\delta \omega_\delta \,\mathrm{d}x + cR^{\widetilde{\gamma}}. \tag{6.35}$$

According to [83], p. 295, this inequality together with $\Theta_\delta \in L^2_{\mathrm{loc}}(\Omega)$ and $\omega_\delta \in W^{1,2}_{\mathrm{loc}}(\Omega)$ (cf. (6.26)) suffices to deduce the following growth estimate for the quantity Θ_δ:

$$\int_{B_R(x_0)} \Theta_\delta^2 \,\mathrm{d}x \le c \frac{1}{\ln\left(\frac{1}{R}\right)^t} \qquad \text{for all } t \ge 1, \tag{6.36}$$

which is valid on all balls $B_R(x_0)$ such that $B_{2R}(x_0) \Subset \Omega$ with radius $0 < R < R_0$, and with a constant c that only depends on the choice of R_0. Observing that for $\sigma_\delta = DF_\delta(\nabla v_\delta)$ (cf. (5.20)) it holds

$$\begin{aligned} |\nabla \sigma_\delta|^2 &= D_i \sigma_\delta : D_i \sigma_\delta = D^2 F_\delta(\nabla v_\delta)(D_i \nabla v_\delta, D_i \sigma_\delta) \\ &\le \Big(D^2 F_\delta(\nabla v_\delta)(D_i \nabla v_\delta, D_i \nabla v_\delta) \Big)^{1/2} \Big(D^2 F_\delta(\nabla v_\delta)(D_i \sigma_\delta, D_i \sigma_\delta) \Big)^{1/2} \\ &\le c \Theta_\delta |\nabla \sigma_\delta|, \end{aligned} \tag{6.37}$$

and thereby

$$|\nabla \sigma_\delta| \le c \Theta_\delta, \tag{6.38}$$

the estimate (6.36) implies

$$\int\limits_{B_R(x_0)} |\nabla\sigma_\delta|^2 \,\mathrm{d}x \le c\frac{1}{\ln\left(\frac{1}{R}\right)^t} \text{ for all } t \ge 1.$$

According to [49] (see (31) on p. 287), this estimate implies the continuity of each component $(\sigma_\delta)_{ij}$ $(i,j \in \{1,2\})$ of the tensor σ_δ on every ball $B_R(x_0)$ with $R < R_0$ (cf. also Lemma 6 and 7 in [86]), with modulus of continuity given by

$$\operatorname{osc}_{B_R(x_0)}(\sigma_\delta)_{ij} := \sup_{x\in B_R(x_0)}(\sigma_\delta)_{ij}(x) - \inf_{y\in B_R(x_0)}(\sigma_\delta)_{ij}(y) \le K|\ln R|^{1-\frac{t}{2}}, \quad (6.39)$$

and where the constant K does not depend on δ. The uniform boundedness of σ_δ in $L^2(\Omega,\mathbb{R}^{2\times 2})$ (cf. (5.25)) along with (6.39) now implies

$$\sup_{\delta\in(0,1)} \|\sigma_\delta\|_{L^\infty(B_R(x_0))} < \infty. \quad (6.40)$$

Furthermore, (6.39) yields the equicontinuity of the σ_δ on any compact subset $\Omega^* \Subset \Omega$. The Arzelà-Ascoli compactness-theorem therefore implies the existence of a continuous function σ such that

$$\sigma_\delta \rightrightarrows \sigma \text{ locally,}$$

at least for a suitable subsequence $\delta \downarrow 0$.

We next observe that, due to (6.25) and the Rellich-Kondrachov embedding theorem, we may (after possibly passing to another subsequence) assume that

$$\nabla v_\delta \to \nabla v \text{ in } L^1_{\mathrm{loc}}(\Omega,\mathbb{R}^{2\times 2}) \text{ and a.e.,}$$

where $(u,v) \in W^{1,1}(\Omega)\times W^{1,1}(\Omega,\mathbb{R}^2)$ is the unique E-minimizer from Theorem 6.1.1. For almost all $x \in \Omega$, we therefore have the equation (note that due to $\delta\nabla v_\delta \to 0$ in $L^2(\Omega,\mathbb{R}^{2\times 2})$ it holds $\delta\nabla v_\delta \to 0$ a.e.)

$$DF(\nabla v(x)) = \lim_{\delta\downarrow 0} DF_\delta(\nabla v_\delta(x)) = \lim_{\delta\downarrow 0}\sigma_\delta(x) = \sigma(x). \quad (6.41)$$

By the strict convexity of F, for all $p \neq q \in \mathbb{R}^{2\times 2}$ it holds that

$$\big[DF(p) - DF(q)\big] : (p-q) > 0,$$

so that the mapping $DF : \mathbb{R}^{2\times 2} \to \operatorname{Im}(DF)$ is one-to-one and the inverse function theorem (recall that $D^2F(p)$ is positive definite for all $p \in \mathbb{R}^{2\times 2}$ by (F2)$'$) it therefore follows that DF is a C^1-diffeomorphism between $\mathbb{R}^{2\times 2}$ and the open set $\operatorname{Im}(DF)$. The equation (6.41) thus gives

$$\nabla v(x) = DF^{-1}(\sigma(x)) \quad \text{for almost all } x \in \Omega, \quad (6.42)$$

i.e. ∇v has a continuous representative on Ω. We now want to estimate the size of the singular set. In the following calculations, we identify ∇v with its precise representative (cf. [75], Definition on p. 46), i.e. for all $x \in \Omega$ we set

$$\nabla v(x) = \begin{cases} \lim_{r \downarrow 0} \fint_{B_r(x_0)} \nabla v \,\mathrm{d}x & \text{if this limit exists and is finite,} \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\Omega_0 := \left\{ x \in \Omega \,:\, \lim_{r \downarrow 0} \fint_{B_r(x_0)} \nabla v \,\mathrm{d}x \text{ exists in } \mathbb{R}^{2\times 2} \right\}.$$

We claim that Ω_0 coincides with the set of all points $x \in \Omega$, for which the equality (6.41) holds. Let therefore x_0 be some point in Ω_0. Since (6.42) holds a.e. on Ω, it follows that

$$\nabla v(x_0) = \lim_{\rho \downarrow 0} \fint_{B_\rho(x_0)} \nabla v(x) \,\mathrm{d}x = \lim_{\rho \downarrow 0} \fint_{B_\rho(x_0)} DF^{-1}(\sigma(x)) \,\mathrm{d}x = DF^{-1}(\sigma(x_0)),$$

since $DF^{-1}(\sigma)$ is continuous. If conversely x_0 is a point for which the equation (6.41) holds, then it is clear that $x_0 \in \Omega_0$. Thus we have

$$\nabla v(x) = DF^{-1}(\sigma(x)) \quad \text{for all } x \in \Omega_0.$$

That $\Omega - \Omega_0$ does indeed have Hausdorff-dimension 0 is now a immediate consequence of Theorem 2.1 on p. 100 of [46] and $v \in W^{2,s}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$, $s \in [1, 2)$. Moreover, the set Ω_0 is an open subset of Ω: let $x_0 \in \Omega_0$ be some point. Our preceding considerations show that

$$x_0 \in \sigma^{-1}(\mathrm{Im}(DF)).$$

Since σ is continuous and $\mathrm{Im}(DF)$ is an open set, we find a small radius $\varepsilon > 0$ such that $B_\varepsilon(x_0) \subset \sigma^{-1}(\mathrm{Im}(DF))$. But then on $B_\varepsilon(x_0)$, $DF^{-1}(\sigma(x))$ is a continuous representative of ∇v, hence $B_\varepsilon(x_0) \subset \Omega_0$.

Now that ∇v is proved to be locally bounded on the set Ω_0, the deduction of the Hölder continuity of ∇v follows from standard results: we first observe that $|\nabla v| \in L^\infty_{\mathrm{loc}}(\Omega_0)$ implies that for any compact subset $\Omega' \Subset \Omega_0$ we have

$$\frac{1}{\big(1 + \|\nabla v\|_{L^\infty(\Omega')}\big)^\mu} \int_{\Omega'} |\nabla^2 v|^2 \,\mathrm{d}x \le \int_{\Omega'} \frac{|\nabla^2 v|^2}{\big(1 + |\nabla v|\big)^\mu} \,\mathrm{d}x \le c \int_{\Omega'} \Theta_\delta^2 \,\mathrm{d}x < \infty,$$

hence

$$v \in W^{2,2}_{\mathrm{loc}}(\Omega_0).$$

Going back to inequality (6.33), fix $B_{2R}(x_0) \subset \Omega_0$. Since $|\nabla v| \in L^\infty_{\mathrm{loc}}(\Omega_0)$ and thereby $|\omega_\delta| \in L^\infty_{\mathrm{loc}}(\Omega_0)$, we obtain from Hölder' inequality

$$\int_{B_{2R}(x_0)} \eta^2 \Theta_\delta^2 \,\mathrm{d}x \le c \left[\int_T \Theta_\delta^2 \,\mathrm{d}x + R^2 \right]^{\frac{1}{2}} \left(\int_T \Theta_\delta^2 \,\mathrm{d}x \right)^{\frac{1}{2}} + cR \left(\int_{B_{2R}(x_0)} \eta^2 \Theta_\delta^2 \,\mathrm{d}x \right)^{\frac{1}{2}}$$

and an application of Young's inequality to the products on the right-hand side then yields (after absorbing terms on the left-hand side)

$$\int\limits_{B_R(x_0)} \Theta_\delta^2 \,\mathrm{d}x \le \int\limits_{B_{2R}(x_0)} \eta^2 \Theta_\delta^2 \,\mathrm{d}x \le c\left[\int\limits_T \Theta_\delta^2 \,\mathrm{d}x + R^2\right].$$

"Filling the hole" on the right-hand side then leads to the estimate

$$\int\limits_{B_R(x_0)} \Theta_\delta^2 \,\mathrm{d}x \le \frac{c}{c+1} \int\limits_{B_{2R}(x_0)} \Theta_\delta^2 \,\mathrm{d}x + cR^2, \tag{6.43}$$

which, according to [46], p. 164, suffices to deduce the following Morrey-type inequality for the quantity Θ_δ^2

$$\int\limits_{B_R(x_0)} \Theta_\delta^2 \,\mathrm{d}x \le c\left(\frac{R}{R_0}\right)^\lambda \left[\int\limits_{B_{2R}(x_0)} \Theta_\delta^2 \,\mathrm{d}x + 2(R_0)^\lambda\right], \tag{6.44}$$

valid for all $R \le R_0$ and with exponent $\lambda = \log_2\left(\frac{c+1}{c}\right) \in (0,1)$. Along with $\Theta_\delta \in L^2_{\mathrm{loc}}(\Omega)$ uniformly (by (6.13)), the inequalities (6.44) and (6.38) imply

$$\int\limits_{B_R(x_0)} |\nabla \sigma_\delta|^2 \,\mathrm{d}x \le cR^\lambda.$$

Passing to the limit $\delta \downarrow 0$ in the above inequality yields

$$\int\limits_{B_R(x_0)} |\nabla \sigma|^2 \,\mathrm{d}x \le \liminf_{\delta\downarrow 0} \int\limits_{B_R(x_0)} |\nabla \sigma_\delta|^2 \,\mathrm{d}x \le cR^\lambda,$$

which, by Theorem 1.1 on p.64 of [46] implies $\sigma \in C^{0,\frac{\lambda}{2}}(B_R(x_0), \mathbb{R}^{2\times 2})$. Since for all $x \in \Omega_0$ it holds

$$\nabla v(x) = DF^{-1}(\sigma(x))$$

and DF^{-1} is locally of class $C^{0,1}$, we thus infer that

$$\nabla v \in C^{0,\frac{\lambda}{2}}(\Omega_0, \mathbb{R}^{2\times 2}). \tag{6.45}$$

Let now $\gamma \in \{1,2\}$ be fixed and set $w := D_\gamma v$. Since (u,v) minimizes E, we have that, for any $\varphi \in C_0^\infty(\Omega_0, \mathbb{R}^2)$, w solves the Euler equation

$$\int\limits_{\Omega_0} \alpha D^2 F(\nabla v)\big(\nabla w, \nabla\varphi\big) \,\mathrm{d}x + \beta \int\limits_{\Omega_0} DG(\nabla u - v)\cdot \varphi \,\mathrm{d}x = 0. \tag{6.46}$$

Setting

$$A^{ij}_{kl}(x) := \frac{\partial^2 F}{\partial p^i_k \partial p^k_l}(\nabla v(x)) \quad \text{as well as} \quad f^i_k(x) = \delta_{\gamma k}\frac{\partial G}{\partial x_i}(\nabla u(x) - v(x))$$

($\delta_{\gamma k}$ being the Kronecker symbol), we therefore find that w is a weak solution of the system

$$D_k\big(A^{ij}_{kl}(x)D_j w^l\big) + D_k f^i_k = 0, \quad i \in \{1,2\}, \tag{6.47}$$

where the sum is taken with respect to the indices $j,k,l \in \{1,2\}$. Let now $\Omega^* \Subset \Omega_0$ be a compact subset of Ω_0. By (6.46), we have that the coefficients $A^{ij}_{kl}(x)$ are uniformly elliptic and continuous on Ω^*. Moreover, the boundedness of DG implies $f^i_k \in L^\infty(\Omega^*)$. We may therefore quote Theorem 3.1 from p. 87 in [46] (see also Footnote 11 on p. 88) to infer that

$$w = D_\gamma v \in C^{0,1-\frac{2}{p}}(\Omega^*) \quad \text{ for any } p > n,$$

hence

$$v \in C^{1,\alpha_1}(\Omega_0) \quad \text{ for any } \alpha_1 \in (0,1).$$

Part 2. Now we discuss the regularity of the function u. In the following calculations, we restrict ourselves to the open subset $\Omega_0 \subset \Omega$ on which we have already established local Hölder-continuity of v. We introduce a (formally) new sequence $(\widetilde{u}_\delta)$ of δ-regularizers which solve

$$\beta \int\limits_{\Omega_0} G(\nabla w - v)\,\mathrm{d}x + \int\limits_{\Omega_0} (w-f)^2\,\mathrm{d}x + \frac{\delta}{2}\int\limits_{\Omega_0} |\nabla w|^2\,\mathrm{d}x \to \min \text{ in } W^{1,2}(\Omega_0),$$

where v is fixed as the Hölder continuous minimizer from above. We claim that the sequence $(\widetilde{u}_\delta, v)$ is E-minimizing. Indeed, for all $w \in W^{1,2}(\Omega_0)$ it holds

$$\begin{aligned}
&\liminf_{\delta\downarrow 0} E(\widetilde{u}_\delta, v) = \liminf_{\delta\downarrow 0} \alpha \int\limits_{\Omega_0} F(\nabla v)\,\mathrm{d}x + \beta \int\limits_{\Omega_0} G(\nabla \widetilde{u}_\delta - v)\,\mathrm{d}x + \int\limits_{\Omega_0} (\widetilde{u}_\delta - f)^2\,\mathrm{d}x \\
&\le \liminf_{\delta\downarrow 0} \alpha \int\limits_{\Omega_0} F(\nabla v)\,\mathrm{d}x + \beta \int\limits_{\Omega_0} G(\nabla \widetilde{u}_\delta - v)\,\mathrm{d}x + \int\limits_{\Omega_0} (\widetilde{u}_\delta - f)^2\,\mathrm{d}x + \frac{\delta}{2}\int\limits_{\Omega_0} |\nabla \widetilde{u}_\delta|^2\,\mathrm{d}x \\
&\le \liminf_{\delta\downarrow 0} \alpha \int\limits_{\Omega_0} F(\nabla v)\,\mathrm{d}x + \beta \int\limits_{\Omega_0} G(\nabla w - v)\,\mathrm{d}x + \int\limits_{\Omega_0} (w - f)^2\,\mathrm{d}x + \frac{\delta}{2}\int\limits_{\Omega_0} |\nabla w|^2\,\mathrm{d}x \\
&= E(w,v).
\end{aligned}$$

If now φ_k denotes a sequence of $W^{1,2}(\Omega_0)$-functions which converges to u in $W^{1,1}(\Omega_0)$, it follows

$$\liminf_{\delta\downarrow 0} E(\widetilde{u}_\delta, v) \le E(\varphi_k, v) \xrightarrow{k\to\infty} E(u,v) = \inf_{\mathcal{V}} E,$$

which proves that (a subsequence of) $(\widetilde{u}_\delta, v)$ in fact minimizes the functional E. Therefore, the uniqueness statement from Theorem 2.1.1 c) implies

$$\widetilde{u}_\delta \to u \text{ in } L^1(\Omega_0) \text{ and a.e.},$$

at least for another subsequence $\delta \downarrow 0$.

Repeating the arguments from the proofs of Lemma 6.1.1 and Lemma 6.2.1 with u_δ replaced by $\widetilde{u}_\delta$ (and v_δ by v) we can show

$$\widehat{\Theta}_\delta := D^2G_\delta(\nabla\widetilde{u}_\delta - v)\big(D_i\nabla\widetilde{u}_\delta, D_i\nabla\widetilde{u}_\delta\big)^{\frac{1}{2}} \in W^{1,2}_{\mathrm{loc}}(\Omega_0), \tag{6.48}$$

$$\widehat{\varphi}_\delta := \big(1+|\nabla\widetilde{u}_\delta|\big)^{1-\frac{\nu}{2}} \in W^{1,2}_{\mathrm{loc}}(\Omega_0), \tag{6.49}$$

$$\widetilde{u}_\delta \in W^{2,s}_{\mathrm{loc}}(\Omega_0) \text{ for all } s\in(1,2), \tag{6.50}$$

where we set $G_\delta(x) = |x|^2 + \beta G(x)$ for $x \in \mathbb{R}^2$. Furthermore, we claim:

Lemma 6.2.2
It holds

$$\widehat{\omega}_\delta := \big(1+|\nabla\widetilde{u}_\delta|\big)^{\frac{\nu}{2}} \in W^{1,2}_{\mathrm{loc}}(\Omega_0).$$

uniformly with respect to the parameter δ

Proof of the lemma. By their minimality, for any $\varphi \in C^\infty_0(\Omega_0)$, $\widetilde{u}_\delta$ satisfies the Euler equation

$$\begin{aligned}&\delta\int_{\Omega_0} D_i\nabla\widetilde{u}_\delta\cdot\nabla\varphi\,\mathrm{d}x+\beta\int_{\Omega_0} D^2G(\nabla\widetilde{u}_\delta - v)\big(D_i\nabla\widetilde{u}_\delta - D_iv, \nabla\varphi\big)\,\mathrm{d}x\\ &= 2\int_{\Omega_0}(\widetilde{u}_\delta - f)D_i\varphi\,\mathrm{d}x,\end{aligned} \tag{6.51}$$

which, by an approximation argument, holds even for all $\varphi \in W^{1,2}(\Omega)$ with compact support. Setting

$$\widetilde{\Gamma}_\delta := 1 + |\nabla\widetilde{u}_\delta|^2,$$

we may therefore choose $\varphi = \eta^2 D_i\widetilde{u}_\delta\widetilde{\Gamma}_\delta^{\nu-1}$ in (6.51), where $\eta \in C^\infty_0(\Omega_0)$ satisfies (4.3) (now with $x_0 \in \Omega_0$ and $B_{2R}(x_0) \Subset \Omega_0$). We obtain:

$$\begin{aligned}&\int_{\Omega_0} D^2G_\delta(\nabla\widetilde{u}_\delta - v)\big(D_i\nabla\widetilde{u}_\delta, \nabla(\eta^2 D_i\widetilde{u}_\delta\widetilde{\Gamma}_\delta^{\nu-1})\big)\,\mathrm{d}x\\ &\quad-\beta\int_{\Omega_0} D^2G\big(D_iv, \nabla(\eta^2 D_i\widetilde{u}_\delta\widetilde{\Gamma}_\delta^{\nu-1})\big)\\ &=2\int_{\Omega_0}(\widetilde{u}_\delta - f)D_i(\eta^2 D_i\widetilde{u}_\delta\widetilde{\Gamma}_\delta^{\nu-1})\,\mathrm{d}x,\end{aligned}$$

which can be expanded to

$$
\begin{aligned}
&\int\limits_{\Omega_0} D^2G_\delta(\nabla\widetilde{u}_\delta - v)\big(D_i\nabla\widetilde{u}_\delta, D_i\nabla\widetilde{u}_\delta\big)\eta^2\widetilde{\Gamma}_\delta^{\nu-1}\,\mathrm{d}x \\
&\qquad = -2\int\limits_{\Omega_0} D^2G_\delta(\nabla\widetilde{u}_\delta - v)\big(D_i\nabla\widetilde{u}_\delta, \eta\nabla\eta\big)D_i\widetilde{u}_\delta\widetilde{\Gamma}_\delta^{\nu-1}\,\mathrm{d}x \\
&\qquad\quad - \int\limits_{\Omega_0} D^2G_\delta(\nabla\widetilde{u}_\delta - v)\big(D_i\nabla\widetilde{u}_\delta, D_i\widetilde{u}_\delta\nabla(\widetilde{\Gamma}_\delta^{\nu-1})\big)\eta^2\,\mathrm{d}x \\
&\qquad\quad + \int\limits_{\Omega_0} D^2G(\nabla\widetilde{u}_\delta - v)\big(D_iv, \nabla(\eta^2D_i\widetilde{u}_\delta\widetilde{\Gamma}_\delta^{\nu-1})\big)\,\mathrm{d}x \\
&\qquad\quad + 2\int\limits_{\Omega_0} (\widetilde{u}_\delta - f)D_i(\eta^2D_i\widetilde{u}_\delta\widetilde{\Gamma}_\delta^{\nu-1})\,\mathrm{d}x. \\
&\qquad\quad =: -J_1 - J_2 + J_3 + J_4.
\end{aligned}
\tag{6.52}
$$

The quantity J_1 can be estimated just like the corresponding integral I_2 from the proof of Lemma 6.2.1. For J_2, we note that

$$
\begin{aligned}
&D^2G_\delta(\nabla\widetilde{u}_\delta - v)\big(D_i\nabla\widetilde{u}_\delta D_i\widetilde{u}_\delta, \nabla(\widetilde{\Gamma}_\delta^{\nu-1})\big) \\
&\quad = (\nu-1)\frac{1}{2}D^2G_\delta(\nabla\widetilde{u}_\delta - v)\big(\nabla|\nabla\widetilde{u}_\delta|^2, \nabla|\nabla\widetilde{u}_\delta|^2\big)\widetilde{\Gamma}_\delta^{\nu-2} > 0,
\end{aligned}
$$

so that we may neglect this term. Since $|D^2G(x)|$ is bounded and $|\nabla v| \in L^\infty_{\mathrm{loc}}(\Omega_0)$, we further have

$$
\begin{aligned}
|J_3| &\le \int\limits_{\Omega_0} |D^2G(\nabla\widetilde{u}_\delta - v)||\nabla v||\nabla(\eta^2D_i\widetilde{u}_\delta\widetilde{\Gamma}_\delta^{\nu-1})|\,\mathrm{d}x \\
&\le c\int\limits_{\Omega_0} |\nabla(\eta^2D_i\widetilde{u}_\delta\widetilde{\Gamma}_\delta^{\nu-1})|\,\mathrm{d}x \le \frac{c}{R}\left[\int\limits_{\Omega_0} \eta|\nabla\widetilde{u}_\delta|\widetilde{\Gamma}_\delta^{\nu-1}\,\mathrm{d}x + \int\limits_{\Omega_0} \eta^2|\nabla^2\widetilde{u}_\delta|\widetilde{\Gamma}_\delta^{\nu-1}\,\mathrm{d}x\right].
\end{aligned}
$$

The first summand is clearly bounded since $|\nabla\widetilde{u}_\delta| \in L^p_{\mathrm{loc}}(\Omega_0)$ uniformly for all $p \in [1,\infty)$ due to (6.49). For the second one, we note

$$
\begin{aligned}
\int\limits_{\Omega_0} \eta^2|\nabla^2\widetilde{u}_\delta|\widetilde{\Gamma}_\delta^{\nu-1}\,\mathrm{d}x &= \int\limits_{\Omega_0} \eta^2\frac{|\nabla^2\widetilde{u}_\delta|}{\big(1+|\nabla\widetilde{u}_\delta - v|\big)^{\frac{\nu}{2}}}\big(1+|\nabla\widetilde{u}_\delta - v|\big)^{\frac{\nu}{2}}\widetilde{\Gamma}_\delta^{\nu-1}\,\mathrm{d}x \\
&\overset{(\mathrm{G2})'}{\le} \int\limits_{\Omega_0} \eta^2\widehat{\Theta}_\delta^2\,\mathrm{d}x + \int\limits_{\Omega_0} \eta^2\big(1+|\nabla\widetilde{u}_\delta - v|\big)^{\nu}\widetilde{\Gamma}_\delta^{2\nu-2}\,\mathrm{d}x \\
&\le \int\limits_{\Omega_0} \eta^2\widehat{\Theta}_\delta^2\,\mathrm{d}x + \int\limits_{\Omega_0} \eta^2\widetilde{\Gamma}_\delta^{\frac{5}{2}\nu-2}\,\mathrm{d}x < \infty,
\end{aligned}
$$

where we have used $|\nabla v| \in L^\infty_{\mathrm{loc}}(\Omega_0)$, again. Now to J_4: an integration by parts yields

$$J_4 = \int\limits_{\Omega_0} \widetilde{u}_\delta D_i(\eta^2 D_i \widetilde{u}_\delta \widetilde{\Gamma}_\delta^{\nu-1})\,\mathrm{d}x - \int\limits_{\Omega_0} f D_i(\eta^2 D_i \widetilde{u}_\delta \widetilde{\Gamma}_\delta^{\nu-1})\,\mathrm{d}x$$
$$= -\int\limits_{\Omega_0} |D_i \widetilde{u}_\delta|^2 \eta^2 \widetilde{\Gamma}_\delta^{\nu-1}\,\mathrm{d}x - \int\limits_{\Omega_0} f D_i(\eta^2 D_i \widetilde{u}_\delta \widetilde{\Gamma}_\delta^{\nu-1})\,\mathrm{d}x.$$

The first term on the right-hand side, being strictly negative, is clearly negligible for the further calculations. For the second term, we note that due to our assumption $f \in L^\infty(\Omega)$ it follows

$$\left| \int\limits_{\Omega_0} f D_i(\eta D_i \widetilde{u}_\delta \widetilde{\Gamma}_\delta^{\nu-1})\,\mathrm{d}x \right| \leq \|f\|_\infty \int\limits_{\Omega_0} \big|D_i(\eta D_i \widetilde{u}_\delta \widetilde{\Gamma}_\delta^{\nu-1})\big|\,\mathrm{d}x$$

which can be treated like J_3 above. We thus infer that the left-hand side of (6.52) is bounded uniformly in δ, so that the statement of Lemma 6.2.2 follows from

$$\begin{aligned}
\int\limits_{B_R(x_0)} |\nabla \widehat{\omega}_\delta|^2\,\mathrm{d}x &\leq c \int\limits_{B_R(x_0)} \big(1+|\nabla \widetilde{u}_\delta|\big)^{\nu-2} |\nabla^2 \widetilde{u}_\delta|^2\,\mathrm{d}x \\
&= c \int\limits_{B_R(x_0)} \big(1+|\nabla \widetilde{u}_\delta|\big)^{-\nu} |\nabla^2 \widetilde{u}_\delta|^2 \big(1+|\nabla \widetilde{u}_\delta|\big)^{2\nu-2}\,\mathrm{d}x \\
&\leq c \int\limits_{B_R(x_0)} \big(1+|\nabla \widetilde{u}_\delta - v|\big)^{-\nu} |\nabla^2 \widetilde{u}_\delta|^2 \big(1+|\nabla \widetilde{u}_\delta|\big)^{2\nu-2}\,\mathrm{d}x \\
&\leq c \int\limits_{B_{2R}(x_0)} \eta^2 D^2 G_\delta(\nabla \widetilde{u}_\delta - v)\big(D_i \nabla \widetilde{u}_\delta, D_i \nabla \widetilde{u}_\delta\big) \widetilde{\Gamma}_\delta^{\nu-1}\,\mathrm{d}x.
\end{aligned}$$

□

Remark 6.2.3
Again we see that if $D \neq \emptyset$, the integration by parts in the estimate of the quantity J_4 above does not work. We therefore have to switch to a fidelity term of linear growth, which eliminates this problem.

Continuing with the proof of Theorem 6.2.1, we choose $\varphi = \eta^2(D_i \widetilde{u}_\delta - \overline{D_i \widetilde{u}_\delta})$ in

(6.51), which gives

$$\begin{aligned}
\int\limits_{\Omega_0} \widehat{\Theta}_\delta^2 \eta^2 \,\mathrm{d}x =& -2\beta \int\limits_{\Omega_0} D^2 G_\delta(\nabla \widetilde{u}_\delta - v)\big(D_i \nabla \widetilde{u}_\delta, \eta(D_i \widetilde{u}_\delta - \overline{D_i \widetilde{u}_\delta}) \otimes \nabla \eta\big) \,\mathrm{d}x \\
&+ \int\limits_{\Omega_0} (\widetilde{u}_\delta - f) D_i\big(\eta^2 (D_i \widetilde{u}_\delta - \overline{D_i \widetilde{u}_\delta})\big) \,\mathrm{d}x \\
&+ \beta \int\limits_{\Omega_0} D^2 G(\nabla \widetilde{u}_\delta - v)(D_i v, D_i \nabla \widetilde{u}_\delta) \eta^2 \,\mathrm{d}x \\
&+ 2\beta \int\limits_{\Omega_0} D^2 G(\nabla \widetilde{u}_\delta - v)\big(D_i v, \eta(D_i \widetilde{u}_\delta - \overline{D_i \widetilde{u}_\delta}) \otimes \nabla \eta\big) \,\mathrm{d}x \\
&=: \widetilde{S}_1 + \widetilde{S}_2 + \widetilde{S}_3 + \widetilde{S}_4.
\end{aligned}$$

We see that the terms $\widetilde{S}_1$ and $\widetilde{S}_2$ can be treated like the corresponding quantities S_1 and S_2 from the first part of the proof, with the result

$$\begin{aligned}
|S_1| &\le \frac{c}{R} \left(\int\limits_{\Omega_0} \eta^2 \widehat{\Theta}_\delta^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \int\limits_T \widehat{\Theta}_\delta \widehat{\omega}_\delta \,\mathrm{d}x, \\
|S_2| &\le c \int\limits_{\Omega_0} \big|\nabla\big(\eta^2 (D_i \widetilde{u}_\delta - \overline{D_i \widetilde{u}_\delta})\big)\big| \,\mathrm{d}x \le c \left[\int\limits_T \widehat{\Theta}_\delta \widehat{\omega}_\delta \,\mathrm{d}x + \int\limits_{\Omega_0} \eta^2 \widehat{\Theta}_\delta \widehat{\omega}_\delta \,\mathrm{d}x \right].
\end{aligned}$$

Here we have used that, due to (6.49) and Sobolev's embedding theorem, $\widetilde{u}_\delta \in L^\infty_{\mathrm{loc}}(\Omega_0)$ uniformly. To $\widetilde{S}_3$ we apply the Cauchy-Schwarz and Young's inequality:

$$\begin{aligned}
|\widetilde{S}_3| &\le c\varepsilon \int\limits_{\Omega_0} \widetilde{\Theta}_\delta^2 \eta^2 \,\mathrm{d}x + c\varepsilon^{-1} \int\limits_{\Omega_0} |\nabla v|^2 \eta^2 \,\mathrm{d}x \\
&\le \varepsilon \int\limits_{\Omega_0} \widetilde{\Theta}_\delta^2 \eta^2 \,\mathrm{d}x + cR^2.
\end{aligned}$$

For $\widetilde{S}_4$, using the Sobolev-Poincaré inequality, we finally obtain

$$\begin{aligned}
|\widetilde{S}_4| &\le \frac{c}{R} \int\limits_T |\nabla v| |\nabla \widetilde{u}_\delta - \overline{\nabla \widetilde{u}_\delta}| \,\mathrm{d}x \\
&\le \frac{c}{R} \left(\int\limits_T |\nabla v|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \left(\int\limits_T \big|\nabla \widetilde{u}_\delta - \overline{\nabla \widetilde{u}_\delta}\big|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \\
&\le c \int\limits_T |\nabla^2 \widetilde{u}_\delta| \,\mathrm{d}x \le c \int\limits_T \widehat{\Theta}_\delta \widehat{\omega}_\delta \,\mathrm{d}x.
\end{aligned}$$

Altogether, this gives the estimate

$$\int\limits_{B_R(x_0)} \widehat{\Theta}_\delta^2 \,\mathrm{d}x \le \frac{c}{R} \left[\int\limits_T \Theta_\delta^2 \,\mathrm{d}x + R^2 \right]^{\frac{1}{2}} \int\limits_T \widehat{\Theta}_\delta \omega_\delta \,\mathrm{d}x + cR^\gamma,$$

($\gamma \in (0,2)$) which, as for Θ_δ from the first part of the proof, implies

$$\int\limits_{B_R(x_0)} \widehat{\Theta}_\delta^2 \,\mathrm{d}x \leq c \frac{1}{\ln\left(\frac{1}{R}\right)^t} \qquad \text{for any } t \geq 1.$$

From this point on, we can just repeat the arguments from the first part to deduce the Hölder continuity of u. However, one should note that we have to replace σ_δ with the quantity $\rho_\delta := DG_\delta(\nabla\widetilde{u}_\delta - v)$ (cf. (5.20)). Then, as in (6.37), we have

$$\begin{aligned}
\int\limits_{B_R(x_0)} |\nabla\rho_\delta|^2 \,\mathrm{d}x &= \int\limits_{B_R(x_0)} D_i\rho_\delta \cdot D_i\rho_\delta \,\mathrm{d}x \\
&= \int\limits_{B_R(x_0)} D^2G_\delta(\nabla\widetilde{u}_\delta - v)\big(D_i\nabla\widetilde{u}_\delta - D_i v, D_i\rho_\delta\big) \,\mathrm{d}x \\
&\leq c\left[\left(\int\limits_{B_R(x_0)} \widehat{\Theta}_\delta^2 \,\mathrm{d}x\right)^{\frac{1}{2}} + \left(\int\limits_{B_R(x_0)} |\nabla v|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}\right]\left(\int\limits_{B_R(x_0)} |\nabla\rho_\delta|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}
\end{aligned}$$

and therefore

$$\int\limits_{B_R(x_0)} |\nabla\rho_\delta|^2 \,\mathrm{d}x \leq c \int\limits_{B_R(x_0)} \widehat{\Theta}_\delta^2 \,\mathrm{d}x + cR^2.$$

Since

$$\frac{1}{\ln\left(\frac{1}{R}\right)^t} + R^2 \leq c\frac{1}{\ln\left(\frac{1}{R}\right)^t} \qquad \text{for } R \downarrow 0,$$

we arrive at

$$\left(\int\limits_{B_R(x_0)} |\nabla\rho_\delta|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \leq c\frac{1}{\ln\left(\frac{1}{R}\right)^{\frac{t}{2}}} \text{ for any } t \geq 1.$$

Arguing as in the first part, we thus infer that there exists a continuous function $\rho \in C^0(\Omega_0, \mathbb{R}^2)$ such that $\rho_\delta \rightrightarrows \rho$ and

$$\nabla u = DG^{-1}(\rho) + v \qquad \text{a.e. on } \Omega_0.$$

In particular it follows that

$$\nabla u \in L^\infty_{\mathrm{loc}}(\Omega_0). \tag{6.53}$$

We further observe that, for any $\varphi \in C_0^\infty(\Omega_0)$, the functions $w_i := D_i u$ ($i \in \{1,2\}$) solve the equation

$$\int\limits_{\Omega_0} D^2G(\nabla u - v)(\nabla w_i, \nabla\varphi) \,\mathrm{d}x = \int\limits_{\Omega_0} D^2G(\nabla u - v)(D_i v, \nabla\varphi) + (u - f)\varphi \,\mathrm{d}x,$$

whose coefficients $a_{ij}(x) := \frac{\partial^2}{\partial x_i \partial x_j} G(\nabla u(x) - v(x))$ are locally uniformly elliptic on Ω_0 due to (6.53) and ((G2)$'$). The Hölder continuity of u therefore follows from the classical results of De Giorgi, Moser and Nash (see, e.g. Theorem 8.22 on p. 200 in [65]), which finishes the proof of Theorem 6.2.1. $\square$

Remark 6.2.4
Note that if we assume $F \in C^{2,1}(\mathbb{R}^{2\times 2})$, then the coefficients $A^{ij}_{kl}(x)$ from (6.47) are even locally Lipschitz continuous. The (local) Hölder continuity of ∇u furthermore implies the Hölder continuity of the functions $f^i_k(x) = \delta_{\gamma k} \frac{\partial G}{\partial x_i}(\nabla u(x) - v(x))$, and we may quote Theorem 3.2 from [46] to infer that $v \in C^{2,\alpha_1}(\Omega_0)$ for some $\alpha_1 \in (0,1)$ in this case.

Appendix

A Function Spaces

Here we give a short overview of the various function spaces that appear in the text. However, the main intent of these remarks is to explain our notation and we refer to the relevant literature for proofs and details, see e.g. [1], [2], [32], [58] and [87]. In the following, let Ω be an open and bounded subset of $\mathbb{R}^n$ with Lipschitz boundary or $\Omega = \mathbb{R}^n$. Note that for all $x = (x_1, ..., x_n) \in \mathbb{R}^n$ we denote the Euclidean norm by

$$|x| := \sqrt{\sum_{i=1}^{n} x_i^2}.$$

We start with the standard Lebesgue space $L^p(\Omega, \mathbb{R}^N)$ (or just $L^p(\Omega)$ if $N = 1$), which consists of all $\mathbb{R}^N$-valued p-integrable functions ($p \in [1, \infty]$) and is endowed with the norm

$$\|u\|_{p;\Omega} := \left(\int_\Omega |u|^p \,\mathrm{d}x \right)^{1/p}, \quad \text{if } p < \infty,$$

and

$$\|u\|_{\infty;\Omega} := \operatorname{ess\,sup} |u| \quad \text{for } p = \infty.$$

By $\mathcal{M}(\Omega, \mathbb{R}^N)$ (or just $\mathcal{M}(\Omega)$, if $N = 1$) we denote the linear space of all $\mathbb{R}^N$-valued Radon measures on Ω, i.e. $\mathbb{R}^N$-valued Borel measures that are finite on compact subsets of Ω. We say that $\mu \in \mathcal{M}(\Omega, \mathbb{R}^N)$ has finite total mass, if

$$|\mu|(\Omega) := \sup \left\{ \sum_{i=1}^{\infty} |\mu(E_i)| \,:\, \Omega = \bigcup_{i=1}^{\infty} E_i \ \begin{array}{c}\text{is a covering of } \Omega \text{ by} \\ \text{pairwise disjoint Borel sets}\end{array} \right\} \tag{A.1}$$

is finite. If a function $u \in L^1(\Omega)$ has a (distributional) gradient in form of a Radon measure of finite total mass, then we say that u has bounded variation (for short $u \in BV(\Omega)$) and write

$$|\nabla u|(\Omega) = \int_\Omega |\nabla u| = \sup \left\{ \int_\Omega u \operatorname{div} \varphi \,\mathrm{d}x \,:\, \varphi \in C_0^1(\Omega, \mathbb{R}^n),\ \|\varphi\|_\infty \le 1 \right\}.$$

The next item is the Sobolev space $W^{m,p}(\Omega)$ which consists of those functions $u \in L^p(\Omega)$, whose partial distributional derivatives up to order k lie in the class L^p, as well. We write $\nabla^k u$ for the tensor

$$\nabla^k u := \big(D_{\alpha_1 \ldots \alpha_k} u\big)_{\alpha_i = 1,\ldots,n}$$

of all k-th order partial (distributional) derivatives, which, for simplicity, is construed as an $\mathbb{R}^{n^k}$-valued function. We define the norm of $u \in W^{m,p}(\Omega)$ by

$$\|u\|_{m,p;\Omega} := \sum_{k=0}^{m} \|\nabla^k u\|_{p;\Omega},$$

where we declare $\nabla^0 u := u$. Of course, $W^{m,p}(\Omega, \mathbb{R}^N)$ is the space of $\mathbb{R}^N$ valued functions whose components lie in $W^{m,p}(\Omega)$. An important tool in the context of Sobolev functions is Poincaré's inequality (see, e.g., [1], Remark 3.50 and Exercise 7.7):

Theorem A.1
Let $E \subset \Omega$ be a measurable subset of Ω with $\mathcal{L}^n(E) > 0$. Then there exists a constant $c > 0$, depending only on Ω and E such that for all $u \in W^{m,p}(\Omega)$ it holds

$$\|u - (u)_E\|_{p;\Omega} \leq c\|\nabla^m u\|_{p;\Omega},$$

with $(u)_E := \fint_E u \, \mathrm{d}x$.

Finally, by $C^{m,\alpha}(\Omega)$ we denote the space of m-times continuously differentiable functions whose derivatives are *locally* Hölder continuous with exponent α, i.e. $u \in C^{m,\alpha}(\Omega)$ iff for every compact subset $\widetilde{\Omega} \Subset \Omega$ and every $k = 0, ..., m$ it holds

$$\sup_{\substack{x,y \in \widetilde{\Omega} \\ x \neq y}} \frac{\big|\nabla^k u(x) - \nabla^k u(y)\big|}{|x-y|^\alpha} \leq c(\widetilde{\Omega}) < \infty.$$

B Convex Functions of a Measure

In this section, we briefly present the concept of convex functions of a measure as introduced in the paper [88] (see also Section 2.6 in [1]). Let first $F : \mathbb{R}^N \to \mathbb{R}$ be a sublinear, positively 1-homogeneous function, i.e.

$$\begin{aligned} &F(x+y) \leq F(x) + F(y) \quad \forall x, y \in \mathbb{R}^N, \\ &F(\lambda x) = \lambda F(x) \quad \forall x \in \mathbb{R}^N, \ \lambda > 0 \end{aligned}$$

and let $\mu \in \mathcal{M}(\Omega, \mathbb{R}^N)$ be Borel measure with values in $\mathbb{R}^N$. Let further

$$F(x) \leq c|x|$$

hold with some constant $c > 0$. Then, based on (A.1), we define for every Borel set $E \subset \Omega$

$$\int_E F(\mu) := \sup \left\{ \sum_{i=1}^{k} F(\mu(E_i)) \,:\, \Omega = \bigcup_{i=1}^{\infty} E_i \begin{array}{c} \text{is a covering of } \Omega \text{ by} \\ \text{pairwise disjoint Borel sets} \end{array} \right\}.$$

Note that if $F(x) = |x|$, then $\int_E F(\mu) = \int_E |\mu|$ is the usual total variation measure. One can show that under the above conditions on F the set function $F(\mu)$ defines a Borel measure on Ω. Now, for a not necessarily 1-homogeneous convex function $F : \mathbb{R}^N \to \mathbb{R}$ that fulfills

$$|F(x)| \leq c(1 + |x|)$$

with some positive constant $c > 0$, we define $F(\mu)$ via its homogenization:

$$F_h : [0, \infty) \times \mathbb{R}^N \to \mathbb{R}, \quad F_h(t, x) := \begin{cases} tF\Big(\dfrac{x}{t}\Big), & \text{if } t \neq 0, \\ F^{\infty}(x), & \text{if } t = 0. \end{cases}$$

Here, $F^{\infty}(x) := \lim_{s \to \infty} \frac{F(sx)}{s}$ is the so called recession function of F. We note that F_h is sublinear and positively 1-homogeneous, which allows us to define

$$F(\mu) := F_h(\mathcal{L}^N, \mu),$$

where $\mathcal{L}^N$ is Lebesgue's measure on $\mathbb{R}^N$. If $\mu = \mu^a \cdot \mathcal{L}^N + \mu^s$ denotes the Lebesgue decomposition of μ, Theorem 2' in [88] gives the representation

$$\int_E F(\mu) = \int_E F_h\big((1, \mu^a) \cdot \mathcal{L}^n + (0, \mu^s)\big) = \int_E F(\mu^a)\, \mathrm{d}x + \int_E F^{\infty}\left(\frac{\mu^s}{|\mu^s|}\right) \mathrm{d}|\mu^s|.$$

An important property is the following continuity Theorem of Reshetnyak (see [89] and cf. [90], Proposition 2.2 as well as [1], Theorem 2.34 and Proposition 3.15):

Theorem B.1
Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let (μ_k) be a sequence of $\mathbb{R}^N$-valued Radon measures, which weakly-$*$ converges to a Radon measure μ. Let further $F : \mathbb{R}^N \to \mathbb{R}$ be a strictly convex function that satisfies the growth estimate

$$0 \leq F(p) \leq c(1 + |p|) \tag{B.1}$$

for some constant $c > 0$. Then:

a) $\displaystyle\int_\Omega F(\mu) \leq \liminf_{k \to \infty} \int_\Omega F(\mu_k)$.

b) If in addition

$$\int_\Omega \sqrt{1+|\mu_k|^2} \to \int_\Omega \sqrt{1+|\mu|^2},$$

then it also holds

$$\int_\Omega F(\mu_k) \to \int_\Omega F(\mu).$$

In this thesis, we employ the formula for $F(\mu)$ to define the relaxation of convex functionals of linear growth on the class $BV(\Omega)$ (see, e.g. Section 5.5 in [1]): let $F : \mathbb{R}^n \to \mathbb{R}$ be a convex function which satisfies (B.1) and let $\Omega \subset \mathbb{R}^n$ be open. Then the functional

$$E[u] := \int_\Omega F(\nabla u)\,\mathrm{d}x$$

is well defined on the Sobolev space $W^{1,1}(\Omega)$. However, due to the lack of reflexivity, E-minimizing sequences need not to be weakly compact in $W^{1,1}(\Omega, \mathbb{R}^N)$. One therefore defines the relaxed functional $\widetilde{E}$ on the space $BV(\Omega)$ in its abstract form by

$$\widetilde{E}[u] := \inf\left\{ \liminf_{k\to\infty} E[u_k] \ : \ (u_k) \subset C^1(\Omega, \mathbb{R}^N),\ u_k \to u \text{ in } L^1(\Omega, \mathbb{R}^N) \right\}.$$

In [91] it was proved (even under the more general condition that F is quasiconvex) that the relaxed functional $\widetilde{E}[u]$ then coincides with $\int_\Omega F(\nabla u)$, which means that $\widetilde{E}$ can be expressed through the formula

$$\widetilde{E}[u] = \int_\Omega F(\nabla u) = \int_\Omega F(\nabla^a u)\,\mathrm{d}x + \int_\Omega F^\infty\left(\frac{\nabla^s u}{|\nabla^s u|}\right) \mathrm{d}|\nabla^s u|,$$

where $\nabla u = \nabla^a u \cdot \mathcal{L}^n + \nabla^s u$ denotes the Lebesgue decomposition of the measure ∇u.

C Proof of Lemma 5.2.1 c)

Let $(u_\delta, v_\delta) \in W^{1,2}(\Omega) \times W^{1,2}(\Omega, \mathbb{R}^2)$ be the unique minimizer of

$$E(u,v) = \frac{\delta}{2}\int_\Omega |\nabla u|^2 + |\nabla v|^2\,\mathrm{d}x + \alpha\int_\Omega F(\nabla v)\,\mathrm{d}x + \beta\int_\Omega G(\nabla u - v)\,\mathrm{d}x + \int_{\Omega - D} (u-f)^2\,\mathrm{d}x$$

with $\Omega, F, G, f, \alpha, \beta$ as specified in Chapter 5. We want to prove part c) of Lemma 5.2.1, i.e.

$$(u_\delta, v_\delta) \in W^{2,2}_{\mathrm{loc}}(\Omega) \times W^{2,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$$

which follows from an application of the so called difference quotient technique. Let $\delta \in (0,1)$ be fixed. We start with the observation, that u_δ minimizes the quadratic functional

$$E_1[u] := \frac{\delta}{2}\int\limits_\Omega |\nabla u|^2 \,\mathrm{d}x + \beta \int\limits_\Omega G(\nabla u - v_\delta)\,\mathrm{d}x + \int\limits_{\Omega - D} (u-f)^2 \,\mathrm{d}x$$

in $W^{1,2}(\Omega)$ and therefore solves the Euler-Lagrange equation (EL2) from p. 104:

$$\delta \int\limits_\Omega \nabla u_\delta \cdot \nabla\varphi \,\mathrm{d}x + \beta \int\limits_\Omega DG(\nabla u_\delta - v_\delta)\cdot\nabla\varphi\,\mathrm{d}x + 2\int\limits_{\Omega - D} (u_\delta - f)\varphi\,\mathrm{d}x = 0 \tag{C.1}$$

for all $\varphi \in W^{1,2}(\Omega)$. Now, for a testing function $\eta \in C_0^\infty(\Omega)$ and some vector $\gamma \in \mathbb{R}^2$, we set $\varphi := \Delta_\gamma^{-h}(\eta^2 \Delta_\gamma^h u_\delta)$, where

$$\Delta_\gamma^h u_\delta(x) := \frac{u_\delta(x+h\gamma) - u_\delta(x)}{h},$$

and $|h|$ is small enough such that $\operatorname{spt}\eta \pm h\gamma \Subset \Omega$. After an integration by parts, (C.1) reads as

$$\begin{aligned}\delta \int\limits_\Omega \Delta_\gamma^h \nabla u_\delta \cdot \nabla(\eta^2\Delta_\gamma^h u_\delta)\,\mathrm{d}x + \int\limits_\Omega \Delta_\gamma^h DG(\nabla u_\delta - v_\delta)\cdot\nabla(\eta^2\Delta_\gamma^h u_\delta)\,\mathrm{d}x& \\ = 2\int\limits_{\Omega-D} (u_\delta - f)\Delta_\gamma^h(\eta^2\Delta_\gamma^h u_\delta)\,\mathrm{d}x&.\end{aligned} \tag{C.2}$$

Next we observe that

$$\begin{aligned}\beta\Delta_\gamma^h DG(\nabla u_\delta - v_\delta) &= \beta\frac{1}{h}\int\limits_0^1 \frac{\mathrm{d}}{\mathrm{d}t} DG\big(\nabla u_\delta - v_\delta + th\Delta_\gamma^h(\nabla u_\delta - v_\delta)\big)\,\mathrm{d}t \\ &= \underbrace{\left(\beta\int\limits_0^1 D^2G\big(\nabla u_\delta - v_\delta + th\Delta_\gamma^h(\nabla u_\delta - v_\delta)\big)\,\mathrm{d}t\right)}_{=:\,B(\cdot,\cdot)}\big(\Delta_\gamma^h(\nabla u_\delta - v_\delta),\ \cdot\ \big),\end{aligned}$$

where due to our assumption (G2) on D^2G we have that $B(\cdot,\cdot)$ is symmetric bilinear form $B(\cdot,\cdot)$ for which there exists a constant $\lambda > 0$ such that

$$B(x,x) \geq \lambda |x|^2 \text{ for all } x \in \mathbb{R}^2.$$

Setting further

$$B_\delta(x,y) := \delta(x\cdot y) + B(x,y) \text{ for } x,y \in \mathbb{R}^2,$$

we may write (C.2) as

$$\begin{aligned}
&\int_\Omega B_\delta\big(\Delta_\gamma^h \nabla u_\delta, \nabla(\eta^2 \Delta_\gamma^h u_\delta)\big)\,\mathrm{d}x \\
&= \underbrace{\int_\Omega B\big(\Delta_\gamma^h v_\delta, \nabla(\eta^2 \Delta_\gamma^h u_\delta)\big)\,\mathrm{d}x}_{=:\,T_1} + \underbrace{2\int_{\Omega-D} (u_\delta - f)\Delta_\gamma^h(\eta^2 \Delta_\gamma^h u_\delta)\,\mathrm{d}x}_{=:\,T_2}
\end{aligned}$$

and thus

$$\lambda \int_\Omega \eta^2 |\Delta_\gamma^h \nabla u_\delta|^2\,\mathrm{d}x + 2\int_\Omega \eta B_\delta\big(\Delta_\gamma^h \nabla u_\delta, \nabla\eta \Delta_\gamma^h u_\delta\big)\,\mathrm{d}x \le T_1 + T_2. \tag{C.3}$$

We note, that (C.3) differs from the corresponding equation (7.1.11) in [41] only through the presence of the term T_1. It will therefore suffice to give an estimate of this quantity. Expanding the differentiation yields

$$T_1 = \int_\Omega \eta^2 B\big(\Delta_\gamma^h v_\delta, \Delta_\gamma^h \nabla u_\delta\big)\,\mathrm{d}x + 2\int_\Omega \eta B\big(\Delta_\gamma^h v_\delta, \nabla\eta \Delta_\gamma^h u_\delta\big)\,\mathrm{d}x$$

and an application of the Cauchy-Schwarz inequality to the bilinear form $B(\cdot,\cdot)$, followed by Young's inequality leads to

$$\begin{aligned}
|T_1| \le\ & \varepsilon \int_\Omega \eta^2 B\big(\Delta_\gamma^h \nabla u_\delta, \Delta_\gamma^h \nabla u_\delta\big)\,\mathrm{d}x + \varepsilon^{-1} \int_\Omega \eta^2 B\big(\Delta_\gamma^h v_\delta, \Delta_\gamma^h v_\delta\big)\,\mathrm{d}x \\
&+ \frac{1}{2}\int_\Omega \eta B\big(\Delta_\gamma^h v_\delta, \Delta_\gamma^h v_\delta\big)\,\mathrm{d}x + \frac{1}{2}\int_\Omega \eta B\big(\nabla\eta \Delta_\gamma^h u_\delta, \nabla\eta \Delta_\gamma^h u_\delta\big)\,\mathrm{d}x.
\end{aligned}$$

We see that (G2) implies the boundedness of the bilinear form B and together with Lemma 7.23 from [65] applied to the difference quotients $\Delta_\gamma^h v_\delta$ and $\Delta_\gamma^h u_\delta$, we obtain

$$|T_1| \le c\left[\varepsilon \int_\Omega \eta^2 |\Delta_\gamma^h \nabla u_\delta|^2\,\mathrm{d}x + \varepsilon^{-1}\int_\Omega |\nabla v_\delta|^2\,\mathrm{d}x + \int_\Omega |\nabla u_\delta|^2\,\mathrm{d}x\right].$$

If we choose ε small enough, the first term in the bracket can be absorbed in the left-hand side of (C.3), whereas the other two terms are clearly bounded (we remind the reader that δ is a fixed number in this calculation!). For the term T_2 we observe that Young's inequality in combination with Lemma 7.23 from [65] implies

$$\begin{aligned}
|T_2| &\le c(\varepsilon) \int_{\Omega-D} (u_\delta - f)^2\,\mathrm{d}x + \varepsilon \int_{\Omega-D} |\nabla(\eta^2 \Delta_\gamma^h u_\delta)|^2\,\mathrm{d}x \\
&\le c + \varepsilon \int_\Omega \eta^2 |\Delta_\gamma^h \nabla u_\delta|^2\,\mathrm{d}x
\end{aligned}$$

and the non-constant term can be absorbed in the left-hand side of (C.3) for ε small enough. From that point on, we may follow the arguments from [41], chapter 7 to obtain a bound for $\int_\Omega \eta^2 |\Delta_\gamma^h \nabla u_\delta|^2 \,\mathrm{d}x$ in terms of $\|\nabla u_\delta\|_{L^2(\Omega)}$, $\|\nabla v_\delta\|_{L^2(\Omega)}$ and $\|\nabla \eta\|_\infty$ which implies $u_\delta \in W^{2,2}_{\mathrm{loc}}(\Omega)$ via Lemma 7.23 from [65].

For the corresponding statement on v_δ we apply the same strategy. Choose $\psi := \Delta_\gamma^h(\eta^2 \Delta_\gamma^h v_\delta)$ in the Euler-Lagrange equation (EL1) on p. 104. Comparing with the computations in the proof of Lemma 7.1.1 (b) in [41], we see that it suffices to estimate the new quantity

$$\widetilde{T} := \beta \int_\Omega DG(\nabla u_\delta - v_\delta) \cdot \Delta_\gamma^h(\eta^2 \Delta_\gamma^h v_\delta) \,\mathrm{d}x,$$

stemming from the coupling term. But since $|DG|$ is bounded by (G1), we have

$$|\widetilde{T}| \leq c \int_\Omega \left|\Delta_\gamma^h(\eta^2 \Delta_\gamma^h v_\delta)\right| \mathrm{d}x,$$

and how this quantity can be treated is shown after equation (7.1.12) in [41]. $\square$

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