# Primal-dual methods for dynamic programming equations arising in non-linear option pricing

Dissertation zur Erlangung des Grades des
Doktors der Naturwissenschaften
der Fakultät Mathematik und Informatik
der Universität des Saarlandes

eingereicht im September 2017
in Saarbrücken

von

**Christian Gärtner**

| | |
|---|---|
| **Tag des Kolloquiums:** | 06.02.2018 |

**Mitglieder des Prüfungsausschusses:**

| | |
|---|---|
| **Vorsitzender:** | Professor Dr. Thomas Schuster |
| **1. Berichterstatter:** | Professor Dr. Christian Bender |
| **2. Berichterstatter:** | PD Dr. John Schoenmakers (während des Kolloquiums vertreten durch Professor Dr. Henryk Zähle) |
| **Protokollführer:** | Dr. Tobias Mai |
| **Dekan:** | Professor Dr. Frank-Olaf Schreyer |

*To my parents and my brother*

*To Robin*

## Acknowledgements

First of all, I would like to express my gratitude to my supervisor Professor Christian Bender for giving me the opportunity to work in the Stochastics group at Saarland university (with financial support by the Deutsche Forschungsgemeinschaft under grant BE3933/5-1 raised by him), for introducing me to this interesting topic, and for sharing his knowledge with me. It would have not been possible to write this thesis without his constant support and encouragement.

Furthermore, I would like to thank my co-author Dr. Nikolaus Schweizer for all the hours, which he spent for helpful discussions even after he left the Stochastics group at Saarland university and for proof-reading parts of thesis. I am also very thankful for all his experience and knowledge concerning numerical implementations, which he shared with me during the last years. It has always been a pleasure to work with him.

I also would like to thank PD John Schoenmakers for being the co-referee of this thesis.

I would like to thank all my present and former colleagues for welcoming me with open arms. I will never forget all the (more or less mathematical) discussions we had throughout the years.

Moreover, I would like to express my deep gratitude to my parents and my brother for always supporting me and for giving me the possibility to achieve all this. Without their constant believe in me, I would have never come so far.

Finally, I want to thank my friends for all the fun and serious discussions we had in the last years.

# Contents

## Abstract

When discretizing non-linear pricing problems, one ends up with stochastic dynamic programs which often possess a concave-convex structure. The key challenge in solving these dynamic programs numerically is the high-order nesting of conditional expectations. In practice, these conditional expectations have to be replaced by some approximation operator, which can be nested several times without leading to exploding computational costs.

In the first part of this thesis, we provide a posteriori criteria for validating approximate solutions to such dynamic programs. To this end, we rely on a primal-dual approach, which takes an approximate solution of the dynamic program as an input and allows the computation of upper and lower bounds to the true solution. The approach proposed here unifies and extends existing results and applies regardless of whether a comparison principle holds or not.

The second part of this thesis establishes an iterative improvement approach for upper and lower bounds in the special case of convex dynamic programs. This approach allows the computation of tight confidence intervals for the true solution, even if the input upper and lower bounds stem from a possibly crude approximate solution to the dynamic program.

The applicability of the presented approaches is demonstrated in various numerical examples.



## Zusammenfassung

Die Diskretisierung nicht linearer Preisprobleme führt typischerweise zu stochastischen dynamischen Programmen, die eine konkav-konvexe Struktur aufweisen. Möchte man solche dynamischen Programme numerisch lösen, stellen die hochgradig verschachtelten bedingten Erwartungen die größte Herausforderung dar. In Anwendungen müssen diese bedingten Erwartungen mit Hilfe eines geeigneten Operators approximiert werden, der mehrfach angewendet werden kann, ohne zu explodierenden Rechenkosten zu führen.

Im ersten Teil dieser Arbeit stellen wir Kriterien zur nachträglichen Validierung approximativer Lösungen solcher dynamischer Programme bereit. Dazu stützen wir uns auf einen primal-dualen Ansatz, der ausgehend von einer approximativen Lösung des dynamischen Programms die Konstruktion oberer und unterer Schranken an die wahre Lösung ermöglicht. Der hier vorgeschlagene Ansatz vereinheitlicht und verallgemeinert bisher bekannte Resultate und kann ungeachtet der Existenz eines Vergleichsprinzips genutzt werden.

Der zweite Teil der Arbeit befasst sich mit einem iterativen Ansatz zur Verbesserung oberer und unterer Schranken im Spezialfall konvexer dynamischer Programme. Dieser Ansatz erlaubt die Konstruktion enger Konfidenzintervalle an die wahre Lösung, selbst wenn die gegebenen Schranken auf einer möglicherweise groben approximativen Lösung des dynamischen Programms beruhen.

In verschiedenen numerischen Beispielen demonstrieren wir die Anwendbarkeit der vorgeschlagenen Ansätze.

# Introduction

In the wake of the financial crisis, non-linear pricing problems received an increased interest in both, academia and practice. These nonlinearities arise, e.g., due to early-exercise features, funding risk (see Bergman (1995); Crépey et al. (2013); Laurent et al. (2014)), counterparty risk (see e.g. Crépey et al. (2013); Brigo et al. (2013)), model uncertainty (see Guyon and Henry-Labordère (2011); Alanko and Avellaneda (2013)), collateralization (see Nie and Rutkowski (2016)) or transaction costs (see Guyon and Henry-Labordère (2011)). In practice, an option is written on risky assets and its payoff is given as a deterministic function of the evolution of these assets over a given time horizon. In order to model the evolution of these risky assets one typically relies on Markovian processes. As a consequence, the value of an option under non-linear pricing can often be described as a solution of a non-linear partial differential equation (PDE). In general, these differential equations do not possess a closed-form solution, so that discretization schemes need to be applied for the computation of an approximate solution of the PDE and, thus, an approximate price. As long as the underlying Markovian process is low-dimensional standard tools for approximately solving PDEs (such as finite-difference schemes) can be applied. For derivatives depending on multiple risk factors, this is however not the case and PDE-methods quickly turn out to be infeasible. This phenomenon is well-known as the curse of dimensionality. A standard trick in mathematical finance to circumvent this problem is to exploit the link between non-linear PDEs and backward stochastic differential equations (BSDEs) established by Pardoux and Peng (1992). This allows the application of Monte Carlo methods which are known to be less sensitive to the dimension of the considered problem. Discretizing the resulting BSDE with respect to time, one typically ends up with concave-convex stochastic dynamic programs of the form

$$\begin{aligned} Y_J &= \xi, \\ Y_j &= G_j(E_j[\beta_{j+1}Y_{j+1}], F_j(E_j[\beta_{j+1}Y_{j+1}])) \end{aligned} \tag{1}$$

for $j = J-1, \ldots, 0$. Here, $E_j[\cdot]$ denotes the conditional expectation with respect to $\mathcal{F}_j$ for a given filtration $(\mathcal{F}_j)_{j=0,\ldots,J}$. Furthermore, the function $G_j$ is concave and increasing in its second argument, while the function $F_j$ is convex. The terminal condition $\xi$ is assumed to be $\mathcal{F}_J$-measurable and reflects the payments of the option that arise at maturity. The process $\beta$ is adapted and allows us to capture possible dependencies of the value process on its Delta and Gamma, i.e., its first- and second-order derivative with respect to the space variable.

Although we consider dynamic programs like (1) mainly in the context of non-linear option pricing, we emphasize that such problems also arise in other applications. Among others, these applications include multistage sequential decision problems under uncertainty (see e.g. Bertsekas (2005); Powell (2011)), evaluation of recursive utility functionals as in Kraft and Seifried (2014) or discretization schemes for fully non-linear second-order parabolic PDEs as discussed in Fahim et al. (2011).

The key challenge in solving dynamic programming equations of this form is the high-order nesting of conditional expectations, which stems from the recursive structure of the problem. Indeed,

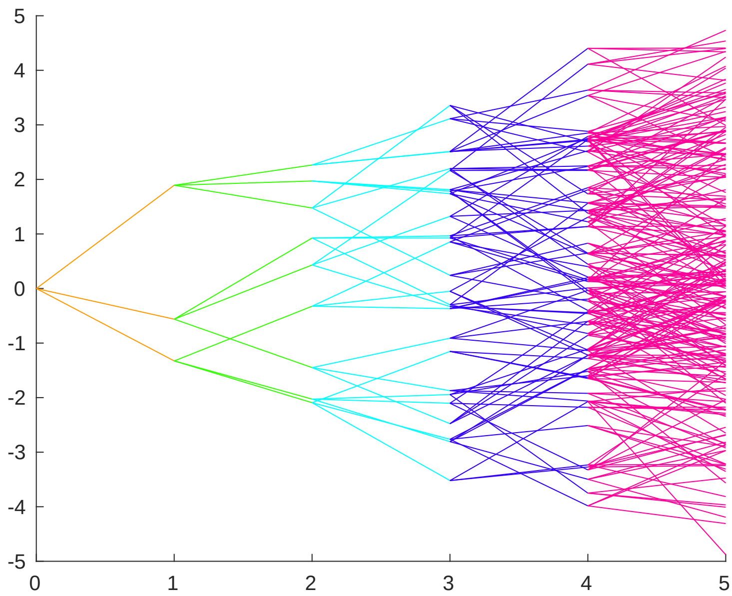


Figure 1: Example of a nested Monte Carlo simulation with $J = 5$ and three sample paths.

the value at a given time point depends on the conditional expectation of the value one time step ahead, which in turn depends on values several time steps ahead. Since we cannot expect in general that the conditional expectations can be evaluated in closed form (or at least up to a negligible error), dynamic programs like (1) need to be solved numerically. This can be done by applying the approximate dynamic programming approach, where the conditional expectations are replaced by some approximation operator. However, due to the high-order nesting of conditional expectations, this operator needs to be nested several times without leading to exploding computational costs. For this reason, a naive plain Monte Carlo approach quickly turns out to be infeasible, even for a moderate number of time steps $J$, since it requires the branching of existing trajectories at every point in time in order to approximate the conditional expectations, see Figure 1. Hence, more sophisticated approximation operators are required for the computation of an approximate solution to (1). In recent years, several approaches have been developed and analyzed in the context of discretization schemes for BSDEs. A non-exhaustive list includes least-squares Monte Carlo (see Gobet et al. (2005); Bender and Denk (2007)), quantization (see Bally and Pagès (2003)), Malliavin Monte Carlo (see Bouchard and Touzi (2004)), sparse grid methods (see Zhang et al. (2013)) or cubature methods on Wiener space as proposed in Crisan and Manolarakis (2012). However, the error stemming from these approximation operators is hard to assess in numerical implementations. Consequently, the derivation of a posteriori criteria for the evaluation of the quality of approximate solutions is desirable.

This thesis consists of two parts. In the first part, we provide a method for the construction of a confidence interval for $Y_0$ using Monte Carlo methods. Such a posteriori criteria have first been developed in the context of optimal stopping problems. The aim of these problems is to stop a reward process $S$ such that the expected reward is maximized. Therefore, following any (possibly non-optimal) stopping strategy obviously results in a lower bound on the value process. This lower bound is complemented by an upper bound which has been proposed independently by Haugh and Kogan (2004) and Rogers (2002). The rationale of their approach is to consider the stopping problem pathwise rather than in conditional expectation, i.e., instead of solving the optimal stopping problem, one maximizes the reward along each path. In order to make this bound tight, the resulting additional information is penalized by subtracting a martingale increment. Taking the infimum over the set of martingales, they prove that the value process possesses a representation as dual minimization problem. Relying on this pair of primal-dual optimization problems, Haugh and Kogan (2004) and Andersen and Broadie (2004) propose a primal-dual approach for the construction of upper and lower bounds: in a first step, one approximately solves the dynamic program associated with this problem, which is given by choosing $G_j(z, y) = y$, $F_j(z) = \max\{S_j, z\}$, $\xi = S_J$, and $\beta \equiv 1$ for an adapted process $(S_j)_{j=0,\ldots,J}$ in (1). Then, an approximate stopping rule and a martingale are constructed from this approximate solution. Taking these suboptimal controls as an input, upper and lower bounds can be constructed from the primal-dual representations.

The information relaxation approach of Haugh and Kogan (2004) and Rogers (2002) was further generalized by Rogers (2007) and Brown et al. (2010) to stochastic control problems in discrete time. While Rogers (2007) only considers perfect information relaxation and martingale penalties as in the optimal stopping problem, Brown et al. (2010) allow for information relaxations to a varying extent and a broader class of penalties.

Bender et al. (2017) extended the primal-dual approach to the class of monotone and convex dynamic programs. Starting from a dynamic programming equation, they derive primal and dual optimization problems with value $Y$ for which optimal controls exist and are given in terms of the true solution $Y$. Following Haugh and Kogan (2004) and Andersen and Broadie (2004) in the numerical implementation, they construct upper and lower bounds by first solving the dynamic program approximately and use this approximate solution to derive suboptimal controls. Taking these suboptimal controls as an input, they recursively compute super- and subsolutions to the dynamic program. Here, a supersolution (respectively subsolution) is an adapted process which satisfies (1) with "$\geq$" (respectively "$\leq$") instead of "$=$". Assuming a comparison principle, which ensures that supersolutions lie above subsolutions, Bender et al. (2017) show that the constructed processes constitute bounds to the solution of the dynamic program.

The first two chapters aim at generalizing the primal-dual approach proposed by Bender et al. (2017) in various directions. In the first chapter, we generalize their approach to the multi-dimensional setting and consider systems of convex dynamic programs. Assuming a componentwise comparison principle, the results of Bender et al. (2017) can be transferred to this new setting in a straightforward way. Since, in general, super- and subsolutions to (1) need not be ordered and, thus, do not constitute bounds, we discuss the comparison principle in more detail. In many one-dimensional applications like the optimal stopping problem or the examples considered in Bender et al. (2017) this assumption is either not an issue or it can be established by mild truncations of the process $\beta$. However, in the context of systems of dynamic programming equations, we show that the existence of a componentwise comparison principle requires that each component does not depend on the space derivative of the other components and that it only depends on the other components in a monotonically increasing way. Consequently, the comparison principle can be a huge drawback in this setting and the remainder of the first chapter is dedicated to remove this assumption.

The main result of this chapter is, thus, concerned with the construction of a pair of super- and subsolutions for which a componentwise comparison principle holds, although it fails to hold in general. This is achieved by a modification of the recursions for upper and lower bounds proposed by Bender et al. (2017). The rationale of the construction is to allow that the lower bound enters the defining recursion for the upper bound and vice versa. Going backwards in time, we check in each recursion step if a violation of the comparison principle occurs on any given path. If the comparison principle is violated, the dependence of each recursion on both bounds applies and ensures the ordering of the bounds. In this way, we end up with coupled recursions for the construction of upper and lower bounds, which need to be computed simultaneously. As a consequence, these bounds cannot be interpreted as stemming from distinct primal and dual optimization problems in general. The applicability of this approach is then demonstrated in two numerical examples, namely pricing under collateralization and pricing under uncertain volatility. To this end, we first provide a general way to implement an algorithm based on this approach in a Markovian framework. For the construction of an approximate solution, we rely on least-squares Monte Carlo (LSMC). In particular, we provide a variant of the regression-later approach by Glasserman and Yu (2004) respectively the martingale basis approach proposed by Bender and Steiner (2012), which is more flexible concerning its applicability.

Thereafter, we pass in Chapter 2 to concave-convex dynamic programs of the form (1). Assuming this structure has essentially two reasons: first, many functions, which are neither convex or concave, can be expressed as a composition of suitable convex and concave functions. Indeed, we show that such a situation arises in the context of pricing under bilateral counterparty risk, i.e., in situations where both parties involved in a contract may default prior to maturity. The second reason is that convex respectively concave structures naturally arise in many maximization respectively minimization problems. Assuming the concave-convex structure, thus allows us to consider dynamic programming equations arising in stochastic two-player games. In mathematical finance a well-known example for such stochastic two-player games is the problem of pricing convertible bonds, see e.g. Beveridge and Joshi (2011).

The aim of this chapter is to transfer the results derived in Chapter 1 to this new setting. In order to simplify the exposition, we restrict ourselves to the case of a single equation, but emphasize that the results can be transferred in a straightforward way to systems of concave-convex dynamic programs. As before, we first derive recursions for the construction of super- and subsolutions in a monotone setting, i.e., when a comparison principle holds. These are obtained by a suitable composition of the upper and lower bounds for the respective concave and convex problems. We further provide sufficient conditions for the comparison principle to hold, but, compared to the convex setting of Chapter 1, we are not able to give equivalent characterizations. This is essentially due to the additional concave structure. Finally, we relax the assumption of a comparison principle and generalize the coupled bounds from Chapter 1 to the concave-convex setting. As in the monotone case, this construction relies on a suitable composition of the coupled bounds for the respective concave and convex problem. Finally, we apply our approach in a numerical example concerned with pricing under bilateral counterparty risk.

The second part of this thesis aims at the derivation of an iterative improvement algorithm for upper and lower bounds in the convex setting of Chapter 1. We call a supersolution (respectively subsolution) to a convex dynamic program an improvement if it lies below (respectively above) a given input supersolution (respectively subsolution). Developing such an improvement approach is motivated by the observation that the width of a confidence interval for $Y_0$ constructed from the primal-dual approach derived in the first two chapters strongly depends on the input approximation. This is due to the derivation of suboptimal controls required for the computation of upper and lower bounds from an approximate solution to the dynamic programming equation.

When computing an approximate solution using LSMC, the resulting error stems to a large part from the so-called projection error, which is hard to control. This error occurs by replacing the projection onto an (in general) infinite-dimensional subspace of $L^2(\Omega, P)$ by the projection onto a finite-dimensional subspace spanned by the basis functions. In order to keep this error moderate, a suitable choice of basis functions is required. Intuitively, a "good" function basis should capture both, the terminal condition $\xi$ and the non-linearities modeled by the functions $G_j$ and $F_j$. As there is no constructive way to obtain such basis functions, searching for these can be rather cumbersome.

In the context of optimal stopping problems, Kolodko and Schoenmakers (2006) propose an iterative improvement approach for lower bounds as an alternative to solving the dynamic program using LSMC. This approach converges to the true solution after finitely many iteration steps and avoids the choice of basis functions. The rationale of this approach is to start from a family of stopping times, and to derive new exercise criteria, from which an increasing sequence of lower bounds is obtained. This kind of policy iteration has first been proposed in the context of stochastic control problems, see Howard (1960); Puterman (1994). Complementing the approach of Kolodko and Schoenmakers (2006), Chen and Glasserman (2007) propose an algorithm which iteratively improves a given upper bound. Taking the martingale part of the Doob decomposition of a given supersolution as an input for the dual approach of Haugh and Kogan (2004) and Rogers (2002), they show that the resulting upper bound lies below the given supersolution.

The aim of the third chapter is to generalize the approaches of Kolodko and Schoenmakers (2006) and Chen and Glasserman (2007) to the class of monotone systems of convex dynamic programs discussed in Chapter 1. For the construction of such an improvement algorithm we rely on the recursions for upper and lower bounds derived in the first chapter. Starting from given super- and subsolutions, the main idea of this construction is to derive controls in terms of the input super- and subsolutions. Taking the resulting controls as an input for the upper and lower bound recursions, we end up with an improvement for the given super- and subsolutions. We further demonstrate that this approach can be iterated in a straightforward way and show that it converges in finitely many iteration steps. Moreover, we show that the true solution $Y$ to the dynamic program is the only fixed point of this iteration. Hence, even when starting with possibly crude super- and subsolutions, this approach does not get stuck in any suboptimal upper and lower bounds.

The results of this thesis are already available in two papers, which are joint work with Christian Bender and Nikolaus Schweizer:

> Christian Bender, Christian Gärtner, and Nikolaus Schweizer. Pathwise Dynamic Programming. *Mathematics of Operations Research.* forthcoming.

> Christian Bender, Christian Gärtner, and Nikolaus Schweizer. Iterative Improvement of Upper and Lower Bounds for Backward SDEs. *SIAM Journal of Scientific Computing.* 39(2):B442-B466, 2017.

Based on these papers, Chapters 1 and 3 are concerned with systems of convex dynamic programs. While Chapter 1 provides a more detailed discussion of such systems compared to the corresponding Section 6 in the first paper, Chapter 3 generalizes the results of the second paper to this multi-dimensional setting wherever possible.

# Notation

In the following, we introduce some notation, which is frequently used:

Let $x \in \mathbb{R}$ be a real number. Then, we denote by $(x)_+$ and $(x)_-$ the positive respectively negative part of $x$, i.e., $(x)_+ := \max\{x, 0\}$ and $(x)_- := \max\{-x, 0\}$. Further, we denote by $|x|$ the absolute value of $x$.

For a vector $y \in \mathbb{R}^D$, we denote by $\|y\|$ the Euclidean norm of $y$. We say that $y_1 \geq y_2$ for two vectors $y_1, y_2 \in \mathbb{R}^D$ if $y_1^{(\nu)} \geq y_2^{(\nu)}$ for all $\nu = 1, \ldots, D$. Moreover, we denote by $\mathbf{1}$ the vector in $\mathbb{R}^D$ consisting of ones and for any matrix $A$, $A^\top$ is the matrix transposition of $A$. For a vector $z \in \mathbb{R}^{ND}$, we denote by $z^{[n]}$ the vector in $\mathbb{R}^D$ consisting of the $((n-1)D+1)$-th up to the $(nD)$-th entry of $z$, i.e. $z = (z^{[1]}, \ldots, z^{[N]})$.

Further let $(\Omega, \mathcal{F}, (\mathcal{F}_j)_{j=0,\ldots,J}, P)$ be a filtered probability space. Then we denote by $L^{\infty-}(\mathbb{R}^m)$, $m \in \mathbb{N}$, the set of $\mathbb{R}^m$-valued random variables that are in $L^p(\Omega, P)$ for all $p \geq 1$. The set of $\mathcal{F}_j$-measurable random variables that are in $L^{\infty-}(\mathbb{R}^m)$ is denoted by $L_j^{\infty-}(\mathbb{R}^m)$. In addition, $L_{ad}^{\infty-}(\mathbb{R}^m)$ denotes the set of adapted processes $Z$ such that $Z_j \in L_j^{\infty-}(\mathbb{R}^m)$ for every $j = 0, \ldots, J$.

For a $D$-dimensional Brownian motion $W$ and a partition $0 = t_0 < t_1 < \ldots < t_J = T$ of the interval $[0, T]$, we denote by $\Delta W_{j+1} := W_{t_{j+1}} - W_{t_j}$, $j = 0, \ldots, J-1$, the increment of the Brownian motion over the interval $[t_j, t_{j+1}]$. The length of the interval $[t_j, t_{j+1}]$ is denoted by $\Delta_{j+1}$. If the partition is assumed to be equidistant, we simply write $\Delta$ instead of $\Delta_{j+1}$ for all $j = 0, \ldots, J-1$.

Moreover, $\mathcal{N}$ and $\varphi$ denote respectively the cumulative distribution function and the density function of the standard normal distribution.

Finally, all equalities and inequalities are meant to hold $P$-a.s, unless otherwise noted.

# Chapter 1

# Systems of convex dynamic programming equations

In this chapter, we consider systems of dynamic programming equations, which arise, e.g., in the context of multiple stopping problems or as discretization schemes for systems of partial differential equations. The scope of this chapter is to derive upper and lower bounds to the solution of such systems. To do this, we generalize the pathwise approach of Bender et al. (2017) to this multi-dimensional setting. Section 1.1 presents some examples for systems of convex dynamic programming equations arising in option pricing. In Section 1.2, we introduce the setting as well as the required definitions and notations. Section 1.3 is dedicated to the pathwise approach of Bender et al. (2017). We recall the main ideas of this approach and, at the same time, generalize them to our multi-dimensional setting. In Section 1.4, we give equivalent characterizations of the comparison principle and explain its restrictiveness by an example. Building on these considerations, we generalize the approach of Bender et al. (2017) in Section 1.5 in such a way that upper and lower bounds to the solution of the dynamic program can be derived without relying on the comparison principle. Section 1.6 provides a first insight in the numerical implementation of the theoretical results presented before. More precisely, we show how the application of approximation methods required in the numerical implementation may lead to an additional bias in the upper and lower bounds. Section 1.7 explains how the theoretical results from Sections 1.3 and 1.5 can be applied in practice. To this end, we first explain how the bounds can be computed in a general setting. Finally, we demonstrate the applicability of our approach with two numerical examples, namely the problem of pricing a European-style option under funding costs and negotiated collateral and pricing under uncertain volatility.

## 1.1 Examples

In this chapter, we focus on systems of dynamics programs of the form

$$\begin{aligned} Y_J^{(\nu)} &= \xi^{(\nu)} \\ Y_j^{(\nu)} &= F_j^{(\nu)}\left(E_j\left[\beta_{j+1}Y_{j+1}^{(1)}\right], \dots, E_j\left[\beta_{j+1}Y_{j+1}^{(N)}\right]\right), \quad \nu = 1, \dots, N,\ j = J-1, \dots, 0, \end{aligned} \tag{1.1}$$

where, $E_j[\cdot]$ denotes the conditional expectation with respect to $\mathcal{F}_j$ for a given filtration $(\mathcal{F}_j)_{j=0,\dots,J}$ and the functions $F_j^{(\nu)}$ are convex. In the following, we present three examples arising in mathematical finance, which motivate the investigation of such systems.

*Example* 1.1.1. We first consider the multiple stopping problem. In mathematical finance, this problem occurs e.g. in the context of swing option pricing problems, see e.g. Carmona and Touzi (2008) and Bender et al. (2015). In the multiple stopping problem, one is interested in stopping a reward process $S \in L^{\infty-}_{ad}(\mathbb{R})$ $N$-times over a given time horizon such that the expected reward is maximized. In this example, we consider a discrete time situation, where all exercise rights need to be executed at different time points and that all remaining rights at maturity need to be executed simultaneously. Hence, the corresponding value process is given by

$$Y_j^{(N)} = \operatorname*{esssup}_{\tau \in \mathcal{S}_j(N)} E_j \left[ \sum_{k=1}^{N} S_{\tau^{(k)}} \right]$$

for every $j = 0, \ldots, J$ and where $\mathcal{S}_j(N)$ is the set of stopping vectors $\tau = (\tau^{(1)}, \ldots, \tau^{(N)})$ such that $j \leq \tau^{(1)} \leq \ldots \leq \tau^{(N)} \leq J$ and $\tau^{(k)} = \tau^{(k+1)}$ implies $\tau^{(k)} = J$. As it is well-known in the literature, this pricing problem can be transferred to solving a system of dynamic programming equations. In our setting, this system is given by

$$Y_j^{(\nu)} = \max \left\{ E_j \left[ Y_{j+1}^{(\nu)} \right], \; S_j + E_j \left[ Y_{j+1}^{(\nu-1)} \right] \right\}, \quad Y_J^{(\nu)} = \nu S_J,$$

for $j = 0, \ldots, J-1$, $\nu = 1, \ldots, N$ and with the convention that $Y^{(0)} \equiv 0$. Here, $Y_j^{(\nu)}$ is the value of the problem at time index $j$ if $\nu$ rights can be executed. For a vector $z \in \mathbb{R}^{ND}$, denote by $z^{[n]}$ the vector in $\mathbb{R}^D$ consisting of the $((n-1)D+1)$-th up to the $(nD)$-th entry of $z$, i.e. $z = (z^{[1]}, \ldots, z^{[N]})$. By taking $D = 1$, the process $\beta \equiv 1$, $\xi^{(\nu)} = \nu S_J$, and $F_j^{(\nu)}(z) = \max\{z^{[\nu]}, \; S_j + z^{[\nu-1]}\}$, we then observe that the multiple stopping problem fits our framework.

*Example* 1.1.2. As a second example, we consider the problem of pricing under negotiated collateralization in the presence of funding costs as discussed in Nie and Rutkowski (2016). Collateralized contracts differ from "standard" contracts in the way that the involved parties not only agree on a payment stream until maturity but also on the collateral posted by both parties. By providing collateral, both parties can reduce the possible loss resulting from a default of the respective counterparty prior to maturity. In the following, we consider the problem of pricing a contract under negotiated collateral, i.e. the imposed collateral depends on the valuations of the contract made by the two parties. More precisely, the party ("hedger") wishes to perfectly hedge the stream of payments consisting of the option payoff and the posted collateral under funding costs, while the counterparty hedges the negative payment stream under funding costs. As hedging under funding costs is known to be non-linear, both hedges do not cancel each other. Hence, one ends up with a coupled system of two equations where the coupling is due to the fact that the counterparty's hedging strategy influences the hedger's payment stream due to the negotiated collateral and vice versa.

We first translate the original backward SDE formulation of the problem in Nie and Rutkowski (2016) into a parabolic PDE setting. To this end let $g : \mathbb{R}^d \to \mathbb{R}$ be a function of polynomial growth which represents the payoff of a European-style option written on $d$ risky assets with maturity $T$. The dynamics of the risky assets $X = (X^{(1)}, \ldots, X^{(d)})$ are given by independent identically distributed Black-Scholes models

$$X_t^{(l)} = x_0 \exp \left\{ \left( R^L - \frac{1}{2}\sigma^2 \right) t + \sigma W_t^{(l)} \right\}, \quad l = 1, \ldots, d,$$

where $R^L \geq 0$ is the risk-free lending rate, $\sigma > 0$ is the assets volatility, and $W = (W^{(1)}, \ldots, W^{(d)})$ is a $d$-dimensional Brownian motion. We, moreover, denote by $R^B$ the risk-free borrowing rate.

Hence, we have that $R^B \geq R^L$. Further, we denote by $R^C$ the collateralization rate, which is the interest that the receiver of the collateral has to pay to the provider of the collateral. As in Example 3.2 in Nie and Rutkowski (2016) we consider the case that the collateral is a convex combination $\bar{q}(v^{(1)}, -v^{(2)}) = \alpha v^{(1)} + (1-\alpha)(-v^{(2)})$ of the hedger's price $v^{(1)}$ (i.e., the party's hedging cost) and the counterparty's price $-v^{(2)}$ (i.e, the negative of the counterparty's hedging cost) for some $\alpha \in [0,1]$. Following Proposition 3.3 in Nie and Rutkowski (2016) with zero initial endowment the system of PDEs then reads as follows:

$$v_t^{(\nu)}(t,x) + \frac{1}{2}\sum_{k,l=1}^{d} v_{x_k,x_l}^{(\nu)}(t,x) = -H^{(\nu)}(v^{(1)}(t,x), \nabla_x v^{(1)}(t,x), v^{(2)}(t,x), \nabla_x v^{(2)}(t,x)), \quad \nu = 1,2,$$

$(t,x) \in [0,T) \times \mathbb{R}^d$, with terminal conditions

$$v^{(\nu)}(T,x) = (-1)^{\nu-1} g\left(\left(x_0 \exp\left\{\left(R^L - \frac{1}{2}\sigma^2\right)t + \sigma x^{(k)}\right\}\right)_{k=1,\dots,d}\right), \; x = (x^{(1)},\dots,x^{(d)}) \in \mathbb{R}^d$$

and non-linearities given by

$$\begin{aligned} &H^{(\nu)}(v^{(1)}(t,x), \nabla_x v^{(1)}(t,x), v^{(2)}(t,x), \nabla_x v^{(2)}(t,x)) \\ &= -R^L a_\nu (v^{(1)}(t,x) + v^{(2)}(t,x)) + (-1)^\nu R^C(\alpha v^{(1)}(t,x) - (1-\alpha)v^{(2)}(t,x)) \\ &\quad +(R^B - R^L)\left(a_\nu(v^{(1)}(t,x) + v^{(2)}(t,x)) - \frac{1}{\sigma}(\nabla_x v^{(\nu)}(t,x))^\top \mathbf{1}\right)_-, \end{aligned}$$

where, $(a_1, a_2) = (1-\alpha, \alpha)$. With this notation, $v^{(1)}(t, W_t)$ and $-v^{(2)}(t, W_t)$ denote the hedger's price and counterparty's price of the collateralized contract at time $t$.

This problem is a special case of general systems of semilinear parabolic PDEs of the form

$$\begin{aligned} &v_t^{(\nu)}(t,x) + \frac{1}{2}\sum_{k,l=1}^{d} (\sigma\sigma^\top)_{k,l}(t,x) v_{x_k,x_l}^{(\nu)}(t,x) + \sum_{k=1}^{d} b_k(t,x) v_{x_k}^{(\nu)}(t,x) \\ &= -H^{(\nu)}(t,x,v^{(1)}(t,x), \sigma(t,x)\nabla_x v^{(1)}(t,x), \dots, v^{(N)}(t,x), \sigma(t,x)\nabla_x v^{(N)}(t,x)), \end{aligned} \tag{1.2}$$

$(t,x) \in [0,T) \times \mathbb{R}^d$, $\nu = 1,\dots,N$ with terminal conditions $v^{(\nu)}(T,x) = g^{(\nu)}(x)$. This system has a unique classical solution, if the coefficients $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$, $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $H^{(\nu)} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^{N(1+d)} \to \mathbb{R}$, and $g^{(\nu)} : \mathbb{R}^d \to \mathbb{R}$ satisfy suitable conditions, see e.g. Friedman (1964). In order to derive a discretization of (1.2), which fits into our framework, we exploit the link between semilinear parabolic PDEs and backward stochastic differential equations (BSDEs) (see e.g. Pardoux, 1998). Let $v$ be a classical solution to (1.2). Then, we have that the process $(Y_s, Z_s)_{0\leq s\leq T} := (v(s,X_s), \sigma(s,X_s)\nabla_x v(s,X_s))_{0\leq s\leq T}$ is a solution to the BSDE

$$Y_s = g(X_T) + \int_s^T H(r, X_r, Y_r, Z_r)\,dr - \int_s^T Z_r^\top\,dW_r, \quad 0 \leq s \leq T. \tag{1.3}$$

Here, $W$ is a $d$-dimensional Brownian motion and the process $(X_s)_{0\leq s\leq T}$ is given by the stochastic differential equation

$$X_s = x + \int_0^s b(r, X_r)\,dr + \int_0^s \sigma(r, X_r)\,dW_r, \quad 0 \leq s \leq T. \tag{1.4}$$

Discretizing (1.3) and (1.4), leads to a discretization scheme for (1.2): To this end, let $\pi = (t_0, \dots, t_J)$ be a partition of $[0,T]$ and denote by $\Delta W_{i+1} := W_{t_{i+1}} - W_{t_i}$ the increments of the

Brownian motion $W$ over time increments of size $\Delta_{i+1} = t_{i+1} - t_i$. Further, let $\mathcal{F}_j$ be the $\sigma$-algebra generated by $W$ up to time $t_j$, $j = 0, \ldots, J$. Then, we consider the Euler-type scheme

$$\begin{aligned}
X_{j+1} &= X_j + b(t_j, X_j)\Delta_{j+1} + \sigma(t_j, X_j)\Delta W_{j+1}, \quad X_0 = x, \\
Y_J^{(\nu)} &= g^{(\nu)}(X_J), \\
Y_j^{(\nu)} &= E_j\left[Y_{j+1}^{(\nu)}\right] + H^{(\nu)}\left(t_j, X_j, E_j\left[\beta_{j+1}Y_{j+1}^{(1)}\right], \ldots, E_j\left[\beta_{j+1}Y_{j+1}^{(N)}\right]\right)\Delta_{j+1}, \\
\beta_{j+1} &= \left(1, \frac{\Delta W_{j+1}^{(1)}}{\Delta_{j+1}}, \ldots, \frac{\Delta W_{j+1}^{(d)}}{\Delta_{j+1}}\right)^{\top},
\end{aligned} \tag{1.5}$$

for $j = 0, \ldots, J-1$ and $\nu = 1, \ldots, N$, where $E_j[\cdot]$ denotes the conditional expectation with respect to $\mathcal{F}_j$. Taking $D = d + 1$, we observe that this scheme is of the form (1.15) for any function $H$, which is convex in the last $ND$ variables, if the coefficients satisfy suitable growth conditions.

Such discretization schemes are well-studied in the BSDE-literature, see e.g. Bouchard and Touzi (2004), Zhang (2004), Gobet and Labart (2007), and Gobet and Makhlouf (2010). Note that, convergence rates for the approximation error $\sup_{\nu=1,\ldots,N} |v^{(\nu)}(0,x) - Y_0^{(\nu)}|$ induced by this kind of approximation schemes are available. Indeed, Zhang (2004) shows that it converges at order 1/2 in the mesh size of the partition, if the non-linearities $H^{(\nu)}$ and the terminal conditions $g^{(\nu)}$ satisfy certain Lipschitz conditions.

*Example* 1.1.3. We finally consider an example for a dynamic program of the form (1.15) with only one equation (i.e. $N = 1$), namely the problem of pricing a European-style option under uncertain volatility. This problem has first been studied in Avellaneda et al. (1995) and Lyons (1995). Hence, let $X^\sigma$ be the value process of a risky asset whose dynamics under the risk-neutral measure and in discounted units are given by

$$X_t^\sigma = x_0 \exp\left\{\int_0^t \sigma_u dW_u - \frac{1}{2}\int_0^t \sigma_u^2 du\right\},$$

where $x_0 \in \mathbb{R}$, $W$ is a Brownian motion and the volatility $\sigma$ is a stochastic process which is adapted to the filtration $(\mathcal{F}_t)_{0 \le t \le T}$ generated by $W$. Further, let $g : \mathbb{R} \to \mathbb{R}$ be the payoff of a European option. Then, the value of this option under uncertain volatility is given by

$$\mathcal{Y}_0 = \sup_\sigma E[g(X_T^\sigma)], \tag{1.6}$$

where the supremum is taken over all nonanticipating volatility processes $\sigma$, which take values in $[\sigma_{low}, \sigma_{up}]$. By considering the supremum over all processes ranging in this interval, $\mathcal{Y}_0$ provides a worst case price which reflects the volatility uncertainty. In the following, we assume that the constants satisfy $0 < \sigma_{low} \le \sigma_{up} < \infty$.

Since (1.6) is a stochastic control problem in continuous time, we can write down the Hamilton-Jacobi-Bellman equation, which is given by

$$\begin{aligned}
&u_t(t,x) + \max_{\sigma \in \{\sigma_{low}, \sigma_{up}\}} \frac{1}{2}\sigma^2 x^2 u_{xx}(t,x) = 0, \quad (t,x) \in [0,T) \times \mathbb{R} \\
&u(T,x) = g(x), \quad x \in \mathbb{R}.
\end{aligned} \tag{1.7}$$

Note that the PDE (1.7) possesses a classical solution, which satisfies appropriate growth conditions, under suitable assumptions on the terminal condition $g$, see Pham (2009).

Similar to Example (1.1.2), we want to derive a discretization scheme for (1.6), which is of the form (1.15), from (1.7). To this end, we fix a constant volatility $\hat{\rho}$ and consider the transformation

$$v(t,x) := u\left(t, x_0 \exp\left\{\hat{\rho}x - \frac{1}{2}\hat{\rho}^2 t\right\}\right), \quad x \in \mathbb{R},$$

in the space variable. Then, (1.7) can be rewritten in the following form:

$$\begin{aligned}
& v_t(t,x) + \frac{1}{2}v_{xx}(t,x) + \max_{\sigma\in\{\sigma_{low},\sigma_{up}\}} \left\{\frac{1}{2}\left(\frac{\sigma^2}{\hat{\rho}^2} - 1\right)(v_{xx}(t,x) - \hat{\rho}v_x(t,x))\right\} = 0, \\
& (t,x) \in [0,T) \times \mathbb{R}, \\
& v(T,x) = g\left(x_0 \exp\left\{\hat{\rho}x - \frac{1}{2}\hat{\rho}^2 T\right\}\right), \quad x \in \mathbb{R}.
\end{aligned} \tag{1.8}$$

In order to derive an approximate solution of (1.8), we apply an operator splitting scheme. Therefore, let $0 = t_0 < t_1 < \ldots < t_J = T$ be an equidistant discretization of the time interval $[0,T]$ with mesh size $\Delta$. Building on this discretization, we consider, for fixed $J$, the system

$$y^J(x) = g\left(x_0 e^{\hat{\rho}x - \frac{1}{2}\hat{\rho}^2 T}\right), \quad x \in \mathbb{R},$$

$$\bar{y}^j_t(t,x) = -\frac{1}{2}\bar{y}^j_{xx}(t,x), \quad (t,x) \in [t_j, t_{j+1}) \times \mathbb{R}, \tag{1.9}$$

$$\bar{y}^j(t_{j+1},x) = y^{j+1}(x), \quad x \in \mathbb{R}, \tag{1.10}$$

$$y^j(x) = \bar{y}^j(t_j,x) + \Delta \max_{\sigma\in\{\sigma_{low},\sigma_{up}\}} \left\{\frac{1}{2}\left(\frac{\sigma^2}{\hat{\rho}^2} - 1\right)\left(\bar{y}^j_{xx}(t_j,x) - \hat{\rho}\bar{y}^j_x(t_j,x)\right)\right\}, \quad x \in \mathbb{R}, \tag{1.11}$$

for $j = J-1, \ldots, 0$. Hence, the idea of this approach is to solve the linear subproblem (1.9) – (1.10), which is a Cauchy problem for the heat equation, of (1.8) on each of the intervals $[t_j, t_{j+1}]$ and to plug the corresponding solution in the non-linearity (1.11). Evaluating $y^j(x)$ along the Brownian paths leads to $Y_j := y^j(W_{t_j})$. A straightforward application of the Feynman-Kac representation for the solution of (1.9) – (1.10), see e.g. Karatzas and Shreve (1991), on each interval, then yields

$$\bar{y}^j(t_j, W_{t_j}) = E_j[y^{j+1}(W_{t_{j+1}})] = E_j[Y_{j+1}],$$

where $E_j[\cdot]$ denotes the conditional expectation with respect to $\mathcal{F}_j$. For the space derivatives $\bar{y}^j_x(t,x)$ and $\bar{y}^j_{xx}(t,x)$, we obtain by integration by parts that

$$\bar{y}^j_x(t_j, W_{t_j}) = E_j\left[\frac{\Delta W_{j+1}}{\Delta} Y_{j+1}\right] \tag{1.12}$$

and

$$\bar{y}^j_{xx}(t_j, W_{t_j}) = E_j\left[\left(\frac{\Delta W_{j+1}^2}{\Delta^2} - \frac{1}{\Delta}\right) Y_{j+1}\right], \tag{1.13}$$

where $\Delta W_j = W_{t_j} - W_{t_{j-1}}$. A detailed derivation of (1.12) and (1.13) can be found in the Appendix A.1. Note that (1.12) and (1.13) are the Malliavin Monte Carlo weights derived in Fournié et al. (1999).

Finally, we end up with the following discrete-time dynamic programming equation

$$Y_J = g(X_T^{\hat{\rho}}),$$

$$Y_j = E_j[Y_{j+1}] + \Delta \max_{\sigma\in\{\sigma_{low},\sigma_{up}\}} \left( \frac{1}{2}\left(\frac{\sigma^2}{\hat{\rho}^2} - 1\right) E_j\left[\left(\frac{\Delta W_{j+1}^2}{\Delta^2} - \hat{\rho}\frac{\Delta W_{j+1}}{\Delta} - \frac{1}{\Delta}\right) Y_{j+1}\right]\right), \quad (1.14)$$

where $X_T^{\hat{\rho}}$ denotes the price of the asset at time $T$ under the constant reference volatility $\hat{\rho}$. Such type of time-discretization scheme is proposed and analyzed for a general class of fully non-linear parabolic PDEs by Fahim et al. (2011). In the particular case of the uncertain volatility model, the scheme was suggested by Guyon and Henry-Labordère (2011) by a slightly different derivation. They rely on the connection between fully non-linear parabolic PDEs and second order backward stochastic differential equations, see Cheridito et al. (2007). Choosing

$$F_j(z) = z^{(1)} + \Delta \max_{s\in\{s_{low},s_{up}\}} sz^{(2)},$$

where $s_\iota = \frac{1}{2}(\frac{\sigma_\iota^2}{\hat{\rho}^2} - 1)$ for $\iota \in \{up, low\}$, and

$$\beta_j = \left(1, \frac{\Delta W_j^2}{\Delta^2} - \hat{\rho}\frac{\Delta W_j}{\Delta} - \frac{1}{\Delta}\right)^\top, \quad j = 1,\dots,J,$$

we observe that (1.14) is of the form (1.15) with $N = 1$ and $D = 2$.

## 1.2 Setup

Let $(\Omega, \mathcal{F}, (\mathcal{F}_j)_{j=0,\dots,J}, P)$ be a complete filtered probability space. Throughout the chapter we consider systems of convex dynamic programs of the form

$$\begin{aligned} Y_J^{(\nu)} &= \xi^{(\nu)} \\ Y_j^{(\nu)} &= F_j^{(\nu)}\left(E_j\left[\beta_{j+1}Y_{j+1}^{(1)}\right], \dots, E_j\left[\beta_{j+1}Y_{j+1}^{(N)}\right]\right), \quad \nu = 1,\dots,N,\ j = J-1,\dots,0, \end{aligned} \quad (1.15)$$

where $E_j[\cdot]$ denotes the conditional expectation with respect to $\mathcal{F}_j$. If this system is one-dimensional, i.e. if $N = 1$, we use the shorthand notation $Y := Y^{(1)}$. For our considerations, the following convexity and regularity assumptions are required:

**Assumption 1.2.1.** *(i) For every $j = 0,\dots,J-1$ and $\nu = 1,\dots,N$, $F_j^{(\nu)} : \Omega \times \mathbb{R}^{ND} \to \mathbb{R}$ is measurable and, for every $z \in \mathbb{R}^{ND}$, the process $(j,\omega) \mapsto F_j^{(\nu)}(\omega, z)$ is adapted.*

*(ii) The map $z \mapsto F_j^{(\nu)}(\omega, z)$ is convex in $z$ for every $j = 0,\dots,J-1$, $\nu = 1,\dots,N$ and $\omega \in \Omega$.*

*(iii) For every $\nu = 1,\dots,N$, $F^{(\nu)}$ is of polynomial growth in $z$ in the following sense: There exist a constant $q \geq 0$ and a non-negative adapted process $(\alpha_j^{(\nu)})_{j=0,\dots,J-1} \in L_{ad}^{\infty-}(\mathbb{R})$ such that for all $z \in \mathbb{R}^{ND}$ and $j = 0,\dots,J-1$*

$$\left|F_j^{(\nu)}(z)\right| \leq \alpha_j^{(\nu)}\left(1 + \sum_{n=1}^N \left\|z^{[n]}\right\|^q\right), \quad P\text{-a.s..}$$

*(iv) The process $\beta = (\beta_j)_{j=1,\dots,J}$ is an element of $L_{ad}^{\infty-}(\mathbb{R}^D)$.*

*(v) For each $\nu = 1,\dots,N$, the terminal conditions $\xi^{(\nu)}$ are elements of $L_J^{\infty-}(\mathbb{R})$.*

From these assumptions, we obtain immediately the following lemma.

**Lemma 1.2.2.** *Under Assumption 1.2.1 the $P$-almost surely unique solution $Y$ to* (1.15) *is an element of $L^{\infty-}_{ad}(\mathbb{R}^N)$.*

*Proof.* The proof is by backward induction on $j = J, \dots, 0$. For $j = J$ the assertion is trivially true as $\xi = (\xi^{(1)}, \dots, \xi^{(N)}) \in L^{\infty-}_J(\mathbb{R}^N)$ by assumption. Now suppose that the assertion is true for $j+1$. Then, $Y_j$ is $\mathcal{F}_j$-measurable, since $E_j[\beta_{j+1} Y^{(\nu)}_{j+1}]$ and $F^{(\nu)}_j(z)$ are $\mathcal{F}_j$-measurable for every $\nu = 1, \dots, N$ and $z \in \mathbb{R}^{ND}$.

For the integrability, we first note, that the case $q = 0$ is trivial, since this corresponds to the situation, where the functions $F^{(\nu)}_j$, and thus the solution $Y$, are bounded by a sufficiently integrable process. Hence, we suppose in the following that $q > 0$. Moreover, we assume without loss of generality that $p \geq 1$ satisfies $2pq \geq 1$. From the polynomial growth condition on $F^{(\nu)}_j$, we first observe that

$$\begin{aligned} E\left[\left|Y^{(\nu)}_j\right|^p\right]^{\frac{1}{p}} &= E\left[\left|F^{(\nu)}_j\left(E_j\left[\beta_{j+1}Y^{(1)}_{j+1}\right], \dots, E_j\left[\beta_{j+1}Y^{(N)}_{j+1}\right]\right)\right|^p\right]^{\frac{1}{p}} \\ &\leq E\left[\left|\alpha^{(\nu)}_j\left(1 + \sum_{n=1}^N \left\|E_j\left[\beta_{j+1}Y^{(n)}_{j+1}\right]\right\|^q\right)\right|^p\right]^{\frac{1}{p}}. \end{aligned}$$

Applying Hölder's inequality and the Minkowski inequality twice then yields

$$\begin{aligned} E\left[\left|Y^{(\nu)}_j\right|^p\right]^{\frac{1}{p}} &\leq E\left[\left|\alpha^{(\nu)}_j\right|^{2p}\right]^{\frac{1}{2p}} E\left[\left|1 + \sum_{n=1}^N \left\|E_j\left[\beta_{j+1}Y^{(n)}_{j+1}\right]\right\|^q\right|^{2p}\right]^{\frac{1}{2p}} \\ &\leq E\left[\left|\alpha^{(\nu)}_j\right|^{2p}\right]^{\frac{1}{2p}} \left(1 + E\left[\left|\sum_{n=1}^N \left\|E_j\left[\beta_{j+1}Y^{(n)}_{j+1}\right]\right\|^q\right|^{2p}\right]^{\frac{1}{2p}}\right) \\ &\leq E\left[\left|\alpha^{(\nu)}_j\right|^{2p}\right]^{\frac{1}{2p}} \left(1 + \sum_{n=1}^N E\left[\left\|E_j\left[\beta_{j+1}Y^{(n)}_{j+1}\right]\right\|^{2qp}\right]^{\frac{1}{2p}}\right). \end{aligned}$$

Finally, we obtain by Jensen's inequality (applied to the convex function $y \mapsto \|y\|^{2pq}$) that

$$\begin{aligned} E\left[\left|Y^{(\nu)}_j\right|^p\right]^{\frac{1}{p}} &\leq E\left[\left|\alpha^{(\nu)}_j\right|^{2p}\right]^{\frac{1}{2p}} \left(1 + \sum_{n=1}^N E\left[\left\|\beta_{j+1}Y^{(n)}_{j+1}\right\|^{2qp}\right]^{\frac{1}{2p}}\right) \\ &< \infty. \end{aligned}$$

Here, the last inequality is a consequence of the Assumption 1.2.1 and the induction hypothesis. □

The aim of this chapter is to construct upper and lower bounds to the solution $Y$, which can be computed pathwise. These build on the concept of super- and subsolutions to (1.15).

**Definition 1.2.3.** A process $Y^{up}$ (resp. $Y^{low}$) $\in L^{\infty-}_{ad}(\mathbb{R}^N)$ is called *supersolution* (resp. *subsolution*) to the dynamic program (1.15) if $Y^{up}_J \geq Y_J$ (resp. $Y^{low}_J \leq Y_J$) and for every $\nu = 1, \dots, N$ and $j = 0, \dots, J-1$ it holds that

$$Y^{(up,\nu)}_j \geq F^{(\nu)}_j\left(E_j\left[\beta_{j+1}Y^{(up,1)}_{j+1}\right], \dots, E_j\left[\beta_{j+1}Y^{(up,N)}_{j+1}\right]\right) \quad P\text{-a.s.},$$

(and with "$\geq$" replaced by "$\leq$" for a subsolution).

In what follows, the construction of supersolutions builds on the choice of a suitable martingale. We thus denote in the following by $\mathcal{M}_{ND}$ the set of martingales $M$, which satisfy $M \in L^{\infty-}_{ad}(\mathbb{R}^{ND})$. For a process $U \in L^{\infty-}_{ad}(\mathbb{R}^m)$, we refer to the martingale part of the Doob decomposition of $U$, which is given by

$$\sum_{i=0}^{j-1} U_{i+1} - E_i[U_{i+1}], \quad j = 0, \ldots, J,$$

as *Doob martingale of* $U$. In particular, we get from Assumption 1.2.1 that the Doob martingale of the process $\beta\bar{U}$ is in $\mathcal{M}_D$ for any $\bar{U} \in L^{\infty-}_{ad}(\mathbb{R})$.

In contrast to supersolutions, subsolutions are constructed by rewriting (1.15) as a stochastic control problem using convex duality techniques and taking an admissible control. To this end, recall that the convex conjugate of $F_j^{(\nu)}$ is, for every $\omega \in \Omega$, given by

$$F_j^{(\nu,\#)}(\omega, u) := \sup_{z \in \mathbb{R}^{ND}} \left( \sum_{n=1}^{N} \left( u^{[n]} \right)^\top z^{[n]} - F_j^{(\nu)}(\omega, z) \right), \tag{1.16}$$

with effective domain

$$D^{(j,\omega)}_{F^{(\nu,\#)}} = \left\{ u \in \mathbb{R}^{ND} \;\middle|\; F_j^{(\nu,\#)}(\omega, u) < \infty \right\}.$$

As we will see below, the sets of admissible controls in our problem are given by

$$\mathcal{A}_j^{F^{(\nu)}} = \left\{ \left( r_i^{(\nu)} \right)_{i=j,\ldots,J-1} \middle| r_i^{(\nu)} \in L_i^{\infty-}\left(\mathbb{R}^{ND}\right), F_i^{(\nu,\#)}\left( r_i^{(\nu)} \right) \in L^{\infty-}(\mathbb{R}) \text{ for } i = j, \ldots, J-1 \right\},$$

where $j = 0, \ldots, J-1$ and $\nu = 1, \ldots, N$. By continuity of $F_i^{(\nu)}$, we obtain that

$$F_i^{(\nu,\#)}(r_i) = \sup_{z \in \mathbb{Q}^{ND}} \left( \sum_{n=1}^{N} \left( r_i^{(\nu),[n]} \right)^\top z^{[n]} - F_i^{(\nu)}(z) \right)$$

is $\mathcal{F}_i$-measurable for every $r^{(\nu)} \in \mathcal{A}_j^{F^{(\nu)}}$ and $i = j, \ldots, J-1$. Moreover, from the integrability condition on the controls we deduce that $F_i^{(\nu,\#)}(r_i^{(\nu)}) < \infty$, i.e., controls take values in the effective domain of the convex conjugate of $F_i^{(\nu)}$. The following lemma shows that the set $\mathcal{A}_j^{F^{(\nu)}}$ is nonempty for every $j = 0, \ldots, J-1$ and $\nu = 1, \ldots, N$ under the given assumptions.

**Lemma 1.2.4.** *Fix $j \in \{0, \ldots, J-1\}$ and let $f_j : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a mapping such that, for every $\omega \in \Omega$, the map $x \mapsto f_j(\omega, x)$ is convex, and for every $x \in \mathbb{R}^d$, the map $\omega \mapsto f_j(\omega, x)$ is $\mathcal{F}_j$-measurable. Moreover, suppose that $f_j$ satisfies the following polynomial growth condition: There are a constant $q \geq 0$ and a non-negative random variable $\alpha_j \in L_j^{\infty-}(\mathbb{R})$ such that*

$$|f_j(x)| \leq \alpha_j (1 + \|x\|^q), \quad P\text{-a.s.},$$

*for every $x \in \mathbb{R}^d$. Then, for every $\bar{Z} \in L^{\infty-}(\mathbb{R}^d)$ there exists a random variable $\bar{\rho}_j \in L^{\infty-}(\mathbb{R}^d)$ such that $f_j^{\#}(\bar{\rho}_j) \in L^{\infty-}(\mathbb{R})$ and*

$$f_j(\bar{Z}) = \bar{\rho}_j^\top \bar{Z} - f_j^{\#}(\bar{\rho}_j), \quad P\text{-a.s.} \tag{1.17}$$

*If, additionally, $\bar{Z}$ is $\mathcal{F}_j$-measurable, then we can take $\bar{\rho}_j$ $\mathcal{F}_j$-measurable.*

*Proof.* Let $\bar{Z} \in L^{\infty-}(\mathbb{R}^d)$. Notice first that, since $f_j$ is convex and closed, we have $f_j^{\#\#} = f_j$ by Theorem 12.2 in Rockafellar (1970) and thus

$$f_j(\bar{Z}) = \sup_{u \in \mathbb{R}^d} u^\top \bar{Z} - f_j^{\#}(u) \geq \rho^\top \bar{Z} - f_j^{\#}(\rho) \tag{1.18}$$

holds $\omega$-wise for any random variable $\rho$. We next show that there exists a random variable $\bar{\rho}_j$ for which (1.18) holds with $P$-almost sure equality. To this end, we apply Theorem 7.4 in Cheridito et al. (2015) which yields the existence of a measurable subgradient to $f_j$, i.e., existence of a random variable $\bar{\rho}_j$ such that for all $\mathbb{R}^d$-valued random variables $Z$

$$f_j\left(\bar{Z} + Z\right) - f_j\left(\bar{Z}\right) \geq \bar{\rho}_j^\top Z, \quad P\text{-a.s.} \tag{1.19}$$

Choosing $Z = z - \bar{Z}$ for $z \in \mathbb{Q}^d$ in (1.19), we conclude that

$$\bar{\rho}_j^\top \bar{Z} - f_j\left(\bar{Z}\right) \geq \bar{\rho}_j^\top z - f_j\left(z\right). \tag{1.20}$$

Since (1.20) holds for any $z \in \mathbb{Q}^d$, we obtain

$$\bar{\rho}_j^\top \bar{Z} - f_j\left(\bar{Z}\right) \geq \sup_{z \in \mathbb{Q}^d} \bar{\rho}_j^\top z - f_j\left(z\right) = f_j^{\#}(\bar{\rho}_j), \quad P\text{-a.s.}, \tag{1.21}$$

by continuity of $f_j$, which is the converse of (1.18), proving $P$-almost sure equality for $\rho = \bar{\rho}_j$ and thus (1.17).

We next show that $\bar{\rho}_j$ satisfies the required integrability conditions, i.e., $\bar{\rho}_j \in L^{\infty-}(\mathbb{R}^d)$ and $f_j^{\#}(\bar{\rho}_j) \in L^{\infty-}(\mathbb{R})$. To this end, we first prove that $\bar{\rho}_j^\top Z \in L^{\infty-}(\mathbb{R})$ for any $Z \in L^{\infty-}(\mathbb{R}^d)$. Due to (1.19) and the Minkowski inequality and since $a \leq b$ implies $a_+ \leq |b|$, it follows for $Z \in L^{\infty-}(\mathbb{R}^d)$ that, for every $p \geq 1$,

$$\left(E\left[\left|\left(\bar{\rho}_j^\top Z\right)_+\right|^p\right]\right)^{\frac{1}{p}} \quad \leq \quad \left(E\left[\left|f_j\left(\bar{Z} + Z\right)\right|^p\right]\right)^{\frac{1}{p}} + \left(E\left[\left|f_j\left(\bar{Z}\right)\right|^p\right]\right)^{\frac{1}{p}} < \infty,$$

since $f_j$ is of polynomial growth with 'random constant' $\alpha_j \in L_j^{\infty-}(\mathbb{R})$ and $\bar{Z}, Z$ are elements of $L^{\infty-}(\mathbb{R}^d)$ by assumption. Applying the same argument to $\tilde{Z} = -Z$ yields

$$E\left[\left|\left(\bar{\rho}_j^\top Z\right)_-\right|^p\right] = E\left[\left|\left(\bar{\rho}_j^\top \tilde{Z}\right)_+\right|^p\right] < \infty,$$

since (1.19) holds for all random variables $Z$ and $\tilde{Z}$ inherits the integrability of $Z$. We thus conclude that

$$E\left[\left|\bar{\rho}_j^\top Z\right|^p\right] < \infty \quad \text{and} \quad E\left[|\bar{\rho}_j|^p\right] < \infty,$$

where the second claim follows from the first by taking $Z = \text{sgn}(\bar{\rho}_j)$ with the sign function applied componentwise. In order to show that $f_j^{\#}(\bar{\rho}_j) \in L^{\infty-}(\mathbb{R})$, we start with (1.17) and apply the Minkowski inequality to conclude that

$$\left(E\left[\left|f_j^{\#}\left(\bar{\rho}_j\right)\right|^p\right]\right)^{\frac{1}{p}} \leq \left(E\left[\left|\bar{\rho}_j^\top \bar{Z}\right|^p\right]\right)^{\frac{1}{p}} + \left(E\left[\left|f_j\left(\bar{Z}\right)\right|^p\right]\right)^{\frac{1}{p}} < \infty.$$

Finally, we show that for $\mathcal{F}_j$-measurable random variables $\bar{Z}$ there exists an $\mathcal{F}_j$-measurable random variable $\rho_j$ satisfying (1.17). To this end, let $\bar{Z} \in L_j^{\infty-}(\mathbb{R}^d)$ and let $\bar{\rho}_j$ be the possibly not $\mathcal{F}_j$-measurable random variable for which (1.17) holds and whose existence is already shown. We show that $\rho_j = E_j[\bar{\rho}_j]$ is the asserted random variable. By taking the conditional expectation of (1.21) and applying Jensen's inequality to the convex function $f_j^{\#}$, we conclude that

$$\begin{aligned} E_j\left[\bar{\rho}_j\right]^\top \bar{Z} &\geq f_j\left(\bar{Z}\right) + E_j\left[f_j^{\#}(\bar{\rho}_j)\right] \\ &\geq f_j\left(\bar{Z}\right) + f_j^{\#}\left(E_j\left[\bar{\rho}_j\right]\right). \end{aligned}$$

In combination with (1.18), we thus end up with

$$f_j(\bar{Z}) = E_j\left[\bar{\rho}_j\right]^\top \bar{Z} - f_j^{\#}\left(E_j\left[\bar{\rho}_j\right]\right) = \rho_j^\top \bar{Z} - f_j^{\#}\left(\rho_j\right)$$

as claimed. The integrability of $\rho_j \in L^{\infty-}(\mathbb{R}^d)$ and $f_j^{\#}(\rho_j) \in L^{\infty-}(\mathbb{R})$ follows by the same arguments applied before. □

## 1.3 The monotone case

In this section, we construct upper and lower bounds to the solution $Y$ to (1.15). To do this, we rely on the pathwise approach proposed by Bender et al. (2017) in the context of one-dimensional convex dynamic programs. This approach builds on the construction of super- and subsolutions to (1.15) and requires an additional monotonicity assumption on the functions $F^{(\nu)}$ in the sense that a comparison principle holds. We begin this section by imposing the comparison principle. Then, we briefly recall the main ideas of Bender et al. (2017) and generalize them at the same time to our present setting.

In general, it is not clear that super- and subsolutions are ordered, i.e., it need not hold, that $Y_j^{up} \geq Y_j \geq Y_j^{low}$ for all $j = 0, \ldots, J$ and, hence, they typically do not constitute bounds. The following assumption, to which we refer as *comparison principle*, ensures this.

**Assumption 1.3.1.** *For every supersolution $Y^{up}$ and every subsolution $Y^{low}$ to the dynamic program (1.15) it holds that*

$$Y_j^{up} \geq Y_j^{low}, \quad P\text{-a.s.},$$

*for every $j = 0, \ldots, J$.*

The main idea of Bender et al. (2017) in the construction of the upper bound is to drop the conditional expectations in (1.15) and instead subtract a martingale increment. Hence, let $j \in \{0, \ldots, J-1\}$ be fixed. Then, for a given martingale $M \in \mathcal{M}_{ND}$, we define the typically non-adapted process $\Theta^{up} := \Theta^{up}(M)$ recursively by

$$\begin{aligned} \Theta_J^{(up,\nu)} &= \xi^{(\nu)} \\ \Theta_i^{(up,\nu)} &= F_i^{(\nu)}\left(\beta_{i+1}\Theta_{i+1}^{(up,1)} - \Delta M_{i+1}^{[1]}, \ldots, \beta_{i+1}\Theta_{i+1}^{(up,N)} - \Delta M_{i+1}^{[N]}\right), \quad i = J-1, \ldots, j,\ \nu = 1, \ldots, N, \end{aligned} \tag{1.22}$$

where $\Delta M_{i+1}^{[n]} := M_{i+1}^{[n]} - M_i^{[n]}$.

**Lemma 1.3.2.** *Suppose Assumptions 1.2.1. Then, for every $j \in \{0, \ldots, J\}$ and $M \in \mathcal{M}_{ND}$, the process $\Theta^{up}(M)$ defined by (1.22) satisfies $\Theta_i^{up}(M) \in L^{\infty-}(\mathbb{R}^N)$ for all $i = j, \ldots, J$.*

The proof of this lemma follows the same lines of reasoning as the one of Lemma 1.2.2, so that we omit the details here.

Based on the recursion (1.22), we define the adapted process $Y^{up}$ by

$$Y_j^{up} := E_j\left[\Theta_j^{up}\right], \quad j = 0, \dots, J,$$

which is well-defined by Lemma 1.3.2. Then, $Y^{up}$ is a supersolution to (1.15). To see this, we first apply Jensen's inequality and obtain

$$\begin{aligned} Y_j^{(up,\nu)} &= E_j\left[\Theta_j^{(up,\nu)}\right] \\ &= E_j\left[F_j^{(\nu)}\left(\beta_{j+1}\Theta_{j+1}^{(up,1)} - \Delta M_{j+1}^{[1]}, \dots, \beta_{j+1}\Theta_{j+1}^{(up,N)} - \Delta M_{j+1}^{[N]}\right)\right] \\ &\geq F_j^{(\nu)}\left(E_j\left[\beta_{j+1}\Theta_{j+1}^{(up,1)} - \Delta M_{j+1}^{[1]}\right], \dots, E_j\left[\beta_{j+1}\Theta_{j+1}^{(up,N)} - \Delta M_{j+1}^{[N]}\right]\right). \end{aligned}$$

From the martingale property of $M$ and the tower property of the conditional expectation, we finally conclude that

$$\begin{aligned} Y_j^{(up,\nu)} &\geq F_j^{(\nu)}\left(E_j\left[\beta_{j+1}\Theta_{j+1}^{(up,1)}\right], \dots, E_j\left[\beta_{j+1}\Theta_{j+1}^{(up,N)}\right]\right) \\ &= F_j^{(\nu)}\left(E_j\left[\beta_{j+1}E_{j+1}\left[\Theta_{j+1}^{(up,1)}\right]\right], \dots, E_j\left[\beta_{j+1}E_{j+1}\left[\Theta_{j+1}^{(up,N)}\right]\right]\right) \\ &= F_j^{(\nu)}\left(E_j\left[\beta_{j+1}Y_{j+1}^{(up,1)}\right], \dots, E_j\left[\beta_{j+1}Y_{j+1}^{(up,N)}\right]\right) \end{aligned}$$

for every $j = 0, \dots, J-1$ and $\nu = 1, \dots, N$ showing the supersolution property for the process $Y^{up}$.

In order to construct a subsolution to (1.15), we rely on duality techniques from convex analysis. More precisely, we linearize the dynamic programming equation (1.15) in the following way: By convexity and closedness of $F_j^{(\nu)}$, we have due to Theorem 12.2 in Rockafellar (1970) that $F_j^{(\nu,\#\#)} = F_j^{(\nu)}$ for every $j = 0, \dots, J-1$, $\nu = 1, \dots, N$ and $\omega \in \Omega$. Hence, for every $j = 0, \dots, J-1$, $\nu = 1, \dots, N$, $\omega \in \Omega$, and $z \in \mathbb{R}^{ND}$, it holds that

$$F_j^{(\nu)}(\omega, z) = \sup_{u \in \mathbb{R}^{ND}} \sum_{n=1}^{N} \left(u^{[n]}\right)^\top z^{[n]} - F_j^{(\nu,\#)}(\omega, u), \tag{1.23}$$

where $F_j^{(\nu,\#)}$ denotes the convex conjugate of $F_j^{(\nu)}$ defined in (1.16). From Lemma 1.2.4, we get existence of an adapted process $r^{(\nu,*)} \in \mathcal{A}_0^{F^{(\nu)}}$ which solves

$$\sum_{n=1}^{N} \left(r_j^{(\nu,*),[n]}\right)^\top E_j\left[\beta_{j+1}Y_{j+1}^{(n)}\right] - F_j^{(\nu,\#)}\left(r_j^{(\nu,*)}\right) = F_j^{(\nu)}\left(E_j\left[\beta_{j+1}Y_{j+1}^{(1)}\right], \dots, E_j\left[\beta_{j+1}Y_{j+1}^{(N)}\right]\right) \tag{1.24}$$

for every $j = 0, \dots, J-1$ and $\nu = 1, \dots, N$.

Following Bender et al. (2017), we now fix admissible controls $r^{(\nu)} \in \mathcal{A}_j^{F^{(\nu)}}$, $\nu = 1, \dots, N$, and define the typically non-adapted process $\Theta^{low} := \Theta^{low}(r^{(1)}, \dots, r^{(N)})$ by

$$\begin{aligned} \Theta_J^{(low,\nu)} &= \xi^{(\nu)}, \\ \Theta_i^{(low,\nu)} &= \sum_{n=1}^{N} \left(r_i^{(\nu),[n]}\right)^\top \beta_{i+1}\Theta_{i+1}^{(low,n)} - F_i^{(\nu,\#)}\left(r_i^{(\nu)}\right), \quad i = J-1, \dots, j, \ \nu = 1, \dots, N, \end{aligned} \tag{1.25}$$

for $j \in \{0, \dots, J-1\}$.

**Lemma 1.3.3.** *Suppose Assumptions 1.2.1. Then, for every $j \in \{0, \dots, J\}$ and any admissible controls $r^{(\nu)} \in \mathcal{A}_j^{F^{(\nu)}}$, $\nu = 1, \dots, N$, the process $\Theta^{low}(r^{(1)}, \dots, r^{(N)})$ defined by (1.25) satisfies $\Theta_i^{low}(r^{(1)}, \dots, r^{(N)}) \in L^{\infty-}(\mathbb{R}^N)$ for all $i = j, \dots, J$.*

*Proof.* Let $j \in \{0, \dots, J-1\}$ and $r^{(\nu)} \in \mathcal{A}_j^{F^{(\nu)}}$, $\nu = 1, \dots, N$, be fixed from now on and define $\Theta^{low} := \Theta^{low}(r^{(1)}, \dots, r^{(N)})$ by (1.25). The proof is by backward on induction on $i = j, \dots, J-1$ with the case $i = J$ being trivial, since $\xi^{(\nu)} \in L_J^{\infty-}(\mathbb{R})$ by assumption for each $\nu$. Now suppose that the assertion is true for $i+1$. Then, the Minkowski inequality and the Hölder inequality yield

$$\begin{aligned}
E\left[\left|\Theta_i^{(low,\nu)}\right|^p\right]^{\frac{1}{p}} &= E\left[\left|\sum_{n=1}^N \left(r_i^{(\nu),[n]}\right)^\top \beta_{i+1}\Theta_{i+1}^{(low,n)} - F_i^{(\nu,\#)}\left(r_i^{(\nu)}\right)\right|^p\right]^{\frac{1}{p}} \\
&\leq E\left[\left|\sum_{n=1}^N \left(r_i^{(\nu),[n]}\right)^\top \beta_{i+1}\Theta_{i+1}^{(low,n)}\right|^p\right]^{\frac{1}{p}} + E\left[\left|F_i^{(\nu,\#)}\left(r_i^{(\nu)}\right)\right|^p\right]^{\frac{1}{p}} \\
&\leq \sum_{n=1}^N E\left[\left|\left(r_i^{(\nu),[n]}\right)^\top \beta_{i+1}\Theta_{i+1}^{(low,n)}\right|^p\right]^{\frac{1}{p}} + E\left[\left|F_i^{(\nu,\#)}\left(r_i^{(\nu)}\right)\right|^p\right]^{\frac{1}{p}} \\
&\leq \sum_{n=1}^N E\left[\left|\left(r_i^{(\nu),[n]}\right)^\top \beta_{i+1}\right|^{2p}\right]^{\frac{1}{2p}} E\left[\left|\Theta_{i+1}^{(low,n)}\right|^{2p}\right]^{\frac{1}{2p}} + E\left[\left|F_i^{(\nu,\#)}\left(r_i^{(\nu)}\right)\right|^p\right]^{\frac{1}{p}}
\end{aligned}$$

From the admissibility of the controls $r^{(\nu)}$, $\nu = 1, \dots, N$, the integrability assumptions on $\beta$, and the induction hypothesis we obtain that $E[|\Theta_i^{(low,\nu)}|^p]^{\frac{1}{p}} < \infty$ and the proof is complete. □

As in the case of supersolutions, we rely on (1.25) to define a subsolution $Y^{low}$ to (1.15). To this end, let $r^{(\nu)} \in \mathcal{A}_0^{F^{(\nu)}}$, $\nu = 1, \dots, N$ and let $\Theta^{low} := \Theta^{low}(r^{(1)}, \dots, r^{(N)})$ be given by (1.25) with $j = 0$. Then, we define the adapted process $Y^{low}$ by

$$Y_j^{low} := E_j\left[\Theta_j^{low}\right], \quad j = 0, \dots, J.$$

By Lemma 1.3.3, this process is well-defined. From the adaptedness of the controls $r^{(\nu)}$, we observe that

$$\begin{aligned}
Y_j^{(low,\nu)} &= E_j\left[\Theta_j^{(low,\nu)}\right] \\
&= E_j\left[\sum_{n=1}^N \left(r_j^{(\nu),[n]}\right)^\top \beta_{j+1}\Theta_{j+1}^{(low,n)} - F_j^{(\nu,\#)}\left(r_j^{(\nu)}\right)\right] \\
&= \sum_{n=1}^N \left(r_j^{(\nu),[n]}\right)^\top E_j\left[\beta_{j+1}\Theta_{j+1}^{(low,n)}\right] - F_j^{(\nu,\#)}\left(r_j^{(\nu)}\right).
\end{aligned}$$

A straightforward application of the tower property of the conditional expectation and (1.23) shows that

$$\begin{aligned}
Y_j^{(low,\nu)} &= \sum_{n=1}^N \left(r_j^{(\nu),[n]}\right)^\top E_j\left[\beta_{j+1}E_{j+1}\left[\Theta_{j+1}^{(low,n)}\right]\right] - F_j^{(\nu,\#)}\left(r_j^{(\nu)}\right) \\
&= \sum_{n=1}^N \left(r_j^{(\nu),[n]}\right)^\top E_j\left[\beta_{j+1}Y_{j+1}^{(low,n)}\right] - F_j^{(\nu,\#)}\left(r_j^{(\nu)}\right)
\end{aligned}$$

$$\leq F_j^{(\nu)}\left(E_j\left[\beta_{j+1}Y_{j+1}^{(low,1)}\right],\ldots,E_j\left[\beta_{j+1}Y_{j+1}^{(low,N)}\right]\right),$$

for every $j = 0,\ldots,J-1$ and $\nu = 1,\ldots,N$ and, thus, $Y^{low}$ is a subsolution to (1.15).

Summarizing, we obtain by the comparison principle that

$$E_j\left[\Theta_j^{low}\left(r^{(1)},\ldots,r^{(N)}\right)\right] \leq Y_j \leq E_j\left[\Theta_j^{up}(M)\right]$$

for every $j = 0,\ldots,J$, $M \in \mathcal{M}_{ND}$ and all admissible controls $r^{(\nu)} \in \mathcal{A}_0^{F^{(\nu)}}$, $\nu = 1,\ldots,N$. In particular, we have that

$$\operatorname*{esssup}_{r^{(1)}\in\mathcal{A}_0^{F^{(1)}},\ldots,r^{(N)}\in\mathcal{A}_0^{F^{(N)}}} E_0\left[\Theta_0^{(low,\nu)}\left(r^{(1)},\ldots,r^{(N)}\right)\right] \leq Y_0^{(\nu)} \leq \operatorname*{essinf}_{M\in\mathcal{M}_{ND}} E_0\left[\Theta_0^{(up,\nu)}(M)\right] \qquad (1.26)$$

for every $\nu = 1,\ldots,N$. We emphasize that the essential supremum is taken over all admissible controls $r^{(1)},\ldots,r^{(N)}$, since $\Theta^{(low,\nu)}$ depends on $r^{(n)}$, $n \neq \nu$, implicitly through the processes $\Theta^{(low,n)}$. The following theorem generalizes (1.26) to arbitrary $j \in \{0,\ldots,J-1\}$ and establishes, at the same time, existence of optimal controls and martingales for these inequalities.

**Theorem 1.3.4.** *Suppose Assumptions 1.2.1 and 1.3.1. Then, for every $j = 0,\ldots,J$ and $\nu = 1,\ldots,N$,*

$$\begin{aligned} Y_j^{(\nu)} &= \operatorname*{essinf}_{M\in\mathcal{M}_{ND}} E_j\left[\Theta_j^{(up,\nu)}(M)\right] \\ &= \operatorname*{esssup}_{r^{(1)}\in\mathcal{A}_j^{F^{(1)}},\ldots,r^{(N)}\in\mathcal{A}_j^{F^{(N)}}} E_j\left[\Theta_j^{(low,\nu)}\left(r^{(1)},\ldots,r^{(N)}\right)\right], \quad P\text{-a.s.} \end{aligned}$$

*Moreover,*

$$Y_j^{(\nu)} = \Theta_j^{(up,\nu)}(M^*) = E_j\left[\Theta_j^{(low,\nu)}\left(r^{(1,*)},\ldots,r^{(N,*)}\right)\right]$$

*$P$-almost surely, whenever each $r^{(\nu,*)}$ satisfies the duality relation* (1.24)*, i.e.,*

$$\sum_{n=1}^{N}\left(r_i^{(\nu,*),[n]}\right)^{\top} E_i\left[\beta_{i+1}Y_{i+1}^{(n)}\right] - F_i^{(\nu,\#)}\left(r_i^{(\nu,*)}\right) = F_i^{(\nu)}\left(E_i\left[\beta_{i+1}Y_{i+1}^{(1)}\right],\ldots,E_i\left[\beta_{i+1}Y_{i+1}^{(N)}\right]\right)$$

*$P$-almost surely for every $i = j,\ldots,J-1$ and each $M^{*,[\nu]}$ is the Doob martingale of $\beta Y^{(\nu)}$.*

The following example illustrates the construction of the proposed upper and lower bounds in the context of stopping problems and relates Theorem 1.3.4 to existing results for this kind of problems.

*Example* 1.3.5. (i) Recall that the system of dynamic programming equations for the multiple stopping problem considered in Example 1.1.1 is given by

$$Y_j^{(\nu)} = \max\left\{E_j\left[Y_{j+1}^{(\nu)}\right],\ S_j + E_j\left[Y_{j+1}^{(\nu-1)}\right]\right\}, \quad Y_J^{(\nu)} = \nu S_J,$$

for $j = 0,\ldots,J-1$, $\nu = 1,\ldots,N$, and $Y^{(0)} \equiv 0$. Due to the monotonicity of the maximum, it is straightforward to show, that this system of dynamic programs satisfies the comparison principle. Indeed, let $Y^{up}$ and $Y^{low}$ be a super- respectively subsolution to the dynamic program and suppose that $Y_{j+1}^{up} \geq Y_{j+1}^{low}$ holds by induction hypothesis. Then, the monotonicity of the maximum and the conditional expectation as well as the super- respectively subsolution property of $Y^{up}$ and $Y^{low}$ yield

$$\begin{aligned} Y_j^{(up,\nu)} &\geq \max\left\{E_j\left[Y_{j+1}^{(up,\nu)}\right],\ S_j+E_j\left[Y_{j+1}^{(up,\nu-1)}\right]\right\} \\ &\geq \max\left\{E_j\left[Y_{j+1}^{(low,\nu)}\right],\ S_j+E_j\left[Y_{j+1}^{(low,\nu-1)}\right]\right\} \\ &\geq Y_j^{(low,\nu)} \end{aligned}$$

for every $\nu = 1,\dots,N$. Taking a martingale $M \in \mathcal{M}_N$ (since $D = 1$) and applying (1.22) to this problem, we obtain that the upper bound $\Theta^{up}$ is given by

$$\begin{aligned} \Theta_J^{(up,\nu)} &= \nu S_J \\ \Theta_j^{(up,\nu)} &= \max\left\{\Theta_{j+1}^{(up,\nu)} - \Delta M_{j+1}^{[\nu]}, S_j + \Theta_{j+1}^{(up,\nu-1)} - \Delta M_{j+1}^{[\nu-1]}\right\}, \quad j = J-1,\dots,0 \end{aligned} \tag{1.27}$$

for $\nu = 1,\dots,N$ and with $\Theta^{(up,0)} \equiv 0$. This system of equations can be solved explicitly and we conclude that

$$\Theta_j^{(up,\nu)} = \max_{\substack{j\leq i_1\leq\cdots\leq i_\nu,\\ i_k=i_{k+1}\Rightarrow i_k=J}} \sum_{k=1}^{\nu}\left(S_{i_k} - M_{i_k}^{[\nu-k+1]} + M_{i_{k-1}}^{[\nu-k+1]}\right), \quad i_0 := j.$$

This is indeed the pure martingale dual proposed by Schoenmakers (2012), for which the numerically more tractable recursion (1.27) is due to Balder et al. (2013). This upper bound has also been derived by Chandramouli and Haugh (2012) in the more general context of information relaxation. In the case of single stopping (i.e. $N = 1$), this dual minimization problem collapses to the one derived independently by Rogers (2002) and Haugh and Kogan (2004).

(ii) In the case $N = 1$, we next explain, how the maximization problem in Theorem 1.3.4 relates to optimal stopping. By Appendix A.2, we get that the convex conjugate $F_j^{\#}$ of the function $F_j(z) = \max\{S_j, z\}$ is given by

$$F_j^{\#}(u) = (u-1)S_j$$

on the effective domain $D_{F^\#}^{(j,\omega)} = [0,1]$. Hence, for any $j \in \{0,\dots,J\}$ and $r \in \mathcal{A}_j^F$, one obtains by backward induction, that

$$\Theta_j^{low}(r) = r_j\Theta_{j+1}^{low}(r) + (1-r_j)S_j = S_J\prod_{i=j}^{J-1} r_i + \sum_{i=j}^{J-1}(1-r_i)S_i\prod_{k=j}^{i-1} r_k. \tag{1.28}$$

We thus conclude by Theorem 1.3.4, that

$$Y_j = \operatorname*{esssup}_{r\in\mathcal{A}_j^F} E_j[\Theta_j^{low}(r)],$$

where the set $\mathcal{A}_j^F$ of admissible controls is given by

$$\mathcal{A}_j^F = \{(r_i)_{i=j,\dots,J-1} |\, r_i\ \mathcal{F}_i - \text{measurable},\ r_i \in [0,1]\}.$$

Since the duality relation (1.24) is given by

$$r_i^* E_i[Y_{i+1}] + (1-r_i^*)S_i = \max\{S_i, E_i[Y_{i+1}]\}, \quad i = 0,\dots,J-1,$$

we observe, that the supremum can be restricted to $\{0,1\}$-valued controls. If $r \in \mathcal{A}_j^F$ takes values in $\{0,1\}$, then

$$\tau_r := \inf\{j \le i \le J-1 | r_i = 0\} \wedge J$$

is a stopping time in $\mathcal{S}_j$ and, by (1.28), $\Theta_j^{low}(r) = S_{\tau_r}$. Conversely, given any stopping time $\tau \in \mathcal{S}_j$, we have that $\tau = \tau_r$ for the admissible control $r \in \mathcal{A}_j^F$ given by $r_i = \mathbb{1}_{\{\tau \neq i\}}$, $i = j, \ldots, J-1$. Hence, we obtain that

$$\operatorname*{esssup}_{r \in \mathcal{A}_j^F} E_j[\Theta_j^{low}(r)] = \operatorname*{esssup}_{\tau \in \mathcal{S}_j} E_j[S_\tau],$$

i.e., the primal maximization problem in Theorem 1.3.4 is a reformulation of the original stopping problem. The multiple stopping case, i.e. $N > 1$, can be handled analogously.

We now give the proof of Theorem 1.3.4.

*Proof of Theorem 1.3.4.* Let $j \in \{0, \ldots, J-1\}$ be fixed from now on. Further, let $M \in \mathcal{M}_{ND}$ be a martingale, $r^{(\nu)} \in \mathcal{A}_j^{F^{(\nu)}}$, $\nu = 1, \ldots, N$, be admissible controls and let $\Theta^{up} := \Theta^{up}(M)$ respectively $\Theta^{low} := \Theta^{low}(r^{(1)}, \ldots, r^{(N)})$ be given by (1.22) and (1.25). We first show that

$$E_j\left[\Theta_j^{low}\right] \le Y_j \le E_j\left[\Theta_j^{up}\right]$$

holds by the comparison principle. To this end, we define the processes $Y^{up,j}$ and $Y^{low,j}$ by

$$Y_i^{(up,\nu),j} = \begin{cases} E_i\left[\Theta_i^{(up,\nu)}\right], & i \ge j \\ F_i^{(\nu)}\left(E_i\left[\beta_{i+1} Y_{i+1}^{(up,1),j}\right], \ldots, E_i\left[\beta_{i+1} Y_{i+1}^{(up,N),j}\right]\right), & i < j \end{cases}$$

and

$$Y_i^{(low,\nu),j} = \begin{cases} E_i\left[\Theta_i^{(low,\nu)}\right], & i \ge j \\ F_i^{(\nu)}\left(E_i\left[\beta_{i+1} Y_{i+1}^{(low,1),j}\right], \ldots, E_i\left[\beta_{i+1} Y_{i+1}^{(low,N),j}\right]\right), & i < j \end{cases}$$

for every $\nu = 1, \ldots, N$. Then, $Y^{up,j}$ and $Y^{low,j}$ are super- and subsolutions to (1.15). Indeed, for $i \ge j$, this follows by the same arguments applied at the beginning of this section. For $i < j$, this is an immediate consequence of the definition of $Y^{up,j}$ and $Y^{low,j}$. Hence, we obtain by the comparison principle that

$$Y_i^{(low,\nu),j} \le Y_i^{(\nu)} \le Y_i^{(up,\nu),j}$$

holds for every $i = 0, \ldots, J$ and $\nu = 1, \ldots, N$. In particular, we have that

$$Y_j^{(low,\nu),j} \le Y_j^{(\nu)} \le Y_j^{(up,\nu),j}$$

and thus

$$E_j\left[\Theta_j^{(low,\nu)}\right] \le Y_j^{(\nu)} \le E_j\left[\Theta_j^{(up,\nu)}\right].$$

As this chain of inequalities holds for all admissible controls $r^{(\nu)} \in \mathcal{A}_j^{F^{(\nu)}}$, $\nu = 1, \ldots, N$, and martingales $M \in \mathcal{M}_{ND}$, we conclude that

$$\operatorname*{esssup}_{r^{(1)} \in \mathcal{A}_j^{F^{(1)}}, \ldots, r^{(N)} \in \mathcal{A}_j^{F^{(N)}}} E_j\left[\Theta_j^{(low,\nu)}\left(r^{(1)}, \ldots, r^{(N)}\right)\right] \le Y_j^{(\nu)} \le \operatorname*{essinf}_{M \in \mathcal{M}_{ND}} E_j\left[\Theta_j^{(up,\nu)}(M)\right].$$

It remains to show that

$$Y_j^{(\nu)} = \Theta_j^{(up,\nu)}(M^*) = E_j\left[\Theta_j^{(low,\nu)}\left(r^{(1,*)},\ldots,r^{(N,*)}\right)\right]$$

$P$-almost surely for every $\nu = 1,\ldots,N$. The proof is by backward induction on $i = j,\ldots,J$. Let $M^{*,[\nu]}$ be the Doob martingale of $\beta Y^{(\nu)}$ and let $r^{(\nu,*)} \in \mathcal{A}_j^{F^{(\nu)}}$ satisfy the duality relation (1.24) for every $\nu = 1,\ldots,N$. The case $i = J$ is trivial, since by definition of $\Theta^{up,*} := \Theta^{up}(M^*)$ and $\Theta^{low,*} := \Theta^{low}\left(r^{(1,*)},\ldots,r^{(N,*)}\right)$, we have $Y_J = \Theta_J^{up,*} = \Theta_J^{low,*}$. Now suppose that the assertion is true for $i+1$. Then, it follows from the induction hypothesis and the definition of $M^*$ that

$$\begin{aligned}
\Theta_i^{(up,*,\nu)} &= F_i^{(\nu)}\left(\beta_{i+1}\Theta_{i+1}^{(up,*,1)} - \Delta M_{i+1}^{*,[1]},\ldots,\beta_{i+1}\Theta_{i+1}^{(up,*,N)} - \Delta M_{i+1}^{*,[N]}\right)\\
&= F_i^{(\nu)}\left(\beta_{i+1}Y_{i+1}^{(1)} - \Delta M_{i+1}^{*,[1]},\ldots,\beta_{i+1}Y_{i+1}^{(N)} - \Delta M_{i+1}^{*,[N]}\right)\\
&= F_i^{(\nu)}\left(\beta_{i+1}Y_{i+1}^{(1)} - \left(\beta_{i+1}Y_{i+1}^{(1)} - E_i\left[\beta_{i+1}Y_{i+1}^{(1)}\right]\right),\ldots,\right.\\
&\qquad \left.\beta_{i+1}Y_{i+1}^{(N)} - \left(\beta_{i+1}Y_{i+1}^{(N)} - E_i\left[\beta_{i+1}Y_{i+1}^{(N)}\right]\right)\right)\\
&= F_i^{(\nu)}\left(E_i\left[\beta_{i+1}Y_{i+1}^{(1)}\right],\ldots,E_i\left[\beta_{i+1}Y_{i+1}^{(N)}\right]\right)\\
&= Y_i^{(\nu)}
\end{aligned}$$

for every $\nu = 1,\ldots,N$ and thus $Y_j = \Theta_j^{up,*}$. For the lower bound, we first observe that

$$\begin{aligned}
E_i\left[\Theta_i^{(low,*,\nu)}\right] &= E_i\left[\sum_{n=1}^{N}\left(r_i^{(\nu,*),[n]}\right)^{\top}\beta_{i+1}\Theta_{i+1}^{(low,*,n)} - F_i^{(\nu,\#)}\left(r_i^{(\nu,*)}\right)\right]\\
&= \sum_{n=1}^{N}\left(r_i^{(\nu,*),[n]}\right)^{\top}E_i\left[\beta_{i+1}\Theta_{i+1}^{(low,*,n)}\right] - F_i^{(\nu,\#)}\left(r_i^{(\nu,*)}\right).
\end{aligned}$$

by the admissibility of $r^{(\nu,*)}$. Then, we obtain by the tower property of the conditional expectation and the induction hypothesis that

$$\begin{aligned}
E_i\left[\Theta_i^{(low,*,\nu)}\right] &= \sum_{n=1}^{N}\left(r_i^{(\nu,*),[n]}\right)^{\top}E_i\left[\beta_{i+1}E_{i+1}\left[\Theta_{i+1}^{(low,*,n)}\right]\right] - F_i^{(\nu,\#)}\left(r_i^{(\nu,*)}\right)\\
&= \sum_{n=1}^{N}\left(r_i^{(\nu,*),[n]}\right)^{\top}E_i\left[\beta_{i+1}Y_{i+1}^{(n)}\right] - F_i^{(\nu,\#)}\left(r_i^{(\nu,*)}\right).
\end{aligned}$$

Exploiting the duality relation (1.24), we conclude that

$$E_i\left[\Theta_i^{(low,*,\nu)}\right] = F_i^{(\nu)}\left(E_i\left[\beta_{i+1}Y_{i+1}^{(1)}\right],\ldots,E_i\left[\beta_{i+1}Y_{i+1}^{(N)}\right]\right) = Y_i^{(\nu)}$$

for every $\nu = 1,\ldots,N$ and thus $Y_j = E_j[\Theta_j^{low,*}]$, which completes the proof. □

*Remark* 1.3.6. Note that, we do not require the adaptedness of the martingale $M$ in the proof of Theorem 1.3.4 but only that $E_j[\Delta M_{j+1}] = 0$ for all $j = 0,\ldots,J-1$. Thus, for the construction of upper bounds, we need not restrict ourselves to the set $\mathcal{M}_{ND}$ of martingales. Indeed, we may take any $V$ from the set $\mathcal{V}_{ND}$ of $\mathbb{R}^{ND}$-valued processes which satisfy $V_j \in L^{\infty-}(\mathbb{R}^{ND})$ and $E_{j-1}[V_j] = 0$ for every $j = 1,\ldots,J$ and replace the martingale increment $\Delta M_{j+1}$ in the recursion (1.22) for $\Theta^{up}$ by the random variable $V_{j+1}$.

Besides its theoretical relevance, Theorem 1.3.4 provides some guidance on the numerical implementation of the recursions (1.22) and (1.25). If we are given an approximate solution $\tilde{Y}$ to (1.15), we can obtain approximations $\tilde{M}^{[\nu]}$ and $\tilde{r}^{(\nu)}$ of the Doob martingales $M^{*,[\nu]}$ and the optimal control $r^{(\nu,*)}$, $\nu = 1, \ldots, N$, by replacing the true solution $Y$ by the approximation $\tilde{Y}$ in the definitions. More precisely, for given $\nu = 1, \ldots, N$, we define $\tilde{M}^{[\nu]}$ by

$$\tilde{M}_j^{[\nu]} = \sum_{i=0}^{j-1} \beta_{i+1}\tilde{Y}_{i+1}^{(\nu)} - E_i\left[\beta_{i+1}\tilde{Y}_{i+1}^{(\nu)}\right], \quad j = 0, \ldots, J,$$

and the process $\tilde{r}^{(\nu)}$ is given by a (possibly approximate) solution of

$$\sum_{n=1}^{N} \left(\tilde{r}_j^{(\nu),[n]}\right)^\top E_j\left[\beta_{j+1}\tilde{Y}_{j+1}^{(n)}\right] - F_j^{(\nu,\#)}\left(\tilde{r}_j^{(\nu)}\right) = F_j^{(\nu)}\left(E_j\left[\beta_{j+1}\tilde{Y}_{j+1}^{(1)}\right], \ldots, E_j\left[\beta_{j+1}\tilde{Y}_{j+1}^{(N)}\right]\right)$$

for $j = 0, \ldots, J-1$. With these approximations at hand, we can go through the recursions (1.22) and (1.25) path by path and apply a standard Monte Carlo estimator at the initial time to obtain an upper and lower bound on $Y_0$. Indeed, we obtain by Theorem 1.3.4 that the upper bound estimator should benefit from a low variance if $\mathcal{F}_0$ is trivial (which is typically the case in numerical applications) and the approximate Doob martingales $\tilde{M}^{[\nu]}$ are close to the Doob martingales $M^{*,[\nu]}$. Since we do not have this pathwise optimality for the controls $r^{(\nu,*)}$ in the lower bound, the corresponding estimator typically suffers from a larger variance. This problem is also discussed in Bender et al. (2017) and Brown and Haugh (2016). In order to avoid this problem, Bender et al. (2017) propose the modified recursion $\Theta^{low} := \Theta^{low}(r^{(1)}, \ldots, r^{(N)}, M)$ initiated at $\Theta_J^{(low,\nu)} = \xi^{(\nu)}$ and given by

$$\Theta_j^{(low,\nu)} = \sum_{n=1}^{N} \left(r_j^{(\nu),[n]}\right)^\top \beta_{j+1}\Theta_{j+1}^{(low,n)} - \sum_{n=1}^{N} \left(r_j^{(\nu),[n]}\right)^\top \Delta M_{j+1}^{[n]} - F_j^{(\nu,\#)}\left(r_j^{(\nu)}\right), \tag{1.29}$$

for $j = J-1, \ldots, 0$ and $\nu = 1, \ldots, N$. This recursion mainly coincides with (1.25) but, additionally, it takes martingale increments into account. From now on, we consider the recursion (1.29) for the lower bound and use the shorthand notation $\Theta^{low}(r^{(1)}, \ldots, r^{(N)}) := \Theta^{low}(r^{(1)}, \ldots, r^{(N)}, 0)$ to denote the recursion (1.25). Since we have that $E_j[\Theta_j^{(low,\nu)}] = E_j[\Theta_j^{(low,\nu)}(r^{(1)}, \ldots, r^{(N)})]$ for every $j = 0, \ldots, J$ and $\nu = 1, \ldots, N$ by backward induction, we observe that these increments play the role of control variates. A straightforward modification in the proof of Theorem 1.3.4 then shows that

$$Y_j = \Theta_j^{low}\left(r^{(1,*)}, \ldots, r^{(N,*)}, M^*\right) \quad P\text{-a.s.}$$

for every $j = 0, \ldots, J$, where, for every $\nu = 1, \ldots, N$, $r^{(\nu,*)}$ is given by (1.24) and $M^{*,[\nu]}$ is the Doob martingale of $\beta Y^{(\nu)}$.

## 1.4 Characterizations of the comparison principle

In the previous section, we observed that the comparison principle plays a key role in the pathwise approach of Bender et al. (2017) for the construction of upper and lower bounds. The following theorem states further characterizations of the comparison principle and is the basis for our further considerations.

**Theorem 1.4.1.** *Under Assumptions 1.2.1 the following assertions are equivalent:*

(a) *The comparison principle as stated in Assumption 1.3.1 is satisfied.*

(b) *For every* $\nu = 1, \ldots, N$ *and* $r^{(\nu)} \in \mathcal{A}_0^{F^{(\nu)}}$ *the following positivity condition is fulfilled: For every* $j = 0, \ldots, J-1$ *and* $n = 1, \ldots, N$

$$\left(r_j^{(\nu),[n]}\right)^\top \beta_{j+1} \geq 0, \quad P\text{-a.s.}$$

(c) *For every* $j = 0, \ldots, J-1$, $\nu = 1, \ldots, N$ *and any two random variables* $Y^{(1)}, Y^{(2)} \in L^{\infty-}(\mathbb{R}^N)$ *with* $Y^{(1)} \geq Y^{(2)}$ $P$-a.s., *the following monotonicity condition is satisfied:*

$$F_j^{(\nu)}\left(E_j\left[\beta_{j+1}Y^{(1,1)}\right], \ldots, E_j\left[\beta_{j+1}Y^{(1,N)}\right]\right) \geq F_j^{(\nu)}\left(E_j\left[\beta_{j+1}Y^{(2,1)}\right], \ldots, E_j\left[\beta_{j+1}Y^{(2,N)}\right]\right),$$

*$P$-almost surely.*

*Proof.* $(b) \Rightarrow (c)$ : Fix $j \in \{0, \ldots, J-1\}$ and $\nu \in \{1, \ldots, N\}$. Further, let $Y^{(1)}$ and $Y^{(2)}$ be two random variables which are in $L^{\infty-}(\mathbb{R}^N)$ and satisfy $Y^{(1)} \geq Y^{(2)}$ $P$-a.s. From Lemma 1.2.4, we have existence of a control $r^{(\nu)} \in \mathcal{A}_0^{F^{(\nu)}}$ satisfying

$$F_j^{(\nu)}\left(E_j\left[\beta_{j+1}Y^{(2,1)}\right], \ldots, E_j\left[\beta_{j+1}Y^{(2,N)}\right]\right) = \sum_{n=1}^N \left(r_j^{(\nu),[n]}\right)^\top E_j\left[\beta_{j+1}Y^{(2,n)}\right] - F_j^{(\nu,\#)}\left(r_j^{(\nu)}\right).$$

Hence, $(b)$ and (1.23) yield

$$\begin{aligned}
&F_j^{(\nu)}\left(E_j\left[\beta_{j+1}Y^{(2,1)}\right], \ldots, E_j\left[\beta_{j+1}Y^{(2,N)}\right]\right) \\
&= \sum_{n=1}^N \left(r_j^{(\nu),[n]}\right)^\top E_j\left[\beta_{j+1}Y^{(2,n)}\right] - F_j^{(\nu,\#)}\left(r_j^{(\nu)}\right) \\
&= \sum_{n=1}^N E_j\left[\left(r_j^{(\nu),[n]}\right)^\top \beta_{j+1}Y^{(2,n)}\right] - F_j^{(\nu,\#)}\left(r_j^{(\nu)}\right) \\
&\leq \sum_{n=1}^N E_j\left[\left(r_j^{(\nu),[n]}\right)^\top \beta_{j+1}Y^{(1,n)}\right] - F_j^{(\nu,\#)}\left(r_j^{(\nu)}\right) \\
&\leq F_j^{(\nu)}\left(E_j\left[\beta_{j+1}Y^{(1,1)}\right], \ldots, E_j\left[\beta_{j+1}Y^{(1,N)}\right]\right).
\end{aligned}$$

$(c) \Rightarrow (a)$ : Let $Y^{up}$ and $Y^{low}$ be super- respectively subsolutions to (1.15). The proof is by backward induction on $j = J, \ldots, 0$. The assertion is trivially true for $j = J$, since $Y_J^{up} \geq Y_J \geq Y_J^{low}$ holds by definition of $Y^{up}$ and $Y^{low}$. Now suppose that the assertion is true for $j+1$, i.e. $Y_{j+1}^{up} \geq Y_{j+1}^{low}$ $P$-a.s. Then, we conclude by the definition of super- and subsolutions, $(c)$ and the induction hypothesis that

$$\begin{aligned}
Y_j^{(up,\nu)} &\geq F_j^{(\nu)}\left(E_j\left[\beta_{j+1}Y_{j+1}^{(up,1)}\right], \ldots, E_j\left[\beta_{j+1}Y_{j+1}^{(up,N)}\right]\right) \\
&\geq F_j^{(\nu)}\left(E_j\left[\beta_{j+1}Y_{j+1}^{(low,1)}\right], \ldots, E_j\left[\beta_{j+1}Y_{j+1}^{(low,N)}\right]\right) \\
&\geq Y_j^{(low,\nu)}
\end{aligned}$$

for every $\nu = 1, \ldots, N$ and, thus, $Y_j^{up} \geq Y_j^{low}$.

$(a) \Rightarrow (b):$ We prove the contraposition. Hence, we assume that there exist $j_0 \in \{0, \dots, J-1\}$, $\nu_0, n_0 \in \{1, \dots, N\}$ and $r^{(\nu_0)} \in \mathcal{A}_0^{F^{(\nu_0)}}$ such that

$$P\left(\left\{\left(r_{j_0}^{(\nu_0),[n_0]}\right)^\top \beta_{j_0+1} < 0\right\}\right) > 0.$$

Further, let $r^{(\nu)} \in \mathcal{A}_0^{F^{(\nu)}}$, $\nu = 1, \dots, N$, $\nu \neq \nu_0$, be admissible controls. Based on these controls, we define the process $\bar{Y}$ by

$$\bar{Y}_j^{(n_0)} = \begin{cases} Y_j^{(n_0)}, & j > j_0 + 1 \\ Y_j^{(n_0)} - k\mathbb{1}_{\{(r_{j_0}^{(\nu_0),[n_0]})^\top \beta_{j_0+1} < 0\}}, & j = j_0 + 1 \\ \sum_{n=1}^N \left(r_j^{(n_0),[n]}\right)^\top E_j\left[\beta_{j+1}\bar{Y}_{j+1}^{(n)}\right] - F_j^{(n_0,\#)}\left(r_j^{(n_0)}\right), & j < j_0 + 1, \end{cases}$$

where $k \in \mathbb{N}$ will be fixed later on, and by

$$\bar{Y}_j^{(\nu)} = \begin{cases} Y_j^{(\nu)}, & j \geq j_0 + 1 \\ \sum_{n=1}^N \left(r_j^{(\nu),[n]}\right)^\top E_j\left[\beta_{j+1}\bar{Y}_{j+1}^{(n)}\right] - F_j^{(\nu,\#)}\left(r_j^{(\nu)}\right), & j < j_0 + 1, \end{cases}$$

for $\nu \neq n_0$. Then, the process $\bar{Y}$ is a subsolution to (1.15). To see this, we consider three different cases: For $j > j_0 + 1$ this is obvious as $\bar{Y}^{(\nu)}$ coincides with the solution $Y^{(\nu)}$ for each $\nu$. Next, we consider the case, that $j < j_0 + 1$. From (1.23), we conclude that

$$\begin{aligned} \bar{Y}_j^{(\nu)} &= \sum_{n=1}^N \left(r_j^{(\nu),[n]}\right)^\top E_j\left[\beta_{j+1}\bar{Y}_{j+1}^{(n)}\right] - F_j^{(\nu,\#)}\left(r_j^{(\nu)}\right) \\ &\leq F_j^{(\nu)}\left(E_j\left[\beta_{j+1}\bar{Y}_{j+1}^{(1)}\right], \dots, E_j\left[\beta_{j+1}\bar{Y}_{j+1}^{(N)}\right]\right) \end{aligned}$$

for every $\nu = 1, \dots, N$. Finally, we consider the case $j = j_0 + 1$. For $\nu \neq n_0$, the proof is completely analog to the case $j > j_0 + 1$, so that we only consider the case $\nu = n_0$ in more detail. A straightforward application of the definition of $\bar{Y}$ and $Y$, shows that

$$\begin{aligned} \bar{Y}_j^{(n_0)} &= Y_j^{(n_0)} - k\mathbb{1}_{\{(r_{j_0}^{(\nu_0),[n_0]})^\top \beta_{j_0+1} < 0\}} \\ &\leq Y_j^{(n_0)} \\ &= F_j^{(n_0)}\left(E_j\left[\beta_{j+1}Y_{j+1}^{(1)}\right], \dots, E_j\left[\beta_{j+1}Y_{j+1}^{(N)}\right]\right) \\ &= F_j^{(n_0)}\left(E_j\left[\beta_{j+1}\bar{Y}_{j+1}^{(1)}\right], \dots, E_j\left[\beta_{j+1}\bar{Y}_{j+1}^{(N)}\right]\right), \end{aligned}$$

and, thus, $\bar{Y}$ is a subsolution.

Now, let $r^{(\nu_0,*)} \in \mathcal{A}_0^{F^{(\nu_0)}}$ be given by the duality relation (1.24), i.e.,

$$\begin{aligned} &\sum_{n=1}^N \left(r_j^{(\nu_0,*),[n]}\right)^\top E_j\left[\beta_{j+1}Y_{j+1}^{(n)}\right] - F_j^{(\nu_0,\#)}\left(r_j^{(\nu_0,*)}\right) \\ &\qquad\qquad = F_j^{(\nu_0)}\left(E_j\left[\beta_{j+1}Y_{j+1}^{(1)}\right], \dots, E_j\left[\beta_{j+1}Y_{j+1}^{(N)}\right]\right) \end{aligned}$$

for every $j = 0, \dots, J-1$. From this and the definition of $\bar{Y}$, we conclude that

$$\begin{aligned}
&\bar{Y}_{j_0}^{(\nu_0)} - Y_{j_0}^{(\nu_0)} \\
&= \sum_{n=1}^{N} \left(r_{j_0}^{(\nu_0),[n]}\right)^\top E_{j_0}\left[\beta_{j_0+1}\bar{Y}_{j_0+1}^{(n)}\right] - F_{j_0}^{(\nu_0,\#)}\left(r_{j_0}^{(\nu_0)}\right) \\
&\quad -\sum_{n=1}^{N} \left(r_{j_0}^{(\nu_0,*),[n]}\right)^\top E_{j_0}\left[\beta_{j_0+1}Y_{j_0+1}^{(n)}\right] + F_{j_0}^{(\nu_0,\#)}\left(r_{j_0}^{(\nu_0,*)}\right) \\
&= \sum_{\substack{n=1,\\ n\neq n_0}}^{N} \left(r_{j_0}^{(\nu_0),[n]}\right)^\top E_{j_0}\left[\beta_{j_0+1}Y_{j_0+1}^{(n)}\right] \\
&\quad + \left(r_{j_0}^{(\nu_0),[n_0]}\right)^\top E_{j_0}\left[\beta_{j_0+1}\left(Y_{j_0+1}^{(n_0)} - k\mathbb{1}_{\{(r_{j_0}^{(\nu_0),[n_0]})^\top\beta_{j_0+1}<0\}}\right)\right] - F_{j_0}^{(\nu_0,\#)}\left(r_{j_0}^{(\nu_0)}\right) \\
&\quad -\sum_{n=1}^{N} \left(r_{j_0}^{(\nu_0,*),[n]}\right)^\top E_{j_0}\left[\beta_{j_0+1}Y_{j_0+1}^{(n)}\right] + F_{j_0}^{(\nu_0,\#)}\left(r_{j_0}^{(\nu_0,*)}\right) \\
&= \sum_{n=1}^{N} \left(r_{j_0}^{(\nu_0),[n]} - r_{j_0}^{(\nu_0,*),[n]}\right)^\top E_{j_0}\left[\beta_{j_0+1}Y_{j_0+1}^{(n)}\right] \\
&\quad -kE_{j_0}\left[\left(r_{j_0}^{(\nu_0),[n_0]}\right)^\top \beta_{j_0+1}\mathbb{1}_{\{(r_{j_0}^{(\nu_0),[n_0]})^\top\beta_{j_0+1}<0\}}\right] - F_{j_0}^{(\nu_0,\#)}\left(r_j^{(\nu_0)}\right) + F_{j_0}^{(\nu_0,\#)}\left(r_j^{(\nu_0,*)}\right) \\
&= \sum_{n=1}^{N} \left(r_{j_0}^{(\nu_0),[n]} - r_{j_0}^{(\nu_0,*),[n]}\right)^\top E_{j_0}\left[\beta_{j_0+1}Y_{j_0+1}^{(n)}\right] + kE_{j_0}\left[\left(\left(r_{j_0}^{(\nu_0),[n_0]}\right)^\top \beta_{j_0+1}\right)_-\right] \\
&\quad -F_{j_0}^{(\nu_0,\#)}\left(r_j^{(\nu_0)}\right) + F_{j_0}^{(\nu_0,\#)}\left(r_j^{(\nu_0,*)}\right).
\end{aligned}$$

Based on these considerations, we define the set $A_{j_0,\nu_0,n_0,K}$ by

$$\begin{aligned}
A_{j_0,\nu_0,n_0,K} &= \left\{E_{j_0}\left[\left(\left(r_{j_0}^{(\nu_0),[n_0]}\right)^\top \beta_{j_0+1}\right)_-\right] > \frac{1}{K}\right\} \\
&\quad \cap\left\{\sum_{n=1}^{N} \left(r_{j_0}^{(\nu_0),[n]} - r_{j_0}^{(\nu_0,*),[n]}\right)^\top E_{j_0}\left[\beta_{j_0+1}Y_{j_0+1}^{(n)}\right]\right. \\
&\quad \left. -F_{j_0}^{(\nu_0,\#)}\left(r_j^{(\nu_0)}\right) + F_{j_0}^{(\nu_0,\#)}\left(r_j^{(\nu_0,*)}\right) > -K\right\}.
\end{aligned}$$

Taking $K \in \mathbb{N}$ sufficiently large (which is fixed from now on), we get that $P(A_{j_0,\nu_0,n_0,K}) > 0$ and therefore, for $k > K^2$,

$$\left(\bar{Y}_{j_0}^{(\nu_0)} - Y_{j_0}^{(\nu_0)}\right)\mathbb{1}_{A_{j_0,\nu_0,n_0,K}} > -K + \frac{k}{K} > 0.$$

Hence, the comparison principle is violated for the subsolution $\bar{Y}$ with this choice of $k$ and the supersolution $Y$.

□

The following example further illustrates the restrictiveness of assertion $(b)$, and hence of the comparison principle.

*Example* 1.4.2. We consider the problem of pricing under negotiated collateral introduced in Example 1.1.2. Applying the discretization scheme (1.5) proposed there on an equidistant time grid with increments $\Delta$, we end up with the following system of convex dynamic programs

$$\begin{aligned}
X_{j+1}^{(k)} &= X_j^{(k)} \exp\left\{ \left( R^L - \frac{1}{2}\sigma^2 \right)\Delta + \sigma \Delta W_{j+1}^{(k)} \right\}, \quad X_0^{(k)} = x_0, \quad k = 1, \ldots, d \\
Y_J^{(1)} &= -Y_J^{(2)} = h(X_J) \\
Y_j^{(\nu)} &= E_j\left[Y_{j+1}^{(\nu)}\right] - R^L a_\nu \left( E_j\left[Y_{j+1}^{(1)}\right] + E_j\left[Y_{j+1}^{(2)}\right] \right)\Delta \\
&\quad + (-1)^\nu R^C \left( \alpha E_j\left[Y_{j+1}^{(1)}\right] - (1-\alpha) E_j\left[Y_{j+1}^{(2)}\right] \right)\Delta \\
&\quad + (R^B - R^L)\left( a_\nu \left( E_j\left[Y_{j+1}^{(1)}\right] + E_j\left[Y_{j+1}^{(2)}\right] \right) - \frac{1}{\sigma}\left( E_j\left[ \frac{\Delta W_{j+1}}{\Delta} Y_{j+1}^{(\nu)} \right] \right)^\top \mathbf{1} \right)_- \Delta,
\end{aligned}$$

for $\nu = 1, 2$, where $W$ is a $d$-dimensional Brownian motion, $\alpha \in [0,1]$, $(a_1, a_2) = (1-\alpha, \alpha)$, $R^B, R^C, R^L \geq 0$, and $\mathbf{1}$ is the vector in $\mathbb{R}^d$ consisting of ones. Hence, we observe that this dynamic program fits our framework with $N = 2$, $D = d+1$ and the functions $F_j^{(1)}, F_j^{(2)} : \mathbb{R}^{2(d+1)} \to \mathbb{R}$ given by

$$F_j^{(\nu)}(z_1, z_2) = z_\nu^{(1)} + H^{(\nu)}(z_1, z_2)\Delta$$

for $z_\nu = (z_\nu^{(1)}, \ldots, z_\nu^{(d+1)}) \in \mathbb{R}^{d+1}$. Since the non-linearity $H^{(\nu)}$ is piecewise-linear, we conclude by Appendix A.2 that $F_j^{(\nu,\#)} = 0$ on its effective domain $D_{F^{(\nu,\#)}}^{(j,\cdot)} = \{u^{(\nu)}(R) | R \in [R^L, R^B]\}$, $\nu = 1, 2$, where

$$u^{(1)}(r) = \begin{pmatrix} 1 - r(1-\alpha)\Delta - R^C \alpha \Delta \\ \frac{(r - R^L)\Delta}{\sigma} \cdot \mathbf{1} \\ (R^C - r)(1-\alpha)\Delta \\ 0 \cdot \mathbf{1} \end{pmatrix} \quad \text{and} \quad u^{(2)}(r) = \begin{pmatrix} (R^C - r)\alpha\Delta \\ 0 \cdot \mathbf{1} \\ 1 - r\alpha\Delta - R^C(1-\alpha)\Delta \\ \frac{(r - R^L)\Delta}{\sigma} \cdot \mathbf{1} \end{pmatrix}.$$

Consequently, the duality relation (1.24) reads as follows:

$$\begin{aligned}
&\left(r_j^{(\nu,*),[1]}\right)^\top E_j\left[\beta_{j+1} Y_{j+1}^{(1)}\right] + \left(r_j^{(\nu,*),[2]}\right)^\top E_j\left[\beta_{j+1} Y_{j+1}^{(2)}\right] \\
&= E_j\left[Y_{j+1}^{(\nu)}\right] - R^L a_\nu \left( E_j\left[Y_{j+1}^{(1)}\right] + E_j\left[Y_{j+1}^{(2)}\right] \right)\Delta + (-1)^\nu R^C \left( \alpha E_j\left[Y_{j+1}^{(1)}\right] - (1-\alpha) E_j\left[Y_{j+1}^{(2)}\right] \right)\Delta \\
&\quad + (R^B - R^L)\left( a_\nu \left( E_j\left[Y_{j+1}^{(1)}\right] + E_j\left[Y_{j+1}^{(2)}\right] \right) - \frac{1}{\sigma}\left( E_j\left[ \frac{\Delta W_{j+1}}{\Delta} Y_{j+1}^{(\nu)} \right] \right)^\top \mathbf{1} \right)_- \Delta, \ \nu = 1, 2,
\end{aligned}$$

with $\beta$ as in Example 1.1.2. This equation can be solved explicitly and a solution is given by

$$r_j^{(\nu,*)} = \begin{cases} u^{(\nu)}(R^L), & a_\nu \left( E_j\left[Y_{j+1}^{(1)}\right] + E_j\left[Y_{j+1}^{(2)}\right] \right) - \frac{1}{\sigma}\left( E_j\left[ \frac{\Delta W_{j+1}}{\Delta} Y_{j+1}^{(\nu)} \right] \right)^\top \mathbf{1} \geq 0 \\ u^{(\nu)}(R^B), & a_\nu \left( E_j\left[Y_{j+1}^{(1)}\right] + E_j\left[Y_{j+1}^{(2)}\right] \right) - \frac{1}{\sigma}\left( E_j\left[ \frac{\Delta W_{j+1}}{\Delta} Y_{j+1}^{(\nu)} \right] \right)^\top \mathbf{1} < 0. \end{cases}$$

Let $R^B > R^C$, which is typically the case in this example. Taking the admissible control $r^{(1)} \equiv u^{(1)}(R^B) \in \mathcal{A}_0^{F^{(1)}}$, we observe that

$$P\left( \left\{ \left( r_j^{(1),[2]} \right)^\top \beta_{j+1} < 0 \right\} \right) = P\left( \left\{ (R^C - R^B)(1-\alpha)\Delta < 0 \right\} \right) = 1,$$

for every $j = 0, \ldots, J-1$, so that $(b)$ in Theorem 1.4.1 is violated. Thus, the comparison principle fails to hold in this example for this choice of parameters.

*Remark* 1.4.3. When applying discretization schemes for PDE-systems as proposed in Example 1.1.2, such problems arise, whenever $v^{(\nu)}$ depends on $v^{(n)}$, $n \neq \nu$, in a monotonically decreasing way or if $H^{(\nu)}$ depends on the gradient of $v^{(n)}$ for $n \neq \nu$ (even if the Brownian increments are truncated in a standard way).

## 1.5 The general case

In the previous section we have seen that the comparison principle can be a huge drawback and we are now interested in removing it. More precisely, we want to construct a pair $(\theta^{up}, \theta^{low})$ of upper and lower bounds such that the comparison principle still holds for the corresponding super- and subsolutions, although it may fail to hold in general.

The main idea is to couple the recursions (1.22) and (1.25) in a suitable way: To this end, let $j \in \{0, \ldots, J\}$, $M \in \mathcal{M}_{ND}$ be a martingale and let $r^{(\nu)} \in \mathcal{A}_j^{F^{(\nu)}}$, $\nu = 1, \ldots, N$, be admissible controls. Then, the in general non-adapted processes $\theta^{up} := \theta^{up}(r^{(1)}, \ldots, r^{(N)}, M)$ and $\theta^{low} := \theta^{low}(r^{(1)}, \ldots, r^{(N)}, M)$ are given by the following modified pathwise recursions:

$$\begin{aligned}
\theta_J^{(up,\nu)} &= \theta_J^{(low,\nu)} = \xi^{(\nu)} \\
\theta_i^{(up,\nu)} &= \max_{\iota \in \{up,low\}^N} F_i^{(\nu)} \left( \beta_{i+1}\theta_{i+1}^{(\iota_1,1)} - \Delta M_{i+1}^{[1]}, \ldots, \beta_{i+1}\theta_{i+1}^{(\iota_N,N)} - \Delta M_{i+1}^{[N]} \right) \\
\theta_i^{(low,\nu)} &= \sum_{n=1}^{N} \left( \left( r_i^{(\nu),[n]} \right)^\top \beta_{i+1} \right)_+ \theta_{i+1}^{(low,n)} - \sum_{n=1}^{N} \left( \left( r_i^{(\nu),[n]} \right)^\top \beta_{i+1} \right)_- \theta_{i+1}^{(up,n)} \\
&\quad - \sum_{n=1}^{N} \left( r_i^{(\nu),[n]} \right)^\top \Delta M_{i+1}^{[n]} - F_i^{(\nu,\#)} \left( r_i^{(\nu)} \right), \quad i = J-1, \ldots, j,\ \nu = 1, \ldots, N. \quad (1.30)
\end{aligned}$$

The recursion for $\theta^{low}$ demonstrates the idea of this construction most clearly: As we have seen in Theorem 1.4.1, the sign of the weight $(r_i^{(\nu),[n]})^\top \beta_{i+1}$ determines whether the comparison principle holds or not. Therefore, we split up the weight into its positive and negative part. If the weight is positive, the new recursion step coincides with the recursion (1.29). If, however, the weight becomes negative and the comparison principle is violated, we replace $\theta_{i+1}^{(low,n)}$ by $\theta_{i+1}^{(up,n)}$ in this recursion step. Since we have by induction that $\theta_{i+1}^{up}$ is larger than $\theta_{i+1}^{low}$, as we will see in Proposition 1.5.2, the process $\theta^{(low,\nu)}$ becomes smaller and, thus, the order of the bounds can be maintained. By a straightforward modification of the proofs of Lemma 1.2.2 and Lemma 1.3.3, we obtain the following regularity result for the processes $\theta^{up}$ and $\theta^{low}$.

**Lemma 1.5.1.** *Suppose Assumption 1.2.1. Then, for every $j \in \{0, \ldots, J-1\}$, $M \in \mathcal{M}_{ND}$ and $r^{(\nu)} \in \mathcal{A}_j^{F^{(\nu)}}$, $\nu = 1, \ldots, N$, the processes $\theta^{up}(r^{(1)}, \ldots, r^{(N)}, M)$ and $\theta^{low}(r^{(1)}, \ldots, r^{(N)}, M)$ given by (1.30) satisfy $\theta_i^{up}(r^{(1)}, \ldots, r^{(N)}, M) \in L^{\infty-}(\mathbb{R}^N)$ respectively $\theta_i^{low}(r^{(1)}, \ldots, r^{(N)}, M) \in L^{\infty-}(\mathbb{R}^N)$ for all $i = j, \ldots, J$.*

Although, these recursions are a straightforward generalization of the recursions (1.22) and (1.25), it is not straightforward to show that the processes $Y^{up}$ and $Y^{low}$ given by $Y_j^{up} = E_j[\theta_j^{up}]$ and $Y_j^{low} = E_j[\theta_j^{low}]$, $j = 0, \ldots, J$, are again super- and subsolutions to (1.15), since the arguments applied in Section 1.3 do not apply here. Hence, a more careful analysis is required. The following proposition is the key step in this analysis. On the one hand, it provides an alternative representation for $\theta^{up}$, which turns out to be useful for theoretical considerations. On the other hand, it states that the pair $(\theta^{up}, \theta^{low})$ given by (1.30) is ordered.

**Proposition 1.5.2.** *Suppose Assumptions 1.2.1 and let $M \in \mathcal{M}_{ND}$. Then, for every $j = 0, \ldots, J$, $\nu = 1, \ldots, N$ and $r^{(\nu)} \in \mathcal{A}_j^{F^{(\nu)}}$, we have for all $i = j, \ldots, J-1$ the $P$-almost sure identity*

$$\theta_i^{(up,\nu)}\left(r^{(1)}, \ldots, r^{(N)}, M\right) = \sup_{u \in \mathbb{R}^{ND}} \Phi_{i+1}^{(\nu)}\left(u, \theta_{i+1}^{up}\left(r^{(1)}, \ldots, r^{(N)}, M\right), \theta_{i+1}^{low}\left(r^{(1)}, \ldots, r^{(N)}, M\right), \Delta M_{i+1}\right), \quad (1.31)$$

*where $\Phi_{J+1}^{(\nu)}(u, \vartheta_1, \vartheta_2, m) = \xi^{(\nu)}$ and*

$$\Phi_{i+1}^{(\nu)}(u, \vartheta_1, \vartheta_2, m) = \sum_{n=1}^{N} \left(\left(u^{[n]}\right)^\top \beta_{i+1}\right)_+ \vartheta_1^{(n)} - \sum_{n=1}^{N} \left(\left(u^{[n]}\right)^\top \beta_{i+1}\right)_- \vartheta_2^{(n)} - \sum_{n=1}^{N} \left(u^{[n]}\right)^\top m^{[n]} - F_i^{(\nu,\#)}(u) \quad (1.32)$$

*for $i = j, \ldots, J-1$ and $\nu = 1, \ldots, N$. In particular,*

$$\theta_i^{low}\left(r^{(1)}, \ldots, r^{(N)}, M\right) \le \theta_i^{up}\left(r^{(1)}, \ldots, r^{(N)}, M\right) \quad (1.33)$$

*$P$-almost surely for every $i = j, \ldots, J$.*

*Remark* 1.5.3. (i) In contrast to the recursions proposed in Section 1.3, the modified recursions (1.30) are coupled in the sense that they cannot be computed separately. We have already seen that the lower bound recursion decouples to (1.29), if the comparison principle holds. From (1.31) and (1.32), we, however, observe that this is insufficient for the upper bound to decouple. Indeed, we require that

$$P\left(\left\{\left(r^{(\nu),[n]}\right)^\top \beta_{j+1} \ge 0\right\}\right) = 1 \quad (1.34)$$

for every $j = 0, \ldots, J-1$, $n = 1, \ldots, N$ and any random variable $r^{(\nu)} \in L^{\infty-}(\mathbb{R}^{ND})$ satisfying $F_j^{(\nu,\#)}(r^{(\nu)}) \in L^{\infty-}(\mathbb{R})$, $\nu = 1, \ldots, N$. In this case, it is however preferable to apply the decoupled recursions (1.22) and (1.25) for $\Theta^{up}$ and $\Theta^{low}$ instead of (1.30), since we have by backward induction that $\Theta_j^{up}(M) \le \theta_j^{up}(r^{(1)}, \ldots, r^{(N)}, M)$ and $\Theta_j^{low}(r^{(1)}, \ldots, r^{(N)}, M) \ge \theta_j^{low}(r^{(1)}, \ldots, r^{(N)}, M)$ for every $j = 0, \ldots, J$, $r^{(\nu)} \in \mathcal{A}_0^{F^{(\nu)}}$, $\nu = 1, \ldots, N$, and $M \in \mathcal{M}_{ND}$.

(ii) Proposition 1.5.2 can also turn out to be useful in numerical applications. If the system of dynamic programming equations is, e.g., high-dimensional, i.e., $N$ is large, the valuation of the $F_j^{(\nu)}$ in (1.30) can be burdensome. Since the supremum in (1.31) can be restricted to the effective domain $D_{F^{(\nu,\#)}}^{(j,\cdot)}$ of $F_j^{(\nu,\#)}$, the evaluation of (1.31) may be preferred to (1.30) in such situations, if $D_{F^{(\nu,\#)}}^{(j,\cdot)}$ can be parametrized easily.

*Proof.* First we fix $j \in \{0, \ldots, J-1\}$, $M \in \mathcal{M}_{ND}$ and controls $r^{(\nu)}$ in $\mathcal{A}_j^{F^{(\nu)}}$. Then, we define $\theta^{up} := \theta^{up}(r^{(1)}, \ldots, r^{(N)}, M)$ and $\theta^{low} := \theta^{low}(r^{(1)}, \ldots, r^{(N)}, M)$ by (1.30). To lighten the notation, we set

$$\Phi_{i+1}^{(\nu)}(u) = \Phi_{i+1}^{(\nu)}(u, \theta_{i+1}^{up}(r^{(1)}, \ldots, r^{(N)}, M), \theta_{i+1}^{low}(r^{(1)}, \ldots, r^{(N)}, M), \Delta M_{i+1}).$$

The proof is by backward induction on $i = J, \dots, j$, with the case $i = J$ being trivial, since $\theta_J^{(up,\nu)} = \theta_J^{(low,\nu)} = \Phi_{J+1}^{(\nu)} = \xi^{(\nu)}$ by definition for every $\nu = 1, \dots, N$. Now suppose that $\theta_{i+1}^{up} \geq \theta_{i+1}^{low}$ holds for $i+1$. From (1.23) and since $\theta_{i+1}^{up} \geq \theta_{i+1}^{low}$ by the induction hypothesis, we conclude that

$$\begin{aligned}
\theta_i^{(up,\nu)} &= \max_{\iota \in \{up,low\}^N} F_i^{(\nu)} \left( \beta_{i+1} \theta_{i+1}^{(\iota_1,1)} - \Delta M_{i+1}^{[1]}, \dots, \beta_{i+1} \theta_{i+1}^{(\iota_N,N)} - \Delta M_{i+1}^{[N]} \right) \\
&= \max_{\iota \in \{up,low\}^N} \left\{ \sup_{u \in \mathbb{R}^{ND}} \sum_{n=1}^N \left( u^{[n]} \right)^\top \beta_{i+1} \theta_{i+1}^{(\iota_n,n)} - \sum_{n=1}^N \left( u^{[n]} \right)^\top \Delta M_{i+1}^{[n]} - F_i^{(\nu,\#)}(u) \right\} \\
&= \max_{\iota \in \{up,low\}^N} \left\{ \sup_{u \in \mathbb{R}^{ND}} \sum_{n=1}^N \left( \left( u^{[n]} \right)^\top \beta_{i+1} \right)_+ \theta_{i+1}^{(\iota_n,n)} - \sum_{n=1}^N \left( \left( u^{[n]} \right)^\top \beta_{i+1} \right)_- \theta_{i+1}^{(\iota_n,n)} \right. \\
&\qquad \left. - \sum_{n=1}^N \left( u^{[n]} \right)^\top \Delta M_{i+1}^{[n]} - F_i^{(\nu,\#)}(u) \right\} \\
&\leq \max_{\iota \in \{up,low\}^N} \left\{ \sup_{u \in \mathbb{R}^{ND}} \sum_{n=1}^N \left( \left( u^{[n]} \right)^\top \beta_{i+1} \right)_+ \theta_{i+1}^{(up,n)} - \sum_{n=1}^N \left( \left( u^{[n]} \right)^\top \beta_{i+1} \right)_- \theta_{i+1}^{(low,n)} \right. \\
&\qquad \left. - \sum_{n=1}^N \left( u^{[n]} \right)^\top \Delta M_{i+1}^{[n]} - F_i^{(\nu,\#)}(u) \right\} \\
&= \sup_{u \in \mathbb{R}^{ND}} \sum_{n=1}^N \left( \left( u^{[n]} \right)^\top \beta_{i+1} \right)_+ \theta_{i+1}^{(up,n)} - \sum_{n=1}^N \left( \left( u^{[n]} \right)^\top \beta_{i+1} \right)_- \theta_{i+1}^{(low,n)} \\
&\qquad - \sum_{n=1}^N \left( u^{[n]} \right)^\top \Delta M_{i+1}^{[n]} - F_i^{(\nu,\#)}(u) \\
&= \sup_{u \in \mathbb{R}^{ND}} \Phi_{i+1}^{(\nu)}(u)
\end{aligned}$$

$P$-almost surely for every $\nu = 1, \dots, N$. In order to obtain the converse inequality, we fix $u \in \mathbb{R}^{ND}$. Applying (1.23) yields

$$\begin{aligned}
\Phi_{i+1}^{(\nu)}(u) &= \sum_{n=1}^N \left( \left( u^{[n]} \right)^\top \beta_{i+1} \right)_+ \theta_{i+1}^{(up,n)} - \sum_{n=1}^N \left( \left( u^{[n]} \right)^\top \beta_{i+1} \right)_- \theta_{i+1}^{(low,n)} \\
&\qquad - \sum_{n=1}^N \left( u^{[n]} \right)^\top \Delta M_{i+1}^{[n]} - F_i^{(\nu,\#)}(u) \\
&= \sum_{n=1}^N \left( u^{[n]} \right)^\top \beta_{i+1} \theta_{i+1}^{(up,n)} \mathbb{1}_{\{(u^{[n]})^\top \beta_{i+1} \geq 0\}} + \sum_{n=1}^N \left( u^{[n]} \right)^\top \beta_{i+1} \theta_{i+1}^{(low,n)} \mathbb{1}_{\{(u^{[n]})^\top \beta_{i+1} < 0\}} \\
&\qquad - \sum_{n=1}^N \left( u^{[n]} \right)^\top \Delta M_{i+1}^{[n]} - F_i^{(\nu,\#)}(u) \\
&= \sum_{n=1}^N \left( u^{[n]} \right)^\top \beta_{i+1} \left( \theta_{i+1}^{(up,n)} \mathbb{1}_{\{(u^{[n]})^\top \beta_{i+1} \geq 0\}} + \theta_{i+1}^{(low,n)} \mathbb{1}_{\{(u^{[n]})^\top \beta_{i+1} < 0\}} \right) \\
&\qquad - \sum_{n=1}^N \left( u^{[n]} \right)^\top \Delta M_{i+1}^{[n]} - F_i^{(\nu,\#)}(u)
\end{aligned}$$

$$
\begin{aligned}
&\leq F_i^{(\nu)}\left(\beta_{i+1}\left(\theta_{i+1}^{(up,1)}\mathbb{1}_{\{(u^{[1]})^\top\beta_{i+1}\geq 0\}}+\theta_{i+1}^{(low,1)}\mathbb{1}_{\{(u^{[1]})^\top\beta_{i+1}<0\}}\right)-\Delta M_{i+1}^{[1]},\dots,\right.\\
&\qquad\left.\beta_{i+1}\left(\theta_{i+1}^{(up,N)}\mathbb{1}_{\{(u^{[N]})^\top\beta_{i+1}\geq 0\}}+\theta_{i+1}^{(low,N)}\mathbb{1}_{\{(u^{[N]})^\top\beta_{i+1}<0\}}\right)-\Delta M_{i+1}^{[N]}\right)\\
&\leq \max_{\iota\in\{up,low\}^N} F_i^{(\nu)}\left(\beta_{i+1}\theta_{i+1}^{(\iota_1,1)}-\Delta M_{i+1}^{[1]},\dots,\beta_{i+1}\theta_{i+1}^{(\iota_N,N)}-\Delta M_{i+1}^{[N]}\right)\\
&= \theta_i^{(up,\nu)}
\end{aligned}
$$

for every $\omega\in\Omega$ and $\nu=1,\dots,N$. Hence, we have

$$
\theta_i^{(up,\nu)}=\sup_{u\in\mathbb{R}^{ND}}\Phi_{i+1}^{(\nu)}(u)
$$

$P$-a.s. for every $i=j,\dots,J$ and $\nu=1,\dots,N$. To complete the proof, it remains to show that $\theta_i^{up}\geq\theta_i^{low}$. By the induction hypothesis we conclude that

$$
\begin{aligned}
\theta_i^{(up,\nu)} &= \sup_{u\in\mathbb{R}^{ND}}\sum_{n=1}^N\left(\left(u^{[n]}\right)^\top\beta_{i+1}\right)_+\theta_{i+1}^{(up,n)}-\sum_{n=1}^N\left(\left(u^{[n]}\right)^\top\beta_{i+1}\right)_-\theta_{i+1}^{(low,n)}\\
&\quad-\sum_{n=1}^N\left(u^{[n]}\right)^\top\Delta M_{i+1}^{[n]}-F_i^{(\nu,\#)}(u)\\
&\geq \sup_{u\in\mathbb{R}^{ND}}\sum_{n=1}^N\left(\left(u^{[n]}\right)^\top\beta_{i+1}\right)_+\theta_{i+1}^{(low,n)}-\sum_{n=1}^N\left(\left(u^{[n]}\right)^\top\beta_{i+1}\right)_-\theta_{i+1}^{(up,n)}\\
&\quad-\sum_{n=1}^N\left(u^{[n]}\right)^\top\Delta M_{i+1}^{[n]}-F_i^{(\nu,\#)}(u)\\
&\geq \sum_{n=1}^N\left(\left(r_i^{(\nu),[n]}\right)^\top\beta_{i+1}\right)_+\theta_{i+1}^{(low,n)}-\sum_{n=1}^N\left(\left(r_i^{(\nu),[n]}\right)^\top\beta_{i+1}\right)_-\theta_{i+1}^{(up,n)}\\
&\quad-\sum_{n=1}^N\left(r_i^{(\nu),[n]}\right)^\top\Delta M_{i+1}^{[n]}-F_i^{(\nu,\#)}\left(r_i^{(\nu),[n]}\right)\\
&= \theta_i^{(low,\nu)}
\end{aligned}
$$

$P$-a.s. for every $\nu=1,\dots,N$. □

From Proposition 1.5.2 and the monotonicity of the conditional expectation, we conclude that the processes $Y^{up}$ and $Y^{low}$ are ordered. We next show that $Y^{up}$ and $Y^{low}$ are super- and subsolutions.

**Proposition 1.5.4.** *Under Assumptions 1.2.1 the processes $Y^{up}$ and $Y^{low}$, which are given by $Y_j^{up}=E_j[\theta_j^{up}(r^{(1)},\dots,r^{(N)},M)]$ respectively $Y_j^{low}=E_j[\theta_j^{low}(r^{(1)},\dots,r^{(N)},M)]$, $j=0,\dots,J$, define super- and subsolutions to* (1.15) *for every $M\in\mathcal{M}_{ND}$ and $r^{(\nu)}\in\mathcal{A}_0^{F^{(\nu)}}$, $\nu=1,\dots,N$.*

*Proof.* Let $M\in\mathcal{M}_{ND}$, $r^{(\nu)}\in\mathcal{A}_0^{F^{(\nu)}}$, $\nu=1,\dots,N$, and define $\theta^{up}:=\theta^{up}(r^{(1)},\dots,r^{(N)},M)$ and $\theta^{low}:=\theta^{low}(r^{(1)},\dots,r^{(N)},M)$ according to (1.30). Then, we observe by the definition of $\theta^{up}$ and Jensen's inequality applied to the convex functions max and $F_j^{(\nu)}$ that

$$
Y_j^{(up,\nu)}=E_j\left[\theta_j^{(up,\nu)}\right]
$$

$$= E_j \left[ \max_{\iota \in \{up,low\}^N} F_j^{(\nu)} \left( \beta_{j+1}\theta_{j+1}^{(\iota_1,1)} - \Delta M_{j+1}^{[1]}, \dots, \beta_{j+1}\theta_{j+1}^{(\iota_N,N)} - \Delta M_{j+1}^{[N]} \right) \right]$$
$$\geq \max_{\iota \in \{up,low\}^N} F_j^{(\nu)} \left( E_j \left[ \beta_{j+1}\theta_{j+1}^{(\iota_1,1)} - \Delta M_{j+1}^{[1]} \right], \dots, E_j \left[ \beta_{j+1}\theta_{j+1}^{(\iota_N,N)} - \Delta M_{j+1}^{[N]} \right] \right).$$

Now, the martingale property of $M$ and the tower property of the conditional expectation yield

$$\begin{aligned} Y_j^{(up,\nu)} &\geq \max_{\iota \in \{up,low\}^N} F_j^{(\nu)} \left( E_j \left[ \beta_{j+1}\theta_{j+1}^{(\iota_1,1)} \right], \dots, E_j \left[ \beta_{j+1}\theta_{j+1}^{(\iota_N,N)} \right] \right) \\ &= \max_{\iota \in \{up,low\}^N} F_j^{(\nu)} \left( E_j \left[ \beta_{j+1} E_{j+1} \left[ \theta_{j+1}^{(\iota_1,1)} \right] \right], \dots, E_j \left[ \beta_{j+1} E_{j+1} \left[ \theta_{j+1}^{(\iota_N,N)} \right] \right] \right). \end{aligned}$$

Using the definition of $Y^{up}$ shows that

$$\begin{aligned} Y_j^{(up,\nu)} &\geq \max_{\iota \in \{up,low\}^N} F_j^{(\nu)} \left( E_j \left[ \beta_{j+1} Y_{j+1}^{(\iota_1,1)} \right], \dots, E_j \left[ \beta_{j+1} Y_{j+1}^{(\iota_N,N)} \right] \right) \\ &\geq F_j^{(\nu)} \left( E_j \left[ \beta_{j+1} Y_{j+1}^{(up,1)} \right], \dots, E_j \left[ \beta_{j+1} Y_{j+1}^{(up,N)} \right] \right). \end{aligned}$$

It remains to show that $Y^{low}$ defines a subsolution. We first obtain by the definition of $\theta^{low}$, the martingale property of $M$ and the admissibility of the controls that

$$\begin{aligned} Y_j^{(low,\nu)} &= E_j \left[ \theta_j^{(low,\nu)} \right] \\ &= E_j \left[ \sum_{n=1}^N \left( \left( r_j^{(\nu),[n]} \right)^\top \beta_{j+1} \right)_+ \theta_{j+1}^{(low,n)} - \sum_{n=1}^N \left( \left( r_j^{(\nu),[n]} \right)^\top \beta_{j+1} \right)_- \theta_{j+1}^{(up,n)} \right. \\ &\quad \left. - \sum_{n=1}^N \left( r_j^{(\nu),[n]} \right)^\top \Delta M_{j+1}^{[n]} - F_j^{(\nu,\#)} \left( r_j^{(\nu)} \right) \right] \\ &= \sum_{n=1}^N E_j \left[ \left( \left( r_j^{(\nu),[n]} \right)^\top \beta_{j+1} \right)_+ \theta_{j+1}^{(low,n)} \right] - \sum_{n=1}^N E_j \left[ \left( \left( r_j^{(\nu),[n]} \right)^\top \beta_{j+1} \right)_- \theta_{j+1}^{(up,n)} \right] \\ &\quad - F_j^{(\nu,\#)} \left( r_j^{(\nu)} \right). \end{aligned}$$

In a next step, we exploit the pathwise comparison (1.33) in Proposition 1.5.2 in order to observe that

$$\begin{aligned} Y_j^{(low,\nu)} &\leq \sum_{n=1}^N E_j \left[ \left( \left( r_j^{(\nu),[n]} \right)^\top \beta_{j+1} \right)_+ \theta_{j+1}^{(low,n)} \right] - \sum_{n=1}^N E_j \left[ \left( \left( r_j^{(\nu),[n]} \right)^\top \beta_{j+1} \right)_- \theta_{j+1}^{(low,n)} \right] \\ &\quad - F_j^{(\nu,\#)} \left( r_j^{(\nu)} \right) \\ &= \sum_{n=1}^N \left( r_j^{(\nu),[n]} \right)^\top E_j \left[ \beta_{j+1} \theta_{j+1}^{(low,n)} \right] - F_j^{(\nu,\#)} \left( r_j^{(\nu)} \right). \end{aligned}$$

To complete the proof, we conclude by the tower property of the conditional expectation, the definition of $Y^{low}$, and (1.23) that

$$Y_j^{(low,\nu)} \leq \sum_{n=1}^N \left( r_j^{(\nu),[n]} \right)^\top E_j \left[ \beta_{j+1} E_{j+1} \left[ \theta_{j+1}^{(low,n)} \right] \right] - F_j^{(\nu,\#)} \left( r_j^{(\nu)} \right)$$

$$
\begin{aligned}
&= \sum_{n=1}^{N} \left(r_j^{(\nu),[n]}\right)^\top E_j\left[\beta_{j+1} Y_{j+1}^{(low,n)}\right] - F_j^{(\nu,\#)}\left(r_j^{(\nu)}\right) \\
&\leq F_j^{(\nu)}\left(E_j\left[\beta_{j+1} Y_{j+1}^{(low,1)}\right], \ldots, E_j\left[\beta_{j+1} Y_{j+1}^{(low,N)}\right]\right).
\end{aligned}
$$

Since $j \in \{0, \ldots, J-1\}$ was arbitrary, we conclude that $Y^{up}$ and $Y^{low}$ are super- respectively subsolutions to (1.15). □

We are now in the position to state the main result of this section, which generalizes Theorem 1.3.4 to the coupled bounds (1.30).

**Theorem 1.5.5.** *For every $j = 0, \ldots, J$ and $\nu = 1, \ldots, N$,*

$$
\begin{aligned}
Y_j^{(\nu)} &= \operatorname*{essinf}_{\substack{r^{(1)} \in \mathcal{A}_j^{F^{(1)}}, \ldots, r^{(N)} \in \mathcal{A}_j^{F^{(N)}}, \\ M \in \mathcal{M}_{ND}}} E_j\left[\theta_j^{(up,\nu)}\left(r^{(1)}, \ldots, r^{(N)}, M\right)\right] \\
&= \operatorname*{esssup}_{\substack{r^{(1)} \in \mathcal{A}_j^{F^{(1)}}, \ldots, r^{(N)} \in \mathcal{A}_j^{F^{(N)}}, \\ M \in \mathcal{M}_{ND}}} E_j\left[\theta_j^{(low,\nu)}\left(r^{(1)}, \ldots, r^{(N)}, M\right)\right], \quad P\text{-a.s.}
\end{aligned}
$$

*Moreover,*

$$
Y_j^{(\nu)} = \theta_j^{(up,\nu)}\left(r^{(1,*)}, \ldots, r^{(N,*)}, M^*\right) = \theta_j^{(low,\nu)}\left(r^{(1,*)}, \ldots, r^{(N,*)}, M^*\right) \tag{1.35}
$$

*$P$-almost surely, whenever each $r^{(\nu,*)}$ satisfies the duality relation* (1.24)*, i.e.,*

$$
\sum_{n=1}^{N} \left(r_i^{(\nu,*),[n]}\right)^\top E_i\left[\beta_{i+1} Y_{i+1}^{(n)}\right] - F_i^{(\nu,\#)}\left(r_i^{(\nu,*)}\right) = F_i^{(\nu)}\left(E_i\left[\beta_{i+1} Y_{i+1}^{(1)}\right], \ldots, E_i\left[\beta_{i+1} Y_{i+1}^{(N)}\right]\right)
$$

*$P$-almost surely for every $i = j, \ldots, J-1$ and each $M^{*,[\nu]}$ is the Doob martingale of $\beta Y^{(\nu)}$.*

Before we turn to the proof of Theorem 1.5.5, we should emphasize the role of the martingale increment in the recursion (1.30) for $\theta^{low}$. Recall that the martingale increment only acted as a control variate in the modified recursion (1.29) for $\Theta^{low}$. In this generalized setting, it is, however, crucial, as the pathwise comparison property stated in Proposition 1.5.2, which plays a key role in the proof of Theorem 1.5.5, requires the same choice of martingales in the recursions for $\theta^{up}$ and $\theta^{low}$.

*Proof.* Let $j \in \{0, \ldots, J-1\}$ be fixed from now on. Further, let $M \in \mathcal{M}_{ND}$ be a martingale, $r^{(\nu)} \in \mathcal{A}_j^{F^{(\nu)}}$, $\nu = 1, \ldots, N$, be admissible controls and let $\theta^{up} := \theta^{up}(r^{(1)}, \ldots, r^{(N)}, M)$ respectively $\theta^{low} := \theta^{low}(r^{(1)}, \ldots, r^{(N)}, M)$ be given by (1.30). Further, we denote by $r^{(\nu,*)} \in \mathcal{A}_j^{F^{(\nu)}}$, $\nu = 1, \ldots, N$, optimal controls satisfying the duality relation (1.24). We first show by backward induction on $i$ that

$$
E_i[\theta_i^{low}] \leq Y_i \leq E_i[\theta_i^{up}] \tag{1.36}
$$

holds $P$-a.s. for every $i = j, \ldots, J$. The case $i = J$ is trivial, since it holds that $\theta_J^{(up,\nu)} = \theta_J^{(low,\nu)} = \xi^{(\nu)} = Y_J^{(\nu)}$ for every $\nu = 1, \ldots, N$ by definition of $\theta^{up}$ and $\theta^{low}$. Suppose that (1.36) is true for

$i+1$. From the martingale property of $M$ and the tower property of the conditional expectation, we obtain that

$$
\begin{aligned}
E_i\left[\theta_i^{(low,\nu)}\right] &= E_i\Bigg[\sum_{n=1}^N \left(\left(r_i^{(\nu),[n]}\right)^\top \beta_{i+1}\right)_+ \theta_{i+1}^{(low,n)} - \sum_{n=1}^N \left(\left(r_i^{(\nu),[n]}\right)^\top \beta_{i+1}\right)_- \theta_{i+1}^{(up,n)} \\
&\quad -\sum_{n=1}^N \left(r_i^{(\nu),[n]}\right)^\top \Delta M_{i+1}^{[n]} - F_i^{(\nu,\#)}\left(r_i^{(\nu)}\right)\Bigg] \\
&= \sum_{n=1}^N E_i\left[\left(\left(r_i^{(\nu),[n]}\right)^\top \beta_{i+1}\right)_+ \theta_{i+1}^{(low,n)}\right] - \sum_{n=1}^N E_i\left[\left(\left(r_i^{(\nu),[n]}\right)^\top \beta_{i+1}\right)_- \theta_{i+1}^{(up,n)}\right] \\
&\quad -F_i^{(\nu,\#)}\left(r_i^{(\nu)}\right) \\
&= \sum_{n=1}^N E_i\left[\left(\left(r_i^{(\nu),[n]}\right)^\top \beta_{i+1}\right)_+ E_{i+1}\left[\theta_{i+1}^{(low,n)}\right]\right] \\
&\quad -\sum_{n=1}^N E_i\left[\left(\left(r_i^{(\nu),[n]}\right)^\top \beta_{i+1}\right)_- E_{i+1}\left[\theta_{i+1}^{(up,n)}\right]\right] - F_i^{(\nu,\#)}\left(r_i^{(\nu)}\right).
\end{aligned}
$$

Then, we observe by the induction hypothesis and (1.23) that

$$
\begin{aligned}
E_i\left[\theta_i^{(low,\nu)}\right] &\leq \sum_{n=1}^N E_i\left[\left(\left(r_i^{(\nu),[n]}\right)^\top \beta_{i+1}\right)_+ Y_{i+1}^{(n)}\right] - \sum_{n=1}^N E_i\left[\left(\left(r_i^{(\nu),[n]}\right)^\top \beta_{i+1}\right)_- Y_{i+1}^{(n)}\right] \\
&\quad -F_i^{(\nu,\#)}\left(r_i^{(\nu)}\right) \\
&= \sum_{n=1}^N \left(r_i^{(\nu),[n]}\right)^\top E_i\left[\beta_{i+1}Y_{i+1}^{(n)}\right] - F_i^{(\nu,\#)}\left(r_i^{(\nu)}\right) \\
&\leq F_i^{(\nu)}\left(E_i\left[\beta_{i+1}Y_{i+1}^{(1)}\right], \dots, E_i\left[\beta_{i+1}Y_{i+1}^{(N)}\right]\right) = Y_i^{(\nu)}
\end{aligned}
$$

for every $\nu = 1, \dots, N$, which proves the first inequality in (1.36). By applying the alternative representation for $\theta^{up}$ in Proposition 1.5.2 and essentially the same arguments as before, we conclude that

$$
\begin{aligned}
E_i\left[\theta_i^{(up,\nu)}\right] &= E_i\Bigg[\sup_{u\in\mathbb{R}^{ND}} \sum_{n=1}^N \left(\left(u^{[n]}\right)^\top \beta_{i+1}\right)_+ \theta_{i+1}^{(up,n)} - \sum_{n=1}^N \left(\left(u^{[n]}\right)^\top \beta_{i+1}\right)_- \theta_{i+1}^{(low,n)} \\
&\quad -\sum_{n=1}^N \left(u^{[n]}\right)^\top \Delta M_{i+1}^{[n]} - F_i^{(\nu,\#)}(u)\Bigg] \\
&\geq E_i\Bigg[\sum_{n=1}^N \left(\left(r_i^{(\nu,*),[n]}\right)^\top \beta_{i+1}\right)_+ \theta_{i+1}^{(up,n)} - \sum_{n=1}^N \left(\left(r_i^{(\nu,*),[n]}\right)^\top \beta_{i+1}\right)_- \theta_{i+1}^{(low,n)} \\
&\quad -\sum_{n=1}^N \left(r_i^{(\nu,*),[n]}\right)^\top \Delta M_{i+1}^{[n]} - F_i^{(\nu,\#)}\left(r_i^{(\nu,*)}\right)\Bigg] \\
&= \sum_{n=1}^N E_i\left[\left(\left(r_i^{(\nu,*),[n]}\right)^\top \beta_{i+1}\right)_+ \theta_{i+1}^{(up,n)}\right] - \sum_{n=1}^N E_i\left[\left(\left(r_i^{(\nu,*),[n]}\right)^\top \beta_{i+1}\right)_- \theta_{i+1}^{(low,n)}\right] \\
&\quad -F_i^{(\nu,\#)}\left(r_i^{(\nu,*)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=1}^{N} E_i\left[\left(\left(r_i^{(\nu,*),[n]}\right)^\top \beta_{i+1}\right)_+ E_{i+1}\left[\theta_{i+1}^{(up,n)}\right]\right] \\
&\quad -\sum_{n=1}^{N} E_i\left[\left(\left(r_i^{(\nu,*),[n]}\right)^\top \beta_{i+1}\right)_- E_{i+1}\left[\theta_{i+1}^{(low,n)}\right]\right] - F_i^{(\nu,\#)}\left(r_i^{(\nu,*)}\right) \\
&\geq \sum_{n=1}^{N} E_i\left[\left(\left(r_i^{(\nu,*),[n]}\right)^\top \beta_{i+1}\right)_+ Y_{i+1}^{(n)}\right] - \sum_{n=1}^{N} E_i\left[\left(\left(r_i^{(\nu,*),[n]}\right)^\top \beta_{i+1}\right)_- Y_{i+1}^{(n)}\right] \\
&\quad -F_i^{(\nu,\#)}\left(r_i^{(\nu,*)}\right) \\
&= \sum_{n=1}^{N} \left(r_i^{(\nu,*),[n]}\right)^\top E_i\left[\beta_{i+1}Y_{i+1}^{(n)}\right] - F_i^{(\nu,\#)}\left(r_i^{(\nu,*)}\right) \\
&= F_i^{(\nu)}\left(E_i\left[\beta_{i+1}Y_{i+1}^{(1)}\right], \ldots, E_i\left[\beta_{i+1}Y_{i+1}^{(N)}\right]\right) \\
&= Y_i^{(\nu)}.
\end{aligned}
$$

For the second last equality, we additionally used the duality relation (1.24). Since $\nu \in \{1, \ldots, N\}$ is arbitrary, we obtain the second inequality in (1.36), and thus

$$
\begin{aligned}
\operatorname*{esssup}_{\substack{r^{(n)} \in \mathcal{A}_j^{F^{(n)}},\ n=1,\ldots,N,\\ M \in \mathcal{M}_{ND}}} \theta_j^{(low,\nu)}\left(r^{(1)}, \ldots, r^{(N)}, M\right) &\leq Y_j^{(\nu)} \\
&\leq \operatorname*{essinf}_{\substack{r^{(n)} \in \mathcal{A}_j^{F^{(n)}},\ n=1,\ldots,N,\\ M \in \mathcal{M}_{ND}}} \theta_j^{(up,\nu)}\left(r^{(1)}, \ldots, r^{(N)}, M\right)
\end{aligned}
$$

for all $\nu = 1, \ldots, N$.

To complete the proof, we show that

$$
Y_i^{(\nu)} = \theta_i^{(up,\nu)}\left(r^{(1,*)}, \ldots, r^{(N,*)}, M^*\right) = \theta_i^{(low,\nu)}\left(r^{(1,*)}, \ldots, r^{(N,*)}, M^*\right)
$$

holds for every $i = j, \ldots, J$ and $\nu = 1, \ldots, N$. To this end, let $M^{*,[\nu]}$ be the Doob martingale of $\beta Y^{(\nu)}$. Then, the proof is again by backward induction on $i$. As before, the case $i = J$ is trivially true by definition of $\theta^{up,*} := \theta^{up}(r^{(1,*)}, \ldots, r^{(N,*)}, M^*)$ and $\theta^{low,*} := \theta^{low}(r^{(1,*)}, \ldots, r^{(N,*)}, M^*)$. Now, suppose that the assertion is true for $i+1$. Then, we conclude by the definition of $M^*$, the induction hypothesis, and (1.24) that

$$
\begin{aligned}
\theta_i^{(low,*,\nu)} &= \sum_{n=1}^{N} \left(\left(r_i^{(\nu,*),[n]}\right)^\top \beta_{i+1}\right)_+ \theta_{i+1}^{(low,*,n)} - \sum_{n=1}^{N} \left(\left(r_i^{(\nu,*),[n]}\right)^\top \beta_{i+1}\right)_- \theta_{i+1}^{(up,*,n)} \\
&\quad -\sum_{n=1}^{N} \left(r_i^{(\nu,*),[n]}\right)^\top \Delta M_{i+1}^{*,[n]} - F_i^{(\nu,\#)}\left(r_i^{(\nu,*)}\right) \\
&= \sum_{n=1}^{N} \left(\left(r_i^{(\nu,*),[n]}\right)^\top \beta_{i+1}\right)_+ \theta_{i+1}^{(low,*,n)} - \sum_{n=1}^{N} \left(\left(r_i^{(\nu,*),[n]}\right)^\top \beta_{i+1}\right)_- \theta_{i+1}^{(up,*,n)} \\
&\quad -\sum_{n=1}^{N} \left(r_i^{(\nu,*),[n]}\right)^\top \left(\beta_{i+1}Y_{i+1}^{(n)} - E_i\left[\beta_{i+1}Y_{i+1}^{(n)}\right]\right) - F_i^{(\nu,\#)}\left(r_i^{(\nu,*)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=1}^{N}\left(\left(r_i^{(\nu,*),[n]}\right)^{\top}\beta_{i+1}\right)_{+} Y_{i+1}^{(n)} - \sum_{n=1}^{N}\left(\left(r_i^{(\nu,*),[n]}\right)^{\top}\beta_{i+1}\right)_{-} Y_{i+1}^{(n)} \\
&\quad -\sum_{n=1}^{N}\left(r_i^{(\nu,*),[n]}\right)^{\top}\left(\beta_{i+1}Y_{i+1}^{(n)} - E_i\left[\beta_{i+1}Y_{i+1}^{(n)}\right]\right) - F_i^{(\nu,\#)}\left(r_i^{(\nu,*)}\right) \\
&= \sum_{n=1}^{N}\left(r_i^{(\nu,*),[n]}\right)^{\top} E_i\left[\beta_{i+1}Y_{i+1}^{(n)}\right] - F_i^{(\nu,\#)}\left(r_i^{(\nu,*)}\right) \\
&= F_i^{(\nu)}\left(E_i\left[\beta_{i+1}Y_{i+1}^{(1)}\right],\ldots,E_i\left[\beta_{i+1}Y_{i+1}^{(N)}\right]\right) \\
&= Y_i^{(\nu)}
\end{aligned}
$$

holds for every $\nu = 1,\ldots,N$. For the upper bound, the definition of $M^*$ and the induction hypothesis yield

$$
\begin{aligned}
\theta_i^{(up,*,\nu)} &= \max_{\iota\in\{up,low\}^N} F_i^{(\nu)}\left(\beta_{i+1}\theta_{i+1}^{(\iota_1,*,1)} - \Delta M_{i+1}^{*,[1]},\ldots,\beta_{i+1}\theta_{i+1}^{(\iota_N,*,N)} - \Delta M_{i+1}^{*,[N]}\right) \\
&= \max_{\iota\in\{up,low\}^N} F_i^{(\nu)}\left(\beta_{i+1}\theta_{i+1}^{(\iota_1,*,1)} - \left(\beta_{i+1}Y_{i+1}^{(1)} - E_i\left[\beta_{i+1}Y_{i+1}^{(1)}\right]\right),\ldots,\right. \\
&\qquad \left.\beta_{i+1}\theta_{i+1}^{(\iota_N,*,N)} - \left(\beta_{i+1}Y_{i+1}^{(N)} - E_i\left[\beta_{i+1}Y_{i+1}^{(N)}\right]\right)\right) \\
&= \max_{\iota\in\{up,low\}^N} F_i^{(\nu)}\left(\beta_{i+1}Y_{i+1}^{(1)} - \left(\beta_{i+1}Y_{i+1}^{(1)} - E_i\left[\beta_{i+1}Y_{i+1}^{(1)}\right]\right),\ldots,\right. \\
&\qquad \left.\beta_{i+1}Y_{i+1}^{(N)} - \left(\beta_{i+1}Y_{i+1}^{(N)} - E_i\left[\beta_{i+1}Y_{i+1}^{(N)}\right]\right)\right) \\
&= F_i^{(\nu)}\left(E_i\left[\beta_{i+1}Y_{i+1}^{(1)}\right],\ldots,E_i\left[\beta_{i+1}Y_{i+1}^{(N)}\right]\right) \\
&= Y_i^{(\nu)}
\end{aligned}
$$

for every $\nu = 1,\ldots,N$, and, thus, the proof is complete. □

## 1.6 Influence of martingale approximations

The numerical implementation of the bounds proposed above typically requires the approximation of the optimal martingale $M^*$. In this section, we want to investigate the influence of such approximations on the upper and lower bounds. This investigation is motivated by the following situation arising in numerical applications:

There is an $\mathbb{R}^{\mathcal{D}}$-valued, $\mathcal{D} \geq D$, adapted process $(B_j)_{j=1,\ldots,J}$ such that for every $j = 1,\ldots,J$ the first $D$ components of $B_j$ are given by $\beta_j$ and $B_j$ is independent of $\mathcal{F}_{j-1}$. Moreover, we have an $\mathbb{R}^d$-dimensional Markovian process $X$, whose dynamics are given by

$$X_j = h_j(X_{j-1}, B_j), \quad X_0 = x_0 \in \mathbb{R}, \tag{1.37}$$

for measurable functions $h_j : \mathbb{R}^d \times \mathbb{R}^{\mathcal{D}} \to \mathbb{R}^d$. Furthermore, suppose that we are given an approximate solution $\tilde{Y}$ to (1.15), which is given by

$$\tilde{Y}_j = E[v_j(X_j,\ldots,X_J)|X_j], \quad j = 0,\ldots,J,$$

for measurable functions $v_j$.

Building on this approximate solution, we may take the Doob martingale of $\beta\tilde{Y}$, which is for every $j = 0, \ldots, J$ and $\nu = 1, \ldots, N$ given by

$$\hat{M}_j^{[\nu]} = \sum_{i=0}^{j-1} \beta_{i+1} E\left[ v_{i+1}^{(\nu)}(X_{i+1}, \ldots, X_J) \middle| X_{i+1} \right] - E\left[ \beta_{i+1} v_{i+1}^{(\nu)}(X_{i+1}, \ldots, X_J) \middle| X_i \right], \tag{1.38}$$

as an input for the computation of upper and lower bounds. In general, however, these conditional expectations are not available in closed form and thus need to be approximated. To this end, we apply the following subsampling approach:

For every time point $j \in \{0, \ldots, J-1\}$, we simulate independent copies $(B_i(\lambda^{in}, j))_{i \geq j+1}$, $\lambda^{in} = 1, \ldots, \Lambda^{in}$, of $(B_i)_{i\geq j+1}$ which are independent of $\mathcal{F}_J$. Then, for every $j$, independent copies $(X_i(\lambda^{in}, j))_{i\geq j+1}$ of $(X_i)_{i\geq j+1}$ given $X_j$ are obtained by evaluating (1.37) along these paths, i.e.

$$\begin{aligned} &X_j(\lambda^{in}, j) = X_j, \\ &X_i(\lambda^{in}, j) = h_i(X_{i-1}(\lambda^{in}, j), B_i(\lambda^{in}, j)), \quad i = j+1, \ldots, J. \end{aligned}$$

With these samples at hand, we can replace the conditional expectations in (1.38) for every $\nu = 1, \ldots, N$ by the conditionally unbiased estimators

$$\begin{aligned} \hat{E}_j\left[ v_j^{(\nu)}(X_j, \ldots, X_J) \right] &:= \frac{1}{\Lambda^{in}} \sum_{\lambda^{in}=1}^{\Lambda^{in}} v_j^{(\nu)}(X_j, X_{j+1}(\lambda^{in}, j), \ldots, X_J(\lambda^{in}, j)) \\ \hat{E}_j\left[ \beta_{j+1} v_{j+1}^{(\nu)}(X_{j+1}, \ldots, X_J) \right] &:= \frac{1}{\Lambda^{in}} \sum_{\lambda^{in}=1}^{\Lambda^{in}} \beta_{j+1}(\lambda^{in}, j) v_{j+1}^{(\nu)}(X_{j+1}(\lambda^{in}, j), \ldots, X_J(\lambda^{in}, j)). \end{aligned}$$

However, the resulting process $\hat{M}$ is in general not a martingale, as the estimators are computed along the same set of inner paths, and thus the increments $\Delta\hat{M}_{j+1}$, $j = 0, \ldots, J-1$, are correlated. In light of Remark 1.3.6, the process $\hat{M}$ may still be taken as an input to compute upper and lower bounds, since $E_j[\Delta\hat{M}_{j+1}] = 0$ by construction.

The following result, which is the main result of this section, implies that the application of such a subsampling approach leads to an additional upward respectively downward bias in the upper and lower bounds.

**Theorem 1.6.1.** *Let $j \in \{0, \ldots, J-1\}$, $r^{(\nu)} \in \mathcal{A}_j^{F^{(\nu)}}$, $\nu = 1, \ldots, N$, and let $M \in \mathcal{M}_{ND}$ be a martingale. Furthermore, let $\hat{M}$ be a $\mathcal{F}$-measurable stochastic process which satisfies $\hat{M}_i \in L^{\infty-}(\mathbb{R}^{ND})$ and $E[\hat{M}_i|\mathcal{F}_J] = M_i$ for every $i = 0, \ldots, J$. Then,*

$$E\left[ \theta_i^{up}\left(r^{(1)}, \ldots, r^{(N)}, \hat{M}\right) \middle| \mathcal{F}_J \right] \geq E\left[ \theta_i^{up}\left(r^{(1)}, \ldots, r^{(N)}, M\right) \middle| \mathcal{F}_J \right] \tag{1.39}$$

*and*

$$E\left[ \theta_i^{low}\left(r^{(1)}, \ldots, r^{(N)}, \hat{M}\right) \middle| \mathcal{F}_J \right] \leq E\left[ \theta_i^{low}\left(r^{(1)}, \ldots, r^{(N)}, M\right) \middle| \mathcal{F}_J \right] \tag{1.40}$$

*$P$-almost surely for every $i = j, \ldots, J$. In particular, it holds that*

$$E\left[ \theta_i^{up}\left(r^{(1)}, \ldots, r^{(N)}, \hat{M}\right) \middle| \mathcal{F}_i \right] \geq E\left[ \theta_i^{up}\left(r^{(1)}, \ldots, r^{(N)}, M\right) \middle| \mathcal{F}_i \right]$$

*and*

$$E\left[ \theta_i^{low}\left(r^{(1)}, \ldots, r^{(N)}, \hat{M}\right) \middle| \mathcal{F}_i \right] \leq E\left[ \theta_i^{low}\left(r^{(1)}, \ldots, r^{(N)}, M\right) \middle| \mathcal{F}_i \right]$$

*$P$-almost surely for every $i = j, \ldots, J$.*

The proof of this theorem requires some preparation. For this purpose, we first introduce some further notation.

We denote by $\pi : \{up, low\}^N \to \{1, \ldots, 2^N\}$ a bijection, which assigns a natural number to each $N$-tuple $\iota \in \{up, low\}^N$. We further denote by $\pi^{-1}$ the inverse function of $\pi$ and by $(\pi^{-1}(k))_n$ the $n$-th component of the $N$-tuple $\pi^{-1}(k)$, $k \in \{1, \ldots, 2^N\}$. Moreover, for each $j = 0, \ldots, J-1$ and $\nu = 1, \ldots, N$, we choose a partition $(A_{j,\nu,\iota})_{\iota \in \{up,low\}^N}$ of $\Omega$ such that

$$A_{j,\nu,\iota} \subset \Big\{ F_j^{(\nu)}\left(\beta_{j+1}\theta_{j+1}^{(\iota_1,1)} - \Delta M_{j+1}^{[1]}, \ldots, \beta_{j+1}\theta_{j+1}^{(\iota_N,N)} - \Delta M_{j+1}^{[N]}\right) \\ \geq F_j^{(\nu)}\left(\beta_{j+1}\theta_{j+1}^{(\kappa_1,1)} - \Delta M_{j+1}^{[1]}, \ldots, \beta_{j+1}\theta_{j+1}^{(\kappa_N,N)} - \Delta M_{j+1}^{[N]}\right) \; \forall \kappa \in \{up, low\}^N \Big\}, \quad (1.41)$$

where $\theta^{up} := \theta^{up}(r^{(1)}, \ldots, r^{(N)}, M)$ and $\theta^{low} := \theta^{low}(r^{(1)}, \ldots, r^{(N)}, M)$ are given by (1.30) for admissible controls $r^{(\nu)} \in \mathcal{A}_0^{F^{(\nu)}}$, $\nu = 1, \ldots, N$, and a martingale $M \in \mathcal{M}_{ND}$.

We are now in the position to state the following auxiliary proposition, which provides a representation of the upper bound in terms of (possibly) non-adapted controls.

**Proposition 1.6.2.** *Suppose Assumptions 1.2.1 and let $j \in \{0, \ldots, J-1\}$. Further, let $M \in \mathcal{M}_{ND}$ and $r^{(\nu)} \in \mathcal{A}_j^{F^{(\nu)}}$, $\nu = 1, \ldots, N$, be given and define $\theta^{up} := \theta^{up}(r^{(1)}, \ldots, r^{(N)}, M)$ and $\theta^{low} := \theta^{low}(r^{(1)}, \ldots, r^{(N)}, M)$ by (1.30). Then, for every $i = j, \ldots, J-1$ and $\nu = 1, \ldots, N$, we have the $P$-almost sure identity*

$$\theta_i^{(up,\nu)} = \sum_{n=1}^{N} \left( \left(\rho_i^{(\nu),[n]}\right)^\top \beta_{i+1} \right)_+ \theta_{i+1}^{(up,n)} - \sum_{n=1}^{N} \left( \left(\rho_i^{(\nu),[n]}\right)^\top \beta_{i+1} \right)_- \theta_{i+1}^{(low,n)} \\ - \sum_{n=1}^{N} \left(\rho_i^{(\nu),[n]}\right)^\top \Delta M_{i+1}^{[n]} - F_i^{(\nu,\#)}\left(\rho_i^{(\nu)}\right). \quad (1.42)$$

*The random variable $\rho_i^{(\nu)}$ in (1.42) is for every $i = j, \ldots, J-1$ given by*

$$\rho_i^{(\nu)} = \sum_{k=1}^{2^N} \rho_i^{(\nu),k} \mathbb{1}_{A_{j,\nu,\pi^{-1}(k)}}, \quad (1.43)$$

*where the sets $A_{j,\nu,\pi^{-1}(k)}$ are given by (1.41) and each $\rho_i^{(\nu),k} = \rho_i^{(\nu),\pi(\iota)}$ solves*

$$F_i^{(\nu)}\left(\beta_{i+1}\theta_{i+1}^{(\iota_1,1)} - \Delta M_{i+1}^{[1]}, \ldots, \beta_{i+1}\theta_{i+1}^{(\iota_N,N)} - \Delta M_{i+1}^{[N]}\right) = \\ \sum_{n=1}^{N} \left(\rho_i^{(\nu),\pi(\iota),[n]}\right)^\top \left(\beta_{i+1}\theta_{i+1}^{(\iota_n,n)} - \Delta M_{i+1}^{[n]}\right) - F_i^{(\nu,\#)}\left(\rho_i^{(\nu),\pi(\iota)}\right). \quad (1.44)$$

*Proof.* Let $j \in \{0, \ldots, J-1\}$ be fixed from now on. Furthermore, we fix $M \in \mathcal{M}_{ND}$ and $r^{(\nu)} \in \mathcal{A}_j^{F^{(\nu)}}$, $\nu = 1, \ldots, N$, and define the processes $\theta^{up} := \theta^{up}(r^{(1)}, \ldots, r^{(N)}, M)$ and $\theta^{low} := \theta^{low}(r^{(1)}, \ldots, r^{(N)}, M)$ according to (1.30).

Then, we conclude by Lemma 1.2.4, that for each $i = j, \ldots, J-1$ and $\iota \in \{up, low\}^N$ there exist random variables $\rho_i^{(\nu),\pi(\iota)}$, which solve (1.44) and satisfy $\rho_i^{(\nu),\pi(\iota)} \in L^{\infty-}(\mathbb{R}^{ND})$ as well as $F_i^{(\nu,\#)}(\rho_i^{(\nu),\pi(\iota)}) \in L^{\infty-}(\mathbb{R})$. As a consequence, we obtain from the definition of $\rho_i^{(\nu)}$, that $\rho_i^{(\nu)} \in$

$L^{\infty-}(\mathbb{R}^{ND})$ and $F_i^{(\nu,\#)}(\rho_i^{(\nu)}) \in L^{\infty-}(\mathbb{R})$ for all $i = j, \dots, J-1$. Hence, we observe by Proposition 1.5.2 that

$$\begin{aligned}
\theta_i^{(up,\nu)} &= \sup_{u \in \mathbb{R}^{ND}} \sum_{n=1}^{N} \left( \left(u^{[n]}\right)^\top \beta_{i+1} \right)_+ \theta_{i+1}^{(up,n)} - \sum_{n=1}^{N} \left( \left(u^{[n]}\right)^\top \beta_{i+1} \right)_- \theta_{i+1}^{(low,n)} \\
&\quad - \sum_{n=1}^{N} \left(u^{[n]}\right)^\top \Delta M_{i+1}^{[n]} - F_i^{(\nu,\#)}(u) \\
&\geq \sum_{n=1}^{N} \left( \left(\rho_i^{(\nu),[n]}\right)^\top \beta_{i+1} \right)_+ \theta_{i+1}^{(up,n)} - \sum_{n=1}^{N} \left( \left(\rho_i^{(\nu),[n]}\right)^\top \beta_{i+1} \right)_- \theta_{i+1}^{(low,n)} \\
&\quad - \sum_{n=1}^{N} \left(\rho_i^{(\nu),[n]}\right)^\top \Delta M_{i+1}^{[n]} - F_i^{(\nu,\#)}\left(\rho_i^{(\nu)}\right).
\end{aligned}$$

In order to obtain the converse inequality, we first conclude by the definition of $\rho_i^{(\nu),\pi(\iota)}$, $\pi$, $A_{j,\nu,\iota}$ and $\rho_i^{(\nu)}$ that

$$\begin{aligned}
\theta_i^{(up,\nu)} &= \max_{\iota \in \{up,low\}^N} F_i^{(\nu)}\left(\beta_{i+1}\theta_{i+1}^{(\iota_1,1)} - \Delta M_{i+1}^{[1]}, \dots, \beta_{i+1}\theta_{i+1}^{(\iota_N,N)} - \Delta M_{i+1}^{[N]}\right) \\
&= \max_{\iota \in \{up,low\}^N} \left\{ \sum_{n=1}^{N} \left(\rho_i^{(\nu),\pi(\iota),[n]}\right)^\top \left(\beta_{i+1}\theta_{i+1}^{(\iota_n,n)} - \Delta M_{i+1}^{[n]}\right) - F_i^{(\nu,\#)}\left(\rho_i^{(\nu),\pi(\iota)}\right) \right\} \\
&= \sum_{k=1}^{2^N} \left( \sum_{n=1}^{N} \left(\rho_i^{(\nu),k,[n]}\right)^\top \left(\beta_{i+1}\theta_{i+1}^{((\pi^{-1}(k))_n,n)} - \Delta M_{i+1}^{[n]}\right) - F_i^{(\nu,\#)}\left(\rho_i^{(\nu),k}\right) \right) \mathbb{1}_{A_{j,\nu,\pi^{-1}(k)}} \\
&= \sum_{k=1}^{2^N} \sum_{n=1}^{N} \left(\rho_i^{(\nu),k,[n]}\right)^\top \beta_{i+1}\theta_{i+1}^{((\pi^{-1}(k))_n,n)} \mathbb{1}_{A_{j,\nu,\pi^{-1}(k)}} \\
&\quad - \sum_{k=1}^{2^N} \sum_{n=1}^{N} \left(\rho_i^{(\nu),k,[n]}\right)^\top \Delta M_{i+1}^{[n]} \mathbb{1}_{A_{j,\nu,\pi^{-1}(k)}} - \sum_{k=1}^{2^N} F_i^{(\nu,\#)}\left(\rho_i^{(\nu),k}\right) \mathbb{1}_{A_{j,\nu,\pi^{-1}(k)}} \\
&= \sum_{k=1}^{2^N} \sum_{n=1}^{N} \left(\rho_i^{(\nu),k,[n]}\right)^\top \beta_{i+1}\theta_{i+1}^{((\pi^{-1}(k))_n,n)} \mathbb{1}_{A_{j,\nu,\pi^{-1}(k)}} \\
&\quad - \sum_{n=1}^{N} \left(\rho_i^{(\nu),[n]}\right)^\top \Delta M_{i+1}^{[n]} - F_i^{(\nu,\#)}\left(\rho_i^{(\nu)}\right).
\end{aligned}$$

By exploiting the pathwise ordering of $\theta^{up}$ and $\theta^{low}$ established in Proposition 1.5.2 and the definition of $\rho_i^{(\nu)}$ once more, we finally deduce that

$$\begin{aligned}
\theta_i^{(up,\nu)} &= \sum_{n=1}^{N} \sum_{k=1}^{2^N} \left( \left(\rho_i^{(\nu),k,[n]}\right)^\top \beta_{i+1} \right)_+ \theta_{i+1}^{((\pi^{-1}(k))_n,n)} \mathbb{1}_{A_{j,\nu,\pi^{-1}(k)}} \\
&\quad - \sum_{n=1}^{N} \sum_{k=1}^{2^N} \left( \left(\rho_i^{(\nu),k,[n]}\right)^\top \beta_{i+1} \right)_- \theta_{i+1}^{((\pi^{-1}(k))_n,n)} \mathbb{1}_{A_{j,\nu,\pi^{-1}(k)}} \\
&\quad - \sum_{n=1}^{N} \left(\rho_i^{(\nu),[n]}\right)^\top \Delta M_{i+1}^{[n]} - F_i^{(\nu,\#)}\left(\rho_i^{(\nu)}\right)
\end{aligned}$$

$$\le \sum_{n=1}^{N}\sum_{k=1}^{2^N}\left(\left(\rho_i^{(\nu),k,[n]}\right)^\top \beta_{i+1}\right)_+ \theta_{i+1}^{(up,n)} \mathbb{1}_{A_{j,\nu,\pi^{-1}(k)}}$$
$$-\sum_{n=1}^{N}\sum_{k=1}^{2^N}\left(\left(\rho_i^{(\nu),k,[n]}\right)^\top \beta_{i+1}\right)_- \theta_{i+1}^{(low,n)} \mathbb{1}_{A_{j,\nu,\pi^{-1}(k)}}$$
$$-\sum_{n=1}^{N}\left(\rho_i^{(\nu),[n]}\right)^\top \Delta M_{i+1}^{[n]} - F_i^{(\nu,\#)}\left(\rho_i^{(\nu)}\right)$$
$$= \sum_{n=1}^{N}\left(\left(\rho_i^{(\nu),[n]}\right)^\top \beta_{i+1}\right)_+ \theta_{i+1}^{(up,n)} - \sum_{n=1}^{N}\left(\left(\rho_i^{(\nu),[n]}\right)^\top \beta_{i+1}\right)_- \theta_{i+1}^{(low,n)}$$
$$-\sum_{n=1}^{N}\left(\rho_i^{(\nu),[n]}\right)^\top \Delta M_{i+1}^{[n]} - F_i^{(\nu,\#)}\left(\rho_i^{(\nu)}\right),$$

which completes the proof. □

With this proposition at hand, we are now able to prove Theorem 1.6.1.

*Proof of Theorem 1.6.1.* Let $j \in \{0,\dots,J-1\}$ be fixed from now on. In order to simplify the exposition, we rely on the shorthand notation $E_i[\cdot]$ to denote the conditional expectation with respect to $\mathcal{F}_i$. Furthermore, we define the processes $\theta^\iota := \theta^\iota(r^{(1)},\dots,r^{(N)},M)$ and $\hat{\theta}^\iota := \theta^\iota(r^{(1)},\dots,r^{(N)},\hat{M})$, $\iota \in \{up, low\}$, according to (1.30), where each $r^{(\nu)} \in \mathcal{A}_j^{F^{(\nu)}}$. Due to the monotonicity and the tower property of the conditional expectation, it is sufficient to show that

$$E_J\left[\hat{\theta}_j^{up}\right] \ge \theta_j^{up} \quad \text{respectively} \quad E_J\left[\hat{\theta}_j^{low}\right] \le \theta_j^{low}.$$

The proof is by backward induction on $i = J,\dots,j$, with the case $i = J$ being trivial by definition of the processes. Hence, we assume that the assertion is true for $i+1$. From Proposition 1.6.2, we get for every $\nu = 1,\dots,N$ existence of an $\mathcal{F}_J$-measurable random variable $\rho_i^{(\nu)}$ such that

$$\theta_i^{(up,\nu)} = \sum_{n=1}^{N}\left(\left(\rho_i^{(\nu,[n])}\right)^\top \beta_{i+1}\right)_+ \theta_{i+1}^{(up,n)} - \sum_{n=1}^{N}\left(\left(\rho_i^{(\nu,[n])}\right)^\top \beta_{i+1}\right)_- \theta_{i+1}^{(low,n)}$$
$$-\sum_{n=1}^{N}\left(\rho_i^{(\nu),[n]}\right)^\top \Delta M_{i+1} - F_i^{(\nu,\#)}\left(\rho_i^{(\nu)}\right). \qquad (1.45)$$

Furthermore, we emphasize that the proof of the pathwise representation for the upper bound stated in Proposition 1.5.2 does not rely on the martingale property of the input martingale $M$. For this reason, Proposition 1.5.2 also applies for the upper bound $\hat{\theta}^{up}$. Therefore, it follows from Proposition 1.5.2, the assumptions on $\hat{M}$, the induction hypothesis, and (1.45) that

$$E_J\left[\hat{\theta}_i^{(up,\nu)}\right] \ge E_J\Bigg[\sum_{n=1}^{N}\left(\left(\rho_i^{(\nu,[n])}\right)^\top \beta_{i+1}\right)_+ \hat{\theta}_{i+1}^{(up,n)} - \sum_{n=1}^{N}\left(\left(\rho_i^{(\nu,[n])}\right)^\top \beta_{i+1}\right)_- \hat{\theta}_{i+1}^{(low,n)}$$
$$-\sum_{n=1}^{N}\left(\rho_i^{(\nu),[n]}\right)^\top \Delta \hat{M}_{i+1} - F_i^{(\nu,\#)}\left(\rho_i^{(\nu)}\right)\Bigg]$$
$$= \sum_{n=1}^{N}\left(\left(\rho_i^{(\nu,[n])}\right)^\top \beta_{i+1}\right)_+ E_J\left[\hat{\theta}_{i+1}^{(up,n)}\right] - \sum_{n=1}^{N}\left(\left(\rho_i^{(\nu,[n])}\right)^\top \beta_{i+1}\right)_- E_J\left[\hat{\theta}_{i+1}^{(low,n)}\right]$$

$$
\begin{aligned}
&-\sum_{n=1}^{N}\left(\rho_i^{(\nu),[n]}\right)^\top \Delta M_{i+1} - F_i^{(\nu,\#)}\left(\rho_i^{(\nu)}\right)\\
\geq\ & \sum_{n=1}^{N}\left(\left(\rho_i^{(\nu,[n])}\right)^\top \beta_{i+1}\right)_+ \theta_{i+1}^{(up,n)} - \sum_{n=1}^{N}\left(\left(\rho_i^{(\nu,[n])}\right)^\top \beta_{i+1}\right)_- \theta_{i+1}^{(low,n)}\\
&-\sum_{n=1}^{N}\left(\rho_i^{(\nu),[n]}\right)^\top \Delta M_{i+1} - F_i^{(\nu,\#)}\left(\rho_i^{(\nu)}\right)\\
=\ & \theta_i^{(up,\nu)}
\end{aligned}
$$

for every $\nu = 1, \ldots, N$. Similarly, we obtain for the lower bound, that

$$
\begin{aligned}
E_J\left[\hat{\theta}_i^{(low,\nu)}\right] =\ & E_J\Bigg[\sum_{n=1}^{N}\left(\left(r_i^{(\nu,[n])}\right)^\top \beta_{i+1}\right)_+ \hat{\theta}_{i+1}^{(low,n)} - \sum_{n=1}^{N}\left(\left(r_i^{(\nu,[n])}\right)^\top \beta_{i+1}\right)_- \hat{\theta}_{i+1}^{(up,n)}\\
&-\sum_{n=1}^{N}\left(r_i^{(\nu),[n]}\right)^\top \Delta\hat{M}_{i+1} - F_i^{(\nu,\#)}\left(r_i^{(\nu)}\right)\Bigg]\\
=\ & \sum_{n=1}^{N}\left(\left(r_i^{(\nu,[n])}\right)^\top \beta_{i+1}\right)_+ E_J\left[\hat{\theta}_{i+1}^{(low,n)}\right] - \sum_{n=1}^{N}\left(\left(r_i^{(\nu,[n])}\right)^\top \beta_{i+1}\right)_- E_J\left[\hat{\theta}_{i+1}^{(up,n)}\right]\\
&-\sum_{n=1}^{N}\left(r_i^{(\nu),[n]}\right)^\top \Delta M_{i+1} - F_i^{(\nu,\#)}\left(r_i^{(\nu)}\right)\\
\leq\ & \sum_{n=1}^{N}\left(\left(r_i^{(\nu,[n])}\right)^\top \beta_{i+1}\right)_+ \theta_{i+1}^{(low,n)} - \sum_{n=1}^{N}\left(\left(r_i^{(\nu,[n])}\right)^\top \beta_{i+1}\right)_- \theta_{i+1}^{(up,n)}\\
&-\sum_{n=1}^{N}\left(r_i^{(\nu),[n]}\right)^\top \Delta M_{i+1} - F_i^{(\nu,\#)}\left(r_i^{(\nu)}\right)\\
=\ & \theta_i^{(low,\nu)}.
\end{aligned}
$$

□

## 1.7 Implementation

In this section, we explain how to implement an algorithm for the computation of the bounds derived in this chapter in a Markovian framework. Hence, we start with a description of the setting and introduce the required notation. Then, we present two approaches for the construction of approximate solutions to (1.15) which rely on least-squares Monte Carlo. Building on these approximate solutions, we explain the construction of approximate controls and martingales required for the construction of upper and lower bounds. With these inputs at hand, we demonstrate that the implementation of the recursions (1.30) for $\theta^{up}$ and $\theta^{low}$ is straightforward. Finally, we apply this algorithm in two numerical examples, namely pricing under negotiated collateral and uncertain volatility.

Throughout this section, we restrict ourselves to the Markovian framework of Section 1.6, as this is the practically most relevant situation. To this end, we assume that $(B_j)_{j=1,\ldots,J}$ is an $\mathbb{R}^{\mathcal{D}}$-dimensional adapted process (with $\mathcal{D} \geq D$), such that the first $D$ components of $B_j$ are given by $\beta_j$

and $B_j$ is independent of $\mathcal{F}_{j-1}$, for every $j = 1, \ldots, J$. $X$ is supposed to be an $\mathbb{R}^d$-valued Markovian process of the form

$$X_j = h_j(X_{j-1}, B_j), \quad j = 1, \ldots, J, \tag{1.46}$$

for measurable functions $h_j : \mathbb{R}^d \times \mathbb{R}^{\mathcal{D}} \to \mathbb{R}^d$, starting at $X_0 = x_0 \in \mathbb{R}^d$. Forward equations of this form for the state process $X$ typically arise as time discretization schemes for stochastic differential equations. Moreover, for the generator $F_j^{(\nu)}$ of the dynamic program (1.15) we assume existence of measurable functions $f_j^{(\nu)} : \mathbb{R}^d \times \mathbb{R}^{ND} \to \mathbb{R}$ satisfying $F_j^{(\nu)}(\cdot) = f_j^{(\nu)}(X_j, \cdot)$, i.e., $F_j^{(\nu)}$ depends on $\omega$ only through the Markovian process $X$. Then, we consider a Markovian version of the dynamic program (1.15) in the form

$$\begin{aligned} Y_J^{(\nu)} &= g^{(\nu)}(X_J), \\ Y_j^{(\nu)} &= f_j^{(\nu)}\left(X_j, E_j\left[\beta_{j+1} Y_{j+1}^{(1)}\right], \ldots, E_j\left[\beta_{j+1} Y_{j+1}^{(N)}\right]\right), \quad j = 0, \ldots, J-1, \ \nu = 1, \ldots, N, \end{aligned} \tag{1.47}$$

where $g^{(\nu)} : \mathbb{R}^d \to \mathbb{R}$ is measurable for each $\nu$ and satisfies $E[|g^{(\nu)}(X_J)|^p] < \infty$ for all $p \geq 1$. In this framework, $Y_j^{(\nu)}$ is a deterministic function of $X_j$, i.e. there exists $\bar{y}_j^{(\nu)} : \mathbb{R}^d \to \mathbb{R}$ such that $Y_j^{(\nu)} = \bar{y}_j^{(\nu)}(X_j)$. In particular, we have that $Y_0^{(\nu)}$ is a constant. Moreover, in view of (1.46), we obtain existence of a measurable function $y_j^{(\nu)} : \mathbb{R}^d \times \mathbb{R}^{\mathcal{D}} \to \mathbb{R}$ such that $Y_j^{(\nu)} = y_j^{(\nu)}(X_{j-1}, B_j)$ for every $j = 1, \ldots, J$ and $\nu = 1, \ldots, N$. Denoting by $P_{B_j}$ the law of $B_j$, we can, thus, write $E_j[\beta_{j+1} Y_{j+1}^{(\nu)}] = z_j^{[\nu]}(X_j)$ with

$$z_j^{[\nu]}(x) = \left(\int_{\mathbb{R}^{\mathcal{D}}} b_1\, y_{j+1}^{(\nu)}(x, b)\, P_{B_{j+1}}(db), \ldots, \int_{\mathbb{R}^{\mathcal{D}}} b_D\, y_{j+1}^{(\nu)}(x, b)\, P_{B_{j+1}}(db)\right)^{\top}. \tag{1.48}$$

### 1.7.1 Computation of approximate solutions and upper and lower bounds

In order to obtain an approximate solution to (1.47), we rely in the following on two variants of the least-squares Monte Carlo (LSMC) approach. First, we consider the regression-now variant of LSMC proposed by Longstaff and Schwartz (2001) and Tsitsiklis and Van Roy (2001) in the context of Bermudan option pricing. Thereafter, we present a variant which is in the spirit of the regression-later variant of Glasserman and Yu (2004) and the martingale basis approach proposed by Bender and Steiner (2012). The main difference between these two approaches lies in the assumptions regarding the basis functions. While the regression-now approach only requires that the basis functions satisfy suitable integrability conditions, the regression-later approach additionally assumes that certain conditional expectations of the basis functions are available in closed-form. As we will explain in more detail below, this additional assumption enables us to avoid the error stemming from possibly unfavorable regressions involving the process $\beta$.

#### Regression-now vs. regression-later approach

The main idea of the regression-now approach is to approximate the conditional expectations in (1.47) by an orthogonal projection onto a linear subspace of $L^2(\Omega, P)$. This subspace is spanned by a set of predefined basis functions $\eta_j^{(\nu)} = (\eta_{j,1}^{(\nu)}, \ldots, \eta_{j,K}^{(\nu)})$ such that $E[|\eta_j^{(\nu)}(X_j)|^2] < \infty$. Then, the orthogonal projection on this set of basis functions is computed via regression, i.e. one computes

$$\tilde{Y}_J^{(\nu)} = g^{(\nu)}(X_J),$$

$$\tilde{Y}_j^{(\nu)} = f_j\left(X_j, \mathcal{P}_j\left[\beta_{j+1}\tilde{Y}_{j+1}^{(1)}\right], \dots, \mathcal{P}_j\left[\beta_{j+1}\tilde{Y}_{j+1}^{(N)}\right]\right), \quad j = J-1, \dots, 0,\ \nu = 1, \dots, N$$

as an approximation to $Y_j^{(\nu)}$. Here, $\mathcal{P}_j$ denotes the empirical regression onto a set of basis functions for a given set of sample paths. Note that, since the process $\beta$ is $\mathbb{R}^D$-valued, this requires the computation of $ND$ regressions at every time step $j = 0, \dots, J-1$.

In order to formalize this idea, we suppose that approximations $\tilde{y}_{j+1}^{(\nu)}$ of $\bar{y}_{j+1}^{(\nu)}$ have already been computed using LSMC for every $\nu = 1, \dots, N$. Recall that we have by the projection property of the conditional expectation that

$$E_j\left[\beta_{j+1}^{(n)}\tilde{y}_{j+1}^{(\nu)}(X_{j+1})\right] = \operatorname*{argmin}_{\bar{z}} E\left[\left|\bar{z}(X_j) - \beta_{j+1}^{(n)}\tilde{y}_{j+1}^{(\nu)}(X_{j+1})\right|^2\right] \tag{1.49}$$

for every $n = 1, \dots, D$, where the minimum is taken over all measurable functions $\bar{z} : \mathbb{R}^d \to \mathbb{R}$ which satisfy $E[\bar{z}(X_j)^2] < \infty$. This minimization problem is infinite-dimensional. Hence, in a first step, we choose a set of measurable basis functions $\eta_j^{(\nu),n} = (\eta_{j,1}^{(\nu),n}, \dots, \eta_{j,K}^{(\nu),n})^\top$ such that $E[|\eta_{j,k}^{(\nu),n}(X_j)|^p] < \infty$ for all $p \geq 1$ and each $k$. Then, we restrict the minimization problem (1.49) to the linear subspace spanned by these basis functions. In this way, we end up with the finite-dimensional minimization problem

$$a_j^{(\nu),n} = \operatorname*{argmin}_{a \in \mathbb{R}^K} E\left[\left|a^\top \eta_j^{(\nu),n}(X_j) - \beta_{j+1}^{(n)}\tilde{y}_{j+1}^{(\nu)}(X_{j+1})\right|^2\right]. \tag{1.50}$$

Since this problem is in general still not solvable in closed form, we transfer it to a linear least-squares problem by replacing the expectation in (1.50) by the empirical mean. To this end, suppose we are given $\Lambda^{reg}$ independent copies $\{B_j(\lambda);\ j = 1, \dots, J,\ \lambda = 1, \dots, \Lambda^{reg}\}$ of the process $(B_j)_{j=1,\dots,J}$. Then, the coefficients $a_j^{(\nu),n}$ are given by

$$a_j^{(\nu),n} = \operatorname*{argmin}_{a \in \mathbb{R}^K} \frac{1}{\Lambda^{reg}} \sum_{\lambda=1}^{\Lambda^{reg}} \left|a^\top \eta_j^{(\nu),n}(X_j(\lambda)) - \beta_{j+1}^{(n)}(\lambda)\tilde{y}_{j+1}^{(\nu)}(X_{j+1}(\lambda))\right|^2. \tag{1.51}$$

It is well-known that a solution to (1.51) exists and is given by

$$\begin{aligned} a_j^{(\nu),n} &= \frac{1}{\sqrt{\Lambda^{reg}}}\left(A(K, \Lambda^{reg}, \nu, n)^\top A(K, \Lambda^{reg}, \nu, n)\right)^{-1} \\ &\quad \times A(K, \Lambda^{reg}, \nu, n)^\top \begin{pmatrix} \beta_{j+1}^{(n)}(1)\tilde{y}_{j+1}^{(\nu)}(X_{j+1}(1)) \\ \vdots \\ \beta_{j+1}^{(n)}(\Lambda^{reg})\tilde{y}_{j+1}^{(\nu)}(X_{j+1}(\Lambda^{reg})) \end{pmatrix}, \end{aligned}$$

where

$$A(K, \Lambda^{reg}, \nu, n) := \frac{1}{\sqrt{\Lambda^{reg}}}\left(\eta_{j,k}^{(\nu),n}(X_j(\lambda))\right)_{\substack{\lambda=1,\dots,\Lambda^{reg}, \\ k=1,\dots,K}}.$$

If the inverse matrix $(A(K, \Lambda^{reg}, \nu, n)^\top A(K, \Lambda^{reg}, \nu, n))^{-1}$ does not exist, we may instead consider the pseudo-inverse $A(K, \Lambda^{reg}, \nu, n)^+$ of $A(K, \Lambda^{reg}, \nu, n)$ and obtain

$$a_j^{(\nu),n} = \frac{1}{\sqrt{\Lambda^{reg}}} A(K, \Lambda^{reg}, \nu, n)^+ \begin{pmatrix} \beta_{j+1}^{(n)}(1)\tilde{y}_{j+1}^{(\nu)}(X_{j+1}(1)) \\ \vdots \\ \beta_{j+1}^{(n)}(\Lambda^{reg})\tilde{y}_{j+1}^{(\nu)}(X_{j+1}(\Lambda^{reg})) \end{pmatrix}.$$

Applying this approach backwards in time for every $j = J-1, \dots, 0$, we end up with the following algorithm for the computation of an approximate solution:

Let $(B_j(\lambda))_{j=1,\dots,J}$, $\lambda = 1, \dots, \Lambda^{reg}$, be independent copies of the process $(B_j)_{j=1,\dots,J}$. In what follows, we refer to these copies as "regression paths". Further, denote by $\beta(\lambda)$ and $X(\lambda)$ the trajectories of $\beta$ and $X$ along these paths and by $\eta_j^{(\nu),n} = (\eta_{j,1}^{(\nu),n}, \dots, \eta_{j,K}^{(\nu),n})^\top$ the basis functions for the approximation of $E_j[\beta_{j+1}^{(n)} Y_{j+1}^{(\nu)}]$, $n = 1, \dots, D$, $\nu = 1, \dots, N$. Then, approximations $\tilde{y}_j^{(\nu)}(x)$ and $\tilde{z}_j^{[\nu]}(x)$, $j = 0, \dots, J$ can be computed recursively by

$$\begin{aligned}
\tilde{y}_J^{(\nu)}(x) &= g^{(\nu)}(x), \\
\tilde{z}_J^{[\nu],n}(x) &= 0, \quad n = 1, \dots, D, \\
a_j^{[\nu],d} &= \operatorname*{argmin}_{a \in \mathbb{R}^K} \frac{1}{\Lambda^{reg}} \sum_{\lambda=1}^{\Lambda^{reg}} \left| a^\top \eta_j^{(\nu),n}(X_j(\lambda)) - \beta_{j+1}^{(n)}(\lambda) \tilde{y}_{j+1}^{(\nu)}(X_{j+1}(\lambda)) \right|^2 \\
\tilde{z}_j^{[\nu],n}(x) &= \left( a_j^{[\nu],n} \right)^\top \eta_j^{(\nu),n}(x), \quad n = 1, \dots, D, \\
\tilde{y}_j^{(\nu)}(x) &= f_j\left(x, \tilde{z}_j^{[1]}(x), \dots, \tilde{z}_j^{[\nu]}(x)\right), \quad j = J-1, \dots, 0, \ \nu = 1, \dots, N.
\end{aligned} \tag{1.52}$$

The LSMC approach explained above suffers from two error sources, namely the projection error induced by the choice of basis functions and the simulation error. In order to control the simulation error, the number of regression paths has to be chosen properly. Especially, in situations where the process $\beta$ might have large variance, this can lead to a substantial increase in the number of required regression paths. This problem is discussed in Bender and Steiner (2012), where they consider Euler-type approximation schemes for BSDEs. As we have seen in Example 1.1.2, the process $\beta$ is in such situations given by

$$\beta_{j+1} = \left(1, \frac{\Delta W_{j+1}^{(1)}}{\Delta_{j+1}}, \dots, \frac{\Delta W_{j+1}^{(d)}}{\Delta_{j+1}}\right),$$

where $\Delta W_{j+1} := W_{t_{j+1}} - W_{t_j}$ denotes the increments of a $d$-dimensional Brownian motion on a time grid $0 = t_0 < \dots < t_J = T$ with time increments $\Delta_{j+1} := t_{j+1} - t_j$. If the mesh of this partition tends to zero, the variance of the process $\beta$ increases and therefore more regression paths are required to keep the simulation error small. In order to deal with this problem, Bender and Steiner (2012) propose a martingale basis variant of LSMC, which is in the spirit of the regression-later approach presented in Glasserman and Yu (2004) for the Bermudan option pricing problem. The main idea is to choose basis functions which form martingales and for which the conditional expectations are available in closed form. This allows them to skip the regressions for the approximation of $E_j[\beta_{j+1} Y_{j+1}^{(\nu)}]$, and thus to avoid the corresponding simulation error. As a consequence, the number of regression paths can be held at a moderate level, even for fine time discretizations. These assumptions are restrictive and we consider in the following a variant which works under milder assumptions, making the approach more flexible.

To this end, let $\eta_j^{(\nu)} = (\eta_{j,1}^{(\nu)}, \dots, \eta_{j,K}^{(\nu)})$, $j = 0, \dots, J$ be basis functions, where each $\eta_{j,k}^{(\nu)} : \mathbb{R}^d \times \mathbb{R}^{\mathcal{D}} \to \mathbb{R}$ is measurable and satisfies $E[|\eta_{j,k}^{(\nu)}(X_{j-1}, B_j)|^p] < \infty$ for all $p \geq 1$. In contrast to the regression-now approach explained above, we additionally assume that the expectations

$$R_{j,k}^{(\nu)}(x) := \left( \int_{\mathbb{R}^{\mathcal{D}}} b_1 \, \eta_{j+1,k}^{(\nu)}(x,b) \, P_{B_{j+1}}(db), \dots, \int_{\mathbb{R}^{\mathcal{D}}} b_D \, \eta_{j+1,k}^{(\nu)}(x,b) \, P_{B_{j+1}}(db) \right)^\top, \tag{1.53}$$

$x \in \mathbb{R}^d$, are available in closed form or can be computed numerically up to a negligible error. Two things should be noted: first, we do not assume that the basis functions form a set of martingales, which is the key assumption in Glasserman and Yu (2004) and Bender and Steiner (2012). Relaxing this assumption increases the applicability of the regression-later approach presented below. Second, we apply the recursive definition (1.46) of the Markovian process so that, in contrast to the regression-now approach, the basis functions do not necessarily depend on the current value of the Markovian process but rather on the value one time-step before and the current value of the process $B$. As the following example demonstrates, this provides more flexibility in the choice of basis functions satisfying the above assumptions.

*Example* 1.7.1. We assume that the Markovian process $X$ is given by an Euler scheme, i.e.,

$$X_j = X_{j-1} + \mu_{j-1}(X_{j-1})\Delta_j + \sigma_{j-1}(X_{j-1})\Delta W_j, \quad X_0 = x_0$$

where $\Delta W_j := W_{t_j} - W_{t_{j-1}}$ denotes increments of a $d'$-dimensional Brownian motion with time increments $\Delta_j = t_j - t_{j-1}$ for an increasing family of time points $0 = t_0 < t_1 \ldots < t_J$. Moreover, we assume that the coefficient functions $\mu_j : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma_j : \mathbb{R}^{d \times d'} \to \mathbb{R}^d$ are Lipschitz continuous. We consider a discretization scheme for BSDEs as discussed in Example 1.1.2 so that

$$\beta_j = B_j = \left(1, \frac{\Delta W_j}{\Delta_j}\right)^\top,$$

with $D = \mathcal{D} = 1 + d'$.

(i) (Global polynomials) When applying an LSMC approach, one often relies on polynomials of the underlying Markovian process $X$ as basis functions. We thus show in the following, that this kind of basis functions satisfies the above assumptions. To this end, we consider a polynomial $p : \mathbb{R}^d \to \mathbb{R}$ in $X_{j+1}$ as basis function at time $j + 1$ and denote by $w$ the vector consisting of the last $d'$ components of $b \in \mathbb{R}^{1+d'}$, which correspond to the Brownian increments. Exploiting the definition of the process $X$, we observe that the basis function $\eta_{j+1}$ can be expressed in terms of $x$ and $w$ by

$$\eta_{j+1}(x, w) = p(x + \mu_j(x)\Delta_{j+1} + \sigma_j(x)w).$$

Hence, for every $x \in \mathbb{R}^d$, $\eta_j(x, w)$ is a polynomial in $w$. As a consequence, the conditional expectation $E[\eta_{j+1}(x, \Delta W_{j+1})]$ (corresponding to the first component on the right-hand side of (1.53)) can be computed in closed form. From the definition of the process $(B_j)_{j=1,\ldots,J}$, we further observe that the remaining components of the vector on the right-hand side of (1.53) are given by $\Delta_{j+1}^{-1} E[\Delta W_{j+1}^{(l)}\, \eta_{j+1}(x, \Delta W_{j+1})]$, $l = 1, \ldots, d'$. Each component is thus, for fixed $x$, again a polynomial in $\Delta W_{j+1}$, so that $E[\Delta W_{j+1}\, \eta_{j+1}(x, \Delta W_{j+1})]$ is also available in closed form. In contrast, the conditional expectations $E[p(X_{j+1})|X_i]$, $i < j$, several steps ahead are in general not available in closed form. This may only be the case in certain situations, e.g., when $\mu$ and $\sigma$ are linear and, thus, $E[p(X_{j+1})|X_j = x]$ is again a polynomial in $x$. Therefore our assumptions on the function basis are less restrictive than the ones imposed by Glasserman and Yu (2004) and Bender and Steiner (2012).

(ii) (One-step-ahead localization) The following example provides the main motivation for considering basis functions which can depend on $(X_j, \Delta W_{j+1})$, although $Y_{j+1}$ is $\sigma(X_{j+1})$-measurable. In the numerical example of Section 3.4.3, we consider a non-linear option pricing problem with a payoff function on the maximum of a basket of assets. For the basis functions, we rely on functions on the largest asset, as these are known to be very successful in such situations in the context of Bermudan options, see e.g. Andersen and Broadie (2004). For this purpose, we denote by $l_j^{(1)}$ the index of the largest component of $X$ at time $j$. For simplicity, we consider the case that the basis function is a one-dimensional polynomial $p$ of $X_{j+1}^{(l_{j+1}^{(1)})}$. In general, the one-step conditional expectation $E[p(X_{j+1}^{(l_{j+1}^{(1)})})|X_j = x]$ is however not available in closed form. In order to circumvent this problem, we check for the maximal component one time step ahead. Then, we end up with the basis function

$$\eta_{j+1}(x,w) = \sum_{l=1}^{d} \mathbb{1}_{\{x^{(l)} \geq x^{(m)}\ \forall m=1,\dots,d\}} p\left(x^{(l)} + \mu_j(x^{(l)})\Delta_{j+1} + \sigma_j(x^{(l)})w\right)$$

which satisfies (1.53).

Under the given assumptions, we are able to apply the following regression-later variant of the LSMC approach:

$$\begin{aligned}
\tilde{Y}_J^{(\nu)} &= \mathcal{P}_J\left[g^{(\nu)}(X_J)\right], \\
\tilde{Y}_j^{(\nu)} &= \mathcal{P}_j\left[f_j\left(X_j, E_j\left[\beta_{j+1}\tilde{Y}_{j+1}^{(1)}\right], \dots, E_j\left[\beta_{j+1}\tilde{Y}_{j+1}^{(N)}\right]\right)\right], \quad j = J-1,\dots,0,\ \nu = 1,\dots,N.
\end{aligned}$$

Note that, in contrast to the regression-now approach, only $N$ regressions are computed in every time step, since we have inductively that $\tilde{Y}_{j+1}$ is a linear combination of the basis functions for which the other conditional expectations are available in closed form.

More formally, suppose that an approximation $\tilde{y}_{j+1}^{(\nu)}$ is given by a linear combination of the basis functions $\eta_{j+1}^{(\nu)}$, i.e.

$$\tilde{y}_{j+1}^{(\nu)}(x,b) = \sum_{k=1}^{K} a_{j+1,k}^{(\nu)} \eta_{j+1,k}^{(\nu)}(x,b) \tag{1.54}$$

for every $\nu = 1,\dots,N$. Then, we observe by (1.53) that the function $\tilde{z}_j^{(\nu)}$ can be expressed in terms of the coefficients $a_{j+1}^{(\nu)}$. Indeed, we have

$$\begin{aligned}
\tilde{z}_j^{(\nu)}(x) &= E\left[\beta_{j+1}\tilde{y}_{j+1}^{(\nu)}(X_j, B_{j+1})\middle| X_j = x\right] \\
&= \sum_{k=1}^{K} a_{j+1,k}^{(\nu)} E\left[\beta_{j+1}\eta_{j+1,k}^{(\nu)}(x, B_{j+1})\right] \\
&= \sum_{k=1}^{K} a_{j+1,k}^{(\nu)} R_{j,k}^{(\nu)}(x),
\end{aligned}$$

so that no regression is required for the computation of $\tilde{z}^{(\nu)}$. This lead us to the following algorithm for the computation of $\tilde{y}^{(\nu)}$ and $\tilde{z}^{(\nu)}$: Suppose, we are given a set $\{B_j(\lambda);\ j = 1,\dots,J,\ \lambda = 1,\dots,\Lambda^{reg}\}$ of regression paths and denote, as before, by $X(\lambda)$ the corresponding trajectories of the process $X$. Then, for every $j = J-1,\dots,1$, an approximate solution to the system of dynamic programs can be constructed by:

$$a_J^{(\nu)} = \underset{a \in \mathbb{R}^K}{\operatorname{argmin}} \frac{1}{\Lambda^{reg}} \sum_{\lambda=1}^{\Lambda^{reg}} \left| a^\top \eta_J^{(\nu)}(X_{J-1}(\lambda), B_J(\lambda)) - g^{(\nu)}(X_J(\lambda))\right|^2$$

$$
\begin{aligned}
\tilde{y}_J^{(\nu)}(x,b) &= \left(a_J^{(\nu)}\right)^\top \eta_J^{(\nu)}(x,b),\\
\tilde{z}_J^{[\nu]}(x) &= 0,\\
\tilde{z}_j^{[\nu]}(x) &= \sum_{k=1}^{K} a_{j+1,k}^{(\nu)} R_{j,k}^{(\nu)}(x),\\
a_j^{(\nu)} &= \operatorname*{argmin}_{a\in\mathbb{R}^K} \frac{1}{\Lambda^{reg}} \sum_{\lambda=1}^{\Lambda^{reg}} \left| a^\top \eta_j^{(\nu)}(X_{j-1}(\lambda), B_j(\lambda)) - f_j^{(\nu)}\left(X_j(\lambda), \tilde{z}_j^{[1]}(X_j(\lambda)), \dots, \tilde{z}_j^{[N]}(X_j(\lambda))\right)\right|^2\\
\tilde{y}_j^{(\nu)}(x,b) &= \left(a_j^{(\nu)}\right)^\top \eta_j^{(\nu)}(x,b),\\
\tilde{z}_0^{[\nu]}(x) &= \sum_{k=1}^{K} a_{1,k}^{(\nu)} R_{0,k}^{(\nu)}(x),\\
\tilde{y}_0^{(\nu)}(x,b) &= f_0^{(\nu)}\left(x_0, \tilde{z}_0^{[1]}(x_0), \dots, \tilde{z}_0^{[N]}(x_0)\right), \quad \nu = 1,\dots,N. \qquad (1.55)
\end{aligned}
$$

Note that compared to the regression-now approach, this algorithm requires that the terminal condition $g^{(\nu)}$ is regressed on the basis functions as initialization. If however, the function $g^{(\nu)}$ satisfies the above conditions on the basis functions, then we may include $g^{(\nu)}$ to the set of basis functions and no regression is required. We also emphasize, that for $j = 0$, no regression on the basis functions is performed to compute $\tilde{y}_0$, as the algorithm terminates and thus no representation of $\tilde{y}_0$ in terms of basis functions is required at initial time.

#### Computation of Upper and Lower Bounds

In a next step, we explain how upper and lower bounds can be computed if the input approximate solution is obtained from the regression-later approach. To this end, we simulate a second set of independent copies $\{B_j(\lambda^{out});\ j = 1,\dots,J,\ \lambda^{out} = 1,\dots,\Lambda^{out}\}$, called "outer paths", which is additionally independent of the regression paths used to compute the input approximation. The corresponding trajectories of the process $X$ are denoted by $X(\lambda^{out})$. Taking the coefficients $a_j^{(\nu)}$, $j = 1,\dots,J$, $\nu = 1,\dots,N$, from the regression step, we first compute an approximate solution to (1.47) along these new paths. The resulting approximations are given by $\tilde{y}^{(\nu)}(X(\lambda^{out}), B(\lambda^{out}))$ respectively $\tilde{z}^{[\nu]}(X(\lambda^{out}))$, $\nu = 1,\dots,N$.

Based on these approximations, we are now able to derive approximations of the optimal controls $r^{(\nu,*)} \in \mathcal{A}_0^{F^{(\nu)}}$, $\nu = 1,\dots,N$. To this end, we first note that optimal controls $r^{(\nu,*)}(\lambda^{out}) \in \mathcal{A}_0^{F^{(\nu)}}$ along the outer paths $\lambda^{out} = 1,\dots,\Lambda^{out}$ are given by

$$
\begin{aligned}
\sum_{n=1}^{N} \left(r_j^{(\nu,*),[n]}(\lambda^{out})\right)^\top z_j^{[n]}(X_j(\lambda^{out})) - f_j^{(\nu,\#)}\left(X_j(\lambda^{out}), r_j^{(\nu,*)}(\lambda^{out})\right)&\\
= f_j^{(\nu)}\left(X_j(\lambda^{out}), z_j^{[1]}(X_j(\lambda^{out})), \dots, z_j^{[N]}(X_j(\lambda^{out}))\right)& \qquad (1.56)
\end{aligned}
$$

for every $j = 0,\dots,J-1$ and $\nu = 1,\dots,N$. Replacing the functions $z_j^{[\nu]}$ in (1.56) by their respective approximations, we can compute approximations $\tilde{r}^{(\nu)}(\lambda^{out}) \in \mathcal{A}_0^{F^{(\nu)}}$ by solving (approximately) the equation

$$
\sum_{n=1}^{N} \left(\tilde{r}_j^{(\nu),[n]}(\lambda^{out})\right)^\top \tilde{z}_j^{[n]}(X_j(\lambda^{out})) - f_j^{(\nu,\#)}\left(X_j(\lambda^{out}), \tilde{r}_j^{(\nu)}(\lambda^{out})\right)
$$

$$= f_j^{(\nu)}\left(X_j(\lambda^{out}), \tilde{z}_j^{[1]}(X_j(\lambda^{out})), \ldots, \tilde{z}_j^{[N]}(X_j(\lambda^{out}))\right) \quad (1.57)$$

for every $j = 0, \ldots, J-1$, $\nu = 1, \ldots, N$ and $\lambda^{out} = 1, \ldots, \Lambda^{out}$. If the convex conjugate $f_j^{(\nu,\#)}$ cannot be computed exactly, it can, of course, be replaced by a numerical approximation.

For the approximation of the Doob martingales $M^{*,[\nu]}$, we proceed similarly, and replace the functions $y_j(x,b)$ by their approximations $\tilde{y}_j(x,b)$. Then, we observe that we need to compute increments of the form

$$\beta_{j+1}(\lambda^{out})\tilde{y}_{j+1}^{(\nu)}(X_j(\lambda^{out}), B_{j+1}(\lambda^{out})) - E\left[\beta_{j+1}\tilde{y}_{j+1}^{(\nu)}(X_j, B_{j+1})\middle| X_j = X_j(\lambda^{out})\right] \quad (1.58)$$

for every $j = 0, \ldots, J-1$ and $\lambda^{out} = 1, \ldots, \Lambda^{out}$. Since we have by construction that

$$\tilde{z}_j^{[\nu]}\left(X_j(\lambda^{out})\right) = E\left[\beta_{j+1}\tilde{y}_{j+1}^{(\nu)}(X_j, B_{j+1})\middle| X_j = X_j(\lambda^{out})\right],$$

we observe that the martingales $\tilde{M}^{[\nu]}$ are given by

$$\tilde{M}_j^{[\nu]}\left(\lambda^{out}\right) = \sum_{i=0}^{j-1} \beta_{i+1}\left(\lambda^{out}\right)\tilde{y}_{i+1}\left(X_i\left(\lambda^{out}\right), B_{i+1}\left(\lambda^{out}\right)\right) - \tilde{z}_i^{[\nu]}\left(X_i\left(\lambda^{out}\right)\right)$$

for every $j = 0, \ldots, J$, $\nu = 1, \ldots, N$ and any outer path $\lambda^{out}$.

With these approximations at hand, we can go through the coupled recursion (1.30) for $\theta^{up}(\lambda^{out}) := \theta^{up}(\tilde{r}^{(1)}(\lambda^{out}), \ldots, \tilde{r}^{(N)}(\lambda^{out}), \tilde{M}(\lambda^{out}))$ and $\theta^{low}(\lambda^{out}) := \theta^{low}(\tilde{r}^{(1)}(\lambda^{out}), \ldots, \tilde{r}^{(N)}(\lambda^{out}), \tilde{M}(\lambda^{out}))$ given by

$$\begin{aligned}
\theta_J^{(up,\nu)}\left(\lambda^{out}\right) &= \theta_J^{(low,\nu)}\left(\lambda^{out}\right) = g^{(\nu)}\left(X_J(\lambda^{out})\right) \\
\theta_j^{(up,\nu)}\left(\lambda^{out}\right) &= \max_{\iota\in\{up,low\}^N} f_j^{(\nu)}\Big(X_j\left(\lambda^{out}\right), \beta_{j+1}\left(\lambda^{out}\right)\theta_{j+1}^{(\iota_1,1)}\left(\lambda^{out}\right) - \Delta\tilde{M}_{j+1}^{[1]}\left(\lambda^{out}\right), \ldots, \\
&\qquad \beta_{j+1}\left(\lambda^{out}\right)\theta_{j+1}^{(\iota_N,N)}\left(\lambda^{out}\right) - \Delta\tilde{M}_{j+1}^{[N]}\left(\lambda^{out}\right)\Big), \\
\theta_j^{(low,\nu)}\left(\lambda^{out}\right) &= \sum_{n=1}^{N}\left(\left(\tilde{r}_j^{(\nu),[n]}\left(\lambda^{out}\right)\right)^\top \beta_{j+1}\left(\lambda^{out}\right)\right)_+ \theta_{j+1}^{(low,n)}\left(\lambda^{out}\right) \\
&\quad - \sum_{n=1}^{N}\left(\left(\tilde{r}_j^{(\nu),[n]}\left(\lambda^{out}\right)\right)^\top \beta_{j+1}\left(\lambda^{out}\right)\right)_- \theta_{j+1}^{(up,n)}\left(\lambda^{out}\right) \\
&\quad - \sum_{n=1}^{N}\left(\tilde{r}_j^{(\nu),[n]}\left(\lambda^{out}\right)\right)^\top \Delta\tilde{M}_{j+1}^{[n]} - f_j^{(\nu,\#)}\left(X_j\left(\lambda^{out}\right), \tilde{r}_j^{(\nu)}\left(\lambda^{out}\right)\right),
\end{aligned}$$

for $j = J-1, \ldots, 0$, $\nu = 1, \ldots, N$, along each outer path $\lambda^{out} = 1, \ldots, \Lambda^{out}$. If $\theta_0^{up}(\lambda^{out})$ and $\theta_0^{low}(\lambda^{out})$ are computed for every $\lambda^{out} = 1, \ldots, \Lambda^{out}$, we can apply the plain Monte Carlo estimator

$$\hat{Y}_0^{(\iota,\nu)} := \frac{1}{\Lambda^{out}} \sum_{\lambda^{out}=1}^{\Lambda^{out}} \theta_0^{(\iota,\nu)}(\lambda^{out}) \quad (1.59)$$

for every $\nu = 1, \ldots, N$ and $\iota \in \{up, low\}$ to obtain upper and lower bounds. Denoting by $\hat{\sigma}^{(up,\nu)}$ and $\hat{\sigma}^{(low,\nu)}$ the empirical standard deviations of $\hat{Y}_0^{(up,\nu)}$ and $\hat{Y}_0^{(low,\nu)}$ which are given by

$$\hat{\sigma}^{(\iota,\nu)} = \left(\frac{1}{\Lambda^{out}(\Lambda^{out}-1)} \sum_{\lambda^{out}=1}^{\Lambda^{out}} \left(\theta_0^{(\iota,\nu)}(\lambda^{out}) - \hat{Y}_0^{(\iota,\nu)}\right)^2\right)^{\frac{1}{2}} \quad (1.60)$$

for $\iota \in \{up, low\}$, we obtain asymptotic 95%-confidence intervals for $E[\theta_0^{(\iota,\nu)}]$ by

$$\left[\hat{Y}_0^{(\iota,\nu)} - 1.96\hat{\sigma}^{(\iota,\nu)}, \hat{Y}_0^{(\iota,\nu)} + 1.96\hat{\sigma}^{(\iota,\nu)}\right].$$

Combining these two confidence intervals, leads to the asymptotic 95%-confidence interval

$$\left[\hat{Y}_0^{(low,\nu)} - 1.96\hat{\sigma}^{(low,\nu)}, \hat{Y}_0^{(up,\nu)} + 1.96\hat{\sigma}^{(up,\nu)}\right]$$

for $Y_0^{(\nu)}$.

*Remark* 1.7.2. (i) We emphasize that the confidence intervals constructed above are conditional on the regression paths used to determine the coefficients for our approximation. Therefore, we have to enlarge the filtration $(\mathcal{F}_j)_{j=0,\dots,J}$ by the regression paths in order to ensure that our approximate solution is adapted. Denoting by $\Xi$ the random variable used to construct the input approximation, we pass from $\mathcal{F}_j$ to

$$\mathcal{F}_j^0 := \sigma\left(\mathcal{F}_j \cup \Xi\right),$$

for every $j = 0, \dots, J$.

(ii) In contrast to the computation of an approximate solution using LSMC, the construction of upper and lower bounds proceeds pathwise. Hence, the implementation of upper and lower bounds is amenable to massive parallelization. This especially turns out to be useful under memory constraints. For a more involved discussion of this topic, we refer to Gobet et al. (2016).

(iii) In case that the input approximate solution $\tilde{y}(x)$ is computed by the regression-now approach, the conditional expectation in (1.58) is in general not available in closed form, so that a subsampling approach is required to approximate it. This is in the spirit of Andersen and Broadie (2004), who proposed such an approach for the computation of upper bounds in the context of Bermudan option pricing. To this end, we simulate at every point in time $j$ and along each outer path $B(\lambda^{out})$ a set of $\Lambda^{in}$ independent copies $(B_{j+1}(\lambda^{out}, \lambda^{in}))_{\lambda^{in}=1,\dots,\Lambda^{in}}$ of $B_{j+1}$, to which we refer as "inner paths" from now on, see Figure 1.1. Along these inner paths, we can compute $X_{j+1}(\lambda^{out}, \lambda^{in}) := h_{j+1}(X_j(\lambda^{out}), B_{j+1}(\lambda^{in}))$ as well as approximations $\tilde{y}_{j+1}^{(\nu)}(X_{j+1}(\lambda^{out}, \lambda^{in}))$ and apply the conditionally unbiased estimator

$$\hat{E}_j\left[\beta_{j+1}\tilde{y}_{j+1}^{(\nu)}(X_{j+1})\right](\lambda^{out}) := \frac{1}{\Lambda^{in}}\sum_{\lambda^{in}=1}^{\Lambda^{in}} \beta_{j+1}(\lambda^{out}, \lambda^{in})\tilde{y}_{j+1}^{(\nu)}(X_{j+1}(\lambda^{out}, \lambda^{in})). \tag{1.61}$$

Replacing the conditional expectation in (1.58) by the unbiased estimator (1.61), we can compute an approximation $\tilde{M}^{[\nu]}$ of $M^{*,[\nu]}$ by

$$\tilde{M}_j^{[\nu]}(\lambda^{out}) = \sum_{i=0}^{j-1} \beta_{i+1}(\lambda^{out})\tilde{y}_{i+1}^{(\nu)}(X_{i+1}(\lambda^{out})) - \hat{E}_i\left[\beta_{i+1}\tilde{y}_{i+1}^{(\nu)}(X_{i+1})\right](\lambda^{out}), \tag{1.62}$$

for every $j = 0, \dots, J$, $\nu = 1, \dots, N$ and $\lambda^{out} = 1, \dots, \Lambda^{out}$. Note however, that, by Theorem 1.6.1, this subsimulation approach leads to an additional upward respectively downward bias in the upper and lower bounds.

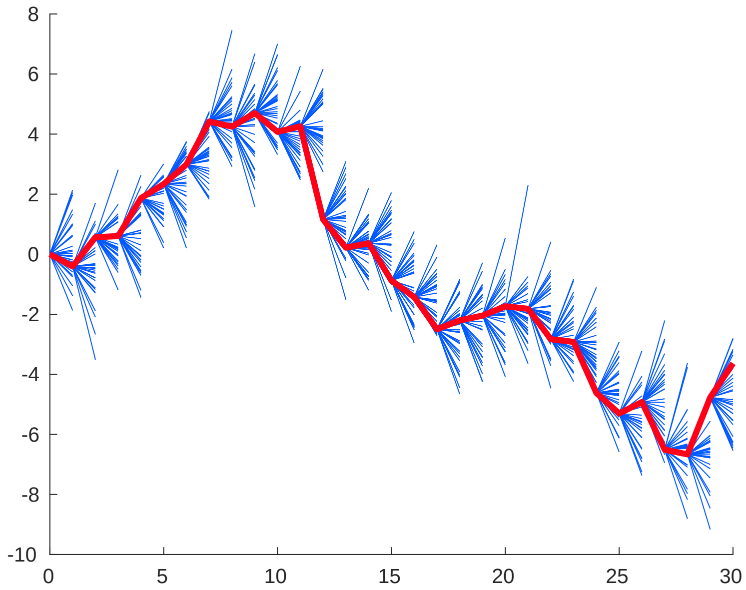


Figure 1.1: Illustration of the subsampling approach with $J = 30$.

(iv) By the law of large numbers, the additional bias from a subsampling approach vanishes when the number of subsamples tends to infinity. In applications, however, there is a trade-off between the reduction of the bias and the computational costs. As a consequence, the bias can still be substantial for a moderate number of samples and the application of variance reduction techniques is advisable. In their paper, Bender et al. (2017) propose to apply control variates building on the process $\beta$ as follows: Suppose that $\beta$ is of the form $\beta_j = (1, \tilde{\beta}_j)$, $j = 1, \dots, J$, for a process $\tilde{\beta}$ which takes values in $\mathbb{R}^{D-1}$ and for which closed form expressions of $E[\tilde{\beta}_j^{(d)}]$ and $E[\tilde{\beta}_j^{(d)} \tilde{\beta}_j^{(d')}]$, $d, d' = 1, \dots, D-1$, are available. Further, define $\boldsymbol{\beta}_j := (E[\tilde{\beta}_j^{(d)} \tilde{\beta}_j^{(d')}])_{d,d'=1,\dots,D-1}$ and denote by $\boldsymbol{\beta}_j^+$ the corresponding Moore-Penrose pseudoinverse. Moreover, we denote by $\bar{y}_j^{(\nu)}$, $q_j^{(\nu)}$ and $z_j^{[\nu]}$ the deterministic functions for which $\bar{y}_j^{(\nu)}(X_j) = Y_j^{(\nu)}$, $q_j^{(\nu)}(X_j) = E_j[Y_{j+1}^{(\nu)}]$ and $z_j^{[\nu]}(X_j) = E_j[\tilde{\beta}_{j+1} Y_{j+1}^{(\nu)}]$ holds and by $\tilde{y}_j^{(\nu)}$, $\tilde{q}_j^{(\nu)}$ and $\tilde{z}_j^{[\nu]}$ their respective approximations. Then, Bender et al. (2017) propose to replace the Monte Carlo estimator (1.61) by

$$\begin{aligned}\hat{E}_j^C[\tilde{y}_{j+1}^{(\nu)}(X_{j+1})](\lambda^{out}) \;&=\; E[\tilde{\beta}_{j+1}]^\top \boldsymbol{\beta}_{j+1}^+ \tilde{z}_j^{[\nu]}(X_j(\lambda^{out})) + \frac{1}{\Lambda^{in}} \sum_{\lambda^{in}=1}^{\Lambda^{in}} \Big( \tilde{y}_{j+1}^{(\nu)}(X_{j+1}(\lambda^{out}, \lambda^{in})) \\ &\quad - \tilde{\beta}_{j+1}^\top(\lambda^{out}, \lambda^{in}) \boldsymbol{\beta}_{j+1}^+ \tilde{z}_j^{[\nu]}(X_j(\lambda^{out})) \Big)\end{aligned}$$

and

$$\begin{aligned}\hat{E}_j^C[\tilde{\beta}_{j+1} \tilde{y}_{j+1}^{(\nu)}(X_{j+1})](\lambda^{out}) \;&=\; E[\tilde{\beta}_{j+1}] \tilde{q}_{j+1}^{(\nu)}(X_j(\lambda^{out})) + \boldsymbol{\beta}_{j+1} \boldsymbol{\beta}_{j+1}^+ \tilde{z}_j^{[\nu]}(X_j(\lambda^{out})) \\ &\quad + \frac{1}{\Lambda^{in}} \sum_{\lambda^{in}=1}^{\Lambda^{in}} \tilde{\beta}_{j+1}(\lambda^{out}, \lambda^{in}) \Big( \tilde{y}_{j+1}^{(\nu)}(X_{j+1}(\lambda^{out}, \lambda^{in})) \\ &\quad - \tilde{q}_j^{(\nu)}(X_j(\lambda^{out})) - \tilde{\beta}_{j+1}^\top(\lambda^{out}, \lambda^{in}) \boldsymbol{\beta}_{j+1}^+ \tilde{z}_j^{[\nu]}(X_j(\lambda^{out})) \Big)\end{aligned}$$

for every $j = 0, \ldots, J-1$ and $\nu = 1, \ldots, N$.

### 1.7.2 Numerical examples

We now apply the pathwise dynamic programming approach in two numerical examples, namely the problem of pricing options under negotiated collateral respectively uncertain volatility.

#### 1.7.2.1 Negotiated collateral

We first consider the problem of pricing under negotiated collateralization in the presence of funding costs as discussed in Example 1.4.2. To this end, let $0 = t_0 < t_1 < \ldots < t_J = T$ be an equidistant partition of the interval $[0, T]$. Moreover, recall that we are given a $d$-dimensional Brownian motion $W$ and that the dynamics of the risky assets $X = (X^{(1)}, \ldots, X^{(d)})$ are given by independent identically distributed Black-Scholes models, i.e.,

$$X_t^{(l)} = x_0 \exp\left\{\left(R^L - \frac{1}{2}\sigma^2\right) t + \sigma W_t^{(l)}\right\}, \quad l = 1, \ldots, d,$$

where $R^L \geq 0$ is the risk-free lending rate, $\sigma > 0$ is the assets volatility. Finally, we denote by $g$ the payoff of a European option written on the risky assets. Then, by Example 1.4.2, we end up with the following dynamic program:

$$\begin{aligned}
X_{j+1}^{(l)} &= X_j^{(l)} \exp\left\{\left(R^L - \frac{1}{2}\sigma^2\right)\Delta + \sigma \Delta W_{j+1}^{(l)}\right\}, \quad X_0^{(l)} = x_0, \quad l = 1, \ldots, d \\
Y_J^{(1)} &= -Y_J^{(2)} = g(X_J) \\
Z_j^{[\nu]} &= E_j\left[\frac{\Delta W_{j+1}}{\Delta} Y_{j+1}^{(\nu)}\right], \quad \nu = 1, 2 \\
Y_j^{(1)} &= E_j[Y_{j+1}^{(1)}] - R^L(1-\alpha)(E_j[Y_{j+1}^{(1)}] + E_j[Y_{j+1}^{(2)}])\Delta - R^C(\alpha E_j[Y_{j+1}^{(1)}] - (1-\alpha)E_j[Y_{j+1}^{(2)}])\Delta \\
&\quad + (R^B - R^L)\left((1-\alpha)(E_j[Y_{j+1}^{(1)}] + E_j[Y_{j+1}^{(2)}]) - \frac{1}{\sigma}\left(Z_j^{[1]}\right)^\top \mathbf{1}\right)_- \Delta \\
Y_j^{(2)} &= E_j[Y_{j+1}^{(2)}] - R^L\alpha(E_j[Y_{j+1}^{(1)}] + E_j[Y_{j+1}^{(2)}])\Delta + R^C(\alpha E_j[Y_{j+1}^{(1)}] - (1-\alpha)E_j[Y_{j+1}^{(2)}])\Delta \\
&\quad + (R^B - R^L)\left(\alpha(E_j[Y_{j+1}^{(1)}] + E_j[Y_{j+1}^{(2)}]) - \frac{1}{\sigma}\left(Z_j^{[2]}\right)^\top \mathbf{1}\right)_- \Delta, \qquad (1.63)
\end{aligned}$$

where $R^B$ and $R^C$ denote the risk-free borrowing rate respectively the collateralization rate and $\alpha \in [0, 1]$. Note that, in a slight abuse of notation, we here changed from time $t_j$ to the time index $j$ in the notation of the stock price models $X^{(l)}$.

Moreover, recall that the functions $F_j^{(1)}, F_j^{(2)} : \mathbb{R}^{2(1+d)} \to \mathbb{R}$ are given by

$$F_j^{(\nu)}(z_1, z_2) = z_\nu^{(1)} + H^{(\nu)}(z_1, z_2)\Delta,$$

for $z_\nu = (z_\nu^{(1)}, \ldots, z_\nu^{(1+d)}) \in \mathbb{R}^{1+d}$ and that the process $B$ is, as in Example 1.1.2, given by

$$B_j = \beta_j = \left(1, \frac{\Delta W_j^{(1)}}{\Delta}, \ldots, \frac{\Delta W_j^{(d)}}{\Delta}\right)^\top, \quad j = 1, \ldots, J.$$

As we have already seen in Example 1.4.2, the duality relation (1.24) reads

$$\begin{aligned}
&\left(r_j^{(\nu,*),[1]}\right)^\top E_j\left[\beta_{j+1}Y_{j+1}^{(1)}\right] + \left(r_j^{(\nu,*),[2]}\right)^\top E_j\left[\beta_{j+1}Y_{j+1}^{(2)}\right] \\
&= E_j\left[Y_{j+1}^{(\nu)}\right] - R^L a_\nu\left(E_j\left[Y_{j+1}^{(1)}\right] + E_j\left[Y_{j+1}^{(2)}\right]\right)\Delta + (-1)^\nu R^C\left(\alpha E_j\left[Y_{j+1}^{(1)}\right] - (1-\alpha)E_j\left[Y_{j+1}^{(2)}\right]\right)\Delta \\
&\qquad + (R^B - R^L)\left(a_\nu\left(E_j\left[Y_{j+1}^{(1)}\right] + E_j\left[Y_{j+1}^{(2)}\right]\right) - \frac{1}{\sigma}\left(E_j\left[\frac{\Delta W_{j+1}}{\Delta}Y_{j+1}^{(\nu)}\right]\right)^\top \mathbf{1}\right)_- \Delta,
\end{aligned}$$

for every $j = 0, \ldots, J-1$ and $\nu = 1, 2$, with solution

$$r_j^{(\nu,*)} = \begin{cases} u^{(\nu)}(R^L), & a_\nu\left(E_j\left[Y_{j+1}^{(1)}\right] + E_j\left[Y_{j+1}^{(2)}\right]\right) - \frac{1}{\sigma}\left(Z_j^{[\nu]}\right)^\top \mathbf{1} \geq 0 \\ u^{(\nu)}(R^B), & a_\nu\left(E_j\left[Y_{j+1}^{(1)}\right] + E_j\left[Y_{j+1}^{(2)}\right]\right) - \frac{1}{\sigma}\left(Z_j^{[\nu]}\right)^\top \mathbf{1} < 0. \end{cases}$$

Here, the functions $u^{(\nu)}(r)$ are defined by

$$u^{(1)}(r) = \begin{pmatrix} 1 - r(1-\alpha)\Delta - R^C\alpha\Delta \\ \frac{(r-R^L)\Delta}{\sigma}\cdot\mathbf{1} \\ (R^C - r)(1-\alpha)\Delta \\ 0\cdot\mathbf{1} \end{pmatrix} \quad \text{and} \quad u^{(2)}(r) = \begin{pmatrix} (R^C - r)\alpha\Delta \\ 0\cdot\mathbf{1} \\ 1 - r\alpha\Delta - R^C(1-\alpha)\Delta \\ \frac{(r-R^L)\Delta}{\sigma}\cdot\mathbf{1} \end{pmatrix}.$$

As a numerical example, we consider the valuation of a European call-spread option on the maximum of $d$ assets with maturity $T$ and payoff

$$g(x) = \left(\max_{l=1,\ldots,d} x^{(l)} - K_1\right)_+ - 2\left(\max_{l=1,\ldots,d} x^{(l)} - K_2\right)_+ .$$

Except for adding the collateralization scheme (and, hence, the coupling between the hedger's and counterparty's valuation), this is the same numerical example as in Bender et al. (2017) and we follow their parameter choices

$$(x_0, d, T, K_1, K_2, \sigma, R^L, R^B, R^C, \alpha) = (100, 5, 0.25, 95, 115, 0.2, 0.01, 0.06, 0.02, 0.5)$$

adding only the values of $\alpha$ and $R^C$. The choice $\alpha = 0.5$ implies that the posted collateral is given by the average of the two parties' value processes $Y^{(1)}$ and $-Y^{(2)}$. Note that, we have $R^B > R^C$ in this example, as this is the practically most relevant case. As discussed in Example 1.4.2, we observe that the system (1.63) fails the componentwise comparison principle with this choice of parameters and, thus, the coupled bounds (1.30) need to be applied.

To do this, we first compute input approximations with the regression-later approach. We run this algorithm with $\Lambda^{reg} = 1{,}000$ regression paths. At time $j+1$ (where $0 \leq j \leq J-1$) we apply the same 7 basis functions for both components (and thus skip the dependence on $\nu$), namely

$$\begin{aligned}
\eta_{j+1,1}(X_j, B_{j+1}) &= 1, \\
\eta_{j+1,l+1}(X_j, B_{j+1}) &= X_{j+1}^{(l)},\ l = 1, \ldots, 5,
\end{aligned}$$

and an approximation to $E_{j+1}[g(X_J)]$. Precisely, this basis function is defined in terms of an optimal $L$-point quantization $\sum_{\kappa=1}^L p_\kappa\,\delta_{z_\kappa}$ of a standard normal distribution by

$$\eta_{j+1,7}(X_j, B_{j+1})$$
$$= \sum_{l=1}^{5}\sum_{\kappa=1}^{L} p_\kappa \sqrt{\frac{T-t_j}{T-t_{j+1}}}\bar{g}\left(X_j^{(l)} e^{\left(R^L-\frac{1}{2}\sigma^2\right)(T-t_j)+\sigma z_\kappa\sqrt{T-t_j}}\right) e^{\frac{z_\kappa^2}{2}-\frac{(\sqrt{T-t_j}z_\kappa-\Delta W_{j+1}^{(l)})^2}{2(T-t_{j+1})}}$$
$$\times \prod_{l'\in\{1,\dots,5\}\setminus\{l\}} \mathcal{N}\left(\frac{1}{\sqrt{T-t_{j+1}}}\left(\sqrt{T-t_j}z_\kappa + \frac{\ln(X_j^{(l)})-\ln(X_j^{(l')})}{\sigma} - \Delta W_{j+1}^{(l')}\right)\right),$$

where, $\delta_z$ denotes the Dirac-measure in $z \in \mathbb{R}$ and $\bar{g}: \mathbb{R} \to \mathbb{R}$ is given by

$$\bar{g}(x) = (x-K_1)_+ - 2(x-K_2)_+.$$

As a trade-off between accuracy and computational time, we choose $L = 25$, but note that this basis function converges to $E_{j+1}[g(X_J)]$, as $L$ tends to infinity. For this choice of basis functions the one-step conditional expectations are available in closed form and can be expressed as $E_j[\eta_{j+1,k}(X_j, B_{j+1})] =: R_{j,k}^{(0)}(X_j)$ respectively $E_j[(\Delta W_{j+1}^{(l)}/\Delta)\eta_{j+1,k}(X_j, B_{j+1})] =: R_{j,k}^{(l)}(X_j)$, $l = 1, \dots, 5$, for deterministic functions $R_{j,k}^{(0)}$ and $R_{j,k}^{(l)}$. Note that, by a slight abuse of notation, the upper index on the functions $R_{j,k}^{(l)}$ does not correspond to the component of the process $Y$ as introduced in Section 1.7 but to the respective component of the process $\beta$, as we do not consider different basis functions for $Y^{(1)}$ and $Y^{(2)}$. Indeed, for the first six basis functions, we observe that

$$R_{j,1}^{(0)}(X_j) = 1,$$
$$R_{j,l+1}^{(0)}(X_j) = e^{R^L\Delta}X_j^{(l)}, \quad l = 1, \dots, 5,$$

respectively

$$R_{j,1}^{(l)}(X_j) = 0,$$
$$R_{j,k+1}^{(l)}(X_j) = \begin{cases} e^{R^L\Delta}\sigma X_j^{(l)}, & l = k \\ 0, & l \neq k \end{cases}$$

for $l, k = 1, \dots, 5$. A straightforward computation (for which we provide the details in Appendix A.3) shows that the respective conditional expectations for the seventh basis function are given by

$$R_{j,7}^{(0)}(X_j) = \sum_{l=1}^{5}\sum_{\kappa=1}^{L} p_\kappa \bar{g}\left(X_j^{(l)} e^{\left(R^L-\frac{1}{2}\sigma^2\right)(T-t_j)+\sigma z_\kappa\sqrt{T-t_j}}\right)$$
$$\times \prod_{l'\in\{1,\dots,5\}\setminus\{l\}} \mathcal{N}\left(z_\kappa + \frac{\ln(X_j^{(l)})-\ln(X_j^{(l')})}{\sigma\sqrt{T-t_j}}\right)$$

and

$$R_{j,7}^{(k)}(X_j) = \sum_{\substack{l=1,\\ l\neq k}}^{5}\sum_{\kappa=1}^{L} p_\kappa \bar{g}\left(X_j^{(l)} e^{\left(R^L-\frac{1}{2}\sigma^2\right)(T-t_j)+\sigma z_\kappa\sqrt{T-t_j}}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2(T-t_j)}\left(\sqrt{T-t_j}z_\kappa+\frac{\ln(X_j^{(l)})-\ln(X_j^{(l')})}{\sigma}\right)^2}$$
$$\times \prod_{l'\in\{1,\dots,5\}\setminus\{l,k\}} \mathcal{N}\left(z_\kappa + \frac{\ln(X_j^{(l)})-\ln(X_j^{(l')})}{\sigma\sqrt{T-t_j}}\right)$$

| $J$ | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|
| $\hat{Y}_0^{(up,1)}$ | 13.8424 (0.0018) | 13.8607 (0.0018) | 13.8677 (0.0019) | 13.8736 (0.0019) | 13.8797 (0.0021) |
| $\hat{Y}_0^{(low,1)}$ | 13.8409 (0.0017) | 13.8568 (0.0018) | 13.8610 (0.0019) | 13.8639 (0.0019) | 13.8673 (0.0019) |
| $-\hat{Y}_0^{(low,2)}$ | 13.2809 (0.0014) | 13.2597 (0.0015) | 13.2510 (0.0016) | 13.2471 (0.0016) | 13.2473 (0.0017) |
| $-\hat{Y}_0^{(up,2)}$ | 13.2798 (0.0014) | 13.2572 (0.0015) | 13.2466 (0.0016) | 13.2406 (0.0016) | 13.2389 (0.0017) |

Table 1.1: Upper and lower bounds with $\Lambda^{reg} = 10^3$ and $\Lambda^{out} = 10^4$ for different time discretizations. Standard deviations are given in brackets.

$$+\sum_{\kappa=1}^{L} p_\kappa \frac{z_\kappa}{\sqrt{T-t_j}} \bar{g}\left(X_j^{(k)} e^{\left(R^L - \frac{1}{2}\sigma^2\right)(T-t_j) + \sigma z_\kappa \sqrt{T-t_j}}\right)$$
$$\times \prod_{l' \in \{1,\ldots,5\}\setminus\{k\}} \mathcal{N}\left(z_\kappa + \frac{\ln(X_j^{(k)}) - \ln(X_j^{(l')})}{\sigma\sqrt{T-t_j}}\right)$$

Note that the conditional expectation $R_{j,7}^{(0)}$ corresponds to the price of a max-call option at time $t_j$ if the quantization is replaced by the respective integral over $\mathbb{R}$. This observation motivates the choice of the basis function $\eta_{j,7}$. We also apply these functions in order to initialize the regression algorithm at

$$\tilde{Y}_J = R_{J-1,7}^{(0)}(X_{J-1}) + \sum_{l=1}^{5} R_{J-1,7}^{(l)}(X_{J-1}))\Delta W_J^{(l)},$$

where the first term approximates the clean price (with zero interest rate) of the payoff at time $t_{J-1}$, while the second one approximates the corresponding Delta hedge on the interval $[t_{J-1}, t_J]$.

In order to compute the upper and lower bounds stated in Table 1.1, we simulate $\Lambda^{out} = 10^4$ outer paths and denote by $\hat{Y}_0^{(up,\nu)}$ and $\hat{Y}_0^{(low,\nu)}$ the Monte Carlo estimators for $E[\theta_0^{(up,\nu)}]$ and $E[\theta_0^{(low,\nu)}]$. Table 1.1 indicates that the quality of the upper and lower bounds is similar for $Y^{(1)}$ and $Y^{(2)}$. This is as expected since the recursions for $Y^{(2)}$ and $Y^{(1)}$ are rather symmetric. With regard to the asymptotic 95%-confidence intervals for $Y_0^{(1)}$ and $Y_0^{(2)}$, we observe two things: First, the relative length of these intervals is about 0.15% for all considered time discretizations, and 25 time steps are quite sufficient in this numerical example. Second, we see that the two parties' valuations differ by about 60 cent, corresponding to about 5 percent of the overall value. So our price bounds are clearly tight enough to distinguish between the two parties' pricing rules.

#### 1.7.2.2 Uncertain volatility model

In this section, we apply our numerical approach to the uncertain volatility model of Example 1.1.3. Let $0 = t_0 < t_1 < \ldots < t_J = T$ be an equidistant partition of the interval $[0, T]$, where $T \in \mathbb{R}_+$. Recall that for an adapted process $(\sigma_t)_t$, the price of the risky asset $X^\sigma$ at time $t$ is given by

$$X_t^\sigma = x_0 \exp\left\{\int_0^t \sigma_u dW_u - \frac{1}{2}\int_0^t \sigma_u^2 du\right\},$$

where $W$ is a Brownian motion. Furthermore, let $g$ be the payoff a European option written on the risky asset. Then, by Example 1.1.3, we consider the following one-dimensional dynamic program

$$\begin{aligned} X^{\hat\rho}_{j+1} &= X^{\hat\rho}_j \exp\left\{\hat\rho \Delta W_{j+1} - \frac{1}{2}\hat\rho^2\Delta\right\},\ X_0 = x_0 \in \mathbb{R}, \\ Y_J &= g\left(X^{\hat\rho}_J\right) \\ \Gamma_j &= E_j\left[\left(\frac{\Delta W^2_{j+1}}{\Delta^2} - \hat\rho\frac{\Delta W_{j+1}}{\Delta} - \frac{1}{\Delta}\right) Y_{j+1}\right] \\ Y_j &= E_j[Y_{j+1}] + \Delta \max_{s\in\{s_{low}, s_{up}\}} s\,\Gamma_j, \end{aligned} \tag{1.64}$$

for $j = J-1, \dots, 0$, where

$$s_\iota = \frac{1}{2}\left(\frac{\sigma^2_\iota}{\hat\rho^2} - 1\right)$$

for $\iota \in \{low, up\}$ and the process $B$ is given by

$$B_j = \left(1, \frac{\Delta W^2_{j+1}}{\Delta^2} - \hat\rho\frac{\Delta W_{j+1}}{\Delta} - \frac{1}{\Delta}, \Delta W_j\right),\ j = 1, \dots, J.$$

Note that $X^{\hat\rho}$ denotes the value process of the risky asset under the constant volatility $\hat\rho$ and that, in a slight abuse of notation, we again changed from time $t_j$ to the time index $j$ in the notation. We emphasize that the reference volatility $\hat\rho$ is a choice parameter in the discretization. The basic idea is to view the uncertain volatility model as a suitable correction of a Black-Scholes model with volatility $\hat\rho$.

As we have already seen in Example 1.1.3, the function $F_j : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$F_j(z) = z^{(1)} + \Delta \max_{s\in\{s_{low}, s_{up}\}} s z^{(2)},$$

in this example. Depending on the choice of the parameters $\sigma_{low}$, $\sigma_{up}$ and $\hat\rho$, this function may fail the monotonicity condition $(c)$ in Theorem 1.4.1. Indeed, in this setting, this condition boils down to the requirement that the prefactor

$$1 + s\left(\frac{\Delta W^2_{j+1}}{\Delta} - \hat\rho\Delta W_{j+1} - 1\right)$$

of $Y_{j+1}$ in equation (1.64) for $Y_j$ is $P$-almost surely non-negative for both of the feasible values of $s$,

$$s \in \left\{\frac{1}{2}\left(\frac{\sigma^2_{low}}{\hat\rho^2} - 1\right), \frac{1}{2}\left(\frac{\sigma^2_{up}}{\hat\rho^2} - 1\right)\right\}.$$

For $s > 1$, this requirement is violated for realizations of $\Delta W_{j+1}$ sufficiently close to zero, while for $s < 0$ violations occur for sufficiently negative realizations of the Brownian increment – and this violation also takes place if one truncates the Brownian increments at $\pm const.\sqrt{\Delta}$ with an arbitrarily large constant. Consequently, we arrive at the necessary condition $s \in [0, 1]$ for comparison to hold. From the possible values for $s$, we deduce that this condition is equivalent to $\hat\rho \in [\sigma_{up}/\sqrt{3}, \sigma_{low}]$. For $\sigma_{low} = 0.1$ and $\sigma_{up} = 0.2$, the numerical test case in Guyon and Henry-Labordère (2011) and Alanko and Avellaneda (2013), these two conditions cannot hold simultaneously, ruling out the possibility of a comparison principle.

By Appendix A.2, we conclude again that $F_j^{\#} = 0$ on the effective domain $D_{F^{\#}}^{(j,\cdot)} = \{1\} \times [s_{low}\Delta, s_{up}\Delta]$, so that the duality relation (1.24) reads as follows:

$$r_j^{*,[1]} E_j[Y_{j+1}] + r_j^{*,[2]} \Gamma_j = E_j[Y_{j+1}] + \Delta \max_{s \in \{s_{low}, s_{up}\}} s\,\Gamma_j, \quad j = 0, \dots, J-1. \tag{1.65}$$

A solution to (1.65) is given by

$$r_j^* = \begin{cases} (1, s_{up}\Delta), & \Gamma_j \geq 0 \\ (1, s_{low}\Delta), & \Gamma_j < 0 \end{cases} \tag{1.66}$$

for every $j = 0, \dots, J-1$.

As a numerical example, we consider a European call-spread option with strikes $K_1$ and $K_2$, i.e.,

$$g(x) = (x - K_1)_+ - (x - K_2)_+,$$

which is also studied in Guyon and Henry-Labordère (2011), Alanko and Avellaneda (2013), and Kharroubi et al. (2014). Following their setting, we choose the maturity $T = 1$, $K_1 = 90$, $K_2 = 110$ and $x_0 = 100$. The reference volatility $\hat{\rho}$ as well as the volatility bounds $\sigma_{low}$ and $\sigma_{up}$ are varied in our numerical experiments.

The input approximation is again computed by the regression-later variant of LSMC. We first simulate $\Lambda^{reg} = 10^5$ regression paths of the process $(B_j)_{j=1,\dots,J}$. For the evaluation of $(X_j^{\hat{\rho}})_{j=0,\dots,J}$ along the regression paths, we do not start all paths at $x_0$. Instead, we rather start $\Lambda^{reg}/200$ trajectories at each of the points $31, \dots, 230$. Since $X$ is a geometric Brownian motion under $\hat{\rho}$, it can be simulated exactly. Starting the regression paths at multiple points allows to reduce the instability of regression coefficients arising at early time points. See Rasmussen (2005) for a discussion of this stability problem and of the method of multiple starting points. For the empirical regression we choose 163 basis functions. For a given point in time $j+1$, the first three basis functions are given by

$$\begin{aligned} \eta_{j+1,1}(X_{j+1}^{\hat{\rho}}) &= 1, \\ \eta_{j+1,2}(X_{j+1}^{\hat{\rho}}) &= X_{j+1}^{\hat{\rho}}, \\ \eta_{j+1,3}(X_{j+1}^{\hat{\rho}}) &= E[g(X_J^{\hat{\rho}})|X_{j+1}^{\hat{\rho}}]. \end{aligned}$$

The third one is, thus, simply the Black-Scholes price (under $\hat{\rho}$) of the spread option $g$. For the remaining 160 basis functions, we also choose Black-Scholes prices of spread options with respective strikes $K^{(l)}$, $K^{(l+1)}$ and $K^{(l+2)}$ for $l = 1, \dots, 160$, where the numbers $K^{(1)}, \dots, K^{(162)}$ increase from 20.5 to 230.5. Precisely, these basis functions are given by

$$\eta_{j+1,k}(X_{j+1}^{\hat{\rho}}) = E[(X_J^{\hat{\rho}} - K^{(k-3)})_+|X_{j+1}^{\hat{\rho}}] - 2E[(X_J^{\hat{\rho}} - K^{(k-2)})_+|X_{j+1}^{\hat{\rho}}] + E[(X_J^{\hat{\rho}} - K^{(k-1)})_+|X_{j+1}^{\hat{\rho}}]$$

for $k = 4, \dots, 163$, where

$$E[(X_J^{\hat{\rho}} - K)_+|X_{j+1}^{\hat{\rho}}] = X_{j+1}^{\hat{\rho}} \mathcal{N}\left(d_+\left(t_J - t_{j+1}, X_{j+1}^{\hat{\rho}}, K\right)\right) - K\mathcal{N}\left(d_-\left(t_J - t_{j+1}, X_{j+1}^{\hat{\rho}}, K\right)\right).$$

Here, $d_+(\tau, x, K)$ and $d_-(\tau, x, K)$ are given by

$$d_\pm(\tau, x, K) = \frac{1}{\hat{\rho}\sqrt{\tau}}\left(\log\left(\frac{x}{K}\right) \pm \frac{1}{2}\hat{\rho}^2\tau\right).$$

Note that, in contrast to the previous example, the basis functions only depend on the value of the risky asset at the given time point using that

$$\eta_{j+1,k}(X_{j+1}^{\hat{\rho}}) = \eta_{j+1,k}\left(X_j^{\hat{\rho}} \exp\left\{\hat{\rho}\Delta W_{j+1} - \frac{1}{2}\hat{\rho}^2\Delta\right\}\right) = \tilde{\eta}_{j+1,k}(X_j^{\hat{\rho}}, B_{j+1}).$$

Under the given assumptions, these basis functions form a set of martingales, for which the conditional expectations are available in closed form. Hence, we have for $R_{j,k}^{(0)}(X_j^{\hat{\rho}}) := E_j[\eta_{j+1,k}(X_{j+1}^{\hat{\rho}})]$ that

$$\begin{aligned} &R_{j,1}^{(0)}(X_j^{\hat{\rho}}) = 1, \quad R_{j,2}^{(0)}(X_j^{\hat{\rho}}) = X_j^{\hat{\rho}}, \quad R_{j,3}^{(0)}(X_j^{\hat{\rho}}) = E[g(X_J^{\hat{\rho}})|X_j^{\hat{\rho}}], \\ &R_{j,k}^{(0)}(X_j^{\hat{\rho}}) = E[(X_J^{\hat{\rho}} - K^{(k-3)})_+|X_j^{\hat{\rho}}] - 2E[(X_J^{\hat{\rho}} - K^{(k-2)})_+|X_j^{\hat{\rho}}] + E[(X_J^{\hat{\rho}} - K^{(k-1)})_+|X_j^{\hat{\rho}}], \end{aligned}$$

for $k = 4, \ldots, 163$. For the one-step conditional expectations $R_{j,k}^{(1)}(X_j^{\hat{\rho}}) := E_j[\beta_{j+1}^{(2)}\eta_{j+1,k}(X_{j+1}^{\hat{\rho}})]$, we conclude by Appendix A.4 that

$$R_{j,k}^{(1)}(x) = E\left[\left(\frac{\Delta W_{j+1}^2}{\Delta^2} - \hat{\rho}\frac{\Delta W_{j+1}}{\Delta} - \frac{1}{\Delta}\right)\eta_{j+1,k}(X_{j+1}^{\hat{\rho}})\middle| X_j^{\hat{\rho}} = x\right] = \hat{\rho}^2 x^2 \frac{d^2}{dx^2}\eta_{j,k}(x)$$

holds. Consequently, these conditional expectations are given by

$$\begin{aligned} R_{j,1}^{(1)}(X_j^{\hat{\rho}}) &= R_{j,2}^{(1)}(X_j^{\hat{\rho}}) \;=\; 0, \\ R_{j,3}^{(1)}(X_j^{\hat{\rho}}) &= \frac{\hat{\rho}X_j^{\hat{\rho}}}{\sqrt{t_J - t_j}}\Big(\varphi(d_+(t_J - t_j, X_j^{\hat{\rho}}, K_1)) - \varphi(d_+(t_J - t_j, X_j^{\hat{\rho}}, K_2))\Big), \\ R_{j,k}^{(1)}(X_j^{\hat{\rho}}) &= \frac{\hat{\rho}X_j^{\hat{\rho}}}{\sqrt{t_J - t_j}}\Big(\varphi(d_+(t_J - t_j, X_j^{\hat{\rho}}, K^{(k-3)})) - 2\varphi(d_+(t_J - t_j, X_j^{\hat{\rho}}, K^{(k-2)})) \\ &\quad +\varphi(d_+(t_J - t_j, X_j^{\hat{\rho}}, K^{(k-1)}))\Big) \end{aligned}$$

for $k = 4, \ldots, 163$, where $\varphi$ denotes the density of a standard normal distribution. Hence, the one-step conditional expectations of the basis functions $\eta_{j+1,k}$, $k \geq 3$, after multiplication with the second derivative weight $\beta^{(2)}$ are essentially (differences of) Black-Scholes Gammas at time $j$. For the computation of upper and lower bounds, we simulate $\Lambda^{out} = 10^5$ outer paths. In contrast to the regression paths, we now take $x_0 = 100$ for the evaluation of $X^{\hat{\rho}}$ along each path. As before, we denote by $\hat{Y}_0^{up}$ and $\hat{Y}_0^{low}$ the corresponding estimators for $E[\theta_0^{up}]$ respectively $E[\theta_0^{low}]$.

We first consider the situation where $\sigma_{low} = 0.1$ and $\sigma_{up} = 0.2$. This example is by now a standard test case for Monte Carlo implementations of Hamilton-Jacobi-Bellman equations. The option price in the continuous time limit can be calculated in closed form and equals 11.2046, see Vanden (2006). Table 1.2 shows the approximated prices $\tilde{Y}_0 := \tilde{y}_0(x_0)$ as well as upper and lower bounds for $\hat{\rho} = 0.2/\sqrt{3} \approx 0.1155$ depending on the time discretization. This is the smallest choice of $\hat{\rho}$, for which the monotonicity condition in Theorem 1.4.1 can only be violated when the absolute values of the Brownian increments are large. The numerical results suggest convergence from below towards the continuous-time limit for finer time discretizations. This is intuitive in this example, since finer time discretizations allow for richer choices of the process $(\sigma_t)_{t\in[0,T]}$ in the maximization problem (1.6). We notice that the bounds are fairly tight (with, e.g., a relative width of 1.9% for the 95% confidence interval with $J = 21$ time discretization points), although the upper bound begins to deteriorate as $\tilde{Y}_0$ approaches its limiting value. The impact of increasing $\hat{\rho}$ to 0.15 (as proposed in Guyon and Henry-Labordère, 2011; Alanko and Avellaneda, 2013) is shown in Table 1.3. The

| J | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 |
|---|---|---|---|---|---|---|---|---|
| $\tilde{Y}_0$ | 10.8549 | 11.0494 | 11.1067 | 11.1336 | 11.1490 | 11.1590 | 11.1659 | 11.1713 |
| $\hat{Y}_0^{up}$ | 10.8604 (0.0001) | 11.0545 (0.0003) | 11.1145 (0.0006) | 11.1472 (0.0010) | 11.1754 (0.0024) | 11.2239 (0.0121) | 11.3172 (0.0312) | 11.4362 (0.0385) |
| $\hat{Y}_0^{low}$ | 10.8544 (0.0001) | 11.0497 (0.0003) | 11.1077 (0.0005) | 11.1341 (0.0003) | 11.1488 (0.0003) | 11.1596 (0.0007) | 11.1665 (0.0008) | 11.1700 (0.0010) |

Table 1.2: Approximated price as well as lower and upper bounds for $\hat{\rho} = 0.1155$ for different time discretizations. Standard deviations are given in brackets

| J | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 |
|---|---|---|---|---|---|---|---|---|
| $\tilde{Y}_0$ | 10.8164 | 10.9981 | 11.0677 | 11.1027 | 11.1241 | 11.1383 | 11.1485 | 11.1561 |
| $\hat{Y}_0^{up}$ | 10.8184 (0.0001) | 11.0041 (0.0001) | 11.0740 (0.0002) | 11.1124 (0.0004) | 11.1561 (0.0160) | 11.1786 (0.0019) | 11.2601 (0.0129) | 11.3691 (0.0143) |
| $\hat{Y}_0^{low}$ | 10.8164 (0.0001) | 10.9982 (0.0001) | 11.0678 (0.0001) | 11.1022 (0.0001) | 11.1230 (0.0002) | 11.1365 (0.0002) | 11.1444 (0.0008) | 11.1507 (0.0006) |

Table 1.3: Approximated price as well as lower and upper bounds for $\hat{\rho} = 0.15$ for different time discretizations. Standard deviations are given in brackets

relative width of the 95%-confidence interval is now about 1.3% for up to $J = 35$ time steps, but also the convergence to the continuous-time limit appears to be slower with this choice of $\hat{\rho}$.

Comparing Table 1.3 with the results in Alanko and Avellaneda (2013), we observe that their point estimates for $Y_0$ at time discretization levels $J = 10$ and $J = 20$ do not lie in our confidence intervals which are given by $[10.9985, 11.0043]$ and $[11.1025, 11.1131]$, indicating that their (regression-now) least-squares Monte Carlo estimator may still suffer from large variances (although they apply control variates). The dependence of the time discretization error on the choice of the reference volatility $\hat{\rho}$ is further illustrated in Table 1.4, which displays the mean and the standard deviation of 30 runs of the regression-later algorithm for different choices of $\hat{\rho}$ and up to 640 time steps. By and large, convergence is faster for smaller choices of $\hat{\rho}$, but the algorithm becomes unstable when the reference volatility is too small.

| J | 10 | 20 | 40 | 80 | 160 | 320 | 640 |
|---|---|---|---|---|---|---|---|
| $\hat{\rho} = 0.06$ | 84.6503 (45.3588) | $1.3012 \cdot 10^5$ $(3.6246 \cdot 10^5)$ | $8.6315 \cdot 10^{11}$ $(4.2492 \cdot 10^{12})$ | $6.4425 \cdot 10^{15}$ $(2.0720 \cdot 10^{16})$ | $3.1259 \cdot 10^{11}$ $(1.0129 \cdot 10^{12})$ | $5.5578 \cdot 10^{18}$ $(2.6571 \cdot 10^{18})$ | $8.1779 \cdot 10^{26}$ $(4.1892 \cdot 10^{27})$ |
| $\hat{\rho} = 0.08$ | 11.6966 (0.0022) | 12.0212 (0.4895) | 45.3317 (106.5248) | 11.5192 (0.2348) | 11.3627 (0.0241) | 160.9274 (819.6279) | 680.9364 $(3.5302 \cdot 10^3)$ |
| $\hat{\rho} = 0.1$ | 11.1546 (0.0002) | 11.1832 (0.0001) | 11.1946 (0.0001) | 11.2002 (0.0001) | 11.2030 (0.0001) | 11.2050 (0.0001) | 11.2061 (0.0001) |
| $\hat{\rho} = 0.15$ | 10.9981 (0.0002) | 11.1030 (0.0001) | 11.1563 (0.0002) | 11.1833 (0.0002) | 11.1969 (0.0002) | 11.2036 (0.0002) | 11.2070 (0.0002) |
| $\hat{\rho} = 0.2$ | 10.8006 (0.0003) | 10.9766 (0.0003) | 11.0846 (0.0002) | 11.1484 (0.0002) | 11.1837 (0.0003) | 11.2023 (0.0002) | 11.2116 (0.0002) |
| $\hat{\rho} = 0.5$ | 9.7087 (0.0001) | 9.9649 (0.0002) | 10.2326 (0.0003) | 10.5020 (0.0008) | 10.7548 (0.0012) | 10.9627 (0.0015) | 11.1103 (0.0018) |

Table 1.4: Mean of $L = 30$ simulations of $\tilde{Y}_0$ for different $\hat{\rho}$ and discretizations. Standard deviations are given in brackets.

In order to gain a better understanding of how the performance of the method depends on the input parameters, we also consider the case $\sigma_{low} = 0.3$ and $\sigma_{up} = 0.4$. Note that, for this choice, the comparison principle is in force if we choose $\hat{\rho} \in [0.4/\sqrt{3}, 0.3]$. Following Vanden (2006), the price of the European call-spread option in the continuous-time limit is 9.7906 in this case. We get qualitatively the same results as for the previous example, in the sense that convergence is faster for the smaller reference volatility and that the upper bound estimators begin to deteriorate as the time partition becomes too fine. However, quantitatively, the numerical results in Table 1.5 and 1.6 are better than in the previous example as the confidence intervals remain tight for finer time partitions. This is quite likely to be connected to the fact that the ratio between $\sigma_{up}$ and $\sigma_{low}$ is smaller in this second example.

| $J$ | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\tilde{Y}_0$ | 9.6169 | 9.7163 | 9.7435 | 9.7568 | 9.7642 | 9.7690 | 9.7721 | 9.7744 | 9.7761 | 9.7775 |
| $\hat{Y}_0^{up}$ | 9.6179 (0.0002) | 9.7192 (0.0002) | 9.7487 (0.0006) | 9.7643 (0.0012) | 9.7744 (0.0014) | 9.7999 (0.0069) | 9.8183 (0.0152) | 9.8262 (0.0170) | 9.8724 (0.0306) | 9.8703 (0.0274) |
| $\hat{Y}_0^{low}$ | 9.6105 (0.0002) | 9.7167 (0.0002) | 9.7434 (0.0003) | 9.7556 (0.0008) | 9.7645 (0.0003) | 9.7695 (0.0007) | 9.7718 (0.0003) | 9.7750 (0.0006) | 9.7761 (0.0006) | 9.7775 (0.0002) |

Table 1.5: Approximated price as well as lower and upper bounds for $\hat{\rho} = 0.23095$ for different time discretizations. Standard deviations are given in brackets

| $J$ | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 |
|---|---|---|---|---|---|---|---|---|
| $\tilde{Y}_0$ | 9.6064 | 9.6922 | 9.7244 | 9.7410 | 9.7509 | 9.7578 | 9.7625 | 9.7660 |
| $\hat{Y}_0^{up}$ | 9.6066 (0.0001) | 9.6929 (0.0001) | 9.7265 (0.0001) | 9.7452 (0.0002) | 9.7602 (0.0004) | 9.7774 (0.0012) | 9.8133 (0.0030) | 9.9123 (0.0077) |
| $\hat{Y}_0^{low}$ | 9.6062 (0.0001) | 9.6917 (0.0001) | 9.7239 (0.0001) | 9.7403 (0.0001) | 9.7504 (0.0001) | 9.7570 (0.0001) | 9.7614 (0.0001) | 9.7648 (0.0001) |

Table 1.6: Approximated price as well as lower and upper bounds for $\hat{\rho} = 0.35$ for different time discretizations. Standard deviations are given in brackets

Finally, we demonstrate the advantage of the regression-later approach over the regression-now variant of LSMC in this example. To this end, we compute the respective approximations of the process $(\Gamma_j)_{j=0,\dots,J}$ for varying time steps and different choices of the parameters $\sigma_{low}$, $\sigma_{up}$ and $\hat{\rho}$. We run the regression with the basis functions described above and $\Lambda^{reg} = 10^5$ regression paths for the regression-later algorithm and $\Lambda^{reg} = 10^7$ paths for the regression-now approach. The resulting approximations are compared with the closed-form expression for $\Gamma$ derived in Vanden (2006) for the continuous-time problem. The approximations as well as the true process are plotted as functions on the real line for three different time points and are presented in Figures 1.2 to 1.7. We emphasize that the scales on the $y$-axis of the plots differ for the different time points.

We first consider the case where $J = 30$, $\sigma_{low} = 0.3$, $\sigma_{up} = 0.4$ and $\hat{\rho} = 0.23095$. The resulting approximations are demonstrated in Figures 1.2 to 1.4 for the time points $t \in \{0.1, 0.5, 0.9\}$. We observe that the regression-now approach provides a less suitable approximation of the true $\Gamma_t$ for all time points, as it is much more oscillating. Recalling that the approximate optimal control depends on the sign of $\tilde{\Gamma}_t$, these oscillations make it more difficult to find a good approximation of the optimal control and the Doob martingale. This, in turn, results in worse bounds compared to those presented above. Going backwards in time, we observe that the approximation becomes worse. This is due to the propagation of the simulation error induced by this approach in every time step. Hence, for the regression-now approach even more than $10^7$ regression paths would be required to reduce this error and, consequently, to obtain a better approximation, see also Bender and Steiner (2012) for an overview of this topic. In contrast, the regression-later approach provides a good approximation for all time points with only $10^5$ regression paths, demonstrating the variance reduction effect of this approach.

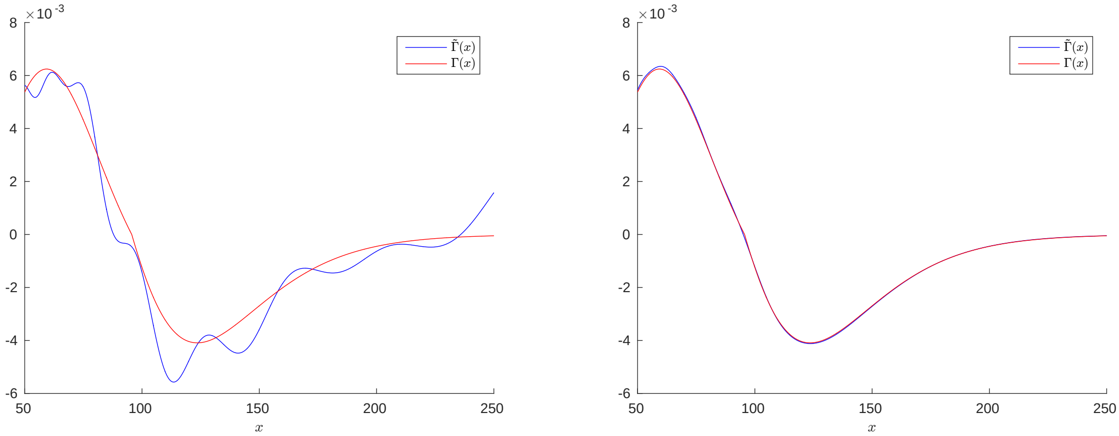


Figure 1.2: Plots of $\tilde{\Gamma}_t$ obtained from the regression-now (left) and the regression-later approach (right) as well as of the true $\Gamma_t$ derived in Vanden (2006) for $\sigma_{low} = 0.3$, $\sigma_{up} = 0.4$ and $\hat{\rho} = 0.23095$ at timepoint $t = 0.1$.

For the sake of completeness, we also consider the situation when $\sigma_{low} = 0.1$, $\sigma_{up} = 0.2$ and $\hat{\rho} = 0.1155$ with $J = 21$ time steps. The resulting approximations for the time points $t \in \{2/21, 11/21, 19/21\}$ are presented in Figures 1.5 to 1.7. All in all, the observations are similar to the first case, i.e. while the regression-later approach provides a good approximation for all time points, the approximation stemming from the regression-now approach suffers from the simulation error. However, the approximations from the regression-now approach appear to become worse for this choice of the parameters as the effect of oscillation is more pronounced.

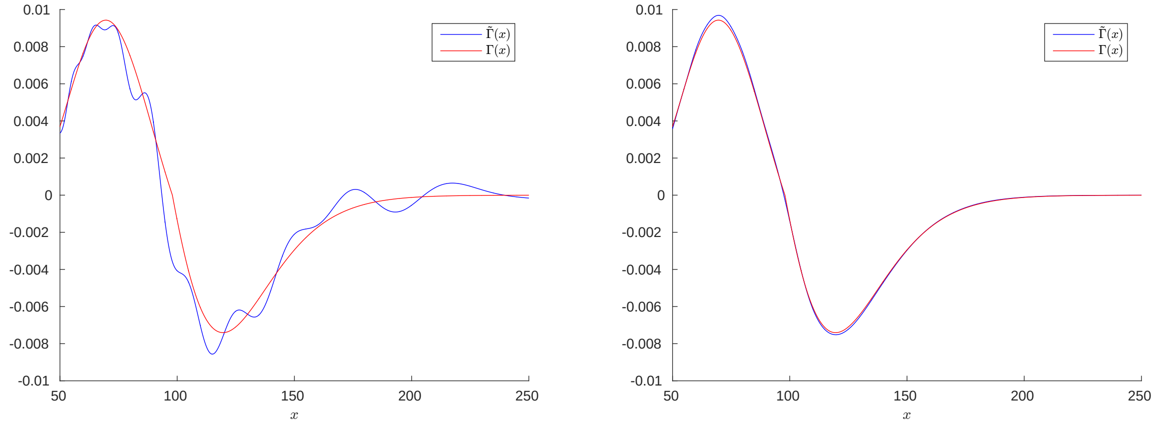


Figure 1.3: Plots of $\tilde{\Gamma}_t$ obtained from the regression-now (left) and the regression-later approach (right) as well as of the true $\Gamma_t$ derived in Vanden (2006) for $\sigma_{low} = 0.3$, $\sigma_{up} = 0.4$ and $\hat{\rho} = 0.23095$ at timepoint $t = 0.5$.

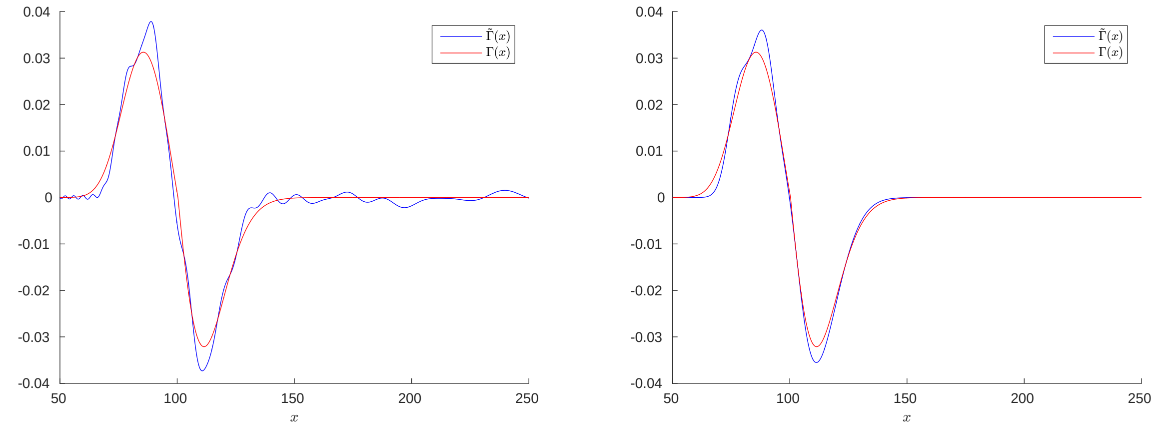


Figure 1.4: Plots of $\tilde{\Gamma}_t$ obtained from the regression-now (left) and the regression-later approach (right) as well as of the true $\Gamma_t$ derived in Vanden (2006) for $\sigma_{low} = 0.3$, $\sigma_{up} = 0.4$ and $\hat{\rho} = 0.23095$ at timepoint $t = 0.9$.

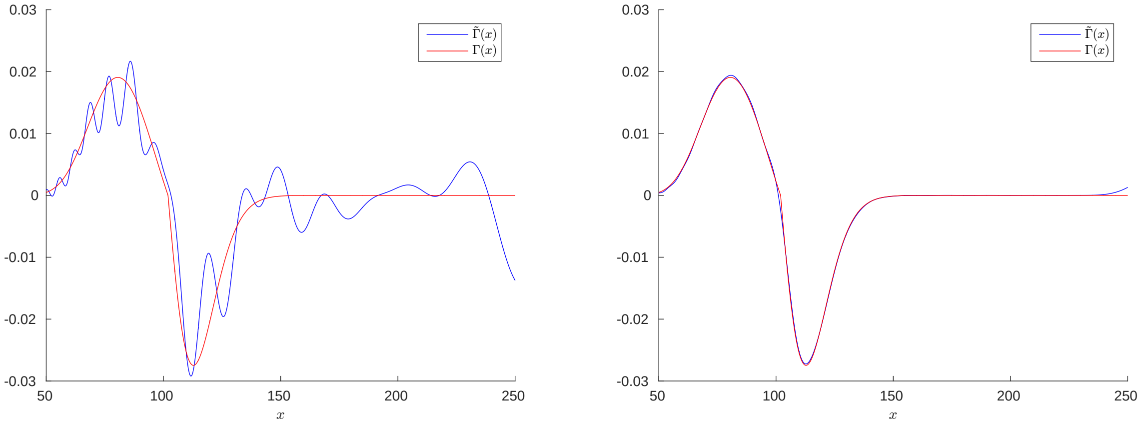


Figure 1.5: Plots of $\tilde{\Gamma}_t$ obtained from the regression-now (left) and the regression-later approach (right) as well as of the true $\Gamma_t$ derived in Vanden (2006) for $\sigma_{low} = 0.1$, $\sigma_{up} = 0.2$ and $\hat{\rho} = 0.1155$ at timepoint $t = 2/21$.

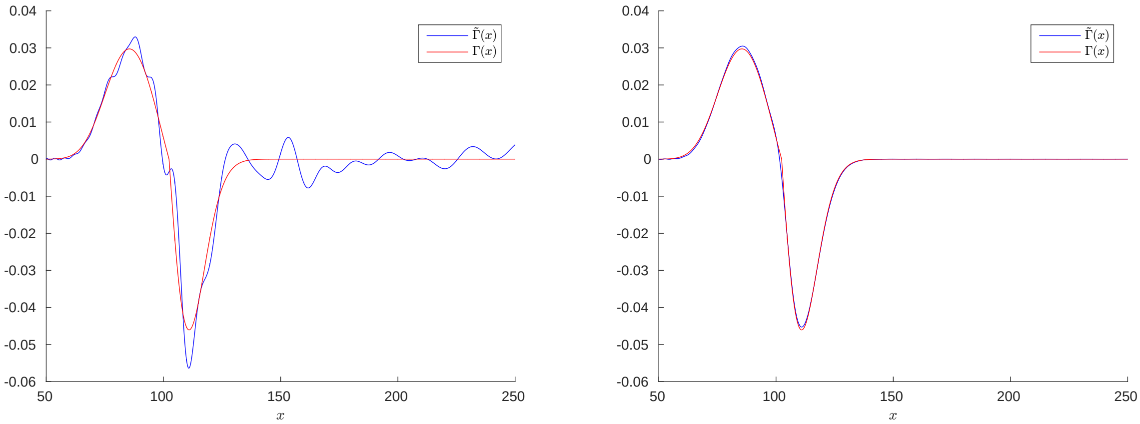


Figure 1.6: Plots of $\tilde{\Gamma}_t$ obtained from the regression-now (left) and the regression-later approach (right) as well as of the true $\Gamma_t$ derived in Vanden (2006) for $\sigma_{low} = 0.1$, $\sigma_{up} = 0.2$ and $\hat{\rho} = 0.1155$ at timepoint $t = 11/21$.

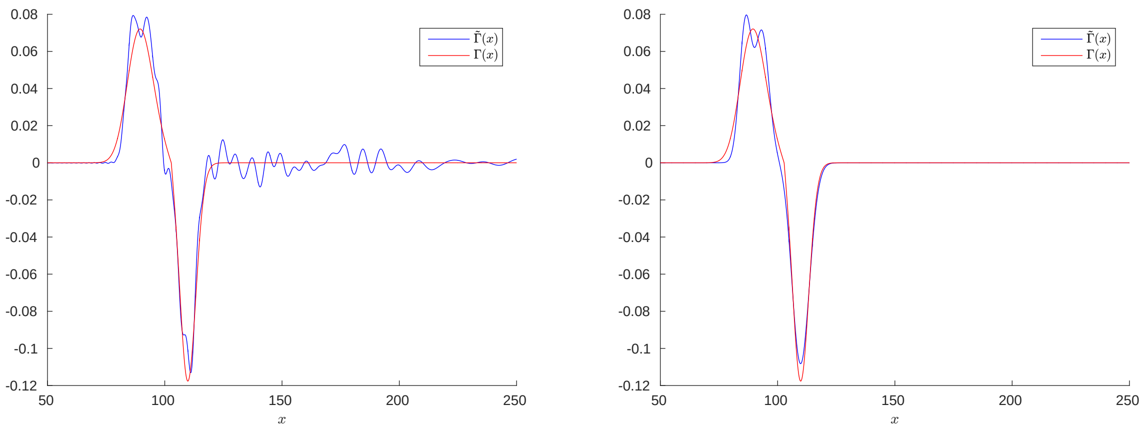


Figure 1.7: Plots of $\tilde{\Gamma}_t$ obtained from the regression-now (left) and the regression-later approach (right) as well as of the true $\Gamma_t$ derived in Vanden (2006) for $\sigma_{low} = 0.1$, $\sigma_{up} = 0.2$ and $\hat{\rho} = 0.1155$ at timepoint $t = 19/21$.

# Chapter 2

# Concave-convex stochastic dynamic programs

In this chapter, we provide a further generalization of the results in Chapter 1 by passing from the convex to a concave-convex structure. This allows us to consider a wider class of applications. It turns out that the constructions of upper and lower bounds are robust in the sense that the results from Chapter 1 can be transferred in a straightforward way to this new framework. Hence, we proceed similar to Chapter 1 by first assuming that a comparison principle holds and then, in a second step, we relax this assumption and consider the general situation. Section 2.1 introduces the assumptions and notations which are required to capture the additional concave part of the dynamic programming equation. Similar to the first chapter, we assume in Section 2.2 that a comparison principle holds and show that upper and lower bounds for concave-convex dynamic programs can be derived by a suitable composition of the bounds for the respective concave and convex problems. In Section 2.3, we show that, in some cases, the solution of the dynamic program is related to a stochastic two-player zero-sum game. Finally, we apply the information relaxation approach of Brown et al. (2010) to this game and show that we end up with the bounds proposed in Section 2.2 if a special class of penalties is considered. In Section 2.4 the comparison principle is relaxed. We first provide a version of Theorem 1.4.1 which states sufficient conditions for the comparison principle to hold. Then, we show that the main ideas and results from Section 1.5 can be transferred immediately to this new setting. In Section 2.5, we consider the problem of pricing a swap under default risk as a numerical example.

## 2.1 Setup

Throughout this section, we consider the following concave-convex dynamic programming equation

$$\begin{aligned} Y_J &= \xi, \\ Y_j &= G_j(E_j[\beta_{j+1}Y_{j+1}], F_j(E_j[\beta_{j+1}Y_{j+1}])), \quad j = J-1, \dots, 0 \end{aligned} \tag{2.1}$$

on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_j)_{j=0,\dots J}, P)$ in discrete time. As before, we denote by $E_j[\cdot]$ the conditional expectation with respect to $\mathcal{F}_j$. In what follows, we rely on the following assumptions:

**Assumption 2.1.1.** *(i) The functions $F_j$, $j = 0, \dots, J-1$, the process $\beta$ and the terminal condition $\xi$ satisfy the Assumptions 1.2.1 (i),(ii),(iv) and (v) with $N = 1$.*

*(ii) For every $j = 0, \dots, J-1$, $G_j : \Omega \times \mathbb{R}^D \times \mathbb{R} \to \mathbb{R}$ is measurable and, for every $(z, y) \in \mathbb{R}^D \times \mathbb{R}$, the process $(j, \omega) \mapsto G_j(\omega, z, y)$ is adapted.*

*(iii) The map $(z, y) \mapsto G_j(\omega, z, y)$ is concave in $(z, y)$ and non-decreasing in $y$ for every $j = 0, \dots, J-1$ and $\omega \in \Omega$.*

*(iv) $G$ and $F$ are of polynomial growth in $(z, y)$ in the following sense: There exist a constant $q \geq 0$ and a non-negative adapted process $(\alpha_j)$ such that for all $(z, y) \in \mathbb{R}^{D+1}$ and $j = 0, \dots, J-1$*

$$|G_j(z, y)| + |F_j(z)| \leq \alpha_j(1 + \|z\|^q + |y|^q), \quad P\text{-a.s.},$$

*and $\alpha_j \in L_j^{\infty-}(\mathbb{R})$.*

**Lemma 2.1.2.** *Under Assumption 2.1.1 the $P$-almost surely unique solution $Y$ to (2.1) is an element of $L_{ad}^{\infty-}(\mathbb{R})$.*

We skip the proof of this lemma as it follows by essentially the same lines of reasoning applied in the proof of Lemma 1.2.2.

*Example* 2.1.3. (i) As a first example, we focus on the Bermudan option pricing problem but with the additional twist that both, the holder and the issuer, of the option have the right to exercise the option prior to maturity. This kind of options are sometimes referred to as Israeli options and arise, e.g., in the context of convertible bonds. Depending on which party exercises the option first, the holder of the option receives either the amount $L_j$ if he exercises first or $H_j$ if the issuer cancels the option first. If both decide to exercise their right at the same time, the holder receives the amount $H_j$. Here, the processes $(L_j)_{j=0,\dots,J}$ and $(H_j)_{j=0,\dots,J}$ are adapted to a filtration $(\mathcal{F}_j)_{j=0,\dots,J}$ and satisfy $0 \leq L_j \leq H_j$ for all $j = 0, \dots, J-1$ and $L_J = H_J$. Since the issuer has to pay the larger amount $H_j$ it is his intention to minimize the expected payoff of the option while the holder of the option tries to maximize it. Hence, the value of the option is given by

$$\begin{aligned} Y_0 &= \underset{\tau \in \mathcal{S}_0}{\operatorname{esssup}}\, \underset{\sigma \in \mathcal{S}_0}{\operatorname{essinf}}\, E\left[\sum_{i=0}^{J} L_i \mathbb{1}_{\{\tau = i < \sigma\}} + H_i \mathbb{1}_{\{\sigma = i \leq \tau\}}\right] \\ &= \underset{\sigma \in \mathcal{S}_0}{\operatorname{essinf}}\, \underset{\tau \in \mathcal{S}_0}{\operatorname{esssup}}\, E\left[\sum_{i=0}^{J} L_i \mathbb{1}_{\{\tau = i < \sigma\}} + H_i \mathbb{1}_{\{\sigma = i \leq \tau\}}\right], \end{aligned} \tag{2.2}$$

where $\mathcal{S}_0$ denotes the set of stopping times with values in $\{0, \dots, J\}$. As it is shown e.g. in Neveu (1975), the value of the option can be represented by the dynamic program

$$\begin{aligned} Y_J &= L_J \\ Y_j &= \min\left\{H_j, \max\left\{L_j, E_j\left[Y_{j+1}\right]\right\}\right\}, \end{aligned} \tag{2.3}$$

where $E_j[\cdot]$ denotes the conditional expectation with respect to $\mathcal{F}_j$. Choosing $G_j(z, y) = \min\{H_j, y\}$ and $F_j(z) = \max\{L_j, z\}$, we observe that (2.3) is of the form (2.1) with $D = 1$ and $\beta \equiv 1$.

(ii) We consider the problem of pricing under credit risk, which is a well-known example in the financial literature, see e.g. Brigo et al. (2013) or Crépey et al. (2014). To this end, suppose that two parties, to which we refer as investor and counterparty, trade several derivatives, which all have the same maturity $T$. Since this is a non-linear pricing problem, the hedging prices for this basket of options is different for the investor and his counterparty. Hence, we focus in the following on the investor's view and denote by $\xi$ the possibly negative payoff of this basket which the investor receives at maturity. The random variable $\xi$ is assumed to be measurable with respect to the market's reference filtration $(\mathcal{F}_t)_{0\leq t\leq T}$. In what follows, we consider the situation of bilateral counterparty risk, so that both parties may default. For simplicity, we rule out the possibility of simultaneous default, so that it is either the investor or the counterparty party that defaults. As an additional difficulty, we also include the funding costs for the investor in our problem. From equations (2.14) and (3.8) in Crépey et al. (2013), which correspond to a CSA recovery scheme with no collateralization, we obtain that the value of this basket is given by the backward stochastic differential equation

$$Y_t = E_t\left[\xi - \int_t^T f(s, Y_s)\, ds\right], \tag{2.4}$$

where $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is defined by

$$f(s, y) = (r_s + \gamma_s(1 - 2p_s)(1 - \mathfrak{r}) + \bar{\lambda})y - (\gamma_s(1 - 3p_s)(1 - \mathfrak{r}) + \bar{\lambda} - \lambda)y_+. \tag{2.5}$$

Here the adapted stochastic process $r$ describes the risk-less short rate and $\gamma_t$ is the rate at which default of either side occurs at time $t$. Further, we denote by $p_t$ the conditional probability that it is the counterparty who defaults, if default occurs at time $t$. Accordingly, $1-p_t$ is the conditional probability that the investor defaults, since we ruled out the possibility of simultaneous default. Moreover, $\mathfrak{r}$ is associated with partial recovery and we assume for simplicity that the free parameters $\rho, \bar{\rho}$ and $\mathfrak{r}$ in Crépey et al. (2013) satisfy $\mathfrak{r} = \rho = \bar{\rho}$. Finally, the constants $\lambda$ and $\bar{\lambda}$ reflect the costs of external lending and borrowing.

Discretizing (2.4) over an equidistant time grid $0 = t_0 < t_1 < \ldots < t_J = T$ with increment $\Delta$ we end up with the dynamic programming equation

$$\begin{aligned} Y_J &= \xi \\ Y_j &= (1 - \Delta(r_{t_j} + \gamma_{t_j}(1 - \mathfrak{r})(1 - 2p_{t_j}) + \bar{\lambda}))E_j[Y_{j+1}] \\ &\quad +\Delta(\gamma_{t_j}(1 - \mathfrak{r})(1 - 3p_{t_j}) + \bar{\lambda} - \lambda)E_j[Y_{j+1}]_+, \quad j = J-1, \ldots, 0, \end{aligned} \tag{2.6}$$

which is of the form (2.1) with $D = 1$ and $\beta \equiv 1$. Indeed, denote by

$$g_j = 1 - \Delta(r_{t_j} + \gamma_{t_j}(1 - \mathfrak{r})(1 - 2p_{t_j}) + \bar{\lambda})$$

and

$$h_j = \Delta(\gamma_{t_j}(1 - \mathfrak{r})(1 - 3p_{t_j}) + \bar{\lambda} - \lambda)$$

the factors in the first and second summand of (2.6) and let $G_j(z, y) : \mathbb{R}^2 \to \mathbb{R}$ and $F_j(z) : \mathbb{R} \to \mathbb{R}$ be given by

$$G_j(z, y) = g_j z + (h_j)_+ y - (h_j)_- z_+,$$

respectively

$$F_j(z) = z_+.$$

Then, the functions $G_j(z, y)$ and $F_j(z)$ are concave respectively convex and the recursion (2.6) can be expressed in terms of $G_j$ and $F_j$, i.e.

$$Y_j = G_j(E_j[Y_{j+1}], F_j(E_j[Y_{j+1}])), \quad j = 0, \ldots, J-1. \tag{2.7}$$

Note that, depending on the choice of the parameters and stochastic processes, $h_j$ may change its sign, so that the dynamic program (2.6) can be both, convex and concave. Hence, the convex structure of Chapter 1 is insufficient to capture this pricing problem and the concave-convex structure (2.7) is required.

For our further considerations, we require an analogue to the set $\mathcal{A}_j^F := \mathcal{A}_j^{F^{(1)}}$ introduced in Section 1.2 for the function $G_j$. Therefore, we recall that the concave conjugate of $G_j$ is, for every $\omega \in \Omega$, given by

$$G_j^{\#}\left(\omega, v^{(1)}, v^{(0)}\right) := \inf_{(z,y)\in\mathbb{R}^{D+1}} \left(\left(v^{(1)}\right)^\top z + v^{(0)}y - G_j(\omega, z, y)\right), \tag{2.8}$$

with effective domain

$$D_{G^{\#}}^{(j,\omega)} = \left\{ \left(v^{(1)}, v^{(0)}\right) \in \mathbb{R}^{D+1} \middle| \, G_j^{\#}\left(\omega, v^{(1)}, v^{(0)}\right) > -\infty \right\}.$$

Note that, similar to the convex case, we can apply Theorem 12.2 in Rockafellar (1970), since $G_j$ is concave and closed, and thus obtain $G_j^{\#\#} = G_j$ for every $j = 0, \ldots, J-1$ and $\omega \in \Omega$. Hence, for every $j = 0, \ldots, J-1$, $\omega \in \Omega$ and $(z, y) \in \mathbb{R}^{D+1}$ it holds that

$$G_j(\omega, z, y) = \inf_{(v^{(1)}, v^{(0)})\in\mathbb{R}^{D+1}} \left(v^{(1)}\right)^\top z + v^{(0)}y - G_j^{\#}\left(\omega, v^{(1)}, v^{(0)}\right). \tag{2.9}$$

Then, we denote the set of admissible controls for the function $G$ by

$$\begin{aligned} \mathcal{A}_j^G \quad = \quad & \left\{ \left(\rho_i^{(1)}, \rho_i^{(0)}\right)_{i=j,\ldots,J-1} \middle| \left(\rho_i^{(1)}, \rho_i^{(0)}\right) \in L_i^{\infty-}\left(\mathbb{R}^{D+1}\right), \right. \\ & \left. G_i^{\#}\left(\rho_i^{(1)}, \rho_i^{(0)}\right) \in L^{\infty-}(\mathbb{R}), \; i = j, \ldots, J-1 \right\} \end{aligned}$$

for every $j = 0, \ldots, J-1$. Applying exactly the same arguments as in Section 1.3, we obtain that $G_i^{\#}(\rho_i^{(1)}, \rho_i^{(0)})$ is $\mathcal{F}_i$-measurable and $G_i^{\#}(\rho_i^{(1)}, \rho_i^{(0)}) > -\infty$ for all admissible controls $(\rho^{(1)}, \rho^{(0)}) \in \mathcal{A}_j^G$ and $i = j, \ldots, J-1$. Moreover, we have that $\rho_i^{(0)} \geq 0$ $P$-a.s. as the map $(z, y) \mapsto G_i(z, y)$ is non-decreasing by assumption for all $i = j, \ldots, J-1$.

## 2.2 The monotone case

As in Chapter 1, we first suppose that a comparison principle holds:

**Assumption 2.2.1.** *For every supersolution $Y^{up}$ and every subsolution $Y^{low}$ to the dynamic program* (2.1) *it holds that*

$$Y_j^{up} \geq Y_j^{low}, \quad P\text{-a.s.},$$

*for every $j = 0, \ldots, J$.*

The main idea in the construction of upper and lower bounds to (2.1) is to consider convex and concave dynamic programs separately and to combine the respective bounds in a suitable way. Hence, the upper bound recursion builds on a linearization of the concave function $G_j$ using Fenchel duality and subtracting a martingale increment in the convex function $F_j$. For the lower bound recursion, we proceed the other way round, i.e. we linearize the convex part in (2.1) and subtract a martingale increment in the concave part. This leads us to the following recursions:

Let $j \in \{0, \ldots, J-1\}$. Then, for a given martingale $M \in \mathcal{M}_D$ and admissible controls $(\rho^{(1)}, \rho^{(0)}) \in \mathcal{A}_j^G$ respectively $r \in \mathcal{A}_j^F$, we define the typically non-adapted processes $\Theta^{up} := \Theta^{up}(\rho^{(1)}, \rho^{(0)}, M)$ and $\Theta^{low} := \Theta^{low}(r, M)$ by

$$\Theta_J^{up} = \Theta_J^{low} = \xi$$

$$\Theta_i^{up} = \left(\rho_i^{(1)}\right)^\top \beta_{i+1}\Theta_{i+1}^{up} - \left(\rho_i^{(1)}\right)^\top \Delta M_{i+1} + \rho_i^{(0)} F_i\left(\beta_{i+1}\Theta_{i+1}^{up} - \Delta M_{i+1}\right) - G_i^{\#}\left(\rho_i^{(1)}, \rho_i^{(0)}\right),$$
$$\Theta_i^{low} = G_i\left(\beta_{i+1}\Theta_{i+1}^{low} - \Delta M_{i+1}, r_i^\top \beta_{i+1}\Theta_{i+1}^{low} - r_i^\top \Delta M_{i+1} - F_i^{\#}(r_i)\right) \tag{2.10}$$

for $i = J-1, \ldots, j$.

**Lemma 2.2.2.** *Suppose Assumption 2.1.1. Then, for every $j \in \{0, \ldots, J-1\}$, $M \in \mathcal{M}_D$, $(\rho^{(1)}, \rho^{(0)}) \in \mathcal{A}_j^G$, and $r \in \mathcal{A}_j^F$, the processes $\Theta^{up}(\rho^{(1)}, \rho^{(0)}, M)$ and $\Theta^{low}(r, M)$ given by (2.10) satisfy $\Theta_i^{up}(\rho^{(1)}, \rho^{(0)}, M) \in L^{\infty-}(\mathbb{R})$ respectively $\Theta_i^{low}(r, M) \in L^{\infty-}(\mathbb{R})$ for all $i = j, \ldots, J$.*

As the proof of this result follows by a straightforward modification of the proofs of Lemma 1.2.2 and Lemma 1.3.3, we omit the details.

Taking admissible controls $(\rho^{(1)}, \rho^{(0)}) \in \mathcal{A}_0^G$ and $r \in \mathcal{A}_0^F$ as well as a martingale $M \in \mathcal{M}_D$, we can define the processes $Y^{up}$ and $Y^{low}$ by $Y_j^{up} := E_j[\Theta_j^{up}(\rho^{(1)}, \rho^{(0)}, M)]$ and $Y_j^{low} := E_j[\Theta_j^{low}(r, M)]$, $j = 0, \ldots, J$. As in Section 1.3, these processes define super- and subsolutions to (2.1). We first show, that $Y^{up}$ is a supersolution. To this end, we apply Jensen's inequality in combination with the non-negativity of $\rho_j^{(0)}$ to obtain

$$\begin{aligned}
Y_j^{up} &= E_j\left[\Theta_j^{up}\right] \\
&= E_j\left[\left(\rho_j^{(1)}\right)^\top \beta_{j+1}\Theta_{j+1}^{up} - \left(\rho_j^{(1)}\right)^\top \Delta M_{j+1} + \rho_j^{(0)} F_j\left(\beta_{j+1}\Theta_{j+1}^{up} - \Delta M_{j+1}\right) - G_j^{\#}\left(\rho_j^{(1)}, \rho_j^{(0)}\right)\right] \\
&\geq \left(\rho_j^{(1)}\right)^\top E_j\left[\beta_{j+1}\Theta_{j+1}^{up} - \Delta M_{j+1}\right] + \rho_j^{(0)} F_j\left(E_j\left[\beta_{j+1}\Theta_{j+1}^{up} - \Delta M_{j+1}\right]\right) - G_j^{\#}\left(\rho_j^{(1)}, \rho_j^{(0)}\right).
\end{aligned}$$

Now, we conclude by the martingale property of $M$ and the tower property of the conditional expectation that

$$\begin{aligned}
Y_j^{up} &\geq \left(\rho_j^{(1)}\right)^\top E_j\left[\beta_{j+1}\Theta_{j+1}^{up}\right] + \rho_j^{(0)} F_j\left(E_j\left[\beta_{j+1}\Theta_{j+1}^{up}\right]\right) - G_j^{\#}\left(\rho_j^{(1)}, \rho_j^{(0)}\right) \\
&= \left(\rho_j^{(1)}\right)^\top E_j\left[\beta_{j+1}E_{j+1}\left[\Theta_{j+1}^{up}\right]\right] + \rho_j^{(0)} F_j\left(E_j\left[\beta_{j+1}E_{j+1}\left[\Theta_{j+1}^{up}\right]\right]\right) - G_j^{\#}\left(\rho_j^{(1)}, \rho_j^{(0)}\right).
\end{aligned}$$

Finally, the definition of $Y^{up}$ and (2.9) yield

$$\begin{aligned}
Y_j^{up} &\geq \left(\rho_j^{(1)}\right)^\top E_j\left[\beta_{j+1}Y_{j+1}^{up}\right] + \rho_j^{(0)} F_j\left(E_j\left[\beta_{j+1}Y_{j+1}^{up}\right]\right) - G_j^{\#}\left(\rho_j^{(1)}, \rho_j^{(0)}\right) \\
&\geq G_j\left(E_j\left[\beta_{j+1}Y_{j+1}^{up}\right], F_j\left(E_j\left[\beta_{j+1}Y_{j+1}^{up}\right]\right)\right)
\end{aligned}$$

for every $j = 0, \ldots, J-1$, from which follows that $Y^{up}$ is indeed a supersolution. Following essentially the same line of reasoning, except that we apply (1.23) instead of (2.9), we conclude that

$$\begin{aligned}
Y_j^{low} &= E_j\left[\Theta_j^{low}\right] \\
&= E_j\left[G_j\left(\beta_{j+1}\Theta_{j+1}^{low} - \Delta M_{j+1}, r_j^\top \beta_{j+1}\Theta_{j+1}^{low} - r_j^\top \Delta M_{j+1} - F_j^{\#}(r_j)\right)\right] \\
&\leq G_j\left(E_j\left[\beta_{j+1}\Theta_{j+1}^{low} - \Delta M_{j+1}\right], r_j^\top E_j\left[\beta_{j+1}\Theta_{j+1}^{low} - \Delta M_{j+1}\right] - F_j^{\#}(r_j)\right) \\
&= G_j\left(E_j\left[\beta_{j+1}\Theta_{j+1}^{low}\right], r_j^\top E_j\left[\beta_{j+1}\Theta_{j+1}^{low}\right] - F_j^{\#}(r_j)\right) \\
&= G_j\left(E_j\left[\beta_{j+1}E_{j+1}\left[\Theta_{j+1}^{low}\right]\right], r_j^\top E_j\left[\beta_{j+1}E_{j+1}\left[\Theta_{j+1}^{low}\right]\right] - F_j^{\#}(r_j)\right)
\end{aligned}$$

$$
\begin{aligned}
&= G_j\left(E_j\left[\beta_{j+1}Y_{j+1}^{low}\right], r_j^\top E_j\left[\beta_{j+1}Y_{j+1}^{low}\right] - F_j^{\#}(r_j)\right)\\
&\le G_j\left(E_j\left[\beta_{j+1}Y_{j+1}^{low}\right], F_j\left(E_j\left[\beta_{j+1}Y_{j+1}^{low}\right]\right)\right).
\end{aligned}
$$

for every $j = 0, \ldots, J-1$, showing that $Y^{low}$ is a subsolution.

From the comparison principle we now conclude that

$$
E_j\left[\Theta_j^{low}(r, M)\right] \le Y_j \le E_j\left[\Theta_j^{up}\left(\rho^{(1)}, \rho^{(0)}, M\right)\right]
$$

holds for every $j = 0, \ldots, J$, $M \in \mathcal{M}_D$ and admissible controls $(\rho^{(1)}, \rho^{(0)}) \in \mathcal{A}_0^G$ and $r \in \mathcal{A}_0^F$. In particular, it follows, similar to Section 1.3, that

$$
\operatorname*{esssup}_{r \in \mathcal{A}_0^F,\, M \in \mathcal{M}_D} E_0\left[\Theta_0^{low}(r, M)\right] \le Y_0 \le \operatorname*{essinf}_{(\rho^{(1)}, \rho^{(0)}) \in \mathcal{A}_0^G,\, M \in \mathcal{M}_D} E_0\left[\Theta_0^{up}\left(\rho^{(1)}, \rho^{(0)}, M\right)\right].
$$

We now provide the analogue of Theorem 1.3.4 for this concave-convex setting.

**Theorem 2.2.3.** *Suppose Assumptions 2.1.1 and 1.3.1. Then, for every $j = 0, \ldots, J$,*

$$
\begin{aligned}
Y_j &= \operatorname*{essinf}_{(\rho^{(1)}, \rho^{(0)}) \in \mathcal{A}_j^G,\, M \in \mathcal{M}_D} E_j\left[\Theta_j^{up}\left(\rho^{(1)}, \rho^{(0)}, M\right)\right]\\
&= \operatorname*{esssup}_{r \in \mathcal{A}_j^F,\, M \in \mathcal{M}_D} E_j\left[\Theta_j^{low}(r, M)\right], \quad P\text{-a.s.}
\end{aligned}
$$

*Moreover,*

$$
Y_j = \Theta_j^{up}\left(\rho^{(1,*)}, \rho^{(0,*)}, M^*\right) = \Theta_j^{low}\left(r^*, M^*\right) \tag{2.11}
$$

*$P$-almost surely, for every $(\rho^{(1,*)}, \rho^{(0,*)}) \in \mathcal{A}_j^G$ and $r^* \in \mathcal{A}_j^F$ satisfying the duality relations*

$$
\begin{aligned}
&\left(\rho_i^{(1,*)}\right)^\top E_i\left[\beta_{i+1}Y_{i+1}\right] + \rho_i^{(0,*)} F_i\left(E_i[\beta_{i+1}Y_{i+1}]\right) - G_i^{\#}\left(\rho_i^{(1,*)}, \rho_i^{(0,*)}\right)\\
&\qquad = G_i\left(E_i\left[\beta_{i+1}Y_{i+1}\right], F_i(E_i[\beta_{i+1}Y_{i+1}])\right)
\end{aligned} \tag{2.12}
$$

*and*

$$
\left(r_i^*\right)^\top E_i\left[\beta_{i+1}Y_{i+1}\right] - F_i^{\#}\left(r_i^*\right) = F_i\left(E_i\left[\beta_{i+1}Y_{i+1}\right]\right) \tag{2.13}
$$

*$P$-almost surely for every $i = j, \ldots, J-1$, and with $M^*$ being the Doob martingale of $\beta Y$.*

We emphasize that the main difference between Theorem 2.2.3 and its convex analogue, Theorem 1.3.4, is the pathwise equality (2.11) for both bounds if optimal controls and an optimal martingale are applied. Recall that in the convex setting of Section 1.3, pathwise equality for the lower bound could only be achieved by the modified recursion (1.29), while for the initial lower bound (1.25) equality only holds after taking conditional expectations. In contrast, it is yet impossible to drop any of the martingale increments in (2.10), since then equality would not even hold after taking conditional expectations. This is due to the fact, that either the convex or the concave part of (2.1) is linearized using Fenchel duality, but not the whole dynamic program.

*Proof.* The overall strategy is similar to the proof of Theorem 1.3.4. We first show that for given $j \in \{0, \dots, J-1\}$ the chain of inequalities

$$E_i\left[\Theta_i^{low}(r,M)\right] \le Y_i \le E_i\left[\Theta_i^{up}\left(\rho^{(1)}, \rho^{(0)}, M\right)\right], \quad i = j, \dots, J$$

holds for all admissible controls and martingales by constructing suitable super- and subsolutions and applying the comparison principle. Finally, we show that pathwise equality holds, if optimal controls and an optimal martingale are taken as an input.

Let $j \in \{0, \dots, J-1\}$ be fixed, $M \in \mathcal{M}_D$, $(\rho^{(1)}, \rho^{(0)}) \in \mathcal{A}_j^G$ and $r \in \mathcal{A}_j^F$. Further, we define the processes $\Theta^{up} := \Theta^{up}(\rho^{(1)}, \rho^{(0)}, M)$ and $\Theta^{low} := \Theta^{low}(r, M)$ according to (2.10). Building on $\Theta^{up}$ and $\Theta^{low}$, we define the two processes $Y^{up,j}$ and $Y^{low,j}$ by

$$Y_i^{up,j} = \begin{cases} E_i\left[\Theta_i^{up}\right], & i \ge j \\ G_i\left(E_i\left[\beta_{i+1} Y_{i+1}^{up,j}\right], F_i\left(E_i\left[\beta_{i+1} Y_{i+1}^{up,j}\right]\right)\right), & i < j \end{cases}$$

and

$$Y_i^{low,j} = \begin{cases} E_i\left[\Theta_i^{low}\right], & i \ge j \\ G_i\left(E_i\left[\beta_{i+1} Y_{i+1}^{low,j}\right], F_i\left(E_i\left[\beta_{i+1} Y_{i+1}^{low,j}\right]\right)\right), & i < j. \end{cases}$$

Then, $Y^{up,j}$ and $Y^{low,j}$ are super- and subsolutions to (1.15). Indeed, for $i \ge j$, this follows by the same arguments applied at the beginning of this section. For $i < j$, this is an immediate consequence of the definition of $Y^{up,j}$ and $Y^{low,j}$.

As an immediate consequence of the comparison principle, we obtain that

$$E_i\left[\Theta_i^{low}\right] \le Y_i \le E_i\left[\Theta_i^{up}\right], \quad i = j, \dots, J.$$

Since this chain of inequalities holds for arbitrary choices of admissible controls and martingales, we have

$$\operatorname*{esssup}_{r \in \mathcal{A}_j^F,\, M \in \mathcal{M}_D} E_j\left[\Theta_j^{low}(r,M)\right] \le Y_j \le \operatorname*{essinf}_{(\rho^{(1)},\rho^{(0)}) \in \mathcal{A}_j^G,\, M \in \mathcal{M}_D} E_j\left[\Theta_j^{up}\left(\rho^{(1)}, \rho^{(0)}, M\right)\right].$$

Finally, we show that these inequalities turn into equalities for $(\rho^{(1,*)}, \rho^{(0,*)}) \in \mathcal{A}_j^G$, $r^* \in \mathcal{A}_j^F$ given by (2.12) respectively (2.13) and with $M^*$ being the Doob martingale of $\beta Y$. To this end, let $\Theta^{up,*} := \Theta^{up}(\rho^{(1,*)}, \rho^{(0,*)}, M^*)$ and $\Theta^{low,*} := \Theta^{low}(r^*, M^*)$. Then, the proof is again by backward induction on $i$. As before, the case $i = J$ is trivial by definition. Suppose that the assertion is true for $i+1$. From the induction hypothesis, the definition of $M^*$, and the duality relation (2.13), we obtain that

$$\begin{aligned}
\Theta_i^{low,*} &= G_i\left(\beta_{i+1}\Theta_{i+1}^{low,*} - \Delta M_{i+1}^*, (r_i^*)^\top \beta_{i+1}\Theta_{i+1}^{low,*} - (r_i^*)^\top \Delta M_{i+1}^* - F_i^{\#}(r_i^*)\right) \\
&= G_i\left(\beta_{i+1} Y_{i+1} - \Delta M_{i+1}^*, (r_i^*)^\top \beta_{i+1} Y_{i+1} - (r_i^*)^\top \Delta M_{i+1}^* - F_i^{\#}(r_i^*)\right) \\
&= G_i\Big(\beta_{i+1} Y_{i+1} - (\beta_{i+1} Y_{i+1} - E_i[\beta_{i+1} Y_{i+1}]), \\
&\qquad (r_i^*)^\top \beta_{i+1} Y_{i+1} - (r_i^*)^\top (\beta_{i+1} Y_{i+1} - E_i[\beta_{i+1} Y_{i+1}]) - F_i^{\#}(r_i^*)\Big) \\
&= G_i\left(E_i[\beta_{i+1} Y_{i+1}], (r_i^*)^\top E_i[\beta_{i+1} Y_{i+1}] - F_i^{\#}(r_i^*)\right) \\
&= G_i\left(E_i[\beta_{i+1} Y_{i+1}], F_i(E_i[\beta_{i+1} Y_{i+1}])\right) = Y_i
\end{aligned}$$

holds $P$-a.s. To complete the proof we apply essentially the same arguments to the upper bound, except that the duality relation (2.12) is required instead of (2.13):

$$\begin{aligned}
\Theta_i^{up,*} &= \left(\rho_i^{(1,*)}\right)^\top \beta_{i+1}\Theta_{i+1}^{up,*} - \left(\rho_i^{(1,*)}\right)^\top \Delta M_{i+1}^* + \rho_i^{(0,*)} F_i\left(\beta_{i+1}\Theta_{i+1}^{up,*} - \Delta M_{i+1}^*\right) \\
&\quad - G_i^\#\left(\rho_i^{(1,*)}, \rho_i^{(0,*)}\right) \\
&= \left(\rho_i^{(1,*)}\right)^\top \beta_{i+1}Y_{i+1} - \left(\rho_i^{(1,*)}\right)^\top \Delta M_{i+1}^* + \rho_i^{(0,*)} F_i\left(\beta_{i+1}Y_{i+1} - \Delta M_{i+1}^*\right) \\
&\quad - G_i^\#\left(\rho_i^{(1,*)}, \rho_i^{(0,*)}\right) \\
&= \left(\rho_i^{(1,*)}\right)^\top \beta_{i+1}Y_{i+1} - \left(\rho_i^{(1,*)}\right)^\top \left(\beta_{i+1}Y_{i+1} - E_i\left[\beta_{i+1}Y_{i+1}\right]\right) \\
&\quad + \rho_i^{(0,*)} F_i\left(\beta_{i+1}Y_{i+1} - \left(\beta_{i+1}Y_{i+1} - E_i\left[\beta_{i+1}Y_{i+1}\right]\right)\right) - G_i^\#\left(\rho_i^{(1,*)}, \rho_i^{(0,*)}\right) \\
&= \left(\rho_i^{(1,*)}\right)^\top E_i\left[\beta_{i+1}Y_{i+1}\right] + \rho_i^{(0,*)} F_i\left(E_i\left[\beta_{i+1}Y_{i+1}\right]\right) - G_i^\#\left(\rho_i^{(1,*)}, \rho_i^{(0,*)}\right) \\
&= G_i\left(E_i\left[\beta_{i+1}Y_{i+1}\right], F_i(E_i[\beta_{i+1}Y_{i+1}])\right) \\
&= Y_i.
\end{aligned}$$

□

## 2.3 Relation to the information relaxation approach

The scope of this section is to relate our upper and lower bound recursions (2.10) to the information relaxation approach in the context of stochastic two-player games. To do this, we first show that the recursions (2.10) can be expressed as pathwise minimization respectively maximization problems. Building on these representations, we prove that the solution $Y$ to (2.1) is the value of a stochastic two-player game. Applying the information relaxation approach to this game shows that the resulting bounds coincide with our upper and lower bound recursions for a certain class of penalties.

We first observe that

$$Y_j = \inf_{(v^{(1)},v^{(0)})\in\mathbb{R}^{D+1}} \left(v^{(1)}\right)^\top E_j\left[\beta_{j+1}Y_{j+1}\right] + v^{(0)}\left(\sup_{u\in\mathbb{R}^D} u^\top E_j[\beta_{j+1}Y_{j+1}] - F_j^\#(u)\right) - G_j^\#\left(v^{(1)}, v^{(0)}\right)$$

for every $j = 0, \ldots, J-1$ by Lemma 1.2.4. As the function $(z, y) \mapsto G_j(z, y)$ is non-decreasing in $y$, we know that $v^{(0)}$ is non-negative, and therefore we obtain that

$$\begin{aligned}
Y_j &= \inf_{(v^{(1)},v^{(0)})\in\mathbb{R}^{D+1}} \sup_{u\in\mathbb{R}^D} \left(v^{(1)}\right)^\top E_j\left[\beta_{j+1}Y_{j+1}\right] + v^{(0)}u^\top E_j[\beta_{j+1}Y_{j+1}] - v^{(0)}F_j^\#(u) \\
&\quad - G_j^\#\left(v^{(1)}, v^{(0)}\right) \\
&= \inf_{(v^{(1)},v^{(0)})\in\mathbb{R}^{D+1}} \sup_{u\in\mathbb{R}^D} \left(v^{(1)} + v^{(0)}u\right)^\top E_j\left[\beta_{j+1}Y_{j+1}\right] - v^{(0)}F_j^\#(u) \\
&\quad - G_j^\#\left(v^{(1)}, v^{(0)}\right),
\end{aligned} \tag{2.14}$$

which formally looks like a dynamic programming equation for a two-player game with $\mathcal{F}_{j+1}$-measurable random weight $(v^{(1)} + v^{(0)}u)^\top\beta_{j+1}$. In order to show that (2.14) is indeed the dynamic programming equation of a two-player zero-sum game, the following positivity assumption is required.

**Assumption 2.3.1.** *For every $j = 0, \dots, J-1$, $\omega \in \Omega$, $(v^{(1)}, v^{(0)}) \in D_{G^\#}^{(j,\omega)}$, and $u \in D_{F^\#}^{(j,\omega)}$, we assume that*

$$\left(v^{(1)} + v^{(0)} u\right)^\top \beta_{j+1}(\omega) \geq 0. \tag{2.15}$$

The following theorem states that the solution $Y$ to the concave-convex dynamic program (2.1) might be interpreted as the value of certain two-player stochastic games.

**Theorem 2.3.2.** *Suppose Assumptions 2.1.1 and 2.3.1. Then, the solution $Y$ to* (2.1) *satisfies*

$$\begin{aligned} Y_0 &= \operatorname*{essinf}_{(\rho^{(1)},\rho^{(0)}) \in \mathcal{A}_0^G} \operatorname*{esssup}_{r \in \mathcal{A}_0^F} E_0 \Bigg[ w_J\left(\rho^{(1)}, \rho^{(0)}, r\right) \xi \\ &\qquad - \sum_{j=0}^{J-1} w_j\left(\rho^{(1)}, \rho^{(0)}, r\right) \left(\rho_j^{(0)} F_j^\#(r_j) + G_j^\#(\rho_j^{(1)}, \rho_j^{(0)})\right) \Bigg] \\ &= \operatorname*{esssup}_{r \in \mathcal{A}_0^F} \operatorname*{essinf}_{(\rho^{(1)},\rho^{(0)}) \in \mathcal{A}_0^G} E_0 \Bigg[ w_J\left(\rho^{(1)}, \rho^{(0)}, r\right) \xi \\ &\qquad - \sum_{j=0}^{J-1} w_j\left(\rho^{(1)}, \rho^{(0)}, r\right) \left(\rho_j^{(0)} F_j^\#(r_j) + G_j^\#(\rho_j^{(1)}, \rho_j^{(0)})\right) \Bigg], \end{aligned}$$

*where*

$$w_j\left(\omega, v^{(1)}, v^{(0)}, u\right) = \prod_{i=0}^{j-1} \left(v_i^{(1)} + v_i^{(0)} u_i\right)^\top \beta_{i+1}(\omega) \tag{2.16}$$

*for every $j = 0, \dots, J$.*

From a financial point of view, the weight $w_j(\rho^{(1)}, \rho^{(0)}, r)$ may be interpreted as a discrete-time price deflator or as an approximation of a continuous-time price deflator given in terms of a stochastic exponential which can incorporate both, discounting in the real-world sense and a change of measure. Then, the first term in

$$E_0 \left[ w_J\left(\rho^{(1)}, \rho^{(0)}, r\right) \xi - \sum_{j=0}^{J-1} w_j\left(\rho^{(1)}, \rho^{(0)}, r\right) \left(\rho_j^{(0)} F_j^\#(r_j) + G_j^\#(\rho_j^{(1)}, \rho_j^{(0)})\right) \right]$$

corresponds to the fair price of an option with payoff $\xi$ in the price system determined by the deflator $w_J(\rho^{(1)}, \rho^{(0)}, r)$, which is to be chosen by the two players. The choice may come with an additional running reward or cost which is formulated via the convex conjugates of $F$ and $G$ in the second term of the above expression. With this interpretation, $Y_0$ is the equilibrium price for an option with payoff $\xi$, on which the two players agree.

The key step in the proof of Theorem 2.3.2 are the following alternative representations of the recursions (2.10) as pathwise maximization respectively minimization problems.

**Proposition 2.3.3.** *Suppose Assumptions 2.1.1 and 2.3.1. Further, let $M \in \mathcal{M}_D$ and $(\rho^{(1)}, \rho^{(0)}) \in \mathcal{A}_0^G$ as well as $r \in \mathcal{A}_0^F$ be admissible controls and define the processes $\Theta^{up}(\rho^{(1)}, \rho^{(0)}, M)$ and*

$\Theta^{low}(r, M)$ *by* (2.10). *Then,* $\Theta_0^{up}(\rho^{(1)}, \rho^{(0)}, M)$ *and* $\Theta_0^{low}(r, M)$ *can be expressed by the pathwise maximization and minimization problems*

$$\Theta_0^{up}\left(\rho^{(1)}, \rho^{(0)}, M\right) = \sup_{(u_j) \in \mathbb{R}^D,\, j=0,\ldots,J-1} \Bigg( w_J\left(\rho^{(1)}, \rho^{(0)}, u\right)\xi - \sum_{j=0}^{J-1} w_j\left(\rho^{(1)}, \rho^{(0)}, u\right)\left(\rho_j^{(0)} F_j^{\#}(u_j) + (\rho_j^{(1)} + \rho_j^{(0)} u_j)^\top \Delta M_{j+1} + G_j^{\#}(\rho_j^{(1)}, \rho_j^{(0)})\right)\Bigg) \quad (2.17)$$

*and*

$$\Theta_0^{low}(r, M) = \inf_{(v_j^{(1)}, v_j^{(0)}) \in \mathbb{R}^{D+1},\, j=0,\ldots,J-1} \Bigg( w_J\left(v^{(1)}, v^{(0)}, r\right)\xi - \sum_{j=0}^{J-1} w_j\left(v^{(1)}, v^{(0)}, r\right)\left(v_j^{(0)} F_j^{\#}(r_j) + (v_j^{(1)} + v_j^{(0)} r_j)^\top \Delta M_{j+1} + G_j^{\#}(v_j^{(1)}, v_j^{(0)})\right)\Bigg), \quad (2.18)$$

*where* $w_j(v^{(1)}, v^{(0)}, u)$ *is for every* $j = 0, \ldots, J$ *given by* (2.16).

*Proof.* Let $M \in \mathcal{M}_D$, $(\rho^{(1)}, \rho^{(0)}) \in \mathcal{A}_0^G$ and $r \in \mathcal{A}_0^F$ be fixed from now on. Then, we define the processes $\Theta^{up}(\rho^{(1)}, \rho^{(0)}, M)$ and $\Theta^{low}(r, M)$ according to (2.10). Additionally, we define two processes $\tilde{\Theta}^{up} := \tilde{\Theta}^{up}(\rho^{(1)}, \rho^{(0)}, M)$ and $\tilde{\Theta}^{low} := \tilde{\Theta}^{low}(r, M)$ by

$$\tilde{\Theta}_j^{up} = \sup_{(u_i) \in \mathbb{R}^D,\, i=j,\ldots,J-1} \Bigg( w_{j,J}\left(\rho^{(1)}, \rho^{(0)}, u\right)\xi - \sum_{i=j}^{J-1} w_{j,i}\left(\rho^{(1)}, \rho^{(0)}, u\right)\left(\rho_i^{(0)} F_i^{\#}(u_i) + (\rho_i^{(1)} + \rho_i^{(0)} u_i)^\top \Delta M_{i+1} + G_i^{\#}(\rho_i^{(1)}, \rho_i^{(0)})\right)\Bigg) \quad (2.19)$$

and

$$\tilde{\Theta}_j^{low} = \inf_{(v_i^{(1)}, v_i^{(0)}) \in \mathbb{R}^{D+1},\, i=j,\ldots,J-1} \Bigg( w_{j,J}\left(v^{(1)}, v^{(0)}, r\right)\xi - \sum_{i=j}^{J-1} w_{j,i}\left(v^{(1)}, v^{(0)}, r\right)\left(v_i^{(0)} F_i^{\#}(r_i) + (v_i^{(1)} + v_i^{(0)} r_i)^\top \Delta M_{i+1} + G_i^{\#}(v_i^{(1)}, v_i^{(0)})\right)\Bigg), \quad (2.20)$$

for $j = 0, \ldots, J$. Here, the weight $w_{j,i}(v^{(1)}, v^{(0)}, u)$ is given by

$$w_{j,i}\left(v^{(1)}, v^{(0)}, u\right) = \prod_{k=j}^{i-1}\left(v_k^{(1)} + v_k^{(0)} u_k\right)^\top \beta_{k+1}.$$

From this definition, we obtain immediately the following simple identity

$$w_{j,i}\left(v^{(1)}, v^{(0)}, u\right) = \left(v_j^{(1)} + v_j^{(0)} u_j\right)^\top \beta_{j+1} w_{j+1,i}\left(v^{(1)}, v^{(0)}, u\right). \quad (2.21)$$

We first show that $\Theta_0^{up} = \tilde{\Theta}_0^{up}$. To this end, we first observe by the definition of $\tilde{\Theta}^{up}$ that $\tilde{\Theta}_J^{up} = \xi = \Theta_J^{up}$. Applying (2.21), we obtain for every $j = 0, \ldots, J-1$ that

$$\tilde{\Theta}_j^{up} = \sup_{(u_i) \in \mathbb{R}^D,\, i=j,\ldots,J-1} \Bigg( w_{j,J}\left(\rho^{(1)}, \rho^{(0)}, u\right)\xi$$

$$
\begin{aligned}
& - \sum_{i=j+1}^{J-1} w_{j,i}\left(\rho^{(1)}, \rho^{(0)}, u\right)\left(\rho_i^{(0)} F_i^{\#}(u_i) + (\rho_i^{(1)} + \rho_i^{(0)} u_i)^\top \Delta M_{i+1} + G_i^{\#}(\rho_i^{(1)}, \rho_i^{(0)})\right) \\
& - w_{j,j}\left(\rho^{(1)}, \rho^{(0)}, u\right)\left(\rho_j^{(0)} F_j^{\#}(u_j) + (\rho_j^{(1)} + \rho_j^{(0)} u_j)^\top \Delta M_{j+1} + G_j^{\#}(\rho_j^{(1)}, \rho_j^{(0)})\right)\Bigg) \\
= & \sup_{(u_i)\in\mathbb{R}^D,\, i=j,\dots,J-1} \Bigg(\left(\rho_j^{(1)} + \rho_j^{(0)} u_j\right)^\top \beta_{j+1}\Bigg(w_{j+1,J}\left(\rho^{(1)}, \rho^{(0)}, u\right)\xi \\
& - \sum_{i=j+1}^{J-1} w_{j+1,i}\left(\rho^{(1)}, \rho^{(0)}, u\right)\left(\rho_i^{(0)} F_i^{\#}(u_i) + (\rho_i^{(1)} + \rho_i^{(0)} u_i)^\top \Delta M_{i+1} + G_i^{\#}(\rho_i^{(1)}, \rho_i^{(0)})\right)\Bigg) \\
& - \rho_j^{(0)} F_j^{\#}(u_j) - \left(\rho_j^{(1)} + \rho_j^{(0)} u_j\right)^\top \Delta M_{j+1} - G_j^{\#}\left(\rho_j^{(1)}, \rho_j^{(0)}\right)\Bigg) \\
= & \sup_{u_j\in\mathbb{R}^D} \sup_{(u_i)\in\mathbb{R}^D,\, i=j+1,\dots,J-1} \Bigg(\left(\rho_j^{(1)} + \rho_j^{(0)} u_j\right)^\top \beta_{j+1}\Bigg(w_{j+1,J}\left(\rho^{(1)}, \rho^{(0)}, u\right)\xi \\
& - \sum_{i=j+1}^{J-1} w_{j+1,i}\left(\rho^{(1)}, \rho^{(0)}, u\right)\left(\rho_i^{(0)} F_i^{\#}(u_i) + (\rho_i^{(1)} + \rho_i^{(0)} u_i)^\top \Delta M_{i+1} + G_i^{\#}(\rho_i^{(1)}, \rho_i^{(0)})\right)\Bigg) \\
& - \rho_j^{(0)} F_j^{\#}(u_j) - \left(\rho_j^{(1)} + \rho_j^{(0)} u_j\right)^\top \Delta M_{j+1} - G_j^{\#}\left(\rho_j^{(1)}, \rho_j^{(0)}\right)\Bigg).
\end{aligned}
$$

In a next step, we want to interchange the inner supremum with the factor $(\rho_j^{(1)} + \rho_j^{(0)} u_j)^\top \beta_{j+1}$. To achieve this, we need to restrict the outer supremum to the effective domain $D_{F\#}^{(j,\cdot)}$, so that we can apply the positivity assumption (2.15). Since we have by definition that $F_j^{\#}(u) = +\infty$ for all $u \in \mathbb{R}^D \setminus D_{F\#}^{(j,\cdot)}$, the expression to be maximized would take the value $-\infty$, which cannot be the supremum. Hence, the restriction to the effective domain maintains the equality and we obtain

$$
\begin{aligned}
\tilde{\Theta}_j^{up} = & \sup_{u_j\in D_{F\#}^{(j,\cdot)}} \sup_{(u_i)\in\mathbb{R}^D,\, i=j+1,\dots,J-1} \Bigg(\left(\rho_j^{(1)} + \rho_j^{(0)} u_j\right)^\top \beta_{j+1}\Bigg(w_{j+1,J}\left(\rho^{(1)}, \rho^{(0)}, u\right)\xi \\
& - \sum_{i=j+1}^{J-1} w_{j+1,i}\left(\rho^{(1)}, \rho^{(0)}, u\right)\left(\rho_i^{(0)} F_i^{\#}(u_i) + (\rho_i^{(1)} + \rho_i^{(0)} u_i)^\top \Delta M_{i+1} + G_i^{\#}(\rho_i^{(1)}, \rho_i^{(0)})\right)\Bigg) \\
& - \rho_j^{(0)} F_j^{\#}(u_j) - \left(\rho_j^{(1)} + \rho_j^{(0)} u_j\right)^\top \Delta M_{j+1} - G_j^{\#}\left(\rho_j^{(1)}, \rho_j^{(0)}\right)\Bigg) \\
= & \sup_{u_j\in D_{F\#}^{(j,\cdot)}} \Bigg(\left(\rho_j^{(1)} + \rho_j^{(0)} u_j\right)^\top \beta_{j+1}\Bigg(\sup_{(u_i)\in\mathbb{R}^D,\, i=j+1,\dots,J-1} w_{j+1,J}\left(\rho^{(1)}, \rho^{(0)}, u\right)\xi \\
& - \sum_{i=j+1}^{J-1} w_{j+1,i}\left(\rho^{(1)}, \rho^{(0)}, u\right)\left(\rho_i^{(0)} F_i^{\#}(u_i) + (\rho_i^{(1)} + \rho_i^{(0)} u_i)^\top \Delta M_{i+1} + G_i^{\#}(\rho_i^{(1)}, \rho_i^{(0)})\right)\Bigg) \\
& - \rho_j^{(0)} F_j^{\#}(u_j) - \left(\rho_j^{(1)} + \rho_j^{(0)} u_j\right)^\top \Delta M_{j+1} - G_j^{\#}\left(\rho_j^{(1)}, \rho_j^{(0)}\right)\Bigg)
\end{aligned}
$$

$$= \sup_{u_j \in D_{F^\#}^{(j,\cdot)}} \Bigg( \left(\rho_j^{(1)} + \rho_j^{(0)} u_j\right)^\top \beta_{j+1} \tilde{\Theta}_{j+1}^{up} - \rho_j^{(0)} F_j^\#(u_j) - \left(\rho_j^{(1)} + \rho_j^{(0)} u_j\right)^\top \Delta M_{j+1}$$
$$-G_j^\# \left(\rho_j^{(1)}, \rho_j^{(0)}\right) \Bigg),$$

where the last equality follows from the definition of $\tilde{\Theta}^{up}$. By the same argument as before, we replace the supremum over $D_{F^\#}^{(j,\cdot)}$ by the supremum over $\mathbb{R}^D$ and apply the non-negativity of $\rho_j^{(0)}$ as well as (1.23) to observe that

$$\begin{aligned}
\tilde{\Theta}_j^{up} &= \sup_{u_j \in \mathbb{R}^D} \Bigg( \left(\rho_j^{(1)} + \rho_j^{(0)} u_j\right)^\top \beta_{j+1} \tilde{\Theta}_{j+1}^{up} - \rho_j^{(0)} F_j^\#(u_j) - \left(\rho_j^{(1)} + \rho_j^{(0)} u_j\right)^\top \Delta M_{j+1} \\
&\quad -G_j^\# \left(\rho_j^{(1)}, \rho_j^{(0)}\right) \Bigg) \\
&= \left(\rho_j^{(1)}\right)^\top \left(\beta_{j+1} \tilde{\Theta}_{j+1}^{up} - \Delta M_{j+1}\right) + \rho_j^{(0)} \left( \sup_{u_j \in \mathbb{R}^D} u_j^\top \left(\beta_{j+1} \tilde{\Theta}_{j+1}^{up} - \Delta M_{j+1}\right) - F_j^\#(u_j) \right) \\
&\quad -G_j^\# \left(\rho_j^{(1)}, \rho_j^{(0)}\right) \\
&= \left(\rho_j^{(1)}\right)^\top \left(\beta_{j+1} \tilde{\Theta}_{j+1}^{up} - \Delta M_{j+1}\right) + \rho_j^{(0)} F_j \left(\beta_{j+1} \tilde{\Theta}_{j+1}^{up} - \Delta M_{j+1}\right) - G_j^\# \left(\rho_j^{(1)}, \rho_j^{(0)}\right).
\end{aligned}$$

Hence the recursions for $\Theta^{up}(\rho^{(1)}, \rho^{(0)}, M)$ and $\tilde{\Theta}^{up}$ coincide, showing that $\Theta_j^{up}(\rho^{(1)}, \rho^{(0)}, M) = \tilde{\Theta}_j^{up}$ for all $j = 0, \ldots, J$ and therefore

$$\begin{aligned}
\Theta_0^{up}(\rho^{(1)}, \rho^{(0)}, M) = \sup_{(u_j) \in \mathbb{R}^D,\, j=0,\ldots,J-1} \Bigg( & w_J \left(\rho^{(1)}, \rho^{(0)}, u\right) \xi \\
& - \sum_{j=0}^{J-1} w_j \left(\rho^{(1)}, \rho^{(0)}, u\right) \left(\rho_j^{(0)} F_j^\#(u_j) + (\rho_j^{(1)} + \rho_j^{(0)} u_j)^\top \Delta M_{j+1} + G_j^\#(\rho_j^{(1)}, \rho_j^{(0)})\right) \Bigg).
\end{aligned}$$

Finally, we show that $\Theta_0^{low} = \tilde{\Theta}_0^{low}$ by essentially the same line of reasoning. By definition of $\tilde{\Theta}^{low}$, we have that $\tilde{\Theta}_J^{low} = \xi = \Theta_J^{low}$. Then, an application of (2.21) yields

$$\begin{aligned}
\tilde{\Theta}_j^{low} &= \inf_{(v_i^{(1)}, v_i^{(0)}) \in \mathbb{R}^{D+1},\, i=j,\ldots,J-1} \Bigg( w_{j,J} \left(v^{(1)}, v^{(0)}, r\right) \xi \\
&\quad - \sum_{i=j+1}^{J-1} w_{j,i} \left(v^{(1)}, v^{(0)}, r\right) \left(v_i^{(0)} F_i^\#(r_i) + (v_i^{(1)} + v_i^{(0)} r_i)^\top \Delta M_{i+1} + G_i^\#(v_i^{(1)}, v_i^{(0)})\right) \\
&\quad - w_{j,j} \left(v^{(1)}, v^{(0)}, r\right) \left(v_j^{(0)} F_j^\#(r_j) + (v_j^{(1)} + v_j^{(0)} r_j)^\top \Delta M_{j+1} + G_j^\#(v_j^{(1)}, v_j^{(0)})\right) \Bigg) \\
&= \inf_{(v_i^{(1)}, v_i^{(0)}) \in \mathbb{R}^{D+1},\, i=j,\ldots,J-1} \Bigg( \left(v_j^{(1)} + v_j^{(0)} r_j\right)^\top \beta_{j+1} \Bigg( w_{j+1,J} \left(v^{(1)}, v^{(0)}, r\right) \xi \\
&\quad - \sum_{i=j+1}^{J-1} w_{j+1,i} \left(v^{(1)}, v^{(0)}, r\right) \left(v_i^{(0)} F_i^\#(r_i) + (v_i^{(1)} + v_i^{(0)} u_i)^\top \Delta M_{i+1} + G_i^\#(v_i^{(1)}, v_i^{(0)})\right) \Bigg) \\
&\quad - v_j^{(0)} F_j^\#(r_j) - \left(v_j^{(1)} + v_j^{(0)} r_j\right)^\top \Delta M_{j+1} - G_j^\# \left(v_j^{(1)}, v_j^{(0)}\right) \Bigg)
\end{aligned}$$

$$= \inf_{(v_j^{(1)},v_j^{(0)})\in\mathbb{R}^{D+1}} \inf_{(v_i^{(1)},v_i^{(0)})\in\mathbb{R}^{D+1},\, i=j+1,\dots,J-1} \Bigg(\left(v_j^{(1)}+v_j^{(0)}r_j\right)^\top \beta_{j+1}\Bigg(w_{j+1,J}\left(v^{(1)},v^{(0)},r\right)\xi$$

$$-\sum_{i=j+1}^{J-1} w_{j+1,i}\left(v^{(1)},v^{(0)},r\right)\left(v_i^{(0)}F_i^{\#}(r_i)+(v_i^{(1)}+v_i^{(0)}r_i)^\top\Delta M_{i+1}+G_i^{\#}(v_i^{(1)},v_i^{(0)})\right)\Bigg)$$

$$-v_j^{(0)}F_j^{\#}(r_j)-\left(v_j^{(1)}+v_j^{(0)}r_j\right)^\top\Delta M_{j+1}-G_j^{\#}\left(v_j^{(1)},v_j^{(0)}\right)\Bigg)$$

for every $j=0,\dots,J-1$. By a similar argument as above, the outer infimum can be taken restricted to such $(v_j^{(1)},v_j^{(0)})\in\mathbb{R}^{D+1}$ which belong to $D_{G^\#}^{(j,\cdot)}$. Then, (2.15) implies that the inner infimum can be interchanged with the non-negative factor $(v_j^{(1)}+v_j^{(0)}r_j)^\top\beta_{j+1}$, which yields in combination with the definition of $\tilde\Theta^{low}$ that

$$\tilde\Theta_j^{low} = \inf_{(v_j^{(1)},v_j^{(0)})\in D_{G^\#}^{(j,\cdot)}} \inf_{(v_i^{(1)},v_i^{(0)})\in\mathbb{R}^{D+1},\, i=j+1,\dots,J-1} \Bigg(\left(v_j^{(1)}+v_j^{(0)}r_j\right)^\top \beta_{j+1}\Bigg(w_{j+1,J}\left(v^{(1)},v^{(0)},r\right)\xi$$

$$-\sum_{i=j+1}^{J-1} w_{j+1,i}\left(v^{(1)},v^{(0)},r\right)\left(v_i^{(0)}F_i^{\#}(r_i)+(v_i^{(1)}+v_i^{(0)}r_i)^\top\Delta M_{i+1}+G_i^{\#}(v_i^{(1)},v_i^{(0)})\right)\Bigg)$$

$$-v_j^{(0)}F_j^{\#}(r_j)-\left(v_j^{(1)}+v_j^{(0)}r_j\right)^\top\Delta M_{j+1}-G_j^{\#}\left(v_j^{(1)},v_j^{(0)}\right)\Bigg)$$

$$= \inf_{(v_j^{(1)},v_j^{(0)})\in D_{G^\#}^{(j,\cdot)}} \Bigg(\left(v_j^{(1)}+v_j^{(0)}r_j\right)^\top \beta_{j+1}\Bigg(\inf_{(v_i^{(1)},v_i^{(0)})\in\mathbb{R}^{D+1},\, i=j+1,\dots,J-1} w_{j+1,J}\left(v^{(1)},v^{(0)},r\right)\xi$$

$$-\sum_{i=j+1}^{J-1} w_{j+1,i}\left(v^{(1)},v^{(0)},r\right)\left(v_i^{(0)}F_i^{\#}(r_i)+(v_i^{(1)}+v_i^{(0)}r_i)^\top\Delta M_{i+1}+G_i^{\#}(v_i^{(1)},v_i^{(0)})\right)\Bigg)$$

$$-v_j^{(0)}F_j^{\#}(r_j)-\left(v_j^{(1)}+v_j^{(0)}r_j\right)^\top\Delta M_{j+1}-G_j^{\#}\left(v_j^{(1)},v_j^{(0)}\right)\Bigg)$$

$$= \inf_{(v_j^{(1)},v_j^{(0)})\in D_{G^\#}^{(j,\cdot)}} \Bigg(\left(v_j^{(1)}+v_j^{(0)}r_j\right)^\top \beta_{j+1}\tilde\Theta_{j+1}^{low}-v_j^{(0)}F_j^{\#}(r_j)-\left(v_j^{(1)}+v_j^{(0)}r_j\right)^\top\Delta M_{j+1}$$

$$-G_j^{\#}\left(v_j^{(1)},v_j^{(0)}\right)\Bigg).$$

Passing to the infimum over $\mathbb{R}^{D+1}$ and applying (2.9), we observe that $\Theta^{low}(r,M)$ and $\tilde\Theta^{low}$ can be expressed by the same recursion from which $\Theta_j^{low}(r,M)=\tilde\Theta_j^{low}$ follows for all $j=0,\dots,J$. Therefore,

$$\Theta_0^{low}(r,M) = \inf_{(v_j^{(1)},v_j^{(0)})\in\mathbb{R}^{D+1},\, j=0,\dots,J-1} \Bigg(w_J\left(v^{(1)},v^{(0)},r\right)\xi$$

$$-\sum_{j=0}^{J-1} w_j\left(v^{(1)},v^{(0)},r\right)\left(v_j^{(0)}F_j^{\#}(r_j)+(v_j^{(1)}+v_j^{(0)}r_j)^\top\Delta M_{j+1}+G_j^{\#}(v_j^{(1)},v_j^{(0)})\right)\Bigg).$$

□

Building on Proposition 2.3.3, we are now in the position to state the proof of Theorem 2.3.2.

*Proof of Theorem 2.3.2.* From Theorem 2.2.3 and Proposition 2.3.3, we observe that

$$
\begin{aligned}
Y_0 &= \operatorname*{esssup}_{r\in\mathcal{A}_0^F,\, M\in\mathcal{M}_D} E_0\left[\Theta_0^{low}(r,M)\right] \\
&= \operatorname*{esssup}_{r\in\mathcal{A}_0^F,\, M\in\mathcal{M}_D} E_0\left[\inf_{(v_j^{(1)},v_j^{(0)})\in\mathbb{R}^{D+1},\, j=0,\dots,J-1}\left(w_J\left(v^{(1)},v^{(0)},r\right)\xi\right.\right. \\
&\quad \left.\left.-\sum_{j=0}^{J-1} w_j\left(v^{(1)},v^{(0)},r\right)\left(v_j^{(0)}F_j^{\#}(r_j)+(v_j^{(1)}+v_j^{(0)}r_j)^\top\Delta M_{j+1}+G_j^{\#}(v_j^{(1)},v_j^{(0)})\right)\right)\right] \\
&\le \operatorname*{esssup}_{r\in\mathcal{A}_0^F,\, M\in\mathcal{M}_D}\ \operatorname*{essinf}_{(\rho^{(1)},\rho^{(0)})\in\mathcal{A}_0^G} E_0\left[w_J\left(\rho^{(1)},\rho^{(0)},r\right)\xi\right. \\
&\quad \left.-\sum_{j=0}^{J-1} w_j\left(\rho^{(1)},\rho^{(0)},r\right)\left(\rho_j^{(0)}F_j^{\#}(r_j)+(\rho_j^{(1)}+\rho_j^{(0)}r_j)^\top\Delta M_{j+1}+G_j^{\#}(\rho_j^{(1)},\rho_j^{(0)})\right)\right].
\end{aligned}
$$

Now, the tower property and the admissibility of the controls in combination with the martingale property of $M$ yield

$$
\begin{aligned}
Y_0 &\le \operatorname*{esssup}_{r\in\mathcal{A}_0^F,\, M\in\mathcal{M}_D}\ \operatorname*{essinf}_{(\rho^{(1)},\rho^{(0)})\in\mathcal{A}_0^G} E_0\left[w_J\left(\rho^{(1)},\rho^{(0)},r\right)\xi\right. \\
&\quad \left.-\sum_{j=0}^{J-1} w_j\left(\rho^{(1)},\rho^{(0)},r\right)\left(\rho_j^{(0)}F_j^{\#}(r_j)+(\rho_j^{(1)}+\rho_j^{(0)}r_j)^\top E_j\left[\Delta M_{j+1}\right]+G_j^{\#}(\rho_j^{(1)},\rho_j^{(0)})\right)\right] \\
&= \operatorname*{esssup}_{r\in\mathcal{A}_0^F}\ \operatorname*{essinf}_{(\rho^{(1)},\rho^{(0)})\in\mathcal{A}_0^G} E_0\left[w_J\left(\rho^{(1)},\rho^{(0)},r\right)\xi\right. \\
&\quad \left.-\sum_{j=0}^{J-1} w_j\left(\rho^{(1)},\rho^{(0)},r\right)\left(\rho_j^{(0)}F_j^{\#}(r_j)+G_j^{\#}(\rho_j^{(1)},\rho_j^{(0)})\right)\right] \\
&\le \operatorname*{essinf}_{(\rho^{(1)},\rho^{(0)})\in\mathcal{A}_0^G}\ \operatorname*{esssup}_{r\in\mathcal{A}_0^F} E_0\left[w_J\left(\rho^{(1)},\rho^{(0)},r\right)\xi\right. \\
&\quad \left.-\sum_{j=0}^{J-1} w_j\left(\rho^{(1)},\rho^{(0)},r\right)\left(\rho_j^{(0)}F_j^{\#}(r_j)+G_j^{\#}(\rho_j^{(1)},\rho_j^{(0)})\right)\right].
\end{aligned}
$$

Repeating the previous argument and applying Proposition 2.3.3 as well as Theorem 2.2.3 once more, we obtain

$$
\begin{aligned}
Y_0 &\le \operatorname*{essinf}_{(\rho^{(1)},\rho^{(0)})\in\mathcal{A}_0^G}\ \operatorname*{esssup}_{r\in\mathcal{A}_0^F} E_0\left[w_J\left(\rho^{(1)},\rho^{(0)},r\right)\xi\right. \\
&\quad \left.-\sum_{j=0}^{J-1} w_j\left(\rho^{(1)},\rho^{(0)},r\right)\left(\rho_j^{(0)}F_j^{\#}(r_j)+(\rho_j^{(1)}+\rho_j^{(0)}r_j)^\top E_j\left[\Delta M_{j+1}\right]+G_j^{\#}(\rho_j^{(1)},\rho_j^{(0)})\right)\right] \\
&\le \operatorname*{essinf}_{(\rho^{(1)},\rho^{(0)})\in\mathcal{A}_0^G,\, M\in\mathcal{M}_D} E_0\left[\sup_{(u_j)\in\mathbb{R}^D,\, j=0,\dots,J-1}\left(w_J\left(\rho^{(1)},\rho^{(0)},u\right)\xi\right.\right. \\
&\quad \left.\left.-\sum_{j=0}^{J-1} w_j\left(\rho^{(1)},\rho^{(0)},u\right)\left(\rho_j^{(0)}F_j^{\#}(u_j)+(\rho_j^{(1)}+\rho_j^{(0)}u_j)^\top\Delta M_{j+1}+G_j^{\#}(\rho_j^{(1)},\rho_j^{(0)})\right)\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \operatorname*{essinf}_{(\rho^{(1)},\rho^{(0)})\in\mathcal{A}_0^G,\, M\in\mathcal{M}_D} E_0\left[\Theta_0^{up}(\rho^{(1)},\rho^{(0)},M)\right] \\
&= Y_0,
\end{aligned}
$$

so that all inequalities turn into equalities, which completes the proof. □

The remainder of this section is dedicated to working out the connection between our recursions (2.10) and the information relaxation duals proposed by Brown et al. (2010) for this kind of stochastic two-player games. To this end, we first define the set $\mathfrak{P}$ of all dual-feasible penalties. A dual-feasible penalty $\mathfrak{p}$ is a mapping $\mathfrak{p}:\Omega\times\mathbb{R}^{(D+1)\times J}\times\mathbb{R}^{D\times J}\to\mathbb{R}\cup\{+\infty\}$, such that $E_0[\mathfrak{p}(\rho^{(1)},\rho^{(0)},r)]\leq 0$ holds for all admissible controls $(\rho^{(1)},\rho^{(0)})\in\mathcal{A}_0^G$ and $r\in\mathcal{A}_0^F$.

As we have shown in Theorem 2.3.2, $Y_0$ is the value of the following max-min-problem:

$$
\begin{aligned}
Y_0 \;=\; &\operatorname*{esssup}_{r\in\mathcal{A}_0^F}\; \operatorname*{essinf}_{(\rho^{(1)},\rho^{(0)})\in\mathcal{A}_0^G} E_0\Bigg[w_J(\rho^{(1)},\rho^{(0)},r)\xi \\
&-\sum_{j=0}^{J-1} w_j(\rho^{(1)},\rho^{(0)},r)\left(\rho_j^{(0)}F_j^{\#}(r_j)+G_j^{\#}(\rho_j^{(1)},\rho_j^{(0)})\right)\Bigg].
\end{aligned}
$$

If we now suppose that player 1 fixes a control $(\rho^{(1)},\rho^{(0)})\in\mathcal{A}_0^G$, we observe that

$$
Y_0\leq \operatorname*{esssup}_{r\in\mathcal{A}_0^F} E_0\left[w_J(\rho^{(1)},\rho^{(0)},r)\xi-\sum_{j=0}^{J-1} w_j(\rho^{(1)},\rho^{(0)},r)\left(\rho_j^{(0)}F_j^{\#}(r_j)+G_j^{\#}(\rho_j^{(1)},\rho_j^{(0)})\right)\right]. \tag{2.22}
$$

Applying Theorem 2.1 in Brown et al. (2010), i.e. the information relaxation dual with strong duality, we obtain that the right-hand side of (2.22) can be rewritten as

$$
\begin{aligned}
\operatorname*{essinf}_{\mathfrak{p}\in\mathfrak{P}} E_0\Bigg[\sup_{(u_j)\in\mathbb{R}^D,j=0,\dots,J-1}\Bigg(&w_J(\rho^{(1)},\rho^{(0)},u)\xi \\
&-\sum_{j=0}^{J-1} w_j(\rho^{(1)},\rho^{(0)},u)\left(\rho_j^{(0)}F_j^{\#}(u_j)+G_j^{\#}(\rho_j^{(1)},\rho_j^{(0)})\right)-\mathfrak{p}(\rho^{(1)},\rho^{(0)},u)\Bigg)\Bigg].
\end{aligned}
$$

Hence, Player 2 is allowed to consider the pathwise optimization problem, but at the same time the choice of anticipating controls is penalized by the mapping $\mathfrak{p}$. There is a penalty $\mathfrak{p}^*$, which achieves the infimum and forces that the optimal control for player 2 is adapted.

In a next step, we restrict ourselves to a certain class of penalties, to which we refer as martingale penalties in the following. To this end, let $M\in\mathcal{M}_D$ be a martingale and define the penalty $\mathfrak{p}_{M,\rho}:\Omega\times\mathbb{R}^{D\times J}\to\mathbb{R}\cup\{+\infty\}$ by

$$
\mathfrak{p}_{M,\rho}(u)=\sum_{j=0}^{J-1} w_j\left(\rho^{(1)},\rho^{(0)},u\right)\left(\rho_j^{(1)}+\rho_j^{(0)}u_j\right)^{\top}\Delta M_{j+1},
$$

where $(\rho^{(1)},\rho^{(0)})\in\mathcal{A}_0^G$ is the fixed control of player 1. Then, $\mathfrak{p}_{M,\rho}$ is a dual-feasible penalty, since, for adapted controls $r\in\mathcal{A}_0^F$,

$$
E_0[\mathfrak{p}_{M,\rho}(r)]=\sum_{j=0}^{J-1} E_0\left[w_j\left(\rho^{(1)},\rho^{(0)},r\right)\left(\rho_j^{(1)}+\rho_j^{(0)}r_j\right)^{\top}E_j[\Delta M_{j+1}]\right]=0
$$

by the martingale property of $M$ and the tower property of the conditional expectation. From Proposition 2.3.3, it now follows that

$$\begin{aligned}\Theta_0^{up}(\rho^{(1)},\rho^{(0)},M) &= \sup_{\substack{(u_j)\in\mathbb{R}^D,\\ j=0,\dots,J-1}} \Bigg( w_J\left(\rho^{(1)},\rho^{(0)},u\right)\xi \\ &\quad -\sum_{j=0}^{J-1} w_j\left(\rho^{(1)},\rho^{(0)},u\right)\left(\rho_j^{(0)}F_j^{\#}(u_j)+G_j^{\#}(\rho_j^{(1)},\rho_j^{(0)})\right)-\mathfrak{p}_{M,\rho}(u)\Bigg)\end{aligned}$$

and, thus, by Theorem 2.2.3

$$\begin{aligned}Y_0 &= \operatorname*{essinf}_{(\rho^{(1)},\rho^{(0)})\in\mathcal{A}_j^G,\ M\in\mathcal{M}_D} E_0\Bigg[\sup_{(u_j)\in\mathbb{R}^D,\ j=0,\dots,J-1}\Bigg(w_J\left(\rho^{(1)},\rho^{(0)},u\right)\xi \\ &\quad -\sum_{j=0}^{J-1} w_j\left(\rho^{(1)},\rho^{(0)},u\right)\left(\rho_j^{(0)}F_j^{\#}(u_j)+G_j^{\#}(\rho_j^{(1)},\rho_j^{(0)})\right)-\mathfrak{p}_{M,\rho}(u)\Bigg)\Bigg]. \qquad (2.23)\end{aligned}$$

Hence, under the positivity condition (2.15), the upper bound $E_0[\Theta_0^{up}(\rho^{(1)},\rho^{(0)},M)]$ can be interpreted in such a way that, first, player 1 fixes her strategy $(\rho^{(1)},\rho^{(0)})$ and the penalty by the choice of the martingale $M$, while, then, player 2 is allowed to maximize the penalized problem pathwise.

In order to derive a similar interpretation for the lower bound, we suppose that player 2 fixes her control. Then, we obtain again by the information relaxation dual with strong duality that

$$\begin{aligned}Y_0 \geq \sup_{\mathfrak{p}\in\mathfrak{P}} E_0\Bigg[&\inf_{(v_j^{(1)},v_j^{(0)})\in\mathbb{R}^{D+1},\ j=0,\dots,J-1}\Bigg(w_J\left(v^{(1)},v^{(0)},r\right)\xi \\ &-\sum_{j=0}^{J-1} w_j\left(v^{(1)},v^{(0)},r\right)\left(v_j^{(0)}F_j^{\#}(r_j)+G_j^{\#}(v_j^{(1)},v_j^{(0)})\right)+\mathfrak{p}\left(v^{(1)},v^{(0)},r\right)\Bigg)\Bigg], \qquad (2.24)\end{aligned}$$

where now player 1 is allowed to minimize the penalized problem pathwise. Choosing the dual-feasible penalty $\mathfrak{p}_{M,r}:\Omega\times\mathbb{R}^{(D+1)\times J}\to\mathbb{R}\cup\{+\infty\}$ given by

$$\mathfrak{p}_{M,r}\left(v^{(1)},v^{(0)}\right)=\sum_{j=0}^{J-1} w_j\left(v^{(1)},v^{(0)},r\right)\left(v_j^{(1)}+v_j^{(0)}r_j\right)^{\top}\Delta M_{j+1},$$

for a martingale $M\in\mathcal{M}_D$, and applying Proposition 2.3.3 we end up with

$$\begin{aligned}\Theta_0^{low}(r,M) = &\inf_{(v_j^{(1)},v_j^{(0)})\in\mathbb{R}^{D+1},\ j=0,\dots,J-1}\Bigg(w_J\left(v^{(1)},v^{(0)},r\right)\xi \\ &-\sum_{j=0}^{J-1} w_j\left(v^{(1)},v^{(0)},r\right)\left(v_j^{(0)}F_j^{\#}(r_j)+G_j^{\#}(v_j^{(1)},v_j^{(0)})\right)-\mathfrak{p}_{M,r}\left(v^{(1)},v^{(0)}\right)\Bigg). \qquad (2.25)\end{aligned}$$

From Theorem 2.2.3, we conclude that

$$Y_0 = \sup_{r\in\mathcal{A}_j^F,\ M\in\mathcal{M}_D} E_0\Bigg[\inf_{(v_j^{(1)},v_j^{(0)})\in\mathbb{R}^{D+1},\ j=0,\dots,J-1}\Bigg(w_J\left(v^{(1)},v^{(0)},r\right)\xi$$

$$-\sum_{j=0}^{J-1} w_j\left(v^{(1)}, v^{(0)}, r\right)\left(v_j^{(0)} F_j^{\#}(r_j) + G_j^{\#}(v_j^{(1)}, v_j^{(0)})\right) - \mathfrak{p}_{M,r}\left(v^{(1)}, v^{(0)}\right)\Bigg)\Bigg], \quad (2.26)$$

showing that a similar interpretation holds for the lower bound. Compared to the upper bound, the situation is now vice versa, as player 2 fixes a strategy and the penalty (by choosing the martingale) and player 1 may optimize the penalized problem path by path.

In this way, we end up with the information relaxation dual of Brown et al. (2010) for each player given that the other player has fixed a control. Moreover, we emphasize that the above approach is analogous to the recent information relaxation approach by Haugh and Wang (2015) for two-player games in a classical Markovian framework which dates back to Shapley (1953).

*Remark* 2.3.4. We also showed by (2.23) and (2.26) that strong duality still applies when the minimization respectively maximization is restricted from $\mathfrak{P}$ to the corresponding subsets $\{\mathfrak{p}_{M,\rho} | M \in \mathcal{M}_D,\ (\rho^{(1)}, \rho^{(0)}) \in \mathcal{A}_0^G\}$ and $\{\mathfrak{p}_{M,r} | M \in \mathcal{M}_D,\ r \in \mathcal{A}_0^F\}$. This can turn out to be useful in numerical implementations. Indeed, as discussed, e.g., in Section 4.2 of Brown and Smith (2011) and in Section 2.3 of Haugh and Lim (2012), choosing a dual-feasible penalty from $\mathfrak{P}$ can make it more difficult to solve the pathwise optimization problems in (2.23) and (2.26). This, however, is the key step in the information relaxation approach. In contrast, the implementation of the approach presented in Section 2.2 is straightforward: After a ($D$-dimensional) martingale $M$ is chosen, we can solve the pathwise maximization respectively minimization problem in (2.23) and (2.26) for the penalties $\mathfrak{p}_{M,\rho}$ and $\mathfrak{p}_{M,r}$ by computing the pathwise recursions for $\Theta_0^{up}(\rho^{(1)}, \rho^{(0)}, M)$ and $\Theta_0^{low}(r, M)$ in Theorem 2.2.3.

## 2.4 The general case

Similar to Section 1.5, we now consider the case when the comparison principle fails to hold. As we will see below, the main idea in the construction of coupled upper and lower bounds from Section 1.5 does not transfer immediately to the concave-convex framework. This is due to the following analogue of Theorem 1.4.1 in this setting.

**Proposition 2.4.1.** *Suppose Assumption 2.1.1, and consider the following assertions:*

*(a) The comparison principle as stated in Assumption 2.2.1 holds.*

*(b) For every $(\rho^{(1)}, \rho^{(0)}) \in \mathcal{A}_0^G$ and $r \in \mathcal{A}_0^F$ the following positivity condition is fulfilled: For every $j = 0, \ldots, J-1$*

$$\left(\rho_j^{(1)} + \rho_j^{(0)} r_j\right)^\top \beta_{j+1} \geq 0, \quad P\text{-a.s.}$$

*(c) For every $j = 0, \ldots, J-1$ and any two random variables $Y^{(1)}, Y^{(2)} \in L^{\infty-}(\mathbb{R})$ with $Y^{(1)} \geq Y^{(2)}$ $P$-a.s., the following monotonicity condition is satisfied:*

$$G_j\left(E_j\left[\beta_{j+1} Y^{(1)}\right], F_j\left(E_j\left[\beta_{j+1} Y^{(1)}\right]\right)\right) \geq G_j\left(E_j\left[\beta_{j+1} Y^{(2)}\right], F_j\left(E_j\left[\beta_{j+1} Y^{(2)}\right]\right)\right), \quad P\text{-a.s.}$$

*Then, (b) ⇒ (c) ⇒ (a).*

*Proof.* (b) ⇒ (c): Fix $j \in \{0, \ldots, J-1\}$ and let $Y^{(1)}$ and $Y^{(2)}$ be random variables which are in $L^{\infty-}(\mathbb{R})$ and satisfy $Y^{(1)} \geq Y^{(2)}$. By Lemma 1.2.4, there are $r \in \mathcal{A}_0^F$ and $(\rho^{(1)}, \rho^{(0)}) \in \mathcal{A}_0^G$ such that

$$F_j\left(E_j\left[\beta_{j+1} Y^{(2)}\right]\right) = r_j^\top E_j\left[\beta_{j+1} Y^{(2)}\right] - F_j^{\#}(r_j)$$

and

$$G_j\left(E_j\left[\beta_{j+1}Y^{(1)}\right], F_j\left(E_j\left[\beta_{j+1}Y^{(1)}\right]\right)\right)$$
$$= \left(\rho_j^{(1)}\right)^\top E_j\left[\beta_{j+1}Y^{(1)}\right] + \rho_j^{(0)} F_j\left(E_j\left[\beta_{j+1}Y^{(1)}\right]\right) - G_j^{\#}\left(\rho_j^{(1)}, \rho_j^{(0)}\right),$$

$P$-almost surely. Hence, by (2.9), $(b)$ and (1.23) we obtain

$$\begin{aligned}
& G_j\left(E_j\left[\beta_{j+1}Y^{(2)}\right], F_j\left(E_j\left[\beta_{j+1}Y^{(2)}\right]\right)\right) \\
\leq\ & \left(\rho_j^{(1)}\right)^\top E_j\left[\beta_{j+1}Y^{(2)}\right] + \rho_j^{(0)} F_j\left(E_j\left[\beta_{j+1}Y^{(2)}\right]\right) - G_j^{\#}\left(\rho_j^{(1)}, \rho_j^{(0)}\right) \\
=\ & E_j\left[\left(\rho_j^{(1)} + \rho_j^{(0)} r_j\right)^\top \beta_{j+1}Y^{(2)} - \rho_j^{(0)} F_j^{\#}(r_j) - G_j^{\#}\left(\rho_j^{(1)}, \rho_j^{(0)}\right)\right] \\
\leq\ & E_j\left[\left(\rho_j^{(1)} + \rho_j^{(0)} r_j\right)^\top \beta_{j+1}Y^{(1)} - \rho_j^{(0)} F_j^{\#}(r_j) - G_j^{\#}\left(\rho_j^{(1)}, \rho_j^{(0)}\right)\right] \\
\leq\ & \left(\rho_j^{(1)}\right)^\top E_j\left[\beta_{j+1}Y^{(1)}\right] + \rho_j^{(0)} F_j\left(E_j\left[\beta_{j+1}Y^{(1)}\right]\right) - G_j^{\#}\left(\rho_j^{(1)}, \rho_j^{(0)}\right) \\
=\ & G_j\left(E_j\left[\beta_{j+1}Y^{(1)}\right], F_j\left(E_j\left[\beta_{j+1}Y^{(1)}\right]\right)\right).
\end{aligned}$$

$(c) \Rightarrow (a)$: We prove this implication by backward induction. Let $Y^{up}$ and $Y^{low}$ respectively be super- and subsolutions of (2.1). Then, the assertion is trivially true for $j = J$, since $Y_J^{low} \leq Y_J \leq Y_J^{up}$ by definition of super- and subsolutions. Now assume, that the assertion is true for $j+1$. It follows by $(c)$, the induction hypothesis and the definition of a sub- and supersolution that

$$\begin{aligned}
Y_j^{up} &\geq G_j\left(E_j\left[\beta_{j+1}Y_{j+1}^{up}\right], F_j\left(E_j\left[\beta_{j+1}Y_{j+1}^{up}\right]\right)\right) \\
&\geq G_j\left(E_j\left[\beta_{j+1}Y_{j+1}^{low}\right], F_j\left(E_j\left[\beta_{j+1}Y_{j+1}^{low}\right]\right)\right) \\
&\geq Y_j^{low}.
\end{aligned}$$

□

Compared to Theorem 1.4.1, Proposition 2.4.1 does not provide equivalent characterizations but sufficient conditions for the comparison principle to hold. Recalling that the coupled recursions (1.30) in Section 1.5 relied on the equivalence of the comparison principle and the positivity statement $(b)$ in Theorem 1.4.1, we observe that upper and lower bounds cannot be constructed in the same way in the current setting. As in Section 2.2, the rationale of the following construction is, thus, to consider the concave and the convex part of the dynamic programming equation (2.1) separately. This allows us to rely on the equivalent characterizations of the comparison principle in Theorem 1.4.1 and to apply the coupled bounds for the concave respectively convex part. Finally, a straightforward composition of these bounds leads to the following recursion:

Let $j \in \{0, \ldots, J-1\}$ and admissible controls $(\rho^{(1)}, \rho^{(0)}) \in \mathcal{A}_j^G$, $r \in \mathcal{A}_j^F$ and a martingale $M \in \mathcal{M}_D$ be given. Then, we define the in general non-adapted processes $\theta_i^{up} = \theta_i^{up}(\rho^{(1)}, \rho^{(0)}, r, M)$ and $\theta_i^{low} = \theta_i^{low}(\rho^{(1)}, \rho^{(0)}, r, M)$, $i = j, \ldots, J$, via the pathwise dynamic program

$$\theta_J^{up} = \theta_J^{low} = \xi,$$

$$
\begin{aligned}
\theta_i^{up} &= \left(\left(\rho_i^{(1)}\right)^\top \beta_{i+1}\right)_+ \theta_{i+1}^{up} - \left(\left(\rho_i^{(1)}\right)^\top \beta_{i+1}\right)_- \theta_{i+1}^{low} - \left(\rho_i^{(1)}\right)^\top \Delta M_{i+1} \\
&\quad + \rho_i^{(0)} \max_{\iota \in \{up,low\}} F_i(\beta_{i+1}\theta_{i+1}^{\iota} - \Delta M_{i+1}) - G_i^{\#}\left(\rho_i^{(1)}, \rho_i^{(0)}\right) \\
\theta_i^{low} &= \min_{\iota \in \{up,low\}} G_i \left(\beta_{i+1}\theta_{i+1}^{\iota} - \Delta M_{i+1}, \left(r_i^\top \beta_{i+1}\right)_+ \theta_{i+1}^{low} - \left(r_i^\top \beta_{i+1}\right)_- \theta_{i+1}^{up} \right. \\
&\quad \left. - r_i^\top \Delta M_{i+1} - F_i^{\#}(r_i)\right), \qquad (2.27)
\end{aligned}
$$

for $i = J-1, \ldots, j$. This leads to the following regularity result for which we omit the details of the straightforward proof.

**Lemma 2.4.2.** *Suppose Assumption 2.1.1. Then, for every $j \in \{0, \ldots, J-1\}$, $M \in \mathcal{M}_D$, $(\rho^{(1)}, \rho^{(0)}) \in \mathcal{A}_j^G$, and $r \in \mathcal{A}_j^F$, the processes $\theta^{up}(\rho^{(1)}, \rho^{(0)}, r, M)$ and $\theta^{low}(\rho^{(1)}, \rho^{(0)}, r, M)$ which are given by (2.27) satisfy $\theta_i^{up}(\rho^{(1)}, \rho^{(0)}, r, M) \in L^{\infty-}(\mathbb{R})$ respectively $\theta_i^{low}(\rho^{(1)}, \rho^{(0)}, r, M) \in L^{\infty-}(\mathbb{R})$ for all $i = j, \ldots, J$.*

Therefore, we next have to show that the processes $Y^{up}$ and $Y^{low}$ defined by $Y_j^{up} := E_j[\theta_j^{up}]$ and $Y_j^{low} := E_j[\theta_j^{low}]$, $j = 0, \ldots, J$, are super- and subsolutions to (2.1), which satisfy the comparison principle. To do this, we require a generalization of Proposition 1.5.2 which provides representations of the recursions (2.27) and states that $\theta^{up}$ and $\theta^{low}$ are ordered.

**Proposition 2.4.3.** *Suppose Assumption 2.1.1 and let $M \in \mathcal{M}_D$. Then, for every $j = 0, \ldots, J$, $\left(\rho^{(1)}, \rho^{(0)}\right) \in \mathcal{A}_j^G$, and $r \in \mathcal{A}_j^F$, we have for all $i = j, \ldots, J$ the $P$-almost sure identities*

$$
\theta_i^{up}\left(\rho^{(1)}, \rho^{(0)}, r, M\right) = \sup_{u \in \mathbb{R}^D} \Phi_{i+1}\left(\rho_i^{(1)}, \rho_i^{(0)}, u, \theta_{i+1}^{up}(\rho^{(1)}, \rho^{(0)}, r, M), \theta_{i+1}^{low}(\rho^{(1)}, \rho^{(0)}, r, M), \Delta M_{i+1}\right)
$$

*and*

$$
\begin{aligned}
&\theta_i^{low}\left(\rho^{(1)}, \rho^{(0)}, r, M\right) \\
&\quad = \inf_{\left(v^{(1)}, v^{(0)}\right) \in \mathbb{R}^{D+1}} \Phi_{i+1}\left(v^{(1)}, v^{(0)}, r_i, \theta_{i+1}^{low}(\rho^{(1)}, \rho^{(0)}, r, M), \theta_{i+1}^{up}(\rho^{(1)}, \rho^{(0)}, r, M), \Delta M_{i+1}\right),
\end{aligned}
$$

*where $\Phi_{J+1}(v^{(1)}, v^{(0)}, u, \vartheta_1, \vartheta_2, m) = \xi$ and*

$$
\begin{aligned}
&\Phi_{i+1}\left(v^{(1)}, v^{(0)}, u, \vartheta_1, \vartheta_2, m\right) \\
&\quad = \left(\left(v^{(1)}\right)^\top \beta_{i+1}\right)_+ \vartheta_1 - \left(\left(v^{(1)}\right)^\top \beta_{i+1}\right)_- \vartheta_2 - \left(v^{(1)}\right)^\top m \\
&\qquad + v^{(0)}\left(\left(u^\top \beta_{i+1}\right)_+ \vartheta_1 - \left(u^\top \beta_{i+1}\right)_- \vartheta_2 - u^\top m - F_i^{\#}(u)\right) - G_i^{\#}\left(v^{(1)}, v^{(0)}\right)
\end{aligned}
$$

*for $i = j, \ldots, J-1$. In particular,*

$$
\theta_i^{low}\left(\rho^{(1)}, \rho^{(0)}, r, M\right) \leq \theta_i^{up}\left(\rho^{(1)}, \rho^{(0)}, r, M\right) \qquad (2.28)
$$

*for every $i = j, \ldots, J$.*

*Proof.* First we fix $j \in \{0, \ldots, J-1\}$, $M \in \mathcal{M}_D$ and controls $\left(\rho^{(1)}, \rho^{(0)}\right)$ and $r$ in $\mathcal{A}_j^G$ respectively $\mathcal{A}_j^F$ and define $\theta^{up}$ and $\theta^{low}$ by (2.27). To lighten the notation, we set

$$\Phi_{i+1}^{low}\left(v^{(1)}, v^{(0)}, r_i\right) = \Phi_{i+1}\left(v^{(1)}, v^{(0)}, r_i, \theta_{i+1}^{low}(\rho^{(1)}, \rho^{(0)}, r, M), \theta_{i+1}^{up}(\rho^{(1)}, \rho^{(0)}, r, M), \Delta M_{i+1}\right)$$

for $i = j, \ldots, J$ and define $\Phi_{i+1}^{up}$ accordingly (interchanging the roles of $\theta^{up}$ and $\theta^{low}$). We show the assertion by backward induction on $i = J, \ldots, j$ with the case $i = J$ being trivial since $\theta_J^{up} = \theta_J^{low} = \Phi_{J+1} = \xi$ by definition. Now suppose that the assertion is true for $i+1$. For any $(v^{(1)}, v^{(0)}) \in \mathbb{R}^{D+1}$ we obtain, by (2.9), the following upper bound for $\theta_i^{low}$:

$$\begin{aligned}
&\Phi_{i+1}^{low}\left(v^{(1)}, v^{(0)}, r_i\right) \\
&= \left(v^{(1)}\right)^\top \left(\beta_{i+1}\left(\theta_{i+1}^{low}\mathbb{1}_{\{(v^{(1)})^\top\beta_{i+1}\geq 0\}} + \theta_{i+1}^{up}\mathbb{1}_{\{(v^{(1)})^\top\beta_{i+1}<0\}}\right) - \Delta M_{i+1}\right) \\
&\quad + v^{(0)}\left(\left(r_i^\top\beta_{i+1}\right)_+\theta_{i+1}^{low} - \left(r_i^\top\beta_{i+1}\right)_-\theta_{i+1}^{up} - r_i^\top\Delta M_{i+1} - F_i^{\#}(r_i)\right) - G_i^{\#}\left(v^{(1)}, v^{(0)}\right) \\
&\geq G_i\Bigg(\beta_{i+1}\left(\theta_{i+1}^{low}\mathbb{1}_{\{(v^{(1)})^\top\beta_{i+1}\geq 0\}} + \theta_{i+1}^{up}\mathbb{1}_{\{(v^{(1)})^\top\beta_{i+1}<0\}}\right) - \Delta M_{i+1}, \\
&\qquad \left(r_i^\top\beta_{i+1}\right)_+\theta_{i+1}^{low} - \left(r_i^\top\beta_{i+1}\right)_-\theta_{i+1}^{up} - r_i^\top\Delta M_{i+1} - F_i^{\#}(r_i)\Bigg) \\
&\geq \min_{\iota\in\{up,low\}} G_i\Bigg(\beta_{i+1}\theta_{i+1}^{\iota} - \Delta M_{i+1}, \left(r_i^\top\beta_{i+1}\right)_+\theta_{i+1}^{low} - \left(r_i^\top\beta_{i+1}\right)_-\theta_{i+1}^{up} \\
&\qquad - r_i^\top\Delta M_{i+1} - F_i^{\#}(r_i)\Bigg) \\
&= \theta_i^{low}.
\end{aligned}$$

We emphasize that this chain of inequalities holds for every $\omega \in \Omega$. Hence,

$$\inf_{(v^{(1)}, v^{(0)})\in\mathbb{R}^{D+1}} \Phi_{i+1}^{low}\left(v^{(1)}, v^{(0)}, r_i\right) \geq \theta_i^{low}$$

for every $\omega \in \Omega$. To conclude the argument for $\theta_i^{low}$, it remains to show that the converse inequality holds $P$-almost surely. Thanks to (2.9), we get

$$\begin{aligned}
&G_i\left(\beta_{i+1}\theta_{i+1}^{\iota} - \Delta M_{i+1}, \left(r_i^\top\beta_{i+1}\right)_+\theta_{i+1}^{low} - \left(r_i^\top\beta_{i+1}\right)_-\theta_{i+1}^{up} - r_i^\top\Delta M_{i+1} - F_i^{\#}(r_i)\right) \\
&= \inf_{\left(v^{(1)}, v^{(0)}\right)\in\mathbb{R}^{D+1}} \left(v^{(1)}\right)^\top\left(\beta_{i+1}\theta_{i+1}^{\iota} - \Delta M_{i+1}\right) \\
&\quad + v^{(0)}\left(\left(r_i^\top\beta_{i+1}\right)_+\theta_{i+1}^{low} - \left(r_i^\top\beta_{i+1}\right)_-\theta_{i+1}^{up} - r_i^\top\Delta M_{i+1} - F_i^{\#}(r_i)\right) - G_i^{\#}\left(v^{(1)}, v^{(0)}\right).
\end{aligned}$$

Together with $\theta_{i+1}^{up} \geq \theta_{i+1}^{low}$ $P$-a.s. (by the induction hypothesis) we obtain

$$\begin{aligned}
\theta_i^{low} &= \min_{\iota\in\{up,low\}} G_i\Bigg(\beta_{i+1}\theta_{i+1}^{\iota} - \Delta M_{i+1}, \left(r_i^\top\beta_{i+1}\right)_+\theta_{i+1}^{low} - \left(r_i^\top\beta_{i+1}\right)_-\theta_{i+1}^{up} \\
&\quad - r_i^\top\Delta M_{i+1} - F_i^{\#}(r_i)\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= \min_{\iota\in\{up,low\}}\Bigg\{\inf_{\left(v^{(1)},v^{(0)}\right)\in\mathbb{R}^{D+1}}\left(v^{(1)}\right)^\top\beta_{i+1}\theta_{i+1}^{\iota}-\left(v^{(1)}\right)^\top\Delta M_{i+1}\\
&\quad +v^{(0)}\left(\left(r_i^\top\beta_{i+1}\right)_+\theta_{i+1}^{low}-\left(r_i^\top\beta_{i+1}\right)_-\theta_{i+1}^{up}-r_i^\top\Delta M_{i+1}-F_i^{\#}\left(r_i\right)\right)-G_i^{\#}\left(v^{(1)},v^{(0)}\right)\Bigg\}\\
&\geq \inf_{\left(v^{(1)},v^{(0)}\right)\in\mathbb{R}^{D+1}}\left(\left(v^{(1)}\right)^\top\beta_{i+1}\right)_+\theta_{i+1}^{low}-\left(\left(v^{(1)}\right)^\top\beta_{i+1}\right)_-\theta_{i+1}^{up}-\left(v^{(1)}\right)^\top\Delta M_{i+1}\\
&\quad +v^{(0)}\left(\left(r_i^\top\beta_{i+1}\right)_+\theta_{i+1}^{low}-\left(r_i^\top\beta_{i+1}\right)_-\theta_{i+1}^{up}-r_i^\top\Delta M_{i+1}-F_i^{\#}\left(r_i\right)\right)-G_i^{\#}\left(v^{(1)},v^{(0)}\right)\\
&= \inf_{\left(v^{(1)},v^{(0)}\right)\in\mathbb{R}^{D+1}}\Phi_{i+1}^{low}\left(v^{(1)},v^{(0)},r_i\right),\quad P\text{-a.s.}
\end{aligned}
$$

We next turn to $\theta_i^{up}$ where the overall strategy of proof is similar. Recall first that the monotonicity of $G$ in the $y$-component implies existence of a set $\bar{\Omega}_\rho$ (depending on $\rho^{(0)}$) of full $P$-measure such that $\rho_k^{(0)}(\omega)\geq 0$ for every $\omega\in\bar{\Omega}_\rho$ and $k=j,\ldots,J-1$. By (1.23) we find that, for any $u\in\mathbb{R}^D$, $\Phi_{i+1}^{up}(\rho_i^{(0)},\rho_i^{(1)},u)$ is a lower bound for $\theta_i^{up}$ on $\bar{\Omega}_\rho$:

$$
\begin{aligned}
&\Phi_{i+1}^{up}\left(\rho_i^{(1)},\rho_i^{(0)},u\right)\\
&= \left(\left(\rho_i^{(1)}\right)^\top\beta_{i+1}\right)_+\theta_{i+1}^{up}-\left(\left(\rho_i^{(1)}\right)^\top\beta_{i+1}\right)_-\theta_{i+1}^{low}-\left(\rho_i^{(1)}\right)^\top\Delta M_{i+1}\\
&\quad +\rho_i^{(0)}\left(\left(u^\top\beta_{i+1}\right)_+\theta_{i+1}^{up}-\left(u^\top\beta_{i+1}\right)_-\theta_{i+1}^{low}-u^\top\Delta M_{i+1}-F_i^{\#}\left(u\right)\right)-G_i^{\#}\left(\rho_i^{(1)},\rho_i^{(0)}\right)\\
&\leq \left(\left(\rho_i^{(1)}\right)^\top\beta_{i+1}\right)_+\theta_{i+1}^{up}-\left(\left(\rho_i^{(1)}\right)^\top\beta_{i+1}\right)_-\theta_{i+1}^{low}-\left(\rho_i^{(1)}\right)^\top\Delta M_{i+1}\\
&\quad +\rho_i^{(0)}F_i\left(\beta_{i+1}\left(\theta_{i+1}^{up}\mathbb{1}_{\{u^\top\beta_{i+1}\geq 0\}}+\theta_{i+1}^{low}\mathbb{1}_{\{u^\top\beta_{i+1}<0\}}\right)-\Delta M_{i+1}\right)-G_i^{\#}\left(\rho_i^{(1)},\rho_i^{(0)}\right)\\
&\leq \left(\left(\rho_i^{(1)}\right)^\top\beta_{i+1}\right)_+\theta_{i+1}^{up}-\left(\left(\rho_i^{(1)}\right)^\top\beta_{i+1}\right)_-\theta_{i+1}^{low}-\left(\rho_i^{(1)}\right)^\top\Delta M_{i+1}\\
&\quad +\rho_i^{(0)}\max_{\iota\in\{up,low\}}F_i(\beta_{i+1}\theta_{i+1}^{\iota}-\Delta M_{i+1})-G_i^{\#}\left(\rho_i^{(1)},\rho_i^{(0)}\right)\\
&= \theta_i^{up}.
\end{aligned}
$$

Hence,

$$
\sup_{u\in\mathbb{R}^D}\Phi_{i+1}^{up}\left(\rho_i^{(1)},\rho_i^{(0)},u\right)\leq\theta_i^{up}
$$

on $\bar{\Omega}_\rho$, and, thus, $P$-almost surely. To complete the proof of the proposition, we show the converse inequality. As $\theta_{i+1}^{up}\geq\theta_{i+1}^{low}$ and $\rho_i^{(0)}\geq 0$ $P$-a.s., we conclude, by (1.23),

$$
\begin{aligned}
\theta_i^{up} &= \left(\left(\rho_i^{(1)}\right)^\top\beta_{i+1}\right)_+\theta_{i+1}^{up}-\left(\left(\rho_i^{(1)}\right)^\top\beta_{i+1}\right)_-\theta_{i+1}^{low}-\left(\rho_i^{(1)}\right)^\top\Delta M_{i+1}\\
&\quad +\rho_i^{(0)}\max_{\iota\in\{up,low\}}F_i(\beta_{i+1}\theta_{i+1}^{\iota}-\Delta M_{i+1})-G_i^{\#}\left(\rho_i^{(1)},\rho_i^{(0)}\right)\\
&= \left(\left(\rho_i^{(1)}\right)^\top\beta_{i+1}\right)_+\theta_{i+1}^{up}-\left(\left(\rho_i^{(1)}\right)^\top\beta_{i+1}\right)_-\theta_{i+1}^{low}-\left(\rho_i^{(1)}\right)^\top\Delta M_{i+1}\\
&\quad +\rho_i^{(0)}\max_{\iota\in\{up,low\}}\left\{\sup_{u\in\mathbb{R}^D}\left(u^\top\beta_{i+1}\theta_{i+1}^{\iota}-u^\top\Delta M_{i+1}-F_i^{\#}\left(u\right)\right)\right\}-G_i^{\#}\left(\rho_i^{(1)},\rho_i^{(0)}\right)
\end{aligned}
$$

$$\leq \left(\left(\rho_i^{(1)}\right)^\top \beta_{i+1}\right)_+ \theta_{i+1}^{up} - \left(\left(\rho_i^{(1)}\right)^\top \beta_{i+1}\right)_- \theta_{i+1}^{low} - \left(\rho_i^{(1)}\right)^\top \Delta M_{i+1}$$
$$+\rho_i^{(0)} \sup_{u\in\mathbb{R}^D} \left(\left(u^\top \beta_{i+1}\right)_+ \theta_{i+1}^{up} - \left(u^\top \beta_{i+1}\right)_- \theta_{i+1}^{low} - u^\top \Delta M_{i+1} - F_i^{\#}(u)\right) - G_i^{\#}\left(\rho_i^{(1)}, \rho_i^{(0)}\right)$$
$$= \sup_{u\in\mathbb{R}^D} \Phi_{i+1}^{up}\left(\rho_i^{(1)}, \rho_i^{(0)}, u\right), \quad P\text{-a.s.}$$

As $\Phi_{i+1}\left(v^{(1)}, v^{(0)}, u, \vartheta_1, \vartheta_2, m\right)$ is increasing in $\vartheta_1$ and decreasing in $\vartheta_2$, we finally get

$$\begin{aligned}
\theta_i^{up} &= \sup_{u\in\mathbb{R}^D} \Phi_{i+1}\left(\rho_i^{(1)}, \rho_i^{(0)}, u, \theta_{i+1}^{up}, \theta_{i+1}^{low}, \Delta M_{i+1}\right) \\
&\geq \sup_{u\in\mathbb{R}^D} \Phi_{i+1}\left(\rho_i^{(1)}, \rho_i^{(0)}, u, \theta_{i+1}^{low}, \theta_{i+1}^{up}, \Delta M_{i+1}\right) \\
&\geq \inf_{\left(v^{(1)}, v^{(0)}\right)\in\mathbb{R}^{D+1}} \Phi_{i+1}\left(v^{(1)}, v^{(0)}, r_i, \theta_{i+1}^{low}, \theta_{i+1}^{up}, \Delta M_{i+1}\right) \\
&= \theta_i^{low}, \quad P\text{-a.s.},
\end{aligned}$$

as $\theta_{i+1}^{up} \geq \theta_{i+1}^{low}$ $P$-a.s. by the induction hypothesis. □

Building on this proposition, we are now in the position to show that $Y^{up}$ and $Y^{low}$ are super- and subsolutions which constitute bounds to the solution $Y$ to (2.1).

**Proposition 2.4.4.** *Suppose Assumption 2.1.1. Then, the processes $Y^{up}$ and $Y^{low}$, which are given by $Y_j^{up} = E_j[\theta_j^{up}(\rho^{(1)}, \rho^{(0)}, r, M)]$ and $Y_j^{low} = E_j[\theta_j^{low}(\rho^{(1)}, \rho^{(0)}, r, M)]$, $j = 0, \ldots, J$ are, respectively, super- and subsolutions to* (2.1) *for every $(\rho^{(1)}, \rho^{(0)}) \in \mathcal{A}_0^G$, $r \in A_0^F$, and $M \in \mathcal{M}_D$.*

*Proof.* Let $(\rho^{(1)}, \rho^{(0)}) \in \mathcal{A}_0^G$, $r \in \mathcal{A}_0^F$, and $M \in \mathcal{M}_D$. Moreover, let the processes $\theta^{up} := \theta^{up}(\rho^{(1)}, \rho^{(0)}, r, M)$ and $\theta^{low} := \theta^{low}(\rho^{(1)}, \rho^{(0)}, r, M)$ be given by (2.27) and define $Y^{up}$ and $Y^{low}$ by $Y_j^{up} = E_j[\theta_j^{up}]$ and $Y_j^{low} = E_j[\theta_j^{low}]$, $j = 0, \ldots, J$. From the definition of $\theta^{up}$ and the martingale property of $M$, we then observe that

$$\begin{aligned}
Y_j^{up} &= E_j\left[\theta_j^{up}\right] \\
&= E_j\Bigg[\left(\left(\rho_j^{(1)}\right)^\top \beta_{j+1}\right)_+ \theta_{j+1}^{up} - \left(\left(\rho_j^{(1)}\right)^\top \beta_{j+1}\right)_- \theta_{j+1}^{low} - \left(\rho_j^{(1)}\right)^\top \Delta M_{j+1} \\
&\quad + \rho_j^{(0)} \max_{\iota\in\{up,low\}} F_j(\beta_{j+1}\theta_{j+1}^{\iota} - \Delta M_{j+1}) - G_j^{\#}\left(\rho_j^{(1)}, \rho_j^{(0)}\right)\Bigg] \\
&= E_j\left[\left(\left(\rho_j^{(1)}\right)^\top \beta_{j+1}\right)_+ \theta_{j+1}^{up} - \left(\left(\rho_j^{(1)}\right)^\top \beta_{j+1}\right)_- \theta_{j+1}^{low}\right] \\
&\quad + \rho_j^{(0)} E_j\left[\max_{\iota\in\{up,low\}} F_j(\beta_{j+1}\theta_{j+1}^{\iota} - \Delta M_{j+1})\right] - G_j^{\#}\left(\rho_j^{(1)}, \rho_j^{(0)}\right)
\end{aligned}$$

holds. Since $\rho_j^{(0)} \geq 0$ $P$-almost surely, we obtain by Jensen's inequality, applied to the convex functions max and $F_j$, that

$$Y_j^{up} \geq E_j\left[\left(\left(\rho_j^{(1)}\right)^\top \beta_{j+1}\right)_+ \theta_{j+1}^{up} - \left(\left(\rho_j^{(1)}\right)^\top \beta_{j+1}\right)_- \theta_{j+1}^{low}\right]$$

$$+\rho_j^{(0)} \max_{\iota\in\{up,low\}} F_j(E_j\left[\beta_{j+1}\theta_{j+1}^{\iota} - \Delta M_{j+1}\right]) - G_j^{\#}\left(\rho_j^{(1)}, \rho_j^{(0)}\right).$$

Applying the martingale property of $M$ once more as well as the pathwise comparison (2.28) in Proposition 2.4.1 yields

$$\begin{aligned}
Y_j^{up} &\geq E_j\left[\left(\left(\rho_j^{(1)}\right)^\top \beta_{j+1}\right)_+ \theta_{j+1}^{up} - \left(\left(\rho_j^{(1)}\right)^\top \beta_{j+1}\right)_- \theta_{j+1}^{up}\right] \\
&\quad +\rho_j^{(0)} \max_{\iota\in\{up,low\}} F_j\left(E_j\left[\beta_{j+1}\theta_{j+1}^{\iota}\right]\right) - G_j^{\#}\left(\rho_j^{(1)}, \rho_j^{(0)}\right) \\
&= \left(\rho_j^{(1)}\right)^\top E_j\left[\beta_{j+1}\theta_{j+1}^{up}\right] + \rho_j^{(0)} \max_{\iota\in\{up,low\}} F_j\left(E_j\left[\beta_{j+1}\theta_{j+1}^{\iota}\right]\right) - G_j^{\#}\left(\rho_j^{(1)}, \rho_j^{(0)}\right).
\end{aligned}$$

By the tower property of the conditional expectation, the non-negativity of $\rho_j^{(0)}$ and (2.9), we conclude that

$$\begin{aligned}
Y_j^{up} &\geq \left(\rho_j^{(1)}\right)^\top E_j\left[\beta_{j+1}Y_{j+1}^{up}\right] + \rho_j^{(0)} \max_{\iota\in\{up,low\}} F_j\left(E_j\left[\beta_{j+1}Y_{j+1}^{\iota}\right]\right) - G_j^{\#}\left(\rho_j^{(1)}, \rho_j^{(0)}\right) \\
&\geq \left(\rho_j^{(1)}\right)^\top E_j\left[\beta_{j+1}Y_{j+1}^{up}\right] + \rho_j^{(0)} F_j\left(E_j\left[\beta_{j+1}Y_{j+1}^{up}\right]\right) - G_j^{\#}\left(\rho_j^{(1)}, \rho_j^{(0)}\right) \\
&\geq G_j\left(E_j\left[\beta_{j+1}Y_{j+1}^{up}\right], F_j\left(E_j\left[\beta_{j+1}Y_{j+1}^{up}\right]\right)\right)
\end{aligned}$$

holds for every $j = 0, \ldots, J-1$, showing that $Y^{up}$ is a supersolution to (2.1). For $Y^{low}$ we follow essentially the same line of reasoning. We first apply Jensen's inequality to the concave functions min and $G_j$ and the martingale property of $M$ to obtain

$$\begin{aligned}
Y_j^{low} &= E_j\left[\theta_j^{low}\right] \\
&= E_j\Bigg[\min_{\iota\in\{up,low\}} G_j\Big(\beta_{j+1}\theta_{j+1}^{\iota} - \Delta M_{j+1}, \\
&\quad \left(r_j^\top \beta_{j+1}\right)_+ \theta_{j+1}^{low} - \left(r_j^\top \beta_{j+1}\right)_- \theta_{j+1}^{up} - r_j^\top \Delta M_{j+1} - F_j^{\#}(r_j)\Big)\Bigg] \\
&\leq \min_{\iota\in\{up,low\}} G_j\Bigg(E_j\left[\beta_{j+1}\theta_{j+1}^{\iota} - \Delta M_{j+1}\right], \\
&\quad E_j\left[\left(r_j^\top \beta_{j+1}\right)_+ \theta_{j+1}^{low} - \left(r_j^\top \beta_{j+1}\right)_- \theta_{j+1}^{up} - r_j^\top \Delta M_{j+1} - F_j^{\#}(r_j)\right]\Bigg) \\
&= \min_{\iota\in\{up,low\}} G_j\left(E_j\left[\beta_{j+1}\theta_{j+1}^{\iota}\right], E_j\left[\left(r_j^\top \beta_{j+1}\right)_+ \theta_{j+1}^{low} - \left(r_j^\top \beta_{j+1}\right)_- \theta_{j+1}^{up}\right] - F_j^{\#}(r_j)\right).
\end{aligned}$$

Since the mapping $y \mapsto G_j(z,y)$ is non-decreasing, it follows from the pathwise comparison (2.28) and the tower property of the conditional expectation that

$$\begin{aligned}
Y_j^{low} &\leq \min_{\iota\in\{up,low\}} G_j\left(E_j\left[\beta_{j+1}\theta_{j+1}^{\iota}\right], r_j^\top E_j\left[\beta_{j+1}\theta_{j+1}^{low}\right] - F_j^{\#}(r_j)\right) \\
&= \min_{\iota\in\{up,low\}} G_j\left(E_j\left[\beta_{j+1}Y_{j+1}^{\iota}\right], r_j^\top E_j\left[\beta_{j+1}Y_{j+1}^{low}\right] - F_j^{\#}(r_j)\right).
\end{aligned}$$

Finally, we observe by (1.23) and the monotonicity assumption on $G_j(z,y)$ in the $y$-variable, that

$$Y_j^{low} \leq \min_{\iota\in\{up,low\}} G_j\left(E_j\left[\beta_{j+1}Y_{j+1}^{\iota}\right], F_j\left(E_j\left[\beta_{j+1}Y_{j+1}^{low}\right]\right)\right)$$

$$\leq \quad G_j\left(E_j\left[\beta_{j+1}Y_{j+1}^{low}\right], F_j\left(E_j\left[\beta_{j+1}Y_{j+1}^{low}\right]\right)\right),$$

which completes the proof. □

Finally, we provide the generalization of Theorem 2.2.3 to this non-monotone setting.

**Theorem 2.4.5.** *Suppose Assumption 2.1.1. Then, for every $j = 0, \ldots, J$,*

$$\begin{aligned} Y_j &= \operatorname*{essinf}_{(\rho^{(1)},\rho^{(0)})\in\mathcal{A}_j^G,\, r\in\mathcal{A}_j^F,\, M\in\mathcal{M}_D} E_j[\theta_j^{up}(\rho^{(1)},\rho^{(0)},r,M)] \\ &= \operatorname*{esssup}_{(\rho^{(1)},\rho^{(0)})\in\mathcal{A}_j^G,\, r\in\mathcal{A}_j^F,\, M\in\mathcal{M}_D} E_j[\theta_j^{low}(\rho^{(1)},\rho^{(0)},r,M)], \quad P\text{-a.s.} \end{aligned}$$

*Moreover,*

$$Y_j = \theta_j^{up}(\rho^{(1,*)},\rho^{(0,*)},r^*,M^*) = \theta_j^{low}(\rho^{(1,*)},\rho^{(0,*)},r^*,M^*) \tag{2.29}$$

*$P$-almost surely, for every $(\rho^{(1,*)},\rho^{(0,*)}) \in \mathcal{A}_j^G$ and $r^* \in \mathcal{A}_j^F$ satisfying the duality relations* (2.12) *and* (2.13) *$P$-almost surely for every $i = j, \ldots, J-1$, and with $M^*$ being the Doob martingale of $\beta Y$.*

*Proof.* Let $j \in \{0, \ldots, J-1\}$ be fixed from now on. We first show that $E_i[\theta_i^{low}] \leq Y_i \leq E_i[\theta_i^{up}]$ for $i = j, \ldots, J$. We prove this by backward induction on $i$. To this end, we fix $M \in \mathcal{M}_D$ and controls $\left(\rho^{(1)},\rho^{(0)}\right)$ and $r$ in $\mathcal{A}_j^G$ respectively $\mathcal{A}_j^F$, as well as "optimizers" $\left(\rho^{(1,*)},\rho^{(0,*)}\right)$ and $r^*$ in $\mathcal{A}_j^G$ respectively $\mathcal{A}_j^F$ which satisfy the duality relations (2.12) and (2.13). By definition of $\theta^{up}$ and $\theta^{low}$ the assertion is trivially true for $i = J$. Suppose that the assertion is true for $i+1$. Recalling Proposition 2.4.3 and applying the tower property of the conditional expectation, we get

$$\begin{aligned} E_i\left[\theta_i^{low}\right] &= E_i\Bigg[\inf_{\left(v^{(1)},v^{(0)}\right)\in\mathbb{R}^{D+1}}\Bigg(\left(\left(v^{(1)}\right)^\top\beta_{i+1}\right)_+\theta_{i+1}^{low} - \left(\left(v^{(1)}\right)^\top\beta_{i+1}\right)_-\theta_{i+1}^{up} \\ &\quad - \left(v^{(1)}\right)^\top\Delta M_{i+1} + v^{(0)}\Bigg(\left(r_i^\top\beta_{i+1}\right)_+\theta_{i+1}^{low} - \left(r_i^\top\beta_{i+1}\right)_-\theta_{i+1}^{up} - r_i^\top\Delta M_{i+1} \\ &\quad - F_i^{\#}(r_i)\Bigg) - G_i^{\#}\left(v^{(1)},v^{(0)}\right)\Bigg)\Bigg] \\ &\leq E_i\Bigg[\left(\left(\rho_i^{(1,*)}\right)^\top\beta_{i+1}\right)_+ E_{i+1}\left[\theta_{i+1}^{low}\right] - \left(\left(\rho_i^{(1,*)}\right)^\top\beta_{i+1}\right)_- E_{i+1}\left[\theta_{i+1}^{up}\right] \\ &\quad - \left(\rho_i^{(1,*)}\right)^\top\Delta M_{i+1} + \rho_i^{(0,*)}\Bigg(\left(r_i^\top\beta_{i+1}\right)_+ E_{i+1}\left[\theta_{i+1}^{low}\right] - \left(r_i^\top\beta_{i+1}\right)_- E_{i+1}\left[\theta_{i+1}^{up}\right] \\ &\quad - r_i^\top\Delta M_{i+1} - F_i^{\#}(r_i)\Bigg) - G_i^{\#}\left(\rho_i^{(1,*)},\rho_i^{(0,*)}\right)\Bigg]. \end{aligned}$$

Finally, the martingale property of $M$ and the induction hypothesis yield

$$\begin{aligned} E_i\left[\theta_i^{low}\right] &\leq E_i\left[\left(\rho_i^{(1,*)}\right)^\top\beta_{i+1}Y_{i+1} + \rho_i^{(0,*)}\left(r_i^\top\beta_{i+1}Y_{i+1} - F_i^{\#}(r_i)\right) - G_i^{\#}\left(\rho_i^{(1,*)},\rho_i^{(0,*)}\right)\right] \\ &\leq G_i\left(E_i\left[\beta_{i+1}Y_{i+1}\right], F_i\left(E_i\left[\beta_{i+1}Y_{i+1}\right]\right)\right) \\ &= Y_i. \end{aligned}$$

Here, the last inequality is an immediate consequence of (2.9), the non-negativity of $\rho_i^{(0,*)}$ and the duality relation (2.12). Applying an analogous argument, we obtain that $E_i[\theta_i^{up}] \geq Y_i$. Indeed,

$$\begin{aligned}
E_i[\theta_i^{up}] &= E_i\Bigg[\left(\left(\rho_i^{(1)}\right)^\top \beta_{i+1}\right)_+ \theta_{i+1}^{up} - \left(\left(\rho_i^{(1)}\right)^\top \beta_{i+1}\right)_- \theta_{i+1}^{low} - \left(\rho_i^{(1)}\right)^\top \Delta M_{i+1} \\
&\quad + \rho_i^{(0)} \sup_{u\in\mathbb{R}^D} \left(\left(u^\top \beta_{i+1}\right)_+ \theta_{i+1}^{up} - \left(u^\top \beta_{i+1}\right)_- \theta_{i+1}^{low} - u^\top \Delta M_{i+1} - F_i^{\#}(u)\right) \\
&\quad - G_i^{\#}\left(\rho_i^{(1)}, \rho_i^{(0)}\right)\Bigg] \\
&\geq E_i\Bigg[\left(\left(\rho_i^{(1)}\right)^\top \beta_{i+1}\right)_+ E_{i+1}\left[\theta_{i+1}^{up}\right] - \left(\left(\rho_i^{(1)}\right)^\top \beta_{i+1}\right)_- E_{i+1}\left[\theta_{i+1}^{low}\right] - \left(\rho_i^{(1)}\right)^\top \Delta M_{i+1} \\
&\quad + \rho_i^{(0)} \Bigg(\left((r_i^*)^\top \beta_{i+1}\right)_+ E_{i+1}\left[\theta_{i+1}^{up}\right] - \left((r_i^*)^\top \beta_{i+1}\right)_- E_{i+1}\left[\theta_{i+1}^{low}\right] - (r_i^*)^\top \Delta M_{i+1} \\
&\quad - F_i^{\#}(r_i^*)\Bigg) - G_i^{\#}\left(\rho_i^{(1)}, \rho_i^{(0)}\right)\Bigg] \\
&\geq \left(\rho_i^{(1)}\right)^\top E_i\left[\beta_{i+1} Y_{i+1}\right] + \rho_i^{(0)}\left((r_i^*)^\top E_i\left[\beta_{i+1} Y_{i+1}\right] - F_i^{\#}(r_i^*)\right) - G_i^{\#}\left(\rho_i^{(1)}, \rho_i^{(0)}\right) \\
&= \left(\rho_i^{(1)}\right)^\top E_i\left[\beta_{i+1} Y_{i+1}\right] + \rho_i^{(0)} F_i\left(E_i\left[\beta_{i+1} Y_{i+1}\right]\right) - G_i^{\#}\left(\rho_i^{(1)}, \rho_i^{(0)}\right) \\
&\geq G_i\left(E_i\left[\beta_{i+1} Y_{i+1}\right], F_i\left(E_i\left[\beta_{i+1} Y_{i+1}\right]\right)\right) \\
&= Y_i.
\end{aligned}$$

making now use of the non-negativity of $\rho_i^{(0)}$, the duality relation (2.13), and (2.9). This establishes $E_i[\theta_i^{low}] \leq Y_i \leq E_i[\theta_i^{up}]$, for $i = j, \ldots, J$.

To complete the proof, it remains to show that pathwise equality holds for the Doob martingale $M^*$ and the optimal controls $(\rho^{(1,*)}, \rho^{(0,*)})$ and $r^*$. Therefore, let $\theta^{up,*} := \theta^{up}(\rho^{(1,*)}, \rho^{(0,*)}, r^*, M^*)$ and $\theta^{low,*} := \theta^{low}(\rho^{(1,*)}, \rho^{(0,*)}, r^*, M^*)$ be given by (2.27). The proof is again by backward induction on $i = J, \ldots, j$, with the case $i = J$ being trivial by definition. Now suppose that the assertion is true for $i+1$. For the lower bound $\theta^{low,*}$, we first observe by the induction hypothesis and the definition of $M^*$ that

$$\begin{aligned}
\theta_i^{low,*} &= \min_{\iota\in\{up,low\}} G_i\Bigg(\beta_{i+1}\theta_{i+1}^{\iota,*} - \Delta M_{i+1}^*, \left((r_i^*)^\top \beta_{i+1}\right)_+ \theta_{i+1}^{low,*} - \left((r_i^*)^\top \beta_{i+1}\right)_- \theta_{i+1}^{up,*} \\
&\quad - (r_i^*)^\top \Delta M_{i+1}^* - F_i^{\#}(r_i^*)\Bigg) \\
&= \min_{\iota\in\{up,low\}} G_i\Bigg(\beta_{i+1} Y_{i+1} - (\beta_{i+1} Y_{i+1} - E_i[\beta_{i+1} Y_{i+1}]), \\
&\quad \left((r_i^*)^\top \beta_{i+1}\right)_+ Y_{i+1} - \left((r_i^*)^\top \beta_{i+1}\right)_- Y_{i+1} - (r_i^*)^\top (\beta_{i+1} Y_{i+1} - E_i[\beta_{i+1} Y_{i+1}]) \\
&\quad - F_i^{\#}(r_i^*)\Bigg) \\
&= G_i\left(E_i[\beta_{i+1} Y_{i+1}], (r_i^*)^\top E_i[\beta_{i+1} Y_{i+1}] - F_i^{\#}(r_i^*)\right).
\end{aligned}$$

From the duality relation (2.13) it follows that

$$\theta_i^{low,*} = G_i\left(E_i[\beta_{i+1} Y_{i+1}]), F_i\left(E_i[\beta_{i+1} Y_{i+1}]\right)\right) = Y_i.$$

Similarly, it holds for the upper bound that

$$\begin{aligned}
\theta_i^{up,*} &= \left(\left(\rho_i^{(1,*)}\right)^\top \beta_{i+1}\right)_+ \theta_{i+1}^{up,*} - \left(\left(\rho_i^{(1,*)}\right)^\top \beta_{i+1}\right)_- \theta_{i+1}^{low,*} - \left(\rho_i^{(1,*)}\right)^\top \Delta M_{i+1}^* \\
&\quad +\rho_i^{(0,*)} \max_{\iota\in\{up,low\}} F_i(\beta_{i+1}\theta_{i+1}^{\iota,*} - \Delta M_{i+1}^*) - G_i^{\#}\left(\rho_i^{(1,*)}, \rho_i^{(0,*)}\right) \\
&= \left(\left(\rho_i^{(1,*)}\right)^\top \beta_{i+1}\right)_+ Y_{i+1} - \left(\left(\rho_i^{(1,*)}\right)^\top \beta_{i+1}\right)_- Y_{i+1} - \left(\rho_i^{(1,*)}\right)^\top (\beta_{i+1}Y_{i+1} - E_i[\beta_{i+1}Y_{i+1}]) \\
&\quad +\rho_i^{(0,*)} \max_{\iota\in\{up,low\}} F_i(\beta_{i+1}Y_{i+1} - (\beta_{i+1}Y_{i+1} - E_i[\beta_{i+1}Y_{i+1}])) - G_i^{\#}\left(\rho_i^{(1,*)}, \rho_i^{(0,*)}\right) \\
&= \left(\rho_i^{(1,*)}\right)^\top E_i[\beta_{i+1}Y_{i+1}] + \rho_i^{(0,*)} F_i(E_i[\beta_{i+1}Y_{i+1}]) - G_i^{\#}\left(\rho_i^{(1,*)}, \rho_i^{(0,*)}\right)
\end{aligned}$$

by the induction hypothesis and the definition of the Doob martingale $M^*$. Applying the duality relation (2.12), we finally conclude that

$$\theta_i^{up,*} = G_i\left(E_i[\beta_{i+1}Y_{i+1}]), F_i\left(E_i[\beta_{i+1}Y_{i+1}]\right)\right) = Y_i,$$

and thus (2.29) is established. □

*Remark* 2.4.6. In this chapter, we discussed the construction of super- and subsolutions to one-dimensional concave-convex dynamic programming equations. Similar to Chapter 1, we could also consider systems of concave-convex dynamic programs of the form

$$\begin{aligned}
Y_J^{(\nu)} &= \xi^{(\nu)} \\
Y_j^{(\nu)} &= G_j^{(\nu)}\left(E_j\left[\beta_{j+1}Y_{j+1}^{(1)}\right], \ldots, E_j\left[\beta_{j+1}Y_{j+1}^{(N)}\right], F_j^{(\nu)}\left(E_j\left[\beta_{j+1}Y_{j+1}^{(1)}\right], \ldots, E_j\left[\beta_{j+1}Y_{j+1}^{(N)}\right]\right)\right)
\end{aligned} \tag{2.30}$$

for $j = J-1, \ldots, 0$ and $\nu = 1, \ldots, N$, where the functions $G_j^{(\nu)}$ and $F_j^{(\nu)}$ satisfy the Assumptions 2.1.1 with $D$ replaced by $ND$. While the sets of admissible controls $\mathcal{A}_j^{F^{(\nu)}}$ for the functions $F_j^{(\nu)}$ coincide for every $j = 0, \ldots, J-1$ with those introduced in Chapter 1, the corresponding sets for the functions $G_j^{(\nu)}$ are given by

$$\begin{aligned}
\mathcal{A}_j^{G^{(\nu)}} &= \left\{ \left(\rho_i^{(\nu)}, \rho_i^{(\nu),0}\right)_{i=j,\ldots,J-1} \,\middle|\, \left(\rho_i^{(\nu)}, \rho_i^{(\nu),0}\right) \in L_i^{\infty-}(\mathbb{R}^{ND+1}), \right. \\
&\qquad \left. G_i^{(\nu,\#)}\left(\rho_i^{(\nu)}, \rho_i^{(\nu),0}\right) \in L^{\infty-}(\mathbb{R})\ \forall i = j, \ldots, J-1 \right\}, \quad j = 0, \ldots, J-1.
\end{aligned}$$

Then, the preceding results of this chapter can be transferred to the multi-dimensional situation in a straightforward way. In particular, the coupled recursions (2.27) are generalized in the following way:

$$\begin{aligned}
\theta_J^{(up,\nu)} &= \theta_J^{(low,\nu)} = \xi^{(\nu)}, \\
\theta_j^{(up,\nu)} &= \sum_{n=1}^N \left(\left(\rho_j^{(\nu),[n]}\right)^\top \beta_{j+1}\right)_+ \theta_{j+1}^{(up,n)} - \sum_{n=1}^N \left(\left(\rho_j^{(\nu),[n]}\right)^\top \beta_{j+1}\right)_- \theta_{j+1}^{(low,n)} \\
&\quad +\rho_j^{(\nu),0} \max_{\iota\in\{up,low\}^N} \left\{ F_j^{(\nu)}\left(\beta_{j+1}\theta_{j+1}^{(\iota_1,1)} - \Delta M_{j+1}^{[1]}, \ldots, \beta_{j+1}\theta_{j+1}^{(\iota_N,N)} - \Delta M_{j+1}^{[N]}\right)\right\}
\end{aligned}$$

$$
\begin{aligned}
& -\sum_{n=1}^{N}\left(\rho_j^{(\nu),[n]}\right)^\top \Delta M_{j+1}^{[n]} - G_j^{(\nu,\#)}\left(\rho_j^{(\nu)},\rho_j^{(\nu),0}\right), \\
\theta_j^{(low,\nu)} \;=\; & \min_{\iota\in\{up,low\}^N}\Bigg\{G_j^{(\nu)}\Bigg(\beta_{j+1}\theta_{j+1}^{(\iota_1,1)}-\Delta M_{j+1}^{[1]},\dots,\beta_{j+1}\theta_{j+1}^{(\iota_N,N)}-\Delta M_{j+1}^{[N]}, \\
& \sum_{n=1}^{N}\left(\left(r_j^{(\nu),[n]}\right)^\top\beta_{j+1}\right)_+\theta_{j+1}^{(low,n)} - \sum_{n=1}^{N}\left(\left(r_j^{(\nu),[n]}\right)^\top\beta_{j+1}\right)_-\theta_{j+1}^{(up,n)} \\
& -\sum_{n=1}^{N}\left(r_j^{(\nu),[n]}\right)^\top \Delta M_{j+1}^{[n]} - F_j^{(\nu,\#)}\left(r_j^{(\nu)}\right)\Bigg)\Bigg\}, \quad j=J-1,\dots,0,\ \nu=1,\dots,N,
\end{aligned}
$$

where $(\rho^{(\nu)},\rho^{(\nu),0})\in\mathcal{A}_0^{G^{(\nu)}}$, $r\in\mathcal{A}_0^{F^{(\nu)}}$, $\nu=1,\dots,N$, and $M\in\mathcal{M}_{ND}$. We emphasize that further generalizations like e.g. different processes $\beta^{(\nu)}$, $\nu=1,\dots,N$, may also be easily incorporated.

## 2.5 Numerical example

In Example 2.1.3 (ii), we introduced the problem of pricing a payoff at maturity under bilateral counterparty risk as proposed in Crépey et al. (2013). In this section, we slightly generalize this example by introducing intermediate payments which arise at predetermined points in time. This generalization allows us to consider the problem of pricing a swap contract under bilateral counterparty risk as a numerical example. The rationale of a swap derivative is that an investor and a counterparty agree to exchange payments at given time points, where one party pays a fixed a leg and, in return, receives a variable leg from the other. Due to the variable leg, the signs of the payments are random so that a consistent pricing approach should reflect the default risk of both parties.

To this end, let $0=t_0<t_1<\ldots<t_J=T$ be an equidistant partition of $[0,T]$ with time increments $\Delta$. Then, we have seen in Example 2.1.3 (ii), that this problem is captured by the concave-convex dynamic program

$$
\begin{aligned}
Y_J &= C_{t_J}, \\
Y_j &= (1-\Delta(r_{t_j}+\gamma_{t_j}(1-\mathfrak{r})(1-2p_{t_j})+\overline{\lambda}))E_j[Y_{j+1}] \\
&\quad +\Delta(\gamma_{t_j}(1-\mathfrak{r})(1-3p_{t_j})+\overline{\lambda}-\lambda)E_j[Y_{j+1}]_+ + C_{t_j}. \qquad (2.31)
\end{aligned}
$$

Recall, that the process $(r_t)_{t\in[0,T]}$ denotes the risk-less short rate, and that $\gamma_t$ reflects the rate at which default of either side occurs at time $t$. Moreover, $p_t$ is the conditional probability that the counterparty defaults, if default occurs at time $t$. Finally, the parameters $\mathfrak{r}$, $\lambda$ and $\overline{\lambda}$ are associated with the recovery rate respectively the costs for external lending and borrowing. Note that the dynamic program (2.31) involves, compared to Example 2.1.3 (ii), the additional term $C_{t_j}$, which reflects intermediate payments at fixed time points $t_0,\dots,t_J$.

Following Example 2.1.3 (ii), the dynamic program (2.31) can be represented by the functions $G_j:\mathbb{R}^2\to\mathbb{R}$ and $F_j:\mathbb{R}\to\mathbb{R}$ defined by

$$
G_j(z,y)=g_j\,z+(h_j)_+\,y-(h_j)_-\,z_+ + C_{t_j} \quad \text{and} \quad F_j(z)=z_+,
$$

where, as before,

$$
g_j = 1-\Delta(r_{t_j}+\gamma_{t_j}(1-\mathfrak{r})(1-2p_{t_j})+\overline{\lambda})
$$

and

$$h_j = \Delta(\gamma_{t_j}(1-\mathfrak{r})(1-3p_{t_j}) + \overline{\lambda} - \lambda).$$

Here, we slightly modified the function $G_j$ compared to Example 2.1.3 (ii) to capture the payment stream $(C_{t_j})_{j=0,\dots,J}$. Note that $\beta \equiv 1$ in this example and therefore a sufficient condition for the comparison principle to hold is that the function $G_j$ is increasing in $z$, cp. Proposition 2.4.1 ($c$). This, however, depends on the choice of the stochastic processes $\gamma$, $p$ and $r$, so that the comparison principle is not a generic property of the dynamic program (2.31).

From Appendix A.2, we further conclude that $G_j^{\#}(v_1, v_2) = -C_{t_j}$ and $F_j^{\#} \equiv 0$ on their effective domains $D_{G^{\#}}^{(j,\cdot)} = [g_j - (h_j)_-, g_j] \times \{(h_j)_+\}$ respectively $D_{F^{\#}}^{(j,\cdot)} = [0,1]$. We emphasize that the result in Appendix A.2 still applies for the function $G_j$ by first passing to the convex function $-G_j$ and then using the relation

$$-((-G_j)^{\#}(-v_1, -v_2)) = G_j^{\#}(v_1, v_2)$$

for $(-v_1, -v_2) \in D_{(-G)^{\#}}^{(j,\cdot)}$. We thus obtain that the duality relations (2.12) and (2.13) read as

$$\rho_j^{(1,*)} E_j[Y_{j+1}] + \rho_j^{(0,*)} (E_j[Y_{j+1}])_+ + C_{t_j} = g_j E_j[Y_{j+1}] + (h_j)_+ (E_j[Y_{j+1}])_+ - (h_j)_- (E_j[Y_{j+1}])_+ + C_{t_j}$$

and

$$r_j^* E_j[Y_{j+1}] = (E_j[Y_{j+1}])_+$$

for $j = 0, \dots, J-1$. For these equations, solutions are given by

$$\left(\rho_j^{(1,*)}, \rho_j^{(0,*)}\right) = \begin{cases} (g_j - (h_j)_-, (h_j)_+), & E_j[Y_{j+1}] \geq 0 \\ (g_j, (h_j)_+), & E_j[Y_{j+1}] < 0 \end{cases}$$

respectively

$$r_j^* = \begin{cases} 1, & E_j[Y_{j+1}] \geq 0 \\ 0, & E_j[Y_{j+1}] < 0. \end{cases}$$

In our numerical example, the payment stream $C_{t_j}$ is given by a swap with notional $N$, fixed rate $R$ and an equidistant sequence of tenor dates $\mathcal{T} = \{T_0, \dots, T_K\} \subseteq \{t_0, \dots t_J\}$. Denote by $\delta$ the length of the time interval between $T_i$ and $T_{i+1}$ and by $P(T_{i-1}, T_i)$ the $T_{i-1}$-price of a zero-bond with maturity $T_i$. Then, the payment process $C_{t_j}$ is given by

$$C_{T_i} = N \cdot \left( \frac{1}{P(T_{i-1}, T_i)} - (1 + R\delta) \right)$$

for $T_i \in \mathcal{T} \setminus \{T_0\}$ and $C_{t_j} = 0$ otherwise, see Brigo and Mercurio (2006), Chapter 1.

For $r$ and $\gamma$, we implement the model of Brigo and Pallavicini (2007), assuming that the risk-neutral dynamics of $r$ is given by a two-factor Gaussian short rate model, a reparametrization of the two-factor Hull-White model, while $\gamma$ is a Cox-Ingersoll-Ross process. For the conditional default probabilities $p_t$ we assume $p_t = 0 \wedge \tilde{p}_t \vee 1$ where $\tilde{p}$ is an Ornstein-Uhlenbeck process. In continuous time, this corresponds to the system of stochastic differential equations

$$\begin{aligned} dx_t &= -\kappa_x x_t dt + \sigma_x dW_t^x, \\ dy_t &= -\kappa_y y_t dt + \sigma_y dW_t^y, \\ d\gamma_t &= \kappa_\gamma(\mu_\gamma - \gamma_t)dt + \sigma_\gamma \sqrt{\gamma_t} dW_t^\gamma, \\ d\tilde{p}_t &= \kappa_p(\mu_p - \tilde{p}_t)dt + \sigma_p dW_t^p \end{aligned}$$

with $r_t = r_0+x_t+y_t$, $x_0 = y_0 = 0$. Here, $W^x$, $W^y$ and $W^\gamma$ are Brownian motions with instantaneous correlations $\rho_{xy}$, $\rho_{x\gamma}$ and $\rho_{y\gamma}$. In addition, we assume that $W_t^p = \rho_{\gamma p} W_t^\gamma + \sqrt{1-\rho_{\gamma p}^2} W_t$ where the Brownian motion $W$ is independent of $(W^x, W^y, W^\gamma)$. We choose the filtration generated by the four Brownian motions as the reference filtration.

For the dynamics of $x$, $y$ and $\tilde{p}$, exact time discretizations are available in closed form and are given by

$$x_j = x_{j-1} e^{-\kappa_x \Delta} + \sigma_x \sqrt{\frac{1-e^{-2\kappa_x\Delta}}{2\kappa_x\Delta}} \Delta W_j^x, \quad x_0 = 0,$$

$$y_j = y_{j-1} e^{-\kappa_y \Delta} + \sigma_y \sqrt{\frac{1-e^{-2\kappa_y\Delta}}{2\kappa_y\Delta}} \Delta W_j^y, \quad y_0 = 0,$$

$$\tilde{p}_j = \tilde{p}_{j-1} e^{-\kappa_p \Delta} + \mu_p \left(1-e^{-\kappa_p\Delta}\right) + \sigma_p \sqrt{\frac{1-e^{-2\kappa_p\Delta}}{2\kappa_p\Delta}} \Delta W_j^p, \quad \tilde{p}_0 = p_0,$$

see e.g. Section 3.3 in Glasserman (2004). Note that we passed, at the same time, to the shorthand notation $U_j := U_{t_j}$ for $U \in \{x, y, \tilde{p}\}$. We discretize $\gamma$ by $(\tilde{\gamma}_j)_+$, where $\tilde{\gamma}_j := \tilde{\gamma}_{t_j}$ denotes the fully truncated scheme of Lord et al. (2010), i.e.

$$\tilde{\gamma}_j = \tilde{\gamma}_{j-1} - \kappa_\gamma \Delta \left((\tilde{\gamma}_{j-1})_+ - \mu_\gamma\right) + \sigma_\gamma \sqrt{(\tilde{\gamma}_{j-1})_+} \Delta W_j^\gamma, \quad \tilde{\gamma}_0 = \gamma_0.$$

The bond prices $P(t,s)$ are given as an explicit function of $x_t$ and $y_t$ in this model, namely by

$$P(t,s) = \exp\left\{-r_0(s-t) - \frac{1-e^{-\kappa_x(s-t)}}{\kappa_x} x_t - \frac{1-e^{-\kappa_y(s-t)}}{\kappa_y} y_t + \frac{1}{2} V(t,s)\right\}, \quad t,s \in [0,T],\ t<s.$$

Here, the deterministic function $V$ is defined by

$$\begin{aligned} V(t,s) \;=\; & \frac{\sigma_x^2}{\kappa_x^2}\left(s-t+\frac{2}{\kappa_x} e^{-\kappa_x(s-t)} - \frac{1}{2\kappa_x} e^{-2\kappa_x(s-t)} - \frac{3}{2\kappa_x}\right) \\ & + \frac{\sigma_y^2}{\kappa_y^2}\left(s-t+\frac{2}{\kappa_y} e^{-\kappa_y(s-t)} - \frac{1}{2\kappa_y} e^{-2\kappa_y(s-t)} - \frac{3}{2\kappa_y}\right) \\ & + 2\rho_{xy}\frac{\sigma_x\sigma_y}{\kappa_x\kappa_y}\left(s-t+\frac{e^{-\kappa_x(s-t)}-1}{\kappa_x} + \frac{e^{-\kappa_y(s-t)}-1}{\kappa_y} - \frac{e^{-(\kappa_x+\kappa_y)(s-t)}-1}{\kappa_x+\kappa_y}\right), \end{aligned}$$

see Section 4.2 of Brigo and Mercurio (2006). This implies that the swap's "clean price", i.e., the price in the absence of counterparty risk, is given in closed form as well:

$$S_t = P(t, T_{\tau(t)}) C_{T_{\tau(t)}} + N \cdot \sum_{i=\tau(t)+1}^{K} \left(P(t, T_{i-1}) - (1+R\delta) P(t, T_i)\right),$$

see Section 1.5 of Brigo and Mercurio (2006). Here, $\tau(t) \in \{1, \ldots, K\}$ denotes the index of the first tenor date after $t$ (with $\tau(t) = t$ if $t$ is a tenor date).

We consider 60 half-yearly payments over a horizon of $T = 30$ years, i.e., $\delta = 0.5$. $J$ is always chosen as an integer multiple of 60 so that $\delta$ is an integer multiple of $\Delta = T/J$. For the model parameters, we choose

$$(r_0, \kappa_x, \sigma_x, \kappa_y, \sigma_y) = (0.03, 0.0558, 0.0093, 0.5493, 0.0138),$$

$$(\gamma_0, \mu_\gamma, \kappa_\gamma, \sigma_\gamma, p_0, \mu_p, \kappa_p, \sigma_p) = (0.0165,\ 0.026,\ 0.4, 0.14,\ 0.5,\ 0.5,\ 0.8,\ 0.2),$$
$$(\rho_{xy}, \rho_{x\gamma}, \rho_{y\gamma}, \mathfrak{r}, \lambda, \overline{\lambda}, N) = (-0.7,\ 0.05,\ -0.7,\ 0.4,\ 0.015,\ 0.045,\ 1).$$

We thus largely follow Brigo and Pallavicini (2007) for the parametrization of $r$ and $\gamma$ but leave out their calibration to initial market data and choose slightly different correlations to avoid the extreme cases of a perfect correlation or independence of $r$ and $\gamma$. The remaining parameters $J$, $R$ and $\rho_{\gamma p}$ are varied in the numerical experiments below.

We initialize the regression at $\tilde{Y}_J = S_{t_J} = C_{t_J}$ and choose, at each time step $1 \le j \le J-1$, the four basis functions

$$\eta_{j,1}(X_j) = 1, \quad \eta_{j,2}(X_j) = \tilde{\gamma}_j, \quad \eta_{j,3}(X_j) = \tilde{\gamma}_j \cdot \tilde{p}_j, \quad \eta_{j,4}(X_j) = S_{t_j},$$

where the process $(X_j)_{j=0,\dots,J}$ defined by $X_j := (x_j, y_j, \tilde{\gamma}_j, \tilde{p}_j, x_{T(j)}, y_{T(j)})$ denotes the underlying discrete-time Markov process. Here, $T(j)$ denotes the largest tenor date which is strictly smaller than $t_j$. Note that we require to include the random variables $x_{T(j)}$ and $y_{T(j)}$ in order to obtain a Markovian framework, as the payment $C$ of the swap at the next tenor date following $T(j)$ is a deterministic function of $x_{T(j)}$ and $y_{T(j)}$. As in the numerical examples before, the one-step conditional expectations $R^{(0)}_{j-1,k}(X_j) := E_{j-1}[\eta_{j,k}(X_j)]$ of these basis functions are available in closed form. Straightforward computations yield

$$\begin{aligned}
R_{j-1,1}(X_{j-1}) &= 1, \\
R_{j-1,2}(X_{j-1}) &= \tilde{\gamma}_{j-1} - \kappa_\gamma \Delta \left( (\tilde{\gamma}_{j-1})_+ - \mu_\gamma \right), \\
R_{j-1,3}(X_{j-1}) &= \left( \tilde{\gamma}_{j-1} - \kappa_\gamma \Delta ((\tilde{\gamma}_{j-1})_+ - \mu_\gamma) \right) \left( \tilde{p}_{j-1} e^{-\kappa_p \Delta} + \mu_p (1 - e^{-\kappa_p \Delta}) \right) \\
&\quad + \sigma_\gamma \sigma_p \rho_{\gamma p} \sqrt{(\tilde{\gamma}_{j-1})_+ \Delta} \sqrt{\frac{1 - e^{-2\kappa_p \Delta}}{2\kappa_p}}, \\
R_{j-1,4}(X_{j-1}) &= E_{j-1}\left[ P(t_j, T_{\tau(j)}) \right] C_{T_{\tau(j)}} \\
&\quad + N \sum_{i=\tau(j)+1}^{K} \left( E_{j-1}\left[ P(t_j, T_{i-1}) \right] - (1 + R\delta) E_{j-1}\left[ P(t_j, T_i) \right] \right).
\end{aligned}$$

In Appendix B.1, we provide the closed-form expressions for $E_{j-1}\left[P(t_j, T_i)\right]$ as well as a detailed derivation of these conditional expectations. For the computation of the approximate solution, we simulate $\Lambda^{reg}$ regression paths of the process $(B_j)_{j=1,\dots,J}$, which is given by

$$B_j = \left(1, \Delta W_j^x, \Delta W_j^y, \Delta W_j^\gamma, \Delta W_j^p\right)^\top$$

and apply the regression-later approach. In this example, we vary the number of regression paths so that we can assess the impact on the upper and lower bounds. In order to compute upper and lower bounds, we take $\Lambda^{out} = 5 \cdot 10^5$ outer paths and denote, as before, by $\hat{Y}_0^{up}$ and $\hat{Y}_0^{low}$ the resulting empirical means as Monte Carlo estimators of $E[\theta_0^{up}]$ and $E[\theta_0^{low}]$.

Table 2.1 displays upper and lower bound estimators with their standard deviations for different step sizes of the time discretization, for two choices of the number of regression paths, $\Lambda^{reg} \in \{10^5, 10^6\}$, and for different correlations between $\gamma$ and $p$. Here, $R$ is chosen as the fair swap rate in the absence of default risk, i.e., it is chosen such that the swap's clean price at $j = 0$ is zero. The four choices of $J$ correspond to a quarterly, monthly, bi-weekly, and weekly time discretization, respectively. In all cases, the width of the resulting confidence interval is about 0.6% of the value. We note that the regression estimates $\tilde{Y}_0$ (which we do not report here) are more stable for $10^6$ paths in the case of

| $J$ | Clean Price | $\rho_{\gamma p} = 0.8$ | | $\rho_{\gamma p} = 0$ | | $\rho_{\gamma p} = -0.8$ | |
|---|---|---|---|---|---|---|---|
| 120 ($\Lambda^{reg}=10^5$) | 0 | 21.30 (0.02) | 21.36 (0.02) | 24.89 (0.02) | 24.95 (0.02) | 28.30 (0.02) | 28.38 (0.02) |
| 120 ($\Lambda^{reg}=10^6$) | 0 | 21.32 (0.02) | 21.37 (0.02) | 24.89 (0.02) | 24.95 (0.02) | 28.30 (0.02) | 28.39 (0.02) |
| 360 ($\Lambda^{reg}=10^5$) | 0 | 21.26 (0.02) | 21.31 (0.02) | 24.84 (0.02) | 24.91 (0.02) | 28.25 (0.02) | 28.34 (0.02) |
| 360 ($\Lambda^{reg}=10^6$) | 0 | 21.28 (0.02) | 21.33 (0.02) | 24.86 (0.02) | 24.92 (0.02) | 28.26 (0.02) | 28.35 (0.02) |
| 720 ($\Lambda^{reg}=10^5$) | 0 | 21.25 (0.02) | 21.30 (0.02) | 24.83 (0.02) | 24.90 (0.02) | 28.24 (0.02) | 28.33 (0.02) |
| 720 ($\Lambda^{reg}=10^6$) | 0 | 21.23 (0.02) | 21.28 (0.02) | 24.81 (0.02) | 24.88 (0.02) | 28.23 (0.02) | 28.32 (0.02) |
| 1440 ($\Lambda^{reg}=10^5$) | 0 | 21.25 (0.02) | 21.30 (0.02) | 24.83 (0.02) | 24.90 (0.02) | 28.23 (0.02) | 28.32 (0.02) |
| 1440 ($\Lambda^{reg}=10^6$) | 0 | 21.23 (0.02) | 21.28 (0.02) | 24.81 (0.02) | 24.87 (0.02) | 28.20 (0.02) | 28.29 (0.02) |

Table 2.1: Lower and upper bound estimators for varying values of $\rho_{\gamma p}$, $J$ and $\Lambda^{reg}$ with $R = 275.12$ basis points (b.p.), $\Lambda^{out} = 5 \cdot 10^5$. Prices and standard deviations (in brackets) are given in b.p.

weekly and bi-weekly time discretizations. Nonetheless, the resulting upper and lower confidence bounds do not vary significantly for the two choices of regression paths. Moreover, the differences in the bounds can all be explained by the standard deviations. These results indicate that a monthly time discretization (i.e., 360 discretization steps) and $10^5$ regression are sufficient to accurately price this long-dated swap under bilateral default risk. The effect of varying the correlation parameter of $\gamma$ and $p$ also has the expected direction. Roughly, if $\rho_{\gamma p}$ is positive then larger values of the overall default rate go together with larger conditional default risk of the counterparty and smaller conditional default risk of the party, making the product less valuable to the party. While this effect is not as pronounced as the overall deviation from the clean price, the bounds are easily tight enough to differentiate between the three cases.

We next compare our numerical results with the "generic method" of Section 5 in Bender et al. (2017). While the latter paper focuses on convex non-linearities, it also suggests a generic local approximation of Lipschitz non-linearities by convex non-linearities, which can be applied for the problem of bilateral default risk (after suitable truncations). Based on the same input approximations as above (computed by the regression-later approach with $\Lambda^{reg} = 10^5$ regression paths), this algorithm produced a 95%-confidence interval of $[-0.3874, 1.0966]$ for the case $J = 360$ and $\rho_{\gamma p} = 0$. The length of this confidence interval is several magnitudes wider than the one computed from Table 2.1, and it cannot even significantly distinguish between the clean price and the price under default risk. These results demonstrate the importance of exploiting the concave-convex structure for pricing under bilateral default risk.

Finally, Table 2.2 displays the adjusted fair swap rates accounting for counterparty risk and funding for the three values of $\rho_{\gamma p}$, i.e., the values of $R$ which set the adjusted price to zero in the three different correlation scenarios. To identify these rates, we fix a set of outer and regression paths and define $\mu(R)$ as the midpoint of the confidence interval we obtain when running the algorithm with these paths and rate $R$ for the fixed leg of the swap. We apply a standard bisection method to find the zero of $\mu(R)$. The confidence intervals for the prices in Table 2.2 are then obtained by validating these swap rates with a new set of outer paths. We observe that switching from a clean valuation to the adjusted valuation with $\rho = 0.8$ increases the fair swap rate by 16 basis points (from 275 to 291). Changing $\rho$ from 0.8 to $-0.8$ leads to a further increase by 5 basis points.

| $\rho_{\gamma p}$ | Adjusted Fair Swap Rate | Clean Price | Bounds | |
|---|---|---|---|---|
| 0.8 | 290.82 | −31.53 | −0.02 (0.02) | 0.05 (0.02) |
| 0 | 293.65 | −37.22 | −0.01 (0.02) | 0.08 (0.02) |
| −0.8 | 296.39 | −42.71 | −0.06 (0.02) | 0.04 (0.02) |

Table 2.2: Adjusted fair swap rates and lower and upper bound estimators for varying values of $\rho_{\gamma p}$ with $\Lambda^{reg} = 10^5$, $\Lambda^{out} = 5 \cdot 10^5$ and $J = 360$. Rates, prices and standard deviations (in brackets) are given in b.p.

# Chapter 3

# Iterative improvement of upper and lower bounds for convex dynamic programs

As we have seen in the previous chapters, the quality of upper and lower bounds in numerical applications strongly depends on the quality of the input approximation. Hence, the key challenge in constructing tight upper and lower bounds to the solution of a dynamic program of the form (1.15) or (2.1), is to compute a suitable approximate solution to these dynamic programs. Depending on the considered problem, this can be rather cumbersome. In this chapter, we, thus, present an iterative improvement algorithm for systems of convex dynamic programs which builds on the pathwise approach presented in Section 1.3 and allows us to obtain tight upper and lower bounds even if the approximate solution is rather crude. In Section 3.1 we first explain how a given supersolution can be improved by using the pathwise approach of Section 1.3 for the construction of upper bounds. Section 3.2 is structured similarly and transfers the results of Section 3.1 to the context of subsolutions. Building on these results, we discuss in Section 3.3 an improvement approach for families of super- and subsolutions, if the dynamic program is one-dimensional. Following this, we show that this approach generalizes the improvement approach of Kolodko and Schoenmakers (2006) proposed in the context of Bermudan option pricing. In Section 3.4, we explain how the improvement algorithms presented in the preceding sections can be applied numerically. Finally, we demonstrate the applicability of this approach in the context of pricing under funding cost.

## 3.1 Improvement of supersolutions

The aim of this section is to construct an improvement of a given supersolution to the system of convex dynamic programs (1.15) given by

$$\begin{aligned} Y_J^{(\nu)} &= \xi^{(\nu)} \\ Y_j^{(\nu)} &= F_j^{(\nu)}\left(E_j\left[\beta_{j+1}Y_{j+1}^{(1)}\right],\ldots,E_j\left[\beta_{j+1}Y_{j+1}^{(N)}\right]\right), \quad \nu = 1,\ldots,N,\ j = J-1,\ldots,0. \end{aligned}$$

Intuitively, such an improvement should satisfy two things: first, it should again be a supersolution to (1.15) and second, it should lie below the given supersolution at all points in time. The following definition formalizes this intuition.

**Definition 3.1.1.** Let $\bar{Y}$ be a supersolution (respectively subsolution) to (1.15). A process $Y^{impr} \in L^{\infty-}_{ad}(\mathbb{R}^N)$ is called an improvement of $\bar{Y}$, if $Y^{impr}$ is a supersolution (respectively subsolution) to (1.15) and it holds that

$$Y_j^{impr} \leq \bar{Y}_j$$

$P$-almost surely for every $j = 0, \ldots, J$ (and with "$\leq$" replaced by "$\geq$" for a subsolution).

For our considerations, we have to restrict ourselves to the monotonic situation of Section 1.3, where we assumed that a comparison principle holds. To establish the comparison principle, we make the following monotonicity assumption on the functions $F_j^{(\nu)}$:

**Assumption 3.1.2.** *For every $j = 0, \ldots, J-1$, $\nu = 1, \ldots, N$ and any two random variables $Y^{(1)}, Y^{(2)} \in L^{\infty-}(\mathbb{R}^N)$ with $Y^{(1)} \geq Y^{(2)}$ $P$-a.s., the following monotonicity condition is satisfied:*

$$F_j^{(\nu)}\left(\beta_{j+1}Y^{(1,1)}, \ldots, \beta_{j+1}Y^{(1,N)}\right) \geq F_j^{(\nu)}\left(\beta_{j+1}Y^{(2,1)}, \ldots, \beta_{j+1}Y^{(2,N)}\right), \quad P\text{-a.s.} \tag{3.1}$$

We briefly explain why Assumption 3.1.2 ensures the existence of the comparison principle. Suppose for the moment, that the underlying filtration $(\mathcal{F}_j)_{j=0,\ldots,J}$ is replaced by the full information filtration $(\mathcal{G}_j)_{j=0,\ldots,J}$, where $\mathcal{G}_j = \mathcal{F}$ for all $j = 0, \ldots, J$. Then, Theorem 1.4.1 still holds true for this enlarged filtration due to our measurability assumptions. In particular, we observe that Assumption 3.1.2 coincides with the monotonicity statement $(c)$ in Theorem 1.4.1. This implies that

$$P\left(\left\{\left(\bar{r}^{(\nu),[n]}\right)^{\top} \beta_{j+1} \geq 0\right\}\right) = 1 \tag{3.2}$$

for all $j = 0, \ldots, J-1$, $\nu, n \in \{1, \ldots, N\}$ and every random variable $\bar{r}^{(\nu)} \in L^{\infty-}(\mathbb{R}^{ND})$ satisfying $F_j^{(\nu,\#)}(\bar{r}^{(\nu)}) \in L^{\infty-}(\mathbb{R})$ for each $j$. From this, we conclude that the positivity condition especially holds true for the admissible controls $r^{(\nu)} \in \mathcal{A}_0^{F^{(\nu)}}$, $\nu = 1, \ldots, N$, as they are obviously adapted to the filtration $(\mathcal{G}_j)_{j=0,\ldots,J}$. Applying Theorem 1.4.1 again for the initial filtration $(\mathcal{F}_j)_{j=0,\ldots,J}$ establishes the comparison principle.

Let $\bar{Y}$ be a supersolution to (1.15) and recall that the recursion (1.22) for $\Theta^{up} := \Theta^{up}(M)$ is given by

$$\begin{aligned}
\Theta_J^{(up,\nu)} &= \xi^{(\nu)} \\
\Theta_j^{(up,\nu)} &= F_j^{(\nu)}\left(\beta_{j+1}\Theta_{j+1}^{(up,1)} - \Delta M_{j+1}^{[1]}, \ldots, \beta_{j+1}\Theta_{j+1}^{(up,N)} - \Delta M_{j+1}^{[N]}\right),
\end{aligned} \tag{3.3}$$

for $j = J-1, \ldots, 0$, $\nu = 1, \ldots, N$, and any martingale $M \in \mathcal{M}_{ND}$. The main idea of the improvement approach is now to choose a suitable martingale $\bar{M} \in \mathcal{M}_{ND}$ such that

$$Y_j \leq E_j\left[\Theta_j^{up}\left(\bar{M}\right)\right] \leq \bar{Y}_j \tag{3.4}$$

$P$-almost surely for every $j = 0, \ldots, J$. In the context of Bermudan option pricing, Chen and Glasserman (2007) showed that taking the Doob martingale of a given supersolution as an input leads to an improved upper bound. This idea can be generalized to our setting: Denote by $\bar{M}^{[n]}$, $n = 1, \ldots, N$, the Doob martingale of $\beta\bar{Y}^{(n)}$. Then, the following Theorem states that the process $(E_j[\Theta_j^{up}(\bar{M})])_{j=0,\ldots,J}$ defined by (3.3) is an improvement for $\bar{Y}$. Moreover, it shows that this approach only gets stuck, if the supersolution $\bar{Y}$, which we want to improve, already coincides with the true solution.

**Theorem 3.1.3.** *Suppose Assumptions 1.2.1 and 3.1.2. Let* $j \in \{0,\dots,J-1\}$ *and let* $\bar{Y}$ *be a supersolution to* (1.15). *Further, let* $\bar{M}^{[\nu]} \in \mathcal{M}_D$ *be the Doob martingale of the process* $\beta\bar{Y}^{(\nu)}$ *for every* $\nu = 1,\dots,N$. *Then, the process* $\Theta^{up}(\bar{M})$ *defined by* (3.3) *satisfies*

$$Y_i^{(\nu)} \le E_i\left[\Theta_i^{(up,\nu)}\left(\bar{M}\right)\right] \le F_i^{(\nu)}\left(E_i\left[\beta_{i+1}\bar{Y}_{i+1}^{(1)}\right],\dots,E_i\left[\beta_{i+1}\bar{Y}_{i+1}^{(N)}\right]\right) \le \bar{Y}_i^{(\nu)}, \tag{3.5}$$

*$P$-almost surely for all* $i = 0,\dots,J$ *and* $\nu = 1,\dots,N$. *Moreover, if* $\bar{Y}_i = Y_i$ *for all* $i = j+1,\dots,J$, *then*

$$\Theta_j^{up}(\bar{M}) = Y_j \tag{3.6}$$

*$P$-almost surely.*

*Proof.* First of all, we recall that the process $(E_i[\Theta_i^{up}(M)])_{i=0,\dots,J}$ is a supersolution to the system of convex dynamic programs (1.15) for every martingale $M \in \mathcal{M}_{ND}$ according to Section 1.3. Hence, the first inequality in (3.5) holds by the comparison principle. Furthermore, the last inequality in (3.5) holds by the supersolution property of $\bar{Y}$. Therefore, it only remains to show that

$$E_i\left[\Theta_i^{(up,\nu)}(\bar{M})\right] \le F_i^{(\nu)}\left(E_i\left[\beta_{i+1}\bar{Y}_{i+1}^{(1)}\right],\dots,E_i\left[\beta_{i+1}\bar{Y}_{i+1}^{(N)}\right]\right) \tag{3.7}$$

holds for every $i = 0,\dots,J-1$ and $\nu = 1,\dots,N$. To this end, let $\bar{M}^{[\nu]} \in \mathcal{M}_D$ be the Doob martingale of $\beta\bar{Y}^{(\nu)}$ for every $\nu = 1,\dots,N$ and define $\Theta^{up} := \Theta^{up}(\bar{M})$ by (3.3). In order to prove (3.7), we show the assertion

$$\Theta_i^{up} \le \bar{Y}_i$$

via backward induction on $i = J,\dots,0$. Since we have by definition that $\Theta_J^{(up,\nu)} = \xi^{(\nu)} \le \bar{Y}_J^{(\nu)}$, the case $i = J$ is again trivial and we suppose that the assertion is true for $i+1$, i.e., we have $\Theta_{i+1}^{up} \le \bar{Y}_{i+1}$ $P$-almost surely. Then, we have by Lemma 1.2.4 that

$$\Theta_i^{(up,\nu)} = \sum_{n=1}^N \left(r^{(\nu),[n]}\right)^\top \beta_{i+1}\Theta_{i+1}^{(up,n)} - \sum_{n=1}^N \left(r^{(\nu),[n]}\right)^\top \Delta\bar{M}_{i+1} - F_i^{(\nu,\#)}\left(r^{(\nu)}\right)$$

for a random variable $r^{(\nu)} \in L^{\infty-}(\mathbb{R}^{ND})$ satisfying $F_i^{(\nu,\#)}(r) \in L^{\infty-}(\mathbb{R})$. Since $(r^{(\nu),[n]})^\top\beta_{i+1} \ge 0$ $P$-almost surely by (3.2), we conclude by the induction hypothesis, the definition of $\bar{M}$, and (1.23) that

$$\begin{aligned}
\Theta_i^{(up,\nu)} &= \sum_{n=1}^N \left(r^{(\nu),[n]}\right)^\top \beta_{i+1}\Theta_{i+1}^{(up,n)} - \sum_{n=1}^N \left(r^{(\nu),[n]}\right)^\top \Delta\bar{M}_{i+1} - F_i^{(\nu,\#)}\left(r^{(\nu)}\right) \\
&\le \sum_{n=1}^N \left(r^{(\nu),[n]}\right)^\top \beta_{i+1}\bar{Y}_{i+1}^{(n)} - \sum_{n=1}^N \left(r^{(\nu),[n]}\right)^\top \left(\beta_{i+1}\bar{Y}_{i+1}^{(n)} - E_i\left[\beta_{i+1}\bar{Y}_{i+1}^{(n)}\right]\right) - F_i^{(\nu,\#)}\left(r^{(\nu)}\right) \\
&= \sum_{n=1}^N \left(r^{(\nu),[n]}\right)^\top E_i\left[\beta_{i+1}\bar{Y}_{i+1}^{(n)}\right] - F_i^{(\nu,\#)}\left(r^{(\nu)}\right) \\
&\le F_i^{(\nu)}\left(E_i\left[\beta_{i+1}\bar{Y}_{i+1}^{(1)}\right],\dots,E_i\left[\beta_{i+1}\bar{Y}_{i+1}^{(N)}\right]\right) \\
&\le \bar{Y}_i^{(\nu)}.
\end{aligned} \tag{3.8}$$

Here the last inequality is due to the supersolution property of $\bar{Y}$. Now, the asserted inequality (3.7) follows from (3.8) by the monotonicity of the conditional expectation.

Finally, it remains to show (3.6), i.e.

$$\Theta_j^{up} = Y_j$$

if $\bar{Y}_i = Y_i$ for all $i = j+1, \ldots, J$, where $j \in \{0, \ldots, J-1\}$ is fixed from now on. Since $\bar{Y}_i = Y_i$ for all $i = j+1, \ldots, J$, we conclude by the definition of $\bar{M}$ that

$$\begin{aligned} \bar{M}_{i+1}^{[\nu]} - \bar{M}_i^{[\nu]} &= \beta_{i+1}\bar{Y}_{i+1}^{[\nu]} - E_i\left[\beta_{i+1}\bar{Y}_{i+1}^{[\nu]}\right] \\ &= \beta_{i+1}Y_{i+1}^{[\nu]} - E_i\left[\beta_{i+1}Y_{i+1}^{[\nu]}\right] \end{aligned}$$

for every $i = j, \ldots, J-1$ and $\nu = 1, \ldots, N$. By exploiting that $\Theta_i^{up} = Y_i$ for every $i = j+1, \ldots, J$ by (3.5), we thus obtain that

$$\begin{aligned} \Theta_j^{(up,\nu)} &= F_j^{(\nu)}\left(\beta_{j+1}\Theta_{j+1}^{(up,1)} - \Delta\bar{M}_{j+1}^{(1)}, \ldots, \beta_{j+1}\Theta_{j+1}^{(up,N)} - \Delta\bar{M}_{j+1}^{(N)}\right) \\ &= F_j^{(\nu)}\left(\beta_{j+1}Y_{j+1}^{(1)} - \left(\beta_{j+1}Y_{j+1}^{(1)} - E_j\left[\beta_{j+1}Y_{j+1}^{(1)}\right]\right), \ldots, \right. \\ &\qquad \left. \beta_{j+1}\Theta_{j+1}^{(up,N)} - \left(\beta_{j+1}Y_{j+1}^{(N)} - E_j\left[\beta_{j+1}Y_{j+1}^{(N)}\right]\right)\right) \\ &= F_j^{(\nu)}\left(E_j\left[\beta_{j+1}Y_{j+1}^{(1)}\right], \ldots, E_j\left[\beta_{j+1}Y_{j+1}^{(N)}\right]\right) \\ &= Y_j^{(\nu)}, \end{aligned}$$

which completes the proof. □

When starting with an arbitrary supersolution, we typically do not obtain the solution $Y$ by applying the approach described in Theorem 3.1.3 once. This is typically the case in numerical applications where the "input supersolution" $\bar{Y}$ is computed by the algorithm explained in Section 1.7 and thus may stem from a possibly crude approximate solution to (1.15). Hence, we now show that the above construction can be iterated in a straightforward way such that a decreasing sequence of supersolutions is obtained.

To this end, let $\bar{Y}$ be a supersolution and define $\Theta^{up,0} := \bar{Y}$. Then, we define the $k$-th iteration according to (3.3) by

$$\Theta^{up,k} := \Theta^{up}\left(M^k\right), \quad k \geq 1, \tag{3.9}$$

where each $M^k = (M^{[1],k}, \ldots, M^{[N],k})$ is given by

$$M_j^{[\nu],k} = \sum_{i=0}^{j-1} \beta_{i+1}E_{i+1}\left[\Theta_{i+1}^{(up,\nu),k-1}\right] - E_i\left[\beta_{i+1}\Theta_{i+1}^{(up,\nu),k-1}\right], \quad j = 0, \ldots, J, \ \nu = 1, \ldots, N. \tag{3.10}$$

Applying Theorem 3.1.3 repeatedly, we observe that this iteration decreasingly converges in at most $J+1$ steps as stated in the following corollary.

**Corollary 3.1.4.** *For every $k \geq 1$ and $j = 0, \ldots, J$,*

$$E_j\left[\Theta_j^{up,k}\right] \leq E_j\left[\Theta_j^{up,k-1}\right] \quad P\text{-a.s.} \tag{3.11}$$

*Moreover, for every $i \geq j$,*

$$\Theta_i^{up,J-j+1} = Y_i, \quad P\text{-a.s.} \tag{3.12}$$

*Hence, the upper bound iteration terminates after at most $J+1$ steps.*

*Proof.* First note that inequality (3.11) is an immediate consequence of Theorem 3.1.3 and the definition of $\Theta^{up,k}$, $k \geq 0$. Hence, it only remains to show that (3.12) holds. The proof is by backward induction on $j$, with the case $j = J$ being trivial, because $\Theta_J^{(up,\nu),1} = \xi^{(\nu)} = Y_J^{(\nu)}$ by definition for every $\nu = 1, \dots, N$. Now suppose, that the assertion is true for $j \in \{1, \dots, J\}$. Then, we have by induction hypothesis that

$$\Theta_i^{up,J-(j-1)+1} = \Theta_i^{up,J-j+1} = Y_i$$

$P$-a.s. for every $i = j, \dots, J$. From Theorem 3.1.3 we thus conclude that

$$\Theta_{j-1}^{up,J-(j-1)+1} = \Theta_{j-1}^{up}\left(M^{J-(j-1)+1}\right) = Y_{j-1} \quad P\text{-a.s.},$$

where $M^{J-(j-1)+1}$ is given by (3.10). □

*Remark* 3.1.5. Note that convergence of the above algorithm in at most $J$ steps can be achieved by a slight modification. Let $\bar{Y}$ be an arbitrary supersolution and define $\Theta^{up,0}$ by

$$\Theta_j^{(up,\nu),0} = \begin{cases} \xi^{(\nu)}, & j = J \\ \bar{Y}_j^{(\nu)}, & j < J \end{cases}$$

for every $\nu = 1, \dots, N$. Then, the process $\Theta^{up,0}$ is again a supersolution. This is obvious for $j < J - 1$ by definition of $\Theta^{up,0}$. For $j = J - 1$, we obtain that

$$\begin{aligned} \Theta_{J-1}^{(up,\nu),0} &= \bar{Y}_{J-1}^{(\nu)} \\ &\geq F_{J-1}^{(\nu)}\left(E_{J-1}\left[\beta_J \bar{Y}_J^{(1)}\right], \dots, E_{J-1}\left[\beta_J \bar{Y}_J^{(N)}\right]\right) \\ &\geq F_{J-1}^{(\nu)}\left(E_{J-1}\left[\beta_J \xi^{(1)}\right], \dots, E_{J-1}\left[\beta_J \xi_J^{(N)}\right]\right) \\ &= F_{J-1}^{(\nu)}\left(E_{J-1}\left[\beta_J \Theta_J^{(up,1),0}\right], \dots, E_{J-1}\left[\beta_J \Theta_J^{(up,N),0}\right]\right) \end{aligned}$$

for every $\nu = 1, \dots, N$ by the supersolution property of $\bar{Y}$ and the monotonicity condition (3.1). Since the terminal value of $\Theta^{(up,\nu),0}$ now coincides with the true terminal value $\xi^{(\nu)}$, we are able to reduce the number of iteration steps by one. In particular, the iteration converges in at most $J$ steps, if the input supersolution is computed by the pathwise approach of Section 1.3.

## 3.2 Improvement of subsolutions

After considering the improvement approach for supersolutions, we now explain how the recursion for lower bounds presented in Section 1.3 can be used to improve arbitrary subsolutions to (1.15). To this end, let $\bar{Y}$ be an arbitrary subsolution to (1.15). In order to construct an improvement of $\bar{Y}$, we rely on the modified recursion (1.29) for the lower bound $\Theta^{low} := \Theta^{low}(r^{(1)}, \dots, r^{(N)}, M)$, which is given by

$$\begin{aligned} \Theta_J^{(low,\nu)} &= \xi^{(\nu)} \\ \Theta_j^{(low,\nu)} &= \sum_{n=1}^{N} \left(r_j^{(\nu),[n]}\right)^\top \beta_{j+1} \Theta_{j+1}^{(low,n)} - \sum_{n=1}^{N} \left(r_j^{(\nu),[n]}\right)^\top \Delta M_{j+1}^{[n]} - F_j^{(\nu,\#)}\left(r_j^{(\nu)}\right), \end{aligned} \tag{3.13}$$

for $j = J-1, \dots, 0$, $\nu = 1, \dots, N$, admissible controls $r^{(\nu)} \in \mathcal{A}_0^{F^{(\nu)}}$, and $M \in \mathcal{M}_{ND}$.

Similar to the case of supersolutions, we want to find suitable controls $\bar{r}^{(\nu)} \in \mathcal{A}_0^{F^{(\nu)}}$ such that

$$\bar{Y}_j \leq E_j \left[ \Theta_j^{low} \left( \bar{r}^{(1)}, \ldots, \bar{r}^{(N)}, M \right) \right] \leq Y_j \tag{3.14}$$

holds $P$-almost surely for all $j = 0, \ldots, J$ and $M \in \mathcal{M}_{ND}$. In order to find such a candidate, we first note that the subsolution property of $\bar{Y}$ establishes the inequality

$$\bar{Y}_j^{(\nu)} \leq F_j^{(\nu)} \left( E_j \left[ \beta_{j+1} \bar{Y}_{j+1}^{(1)} \right], \ldots, E_j \left[ \beta_{j+1} \bar{Y}_{j+1}^{(N)} \right] \right)$$

for every $j = 0, \ldots, J-1$ and $\nu = 1, \ldots, N$. By Lemma 1.2.4, we know that there exist controls $\bar{r}^{(\nu)} \in \mathcal{A}_0^{F^{(\nu)}}$, $\nu = 1, \ldots, N$, such that the right hand side of this inequality can be rewritten as

$$F_j^{(\nu)} \left( E_j \left[ \beta_{j+1} \bar{Y}_{j+1}^{(1)} \right], \ldots, E_j \left[ \beta_{j+1} \bar{Y}_{j+1}^{(N)} \right] \right) = \sum_{n=1}^{N} \left( \bar{r}_j^{(\nu),[n]} \right)^\top E_j \left[ \beta_{j+1} \bar{Y}_{j+1}^{(n)} \right] - F_j^{(\nu,\#)} \left( \bar{r}_j^{(\nu)} \right)$$

for every $j = 0, \ldots, J-1$. Hence, solutions to these equations serve naturally as potential candidates to establish the chain of inequalities (3.14). Note that this approach differs from existing policy improvement approaches like the Howard improvement. In contrast to these approaches, our approach takes an arbitrary subsolution, which need not stem from a control, as an input and constructs a control from which an improved subsolution is derived.

The above consideration is confirmed by the following theorem, which is the main result of this section.

**Theorem 3.2.1.** *Suppose Assumptions 1.2.1 and 3.1.2. Let $j \in \{0, \ldots, J-1\}$, let $\bar{Y}$ be a subsolution to* (1.15) *and denote by $\bar{M}^{[\nu]} \in \mathcal{M}_D$ the Doob martingale of $\beta \bar{Y}^{(\nu)}$ for every $\nu = 1, \ldots, N$. Further let $\bar{r}^{(\nu)} \in \mathcal{A}_0^{F^{(\nu)}}$, $\nu = 1, \ldots, N$, be admissible controls that solve*

$$\sum_{n=1}^{N} \left( \bar{r}_i^{(\nu),[n]} \right)^\top E_i \left[ \beta_{i+1} \bar{Y}_{i+1}^{(n)} \right] - F_i^{(\nu,\#)} \left( \bar{r}_i^{(\nu)} \right) = F_i^{(\nu)} \left( E_i \left[ \beta_{i+1} \bar{Y}_{i+1}^{(1)} \right], \ldots, E_i \left[ \beta_{i+1} \bar{Y}_{i+1}^{(N)} \right] \right) \tag{3.15}$$

*$P$-almost surely for every $i = 0, \ldots, J-1$ and $\nu = 1, \ldots, N$. Then, for any $M \in \mathcal{M}_{ND}$, the process $\Theta^{low}(\bar{r}^{(1)}, \ldots, \bar{r}^{(N)}, M)$ defined by* (3.13) *satisfies*

$$Y_i^{(\nu)} \geq E_i \left[ \Theta_i^{(low,\nu)} \left( \bar{r}^{(1)}, \ldots, \bar{r}^{(N)}, M \right) \right] \geq F_i^{(\nu)} \left( E_i \left[ \beta_{i+1} \bar{Y}_{i+1}^{(1)} \right], \ldots, E_i \left[ \beta_{i+1} \bar{Y}_{i+1}^{(N)} \right] \right) \geq \bar{Y}_i^{(\nu)}, \tag{3.16}$$

*$P$-almost surely for all $i = 0, \ldots, J-1$ and $\nu = 1, \ldots, N$. Moreover, if $\bar{Y}_i = Y_i$ for all $i = j+1, \ldots, J$, then*

$$E_j \left[ \Theta_j^{low} \left( \bar{r}^{(1)}, \ldots, \bar{r}^{(N)}, M \right) \right] = \Theta_j^{low} \left( \bar{r}^{(1)}, \ldots, \bar{r}^{(N)}, \bar{M} \right) = Y_j \tag{3.17}$$

*$P$-almost surely.*

*Remark* 3.2.2. (i) As in the context of supersolutions, (3.17) states that this improvement approach only gets stuck, if the input subsolution $\bar{Y}$ already coincides with the true solution.

(ii) By the chain of inequalities (3.16), we have that an improvement is obtained by taking any martingale $M \in \mathcal{M}_{ND}$. Indeed as observed in Section 1.3, the martingale increment only acts as a control variate in this approach.

*Proof.* As we have seen in Section 1.3, the process $(E_i[\Theta_i^{low}(r^{(1)},\dots,r^{(N)},M)])_{i=0,\dots,J}$ defines a subsolution to (1.15) for any admissible controls $r^{(\nu)} \in \mathcal{A}_0^{F^{(\nu)}}$, $\nu = 1,\dots,N$, and $M \in \mathcal{M}_{ND}$, so that the first inequality in (3.16) is already shown. Moreover, the last inequality in (3.16) is immediate, as $\bar{Y}$ is assumed to be a subsolution. Now, let $\bar{r}^{(\nu)} \in \mathcal{A}_0^{F^{(\nu)}}$, $\nu = 1,\dots,N$, be given by (3.15) and denote by $\bar{M}^{[\nu]}$ the Doob martingale of $\beta\bar{Y}^{(\nu)}$ for every $\nu = 1,\dots,N$. Then, we define the process $\Theta^{low} := \Theta^{low}(\bar{r}^{(1)},\dots,\bar{r}^{(N)},\bar{M})$ according to (3.13). In order to prove the remaining inequality, we proceed as in the proof of Theorem 3.1.3 and show the assertion

$$\Theta_i^{(low,\nu)} \geq \bar{Y}_i^{(\nu)} \tag{3.18}$$

for every $\nu = 1,\dots,N$ by backward induction on $i = J,\dots,0$. The case $i = J$ is trivial, since we have $\Theta_J^{(low,\nu)} = \xi^{(\nu)} \geq \bar{Y}_J^{(\nu)}$ for every $\nu = 1,\dots,N$ by definition. Now suppose that the assertion is true for $i+1$, i.e., $\Theta_{i+1}^{low} \geq \bar{Y}_{i+1}$ $P$-almost surely. Then, the definition of $\bar{M}^{[n]}$, $n = 1,\dots,N$ yields

$$\begin{aligned}
\Theta_i^{(low,\nu)} &= \sum_{n=1}^N \left(\bar{r}_i^{(\nu),[n]}\right)^\top \beta_{i+1}\Theta_{i+1}^{(low,n)} - \sum_{n=1}^N \left(\bar{r}_i^{(\nu),[n]}\right)^\top \Delta\bar{M}_{i+1}^{[n]} - F_i^{(\nu,\#)}\left(\bar{r}_i^{(\nu)}\right) \\
&= \sum_{n=1}^N \left(\bar{r}_i^{(\nu),[n]}\right)^\top \beta_{i+1}\Theta_{i+1}^{(low,n)} - \sum_{n=1}^N \left(\bar{r}_i^{(\nu),[n]}\right)^\top \left(\beta_{i+1}\bar{Y}_{i+1}^{(n)} - E_i\left[\beta_{i+1}\bar{Y}_{i+1}^{(n)}\right]\right) - F_i^{(\nu,\#)}\left(\bar{r}_i^{(\nu)}\right) \\
&= \sum_{n=1}^N \left(\bar{r}_i^{(\nu),[n]}\right)^\top \beta_{i+1}\left(\Theta_{i+1}^{(low,n)} - \bar{Y}_{i+1}^{(n)}\right) + \sum_{n=1}^N \left(\bar{r}_i^{(\nu),[n]}\right)^\top E_i\left[\beta_{i+1}\bar{Y}_{i+1}^{(n)}\right] - F_i^{(\nu,\#)}\left(\bar{r}_i^{(\nu)}\right).
\end{aligned}$$

Since $\Theta_{i+1}^{(low,n)} \geq \bar{Y}_{i+1}^{(n)}$ for all $n = 1,\dots,N$ by induction hypothesis and $(\bar{r}_i^{(\nu),[n]})^\top\beta_{i+1} \geq 0$ for each $n$ by (3.2), we conclude that

$$\Theta_i^{(low,\nu)} \geq \sum_{n=1}^N \left(\bar{r}_i^{(\nu),[n]}\right)^\top E_i\left[\beta_{i+1}\bar{Y}_{i+1}^{(n)}\right] - F_i^{(\nu,\#)}\left(\bar{r}_i^{(\nu)}\right).$$

Finally, it follows from (3.15) and the subsolution property of $\bar{Y}$ that

$$\begin{aligned}
\Theta_i^{(low,\nu)} &\geq F_i^{(\nu)}\left(E_i\left[\beta_{i+1}\bar{Y}_{i+1}^{(1)}\right],\dots,E_i\left[\beta_{i+1}\bar{Y}_{i+1}^{(N)}\right]\right) \\
&\geq \bar{Y}_i^{(\nu)},
\end{aligned} \tag{3.19}$$

and, thus, (3.18) is established. Recalling that $E_i[\Theta_i^{low}(r^{(1)},\dots,r^{(N)},M)]$ does not depend on the choice of $M$, (3.16) now follows from (3.19) and the monotonicity of the conditional expectation.

To complete the proof, we fix $j \in \{0,\dots,J-1\}$ and assume that $\bar{Y}_i = Y_i$ for all $i = j+1,\dots,J$. Then, we observe that (3.15) is equivalent to

$$\sum_{n=1}^N \left(\bar{r}_i^{(\nu),[n]}\right)^\top E_i\left[\beta_{i+1}Y_{i+1}^{(n)}\right] - F_i^{(\nu,\#)}\left(\bar{r}_i^{(\nu)}\right) = F_i^{(\nu)}\left(E_i\left[\beta_{i+1}Y_{i+1}^{(1)}\right],\dots,E_i\left[\beta_{i+1}Y_{i+1}^{(N)}\right]\right)$$

for every $i = j,\dots,J-1$, i.e. $\bar{r}_i$ satisfies the optimality condition (1.24) for all $i = j,\dots,J-1$. Moreover, we conclude, similar to the proof of Theorem 3.1.3, that

$$\bar{M}_{i+1}^{[\nu]} - \bar{M}_i^{[\nu]} = \beta_{i+1}Y_{i+1}^{(\nu)} - E_i\left[\beta_{i+1}Y_{i+1}^{(\nu)}\right]$$

for all $\nu = 1, \dots, N$ and $i = j, \dots, J-1$. Hence, we obtain that

$$\begin{aligned}
\Theta_j^{(low,\nu)} &= \sum_{n=1}^{N} \left(\bar{r}_j^{(\nu),[n]}\right)^\top \beta_{j+1}\Theta_{j+1}^{(low,n)} - \sum_{n=1}^{N} \left(\bar{r}_j^{(\nu),[n]}\right)^\top \Delta\bar{M}_{j+1}^{[n]} - F_j^{(\nu,\#)}\left(\bar{r}_j^{(\nu)}\right) \\
&= \sum_{n=1}^{N} \left(\bar{r}_j^{(\nu),[n]}\right)^\top \beta_{j+1}Y_{j+1}^{(n)} - \sum_{n=1}^{N} \left(\bar{r}_j^{(\nu),[n]}\right)^\top \left(\beta_{j+1}Y_{j+1}^{(n)} - E_j\left[\beta_{j+1}Y_{j+1}^{(n)}\right]\right) - F_j^{(\nu,\#)}\left(\bar{r}_j^{(\nu)}\right) \\
&= \sum_{n=1}^{N} \left(\bar{r}_j^{(\nu),[n]}\right)^\top E_j\left[\beta_{j+1}Y_{j+1}^{(n)}\right] - F_j^{(\nu,\#)}\left(\bar{r}_j^{(\nu)}\right) \\
&= F_j^{(\nu)}\left(E_j\left[\beta_{j+1}Y_{j+1}^{(1)}\right], \dots, E_j\left[\beta_{j+1}Y_{j+1}^{(N)}\right]\right) \\
&= Y_j^{(\nu)},
\end{aligned}$$

since $\Theta_i^{low} = Y_i$ for every $i = j+1, \dots, J$ by (3.16). □

As in Section 3.1, this improvement can be iterated several times. For a given subsolution $\bar{Y}$ define $\Theta^{low,0} := \bar{Y}$ and define $\Theta^{low,k}$ according to (3.13) by

$$\Theta^{low,k} := \Theta^{low}\left(r^{(1),k}, \dots, r^{(N),k}, M^k\right), \quad k \geq 1, \tag{3.20}$$

where the processes $r^{(\nu),k} \in \mathcal{A}_0^{F^{(\nu)}}$ are for every $j = 0, \dots, J-1$ and $\nu = 1, \dots, N$ given by

$$\begin{aligned}
&\sum_{n=1}^{N} \left(r_j^{(\nu),[n],k}\right)^\top E_j\left[\beta_{j+1}\Theta_{j+1}^{(low,n),k-1}\right] - F_j^{(\nu,\#)}\left(r_j^{(\nu),k}\right) \\
&\qquad = F_j^{(\nu)}\left(E_j\left[\beta_{j+1}\Theta_{j+1}^{(low,1),k-1}\right], \dots, E_j\left[\beta_{j+1}\Theta_{j+1}^{(low,1),k-1}\right]\right),
\end{aligned} \tag{3.21}$$

and $M^k \in \mathcal{M}_{ND}$ is arbitrary. Then, iterative application of Theorem 3.2.1 yields the following corollary.

**Corollary 3.2.3.** *For every $k \geq 1$ and $j = 0, \dots, J$,*

$$E_j\left[\Theta_j^{low,k}\right] \geq E_j\left[\Theta_j^{low,k-1}\right], \quad P\text{-a.s.} \tag{3.22}$$

*Moreover,*

$$E_i\left[\Theta_i^{low,J-j+1}\right] = Y_i \quad P\text{-a.s.}, \tag{3.23}$$

*whenever $i \geq j$. In the last equation, the conditional expectation on the left-hand side can be removed, when $M^{[\nu],k}$ is taken as the Doob martingale of the process $(\beta_j E_j[\Theta_j^{(low,\nu),k-1}])_{j=0,\dots,J}$ for each $\nu = 1, \dots, N$ and $k \geq 1$ .*

*Proof.* Let $M^k \in \mathcal{M}_{ND}$, $k \geq 1$, be arbitrary martingales. Then, we first note that the inequality (3.22) is an immediate consequence of Theorem 3.2.1 and the definition of $\Theta^{low,k}$, $k \geq 0$. Hence, it only remains to show (3.23). The proof is by backward induction on $j$, with the case $j = J$ being trivial, because $\Theta_J^{(low,\nu),1} = \xi^{(\nu)} = Y_J^{(\nu)}$ by definition. Now suppose, that the assertion is true for $j \in \{1, \dots, J\}$, i.e.

$$E_i\left[\Theta_i^{low,J-j+1}\right] = Y_i$$

for all $i \geq j$. Then, we have by induction hypothesis that

$$\Theta_i^{low,J-(j-1)+1} = \Theta_i^{low,J-j+1} = Y_i$$

$P$-a.s. for every $i = j, \ldots, J$. From Theorem 3.2.1 we thus conclude that

$$\begin{aligned} E_{j-1}\left[\Theta_{j-1}^{low,J-(j-1)+1}\right] &= E_{j-1}\left[\Theta_{j-1}^{low}\left(r^{(1),J-(j-1)+1}, \ldots, r^{(N),J-(j-1)+1}, M^{J-(j-1)+1}\right)\right] \\ &= \Theta_{j-1}^{low}\left(r^{(1),J-(j-1)+1}, \ldots, r^{(N),J-(j-1)+1}, \bar{M}^{J-(j-1)+1}\right) \\ &= Y_{j-1} \quad P\text{-a.s.}, \end{aligned}$$

where each $r^{(\nu),J-(j-1)+1} \in \mathcal{A}_0^{F^{(\nu)}}$, $\nu = 1, \ldots, N$, is given by (3.21) and $\bar{M}^{J-(j-1)+1,[\nu]}$ is the Doob martingale of $(\beta_i E_i[\Theta_i^{low,J-(j-1)+1}])_{i=0,\ldots,J}$. □

*Remark* 3.2.4. As in the context of supersolutions, convergence of the above algorithm in at most $J$ steps can be achieved. Let $\bar{Y}$ be an arbitrary subsolution and define $\Theta^{low,0}$ by

$$\Theta_j^{(low,\nu),0} = \begin{cases} \xi^{(\nu)}, & j = J \\ \bar{Y}_j^{(\nu)}, & j < J \end{cases}$$

for every $\nu = 1, \ldots, N$. Applying the same arguments as in Remark 3.1.5, we observe that $\Theta^{low,0}$ is still a subsolution. Consequently, convergence in at most $J$ steps can be achieved for subsolutions stemming from the pathwise approach of Section 1.3.

## 3.3 Improving families of super- and subsolutions

In Section 3.4 below, we explain that the numerical costs of algorithms based on (3.9) and (3.20) tend to grow exponentially in the number of iterations $k$. For this reason, a moderate number of iterations must suffice in practical implementations. In the case of one-dimensional convex dynamic programs, we can address this issue by improving whole families of super- and subsolutions instead of just one. Therefore, we suppose throughout this section that $N = 1$ and (1.15) reduces to

$$\begin{aligned} Y_J &= \xi, \\ Y_j &= F_j\left(E_j\left[\beta_{j+1} Y_{j+1}\right]\right), \quad j = J-1, \ldots, 0. \end{aligned} \tag{3.24}$$

As before, we first consider the case of supersolutions. To this end, let $(\bar{Y}^{\{l\}})_{l \in I}$ be a family of supersolutions, where $I$ is a finite index set. Further, we denote by $K(j)$, $j = 1, \ldots, J$, a non-decreasing sequence of subsets of $I$, i.e. it holds that $K(j) \subseteq K(j+1)$. Then, we consider the predictable, $I$-valued process

$$l_*(j) = \inf\left\{ l \in K(j) \;\middle|\; \forall \iota \in K(j)\ F_{j-1}\left(E_{j-1}\left[\beta_j \bar{Y}_j^{\{l\}}\right]\right) \leq F_{j-1}\left(E_{j-1}\left[\beta_j \bar{Y}_j^{\{\iota\}}\right]\right) \right\} \tag{3.25}$$

for every $j = 1, \ldots, J$. This means that, at every time point $j = 1, \ldots, J$, we only consider those supersolutions which are represented in the subset $K(j)$ and the random variable $l_*(j)$ returns an index $l \in K(j)$ at which the evaluation of $F_{j-1}$ is minimized. Considering the sets $K(j)$ makes the approach more flexible, but, obviously, the simplest choice is to take $K(j) = I$ for all $j = 1, \ldots, J$. This additional flexibility turns out to be useful in situations where $I$ is large, and thus the computational costs in order to determine the process $l_*$ are high. More sophisticated choices of $K(j)$ then allow to reduce these costs.

Building on $l_*$, we define the process $\bar{Y}$ by

$$\bar{Y}_j = \begin{cases} \bar{Y}_j^{\{l_*(j)\}}, & j > 0 \\ F_0\left(E_0\left[\beta_1 \bar{Y}_1\right]\right), & j = 0 \end{cases} \tag{3.26}$$

for every $j = 0, \ldots, J$. Indeed, this process is a supersolution to (3.24), which allows us to improve the supersolutions $(\bar{Y}^{\{l\}})_{l \in I}$ simultaneously. To examine the supersolution property of $\bar{Y}$, we first observe that the case $j = 0$ is trivial, since we have $\bar{Y}_0 = F_0(E_0[\beta_1 \bar{Y}_1])$ by definition. For the case $j > 0$, we get by the supersolution property of $\bar{Y}^{\{l\}}$ for every $l \in I$ that

$$\begin{aligned} \bar{Y}_j &= \sum_{l \in K(j)} \bar{Y}_j^{\{l\}} \mathbb{1}_{\{l_*(j)=l\}} \\ &\geq \sum_{l \in K(j)} F_j\left(E_j\left[\beta_{j+1} \bar{Y}_{j+1}^{\{l\}}\right]\right) \mathbb{1}_{\{l_*(j)=l\}}. \end{aligned}$$

Since $K(j) \subseteq K(j+1)$ for all $j = 1, \ldots, J-1$ it follows that

$$\begin{aligned} \bar{Y}_j &\geq \sum_{l \in K(j)} F_j\left(E_j\left[\beta_{j+1} \bar{Y}_{j+1}^{\{l_*(j+1)\}}\right]\right) \mathbb{1}_{\{l_*(j)=l\}} \\ &= F_j\left(E_j\left[\beta_{j+1} \bar{Y}_{j+1}\right]\right) \end{aligned}$$

$P$-almost surely, showing that $\bar{Y}$ is a supersolution. Hence, Theorem 3.1.3 can be applied to the process $\bar{Y}$ and implies, for $\Theta^{up} = \Theta^{up}(\bar{M})$,

$$E_j\left[\Theta_j^{up}\right] \leq F_j\left(E_j\left[\beta_{j+1} \bar{Y}_{j+1}\right]\right) = \min_{l \in K(j+1)} F_j\left(E_j\left[\beta_{j+1} \bar{Y}_{j+1}^{\{l\}}\right]\right) \leq \min_{l \in K(j+1)} \bar{Y}_j^{\{l\}}$$

$P$-almost surely for all $j = 0, \ldots, J-1$, where $\bar{M}$ denotes the Doob martingale of $\beta \bar{Y}$. Thus, if $K(j) = I$ for all $j = 1, \ldots, J$, we achieve a simultaneous improvement of all supersolutions $(\bar{Y}^{\{l\}})_{l \in I}$ by improving $\bar{Y}$.

Finally, we turn to the case of subsolutions, where the overall strategy is similar. Hence, let $(\bar{Y}^{\{l\}})_{l \in I}$ be a family of subsolutions, where $I$ is still a finite set. Then, we consider the predictable, $I$-valued process

$$l^*(j) = \inf\left\{ l \in K(j) \,\middle|\, \forall \iota \in K(j)\ F_{j-1}\left(E_{j-1}\left[\beta_j \bar{Y}_j^{\{l\}}\right]\right) \geq F_{j-1}\left(E_{j-1}\left[\beta_j \bar{Y}_j^{\{\iota\}}\right]\right) \right\} \tag{3.27}$$

for every $j = 1, \ldots, J$, where $K(j)$ is again a non-decreasing family of subsets of $I$. Note that, compared to (3.25), the inequality is now the other way round, since we would like to construct a subsolution, which lies above the given subsolutions $(\bar{Y}^{\{l\}})_{l \in I}$ $P$-almost surely. Then, the process $\bar{Y}$ defined by

$$\bar{Y}_j = \begin{cases} \bar{Y}_j^{(l^*(j))}, & j > 0 \\ F_0\left(E_0\left[\beta_1 \bar{Y}_1\right]\right), & j = 0 \end{cases}$$

is, by similar arguments as before, a subsolution to (3.24). The case $j = 0$ is again trivial since $\bar{Y}_0 = F_0(E_0[\beta_1 \bar{Y}_1])$ by definition. For $j > 0$, we apply the subsolution property of the processes $\bar{Y}^{\{l\}}$, $l \in I$, and $K(j) \subseteq K(j+1)$ to obtain

$$\bar{Y}_j = \sum_{l \in K(j)} \bar{Y}_j^{\{l\}} \mathbb{1}_{\{l^*(j)=l\}}$$

$$\begin{aligned}
&\leq \sum_{l \in K(j)} F_j\left(E_j\left[\beta_{j+1}\bar{Y}_{j+1}^{\{l\}}\right]\right) \mathbb{1}_{\{l^*(j)=l\}} \\
&\leq \sum_{l \in K(j)} F_j\left(E_j\left[\beta_{j+1}\bar{Y}_{j+1}^{\{l^*(j+1)\}}\right]\right) \mathbb{1}_{\{l^*(j)=l\}} \\
&= F_j\left(E_j\left[\beta_{j+1}\bar{Y}_{j+1}\right]\right),
\end{aligned}$$

from which we conclude that $\bar{Y}$ is a subsolution. Thus, by Theorem 3.2.1,

$$E_j\left[\Theta_j^{low}(\bar{r}, M)\right] \geq F_j\left(E_j\left[\beta_{j+1}\bar{Y}_{j+1}\right]\right) = \max_{l \in K(j+1)} F_j\left(E_j\left[\beta_{j+1}\bar{Y}_{j+1}^{\{l\}}\right]\right) \geq \max_{l \in K(j+1)} \bar{Y}_j^{\{l\}} \tag{3.28}$$

$P$-a.s. for every $j = 0, \ldots, J-1$, where $\bar{r}$ is for every $j = 0, \ldots, J-1$ given by (3.15) and $M \in \mathcal{M}_D$. Hence, in the case $K(j) = I$ for $j = 1, \ldots, J$, improving $\bar{Y}$ results again in a simultaneous improvement of all subsolutions $(\bar{Y}^{\{l\}})_{l \in I}$.

In the following example, we present a generic way to construct a family of subsolutions from a given admissible control and show that, in the special case of optimal stopping problems, the policy improvement approach of Kolodko and Schoenmakers (2006) can be recovered from our approach.

*Example* 3.3.1. (i) Suppose that we are given an input policy $r \in \mathcal{A}_0^F$. Then, we may choose a reference policy $\hat{r} \in \mathcal{A}_0^F$ and define a family of policies $(r^{\{l\}})_{l=0,\ldots,J-1}$ by

$$r_j^{\{l\}} = \begin{cases} \hat{r}_j, & j < l \\ r_j, & j \geq l \end{cases} \tag{3.29}$$

for $j = 0, \ldots, J-1$. From the definition of $r^{\{l\}}$, we immediately obtain that $r^{\{l\}} \in \mathcal{A}_0^F$ for each $l$. Building on this family of policies, we can define a family of subsolutions $(\bar{Y}^{\{l\}})_{l=0,\ldots,J-1}$ by

$$\bar{Y}_j^{\{l\}} = E_j\left[\Theta_j^{low}\left(r^{\{l\}}\right)\right], \quad j = 0, \ldots, J.$$

Now let the sets $(K(j))_{j=1,\ldots,J}$ be given by

$$K(j) = \{0, \ldots, \min\{j + \kappa - 1, J\}\}$$

for some $\kappa \geq 1$. Then, we observe from the monotonicity assumption on $F_j$ and the definition of $(\bar{Y}^{\{l\}})_{l=0,\ldots,J-1}$, that the improvement condition (3.15) can be rewritten as

$$\bar{r}_j^\top E_j\left[\beta_{j+1}\Theta_{j+1}^{low}\left(r^{\{l^*(j+1)\}}\right)\right] - F_j^{\#}(\bar{r}_j) = \max_{l=j+1,\ldots,(j+\kappa)\wedge J} F_j\left(E_j\left[\beta_{j+1}\Theta_{j+1}^{low}\left(r^{\{l\}}\right)\right]\right) \tag{3.30}$$

for every $j = 0, \ldots, J-1$. Note that the maximum over the set $K(j+1)$ can be restricted to the subset $\{j+1, \ldots, (j+\kappa) \wedge J\}$ in (3.30), as $\Theta_{j+1}^{low}(r^{\{l\}}) = \Theta_{j+1}^{low}(r)$ for all $l \leq j+1$ by definition of the family $(r^{\{l\}})_{l=0,\ldots,J-1}$. By applying (3.28) to this setting, we observe that

$$\begin{aligned}
E_j\left[\Theta_j^{low}(\bar{r})\right] &\geq \max_{l=0,\ldots,(j+\kappa)\wedge J} F_j\left(E_j\left[\beta_{j+1}\Theta_{j+1}^{low}\left(r^{\{l\}}\right)\right]\right) \\
&= \max_{l=j+1,\ldots,(j+\kappa)\wedge J} F_j\left(E_j\left[\beta_{j+1}\Theta_{j+1}^{low}\left(r^{\{l\}}\right)\right]\right),
\end{aligned}$$

where the last equality follows by the same argument as before.

(ii) We now apply the construction from part (i) to the optimal stopping case. To this end, suppose that we are given a family $(\tau^{\{l\}})_{l=0,\dots,J}$ of stopping times such that

$$\tau^{\{l\}} \geq l \quad \text{and} \quad \left(\tau^{\{l\}} > l \;\Rightarrow\; \tau^{\{l\}} = \tau^{\{l+1\}}\right) \tag{3.31}$$

for every $l = 0, \dots, J$. Following Kolodko and Schoenmakers (2006), we call a family of stopping times satisfying (3.31) *consistent*. As explained in Example 1.3.5 (ii), we can derive an admissible control $r \in \mathcal{A}_0^F$ from this stopping family by setting $r_j = \mathbb{1}_{\{\tau^{\{j\}} \neq j\}}$ for every $j = 0, \dots, J-1$. We further choose the reference policy $\hat{r} \equiv 1$, which corresponds to not stopping the process until terminal time. Then, we may derive from part (i) and Example 1.3.5 (ii) that

$$E_j[S_{\bar{\tau}^{\{j\}}}] \geq \max_{l=j+1,\dots,(j+\kappa)\wedge J} \max\{S_j, E_j[S_{\tau^{\{l\}}}]\}, \quad j = 0, \dots, J-1,$$

where the family $(\bar{\tau}^{\{j\}})_{j=0,\dots,J}$ of stopping times is given by

$$\bar{\tau}^{\{j\}} = \inf\left\{ i \geq j \,\middle|\, S_i \geq \max_{l=i+1,\dots,(i+\kappa)\wedge J} E_i[S_{\tau^{\{l\}}}] \right\}$$

for $j = 0, \dots, J$. In this derivation we use that

$$\tau_{r^{\{l\}}} = \inf\left\{ j \geq 0 \,\middle|\, r_j^{\{l\}} = 0 \right\} = \tau^{\{l\}}$$

by consistency of $(\tau^{\{l\}})_{l=0,\dots,J}$ and that

$$\begin{aligned} S_j \geq E_j\left[\Theta_{j+1}^{low}\left(r^{\{l^*(j+1)\}}\right)\right] &\Leftrightarrow S_j \geq \max\left\{S_j, E_j\left[\Theta_{j+1}^{low}\left(r^{\{l^*(j+1)\}}\right)\right]\right\} \\ &\Leftrightarrow S_j \geq \max_{l=j+1,\dots,(j+\kappa)\wedge J} \max\left\{S_j, E_j\left[\Theta_{j+1}^{low}\left(r^{\{l\}}\right)\right]\right\} \\ &\Leftrightarrow S_j \geq \max_{l=j+1,\dots,(j+\kappa)\wedge J} E_j\left[\Theta_{j+1}^{low}\left(r^{\{l\}}\right)\right] \end{aligned}$$

by the definition of $l^*$. Hence, we recover the policy improvement result in Theorem 3.1 of Kolodko and Schoenmakers (2006) as a special case of our approach.

*Remark* 3.3.2. The approaches presented above cannot be generalized to the multi-dimensional setting of Sections 3.1 and 3.2 in a straightforward way. This is mainly due to the fact that in the case of systems of convex dynamic programs the processes $l_*$ and $l^*$ given by (3.25) respectively (3.27) additionally depend on the dimension parameter $\nu$. Indeed, generalizing e.g. the definition of $l^*$ to the multi-dimensional setting leads to

$$\begin{aligned} l^*(j,\nu) = \Big\{ l \in K(j) \,\Big|\, & F_{j-1}^{(\nu)}\left(E_{j-1}\left[\beta_j \bar{Y}_j^{\{l\},(1)}\right], \dots, E_{j-1}\left[\beta_j \bar{Y}_j^{\{l\},(N)}\right]\right) \\ & \geq F_{j-1}^{(\nu)}\left(E_{j-1}\left[\beta_j \bar{Y}_j^{\{\iota\},(1)}\right], \dots, E_{j-1}\left[\beta_j \bar{Y}_j^{\{\iota\},(N)}\right]\right) \; \forall \iota \in K(j) \Big\}, \end{aligned}$$

for every $j = 1, \dots, J$ and $\nu = 1, \dots, N$, where $(\bar{Y}^{\{l\}})_{l \in I}$ is a family of subsolutions to (1.15). Now, the process given by

$$\bar{Y}_j^{(\nu)} = \bar{Y}_j^{(l^*(j,\nu))} \mathbb{1}_{\{j>0\}} + F_0^{(\nu)}\left(E_0\left[\beta_1 \bar{Y}_1^{(1)}\right], \dots, E_0\left[\beta_1 \bar{Y}_1^{(N)}\right]\right) \mathbb{1}_{\{j=0\}}, \quad j = 0, \dots, J,$$

is not a subsolution to (1.15). To see this, we first note that

$$\begin{aligned}
\bar{Y}_j^{(\nu)} &= \sum_{l \in K(j)} \bar{Y}_j^{\{l\},(\nu)} \mathbb{1}_{\{l^*(j,\nu)=l\}} \\
&\leq \sum_{l \in K(j)} F_j^{(\nu)} \left( E_j \left[ \beta_{j+1} \bar{Y}_{j+1}^{\{l\},(1)} \right], \ldots, E_j \left[ \beta_{j+1} \bar{Y}_{j+1}^{\{l\},(N)} \right] \right) \mathbb{1}_{\{l^*(j,\nu)=l\}} \\
&\leq \sum_{l \in K(j)} F_j^{(\nu)} \left( E_j \left[ \beta_{j+1} \bar{Y}_{j+1}^{\{l^*(j+1,\nu)\},(1)} \right], \ldots, E_j \left[ \beta_{j+1} \bar{Y}_{j+1}^{\{l^*(j+1,\nu)\},(N)} \right] \right) \mathbb{1}_{\{l^*(j,\nu)=l\}}
\end{aligned}$$

for any $j = 1, \ldots, J-1$ and $\nu = 1, \ldots, N$. In contrast to the one-dimensional case, we now have in general that

$$\begin{aligned}
& F_j^{(\nu)} \left( E_j \left[ \beta_{j+1} \bar{Y}_{j+1}^{\{l^*(j+1,\nu)\},(1)} \right], \ldots, E_j \left[ \beta_{j+1} \bar{Y}_{j+1}^{\{l^*(j+1,\nu)\},(N)} \right] \right) \\
\not\leq \;& F_j^{(\nu)} \left( E_j \left[ \beta_{j+1} \bar{Y}_{j+1}^{\{l^*(j+1,1)\},(1)} \right], \ldots, E_j \left[ \beta_{j+1} \bar{Y}_{j+1}^{\{l^*(j+1,N)\},(N)} \right] \right) \\
= \;& F_j^{(\nu)} \left( E_j \left[ \beta_{j+1} \bar{Y}_{j+1}^{(1)} \right], \ldots, E_j \left[ \beta_{j+1} \bar{Y}_{j+1}^{(N)} \right] \right)
\end{aligned}$$

and, thus, $\bar{Y}$ is not a subsolution.

## 3.4 Implementation

In this section, we explain how to implement algorithms based on the iterative improvement approaches of Sections 3.1 and 3.2 in the Markovian setting of Section 1.7. This algorithm proceeds in essentially two steps: in a first step input super- and subsolutions are constructed from the algorithm proposed in Chapter 1. Then, in a second step, improved super- and subsolutions are constructed iteratively. The key challenge in the second step is to compute the conditional expectations which are required for the construction of the controls in (3.21) and the Doob martingales in (3.10). For the approximation of these conditional expectations, we rely on a plain Monte Carlo implementation as applied in Kolodko and Schoenmakers (2006). In contrast to a naive plain Monte Carlo implementation for the solution of dynamic programs, this construction does not lead to computational costs which grow exponentially in the number of time steps but rather in the number of iterations. Our numerical example below demonstrates that two improvement steps are feasible if the input super- and subsolutions are constructed from the regression-later approach. We further provide an alternative to the regression-later approach for the construction of approximate solutions, called the martingale minimization approach, if the dynamic program has only one equation. The rationale of this approach is to choose a set of martingales from which a linear combination is constructed such that the resulting upper bound becomes minimal. Finally, we apply the improvement approach to the problem of pricing under funding cost.

### 3.4.1 Martingale minimization approach

Throughout this section, we assume that $N = 1$, i.e. we consider convex dynamic programs of the form

$$\begin{aligned}
Y_J &= g(X_J), \\
Y_j &= f_j \left( X_j, E_j \left[ \beta_{j+1} Y_{j+1} \right] \right), \quad j = 0, \ldots, J-1,
\end{aligned}$$

where $X$ is a time-discrete Markov process. Recall that the process $X$ is of the form

$$X_j = h_j(X_{j-1}, B_j), \quad X_0 = x_0 \in \mathbb{R}^d,$$

with measurable functions $h_j : \mathbb{R}^d \times \mathbb{R}^{\mathcal{D}} \to \mathbb{R}^d$ and an $\mathbb{R}^{\mathcal{D}}$-valued process $(B_j)_{j=1,\dots,J}$, for which the first $D$ components of $B_j$ are given by $\beta_j$ and such that $B_j$ is independent of $\mathcal{F}_{j-1}$ for every $j = 1, \dots, J$. In the LSMC approaches presented in Section 1.7, the idea is to compute coefficients $(a_j)_{j=1,\dots,J}$ by an empirical regression so that an approximate solution to the dynamic program is given by a linear combination of chosen basis functions, i.e.

$$\tilde{y}_{j+1}(x, b) = \sum_{k=1}^{K} a_{j+1,k} \eta_{j+1,k}(x, b), \quad j = 0, \dots, J-1.$$

As we have seen in the previous numerical examples and as discussed in Bender et al. (2017), the construction of meaningful upper bounds from such approximate solutions to the dynamic program is harder than for the lower bounds. The martingale minimization approach tackles this problem directly by computing the coefficients $(a_j)_{j=1,\dots,J}$ differently. The idea of this approach is to choose a set of martingales and to find a linear combination of these, such that the resulting upper bound is minimized. As a consequence the resulting coefficients are global in the sense that they do not depend on time. This is in the spirit of Desai et al. (2012) and Belomestny (2013), who proposed such an approach in the context of Bermudan option pricing.

To be more precisely, let basis functions $\eta_j = (\eta_{j,1}, \dots, \eta_{j,K})$, $j = 1, \dots, J$, be given, which satisfy the assumptions of the regression-later approach, i.e., they are sufficiently integrable and the one-step conditional expectations $R_{j-1}$ are available in closed form. From these basis functions, we can construct a set of martingales $M^{\{k\}}$, $k = 1, \dots, K$, by

$$M_j^{\{k\}} = \sum_{i=0}^{j-1} \beta_{i+1} \eta_{i+1,k}(X_i, B_{i+1}) - R_{i,k}(X_i).$$

Starting from these martingales, we define the martingale $M^a$ by

$$M_j^a = \sum_{k=1}^{K} a_k M_j^{\{k\}}, \tag{3.32}$$

for coefficients $a = (a_1, \dots, a_K) \in \mathbb{R}^K$. The key step in the martingale minimization approach is now to find coefficients $a^*$ such that $E[\Theta_0^{up}(M^a)]$ becomes minimal. Following the approach analyzed in Belomestny (2013), the coefficients $a^*$ are given by

$$a^* = \operatorname*{argmin}_{a \in \mathbb{R}^K} E\left[\Theta_0^{up}(M^a)\right] + \gamma \sqrt{\operatorname{Var}\left(\Theta_0^{up}(M^a)\right)} \tag{3.33}$$

for fixed $\gamma \geq 0$. Note that (3.33) involves a standard deviation penalty whose impact can be controlled by choosing $\gamma$. The idea behind this penalty is, that the resulting upper bound should not only be minimized but that it should also have low variance, since we know that $\operatorname{Var}(\Theta_0^{up}(M^*))$ vanishes for the martingale $M^*$ due to the pathwise optimality.

Since in general neither $E[\Theta_0^{up}(M^a)]$ nor $\operatorname{Var}(\Theta_0^{up}(M^a))$ are available in closed form, we have to replace them by their empirical counterparts in order to obtain an implementable algorithm. Therefore, we simulate $\Lambda^{mini}$ independent copies $\{B_j(\lambda^{mini}),\ j = 1, \dots, J,\ \lambda^{mini} = 1, \dots, \Lambda^{mini}\}$ of the process $B$ to which we refer as minimization paths. Denoting by $M^{\{k\}}(\lambda^{mini})$ the evaluation of the martingales $M^{\{k\}}$ for each $k$ along these paths and computing $\Theta^{up}(M^a(\lambda^{mini}))$ recursively by (1.22), we replace the optimization problem (3.33) by

$$a^* = \operatorname*{argmin}_{a\in\mathbb{R}^K} \hat{E}\left[\Theta_0^{up}(M^a)\right] + \gamma\sqrt{\frac{1}{\Lambda^{mini}-1}\sum_{\lambda^{mini}=1}^{\Lambda^{mini}}\Big(\Theta_0^{up}\left(M^a(\lambda^{mini})\right) - \hat{E}\left[\Theta_0^{up}(M^a)\right]\Big)^2}, \quad (3.34)$$

where

$$\hat{E}\left[\Theta_0^{up}(M^a)\right] = \frac{1}{\Lambda^{mini}}\sum_{\lambda^{mini}=1}^{\Lambda^{mini}}\Theta_0^{up}\left(M^a(\lambda^{mini})\right). \quad (3.35)$$

Then, an approximate solution $\tilde{y}$ to the dynamic program is obtained by

$$\tilde{y}_j(x,b) = \sum_{k=1}^{K} a_k^*\eta_{j,k}(x,b), \quad j = 1,\dots,J.$$

*Remark* 3.4.1. (i) The minimization approach requires the choice of the parameter $\gamma$. In our numerical results presented in Section 3.4.3, we apply a "training and testing" approach to tune this parameter. To this end, we choose a set $\{\gamma_1,\dots,\gamma_L\}$, $L\in\mathbb{N}$, of parameters. For each $\gamma_l$, $l=1,\dots,L$, we compute a vector of coefficients $a^*_{\gamma_l}\in\mathbb{R}^K$ according to (3.34) along the minimization paths $\Lambda^{mini}$. If vectors $a^*_{\gamma_1},\dots,a^*_{\gamma_L}$ are computed, we sample a new set of $\Lambda^{test}$ test paths (independent copies of $B$ which are also independent of the minimization paths). The parameter $\gamma$ is obtained by taking the $\gamma_l$ such that $a^*_{\gamma_l}$ minimizes the expression in (3.35) along the test paths over the set $\{a^*_{\gamma_1},\dots,a^*_{\gamma_L}\}$. We note that in our numerical test case the method's practical performance is not particularly sensitive to the choice of $\gamma$ and actually chooses $\gamma=0$ in the above "training and testing" approach in the majority of test runs. Yet in principle, it may happen that along an "unfavorable" set of minimization paths, the optimal parameter vector without penalization takes rather large absolute values, minimizing (3.35) by creating a small number of very negative $\Theta_0^{up}$-paths. When re-computing (3.35) along an independent set of test paths, the resulting martingale does not perform well in general. Choosing a positive $\gamma$ in (3.34) may counteract such overfitting effects. In that sense, our approach can be viewed as a safety precaution, adding another layer of flexibility to the algorithm.

(ii) The approach presented above does not apply in the multi-dimensional setting considered in the previous sections. This is essentially due to the fact that the process $\Theta^{up}$ becomes $\mathbb{R}^N$-valued and the minimization in (3.34) is not well-defined anymore. In order to circumvent this problem, one could think of replacing the expectation and the variance of $\Theta^{up}$ by $\mathfrak{R}(E[\Theta_0^{(up,1)}],\dots,E[\Theta_0^{(up,N)}])$ for a function $\mathfrak{R}:\mathbb{R}^N\to\mathbb{R}$, which is monotonically increasing in each variable.

### 3.4.2 Iterative improvement algorithm

Suppose that controls $\tilde{r}^{(\nu)}\in\mathcal{A}_0^{F^{(\nu)}}$, $\nu=1,\dots,N$, and a martingale $\tilde{M}\in\mathcal{M}_{ND}$, which can be evaluated in closed form along a given path $B$, are given, cp. the constructions in 1.7 or 3.4.1 in the one-dimensional case. Denote by $Y_j^{low,0}=E_j[\Theta^{low}(\tilde{r}^{(1)},\dots,\tilde{r}^{(N)},\tilde{M})]$ and $Y_j^{up,0}=E_j[\Theta^{up}(\tilde{M})]$ the corresponding input sub- and supersolutions. In order to compute the first iterations $\Theta^{up,1}$ in (3.9) and $\Theta^{low,1}$ in (3.20), we require approximations of the conditional expectations

$E_j[\beta_{j+1}\Theta_{j+1}^{(low,\nu)}(\tilde{r}^{(1)},\ldots,\tilde{r}^{(N)},\tilde{M})]$, $E_j[\beta_{j+1}\Theta_{j+1}^{(up,\nu)}(\tilde{M})]$, and $E_{j+1}[\Theta_{j+1}^{up}(\tilde{M})]$ for each $\nu = 1,\ldots,N$. In the following, we focus on the supersolution case, but note that the improvement for subsolutions can be implemented analogously.

For the approximation of the conditional expectations, we apply a plain Monte Carlo approach. To this end, we first sample $\Lambda^{out}$ independent copies $B(\lambda^{out})$, $\lambda^{out} = 1,\ldots\Lambda^{out}$, of $B$. Moreover, for every time step $j$ and outer path $B(\lambda^{out})$, we apply a subsampling approach and generate a new sample of independent copies $(B_i(\lambda^{mid},j))_{i\geq j+1}$, $\lambda^{mid} = 1,\ldots\Lambda^{mid}$, of $(B_i)_{i\geq j+1}$. We denote by $B(\lambda^{out},\lambda^{mid},j)$ the path given by $(B_1(\lambda^{out}),\ldots,B_j(\lambda^{out}),B_{j+1}(\lambda^{mid},j),\ldots,B_J(\lambda^{mid},j))$, which switches from a given outer path to the corresponding middle path at time $j+1$. Similarly to the notation introduced before, we write $\beta(\lambda^{out},\lambda^{mid},j)$ and $\Theta^{up,0}(\lambda^{out},\lambda^{mid},j)$ for the trajectories of $\beta$ and $\Theta^{up}(\tilde{M})$ along the path $B(\lambda^{out},\lambda^{mid},j)$. Along each outer path, we approximate the martingale $M^1$ in (3.10) with increment

$$M_{j+1}^{[\nu],1} - M_j^{[\nu],1} = \beta_{j+1}E_{j+1}\left[\Theta_{j+1}^{(up,\nu),0}\right] - E_j\left[\beta_{j+1}\Theta_{j+1}^{(up,\nu),0}\right], \quad j = 0,\ldots,J-1,\ \nu = 1,\ldots,N,$$

by the plain Monte Carlo estimator

$$\tilde{M}_{j+1}^{[\nu],1}(\lambda^{out}) - \tilde{M}_j^{[\nu],1}(\lambda^{out}) = \beta_{j+1}(\lambda^{out})\hat{E}_{j+1}\left[\Theta_{j+1}^{(up,\nu),0}\right](\lambda^{out}) - \hat{E}_j\left[\beta_{j+1}\Theta_{j+1}^{(up,\nu),0}\right](\lambda^{out}),$$

where

$$\begin{aligned}\hat{E}_j\left[\Theta_j^{(up,\nu),0}\right](\lambda^{out}) &:= \frac{1}{\Lambda^{mid}}\sum_{\lambda^{mid}=1}^{\Lambda^{mid}}\Theta_j^{(up,\nu),0}(\lambda^{out},\lambda^{mid},j)\\ \hat{E}_j\left[\beta_{j+1}\Theta_{j+1}^{(up,\nu),0}\right](\lambda^{out}) &:= \frac{1}{\Lambda^{mid}}\sum_{\lambda^{mid}=1}^{\Lambda^{mid}}\beta_{j+1}(\lambda^{out},\lambda^{mid},j)\Theta_{j+1}^{(up,\nu),0}(\lambda^{out},\lambda^{mid},j) \qquad (3.36)\end{aligned}$$

for every $\nu = 1,\ldots,N$. We now write $\Theta^{up,1}(\lambda^{out})$ for the realization of $\Theta^{up}(\tilde{M}^1)$ along the $\lambda^{out}$-th outer path. From the estimators (1.59) and (1.60), we can compute a new upper confidence bound for $Y_0$ based on $(\Theta^{up,1}(\lambda^{out}))_{\lambda^{out}=1,\ldots,\Lambda^{out}}$. Since $\tilde{M}^1$ converges to $M^1$ (along each outer path) as the number of middle paths converges to infinity, and since $E_0[\Theta^{up}(M^1)] \leq E_0[\Theta^{up}(\tilde{M})]$ by Theorem 3.1.3, the corresponding upper bound is typically tighter than the one constructed from $(\Theta^{up,0}(\lambda^{out}))_{\lambda^{out}=1,\ldots,\Lambda^{out}}$, when the number of middle paths is sufficiently large.

In case that a second iteration step shall be computed (e.g., because the once improved confidence interval is still not tight enough), the overall procedure is similar. The only difference is that we cannot assume the input process $\tilde{M}^1$ to be available in closed form along a given path. Its evaluation actually requires one layer of nested simulation as described above. The next iteration step yet requires to evaluate $\tilde{M}^1$ along middle paths and not along outer paths. As a consequence, we have to sample a third layer of $\Lambda^{in}$ "inner paths" for which we omit the details of the straightforward implementation. However, we emphasize that a similar procedure is already required in the first iteration step, when the input martingale $\tilde{M}$ is not available in closed form (e.g., when the approximate solution to the dynamic program is computed by the regression-now variant of LSMC).

As discussed in Section 1.7, subsampling leads to an additional upward bias in the upper bound, which can be reduced by increasing the number of middle paths (in the first iteration step) and inner paths (in the second iteration step). Since this, in turn, increases the computational cost, the number of middle and inner paths should be kept at a moderate level. We thus suggest to apply control variates in the plain Monte Carlo estimation (3.36) of the martingale increments. These can, e.g., be based on the closed form expression for $E_j[\Theta^{up}_{j+1}(\tilde{M})]$ and $E_j[\beta_{j+1}\Theta^{(up,\nu)}_{j+1}(\tilde{M})]$ or, like in our actual implementation, as described in Remark 1.7.2 (iv).

Finally, we emphasize that this procedure can be further iterated but that each iteration step requires an additional layer of subsimulations leading to higher computational costs. Hence, at some point, it might be a better idea to put more effort in the construction of a better input approximation than performing an additional iteration step if the confidence interval is still not tight enough. This consideration is confirmed by our numerical example below, where we demonstrate that two iteration steps are feasible and that improving the input approximation can increase the quality of the resulting 95%-confidence interval substantially.

### 3.4.3 Numerical example

We now apply the improvement approach to the problem of pricing a European option under funding constraints, i.e., under different interest rates for borrowing and lending. In the finance literature, this problem goes back to Bergman (1995). The model is also a prominent example in the literature on backward stochastic differential equations starting with El Karoui et al. (1997) and a well-established numerical test case, see Gobet et al. (2005); Lemor et al. (2006); Bender and Steiner (2012); Bender et al. (2017).

There are two riskless interest rates $R^L < R^B \in \mathbb{R}$ for lending respectively borrowing and $d$ risky assets given by geometric Brownian motions $X^{(1)}, \dots, X^{(d)}$ with dynamics

$$X^{(l)}_t = x^{(l)}_0 \exp\left\{\left(\mu - \frac{1}{2}\sum_{n=1}^d \sigma^2_{l,n}\right) t + \sum_{n=1}^d \sigma_{l,n} W^{(n)}_t\right\}, \quad l = 1, \dots, d,$$

at $t \in [0, T]$. Here, $x^{(l)}_0, \mu \in \mathbb{R}$, $\sigma$ is an invertible $d \times d$-matrix with entries in $\mathbb{R}$ and $W^{(1)}, \dots, W^{(d)}$ are independent Brownian motions. We consider the problem of pricing a European option on the assets $X^{(1)}, \dots, X^{(d)}$ with maturity $T$ and payoff $g(X^{(1)}_T, \dots, X^{(d)}_T)$. Following El Karoui et al. (1997), the value $Y$ of the option is then given by the BSDE

$$Y_t = g\left(X^{(1)}_T, \dots, X^{(d)}_T\right) + \int_t^T f(s, Y_s, Z_s)\, dt - \int_t^T Z_s^\top\, dW_s, \quad t \in [0, T], \tag{3.37}$$

where

$$f(t, y, z) = -R^L y - \left(\mu - R^L\right) z^\top \sigma^{-1}\mathbf{1} + \left(R^B - R^L\right)\left(y - z^\top \sigma^{-1}\mathbf{1}\right)_- .$$

Discretizing BSDE (3.37) over an equidistant partition $0 = t_0 < t_1 < \ldots < t_J = T$ of $[0, T]$ with increments $\Delta$ as explained in Example 1.1.2, we end up with the following convex dynamic program:

$$\begin{aligned}
X^{(l)}_j &= x^{(l)}_0 \exp\left\{\left(\mu - \frac{1}{2}\sum_{n=1}^d \sigma^2_{l,n}\right)\Delta + \sum_{n=1}^d \sigma_{l,n}\Delta W^{(n)}_j\right\}, \quad l = 1, \dots, d,\\
Y_J &= g\left(X^{(1)}_J, \dots, X^{(d)}_J\right)\\
Z_j &= E_j\left[\frac{[\Delta W_{j+1}]_c}{\Delta} Y_{j+1}\right],\\
Y_j &= \left(1 - R^L\Delta\right) E_j[Y_{j+1}] - \left(\mu - R^L\right) Z_j^\top \sigma^{-1}\mathbf{1}\Delta + \left(R^B - R^L\right)\Delta\left(E_j[Y_{j+1}] - Z_j^\top\sigma^{-1}\mathbf{1}\right)_-,
\end{aligned} \tag{3.38}$$

where $E_j[\cdot]$ denotes the conditional expectation with respect to the filtration generated by the Brownian motion $W$ up to time $t_j$. As before, we passed at same time to the shorthand notation $X_j := X_{t_j}$. Moreover, $[\cdot]_c$ denotes a componentwise truncation at $\pm c$ for a constant $c > 0$. This truncation is required to ensure that the monotonicity assumption (3.1) holds. We emphasize, however, that the corresponding truncation error becomes small for sufficiently small time increments $\Delta$. The term $Z_j^\top \sigma^{-1}\mathbf{1}$ in (3.38) represents the overall position in the risky assets in the hedging portfolio at time $t_j$. Therefore, $E_j[Y_{j+1}] - Z_j^\top \sigma^{-1}\mathbf{1}$ is an approximation of the position in the bank account at time $t_j$. The sign of this expression determines which interest rate is applicable.

Taking the function $F_j : \mathbb{R}^{d+1} \to \mathbb{R}$ given by

$$F_j(z) = (1 - R^L\Delta)z^{(1)} - (\mu - R^L)\left(z^{(-1)}\right)^\top \sigma^{-1}\mathbf{1}\Delta + \left(R^B - R^L\right)\Delta\left(z^{(1)} - \left(z^{(-1)}\right)^\top \sigma^{-1}\mathbf{1}\right)_-,$$

with $z = (z^{(1)}, z^{(-1)}) = (z^{(1)}, \dots, z^{(d+1)})$, $N = 1$, $D = d + 1$, and

$$B_j = \left(1, \frac{[\Delta W_j]_c}{\Delta}, \Delta W_j\right) \quad j = 1, \dots, J,$$

we observe that (3.38) is of the form (1.15). From the definition of $F_j$, we obtain by Appendix A.2 that $F_j^{\#} \equiv 0$ on its effective domain $D_{F^{\#}}^{(j,\cdot)} = \{u(R) | R \in [R^L, R^B]\}$, with $u : \mathbb{R} \to \mathbb{R}^{d+1}$ given by

$$u^{(1)}(s) = (1 - s\Delta) \quad \text{and} \quad u^{(l+1)}(s) = -(\mu - s)\Delta \sum_{n=1}^{d} \left(\sigma^{-1}\right)_{l,n}, \quad l = 1, \dots, d,$$

Hence, the duality relation (1.24) reads

$$\begin{aligned} &r_j^{(1,*)} E_j[Y_{j+1}] + \sum_{n=2}^{d+1} r_j^{(n,*)} Z_j^{(n-1)} \\ &\quad = (1 - R^L\Delta)E_j[Y_{j+1}] - (\mu - R^L)Z_j^\top \sigma^{-1}\mathbf{1}\Delta + \left(R^B - R^L\right)\Delta\left(E_j[Y_{j+1}] - Z_j^\top \sigma^{-1}\mathbf{1}\right)_-. \end{aligned}$$

A solution to this equation is given by

$$r_j^* = \begin{cases} u(R^L), & E_j[Y_{j+1}] \geq Z_j^\top \sigma^{-1}\mathbf{1} \\ u(R^B), & E_j[Y_{j+1}] < Z_j^\top \sigma^{-1}\mathbf{1}. \end{cases}$$

For our numerical experiments, we consider the example discussed in Bender et al. (2017), but add a non-trivial correlation structure to the problem. This example is a multi-dimensional version of an example going back to Gobet et al. (2005). We compute upper and lower bounds on the price of a European call-spread option with strikes $K_1$ and $K_2$ on the maximum of $d = 5$ assets, i.e.,

$$g\left(x^{(1)}, \dots, x^{(5)}\right) = \left(\max_{l=1,\dots,5} x^{(n)} - K_1\right)_+ - 2\left(\max_{l=1,\dots,5} x^{(n)} - K_2\right)_+, \quad x \in \mathbb{R}^5.$$

The maturity $T$ is set to three months, i.e. $T = 0.25$, and the strikes are $K_1 = 95$ and $K_2 = 115$. The interest rates $R^L$ and $R^B$ are 1% and 6%. For the geometric Brownian motions $X^{(1)}, \dots, X^{(5)}$ we take $x_0^{(l)} = 100$, $l = 1, \dots, 5$, as starting value and choose the drift $\mu$ to be 0.05. In contrast to Bender et al. (2017), we do not assume that $X^{(1)}, \dots, X^{(5)}$ are independent and consider the diffusion matrix $\sigma$ given by

$$\sigma = \tilde{\sigma} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \rho & \sqrt{1-\rho^2} & 0 & 0 & 0 \\ \rho & 0 & \sqrt{1-\rho^2} & 0 & 0 \\ \rho & 0 & 0 & \sqrt{1-\rho^2} & 0 \\ \rho & 0 & 0 & 0 & \sqrt{1-\rho^2} \end{pmatrix},$$

where $\tilde{\sigma} = 0.2$. In our numerical experiments below, the correlation parameter $\rho$ is varied between $-0.3$ and $0.3$ and the time discretization $J$ takes values in $\{20, 30, 40\}$. With this choice of parameters, we observe that the monotonicity condition (3.1) is satisfied with a truncation level of $c = 0.77$ at the roughest time discretization level $J = 20$. Truncating the Brownian increments with standard deviation $\sqrt{\Delta} \approx 0.112$ at $0.77$ is the same as truncating a standard normal random variable at 6.88, corresponding to truncating a probability mass of $3 \cdot 10^{-12}$ in both tails.

**Generic minimization algorithm**

For the construction of the input approximation, we first run the martingale minimization algorithm with the single and completely generic basis function $\eta_{j,1}(x, b) := 1$, i.e., we initially approximate $Y_j$ by a constant and the $Z_j^{(l)}$ by zero, $l = 1, \dots, 5$. Then, in the minimization approach presented in Section 3.4.1 we have a single 6-dimensional martingale $M^{\{1\}}$ given by $\tilde{M}_{j+1}^{\{1\},(0)} - \tilde{M}_j^{\{1\},(0)} = 0$ and

$$\tilde{M}_{j+1}^{\{1\},(l)} - \tilde{M}_j^{\{1\},(l)} = \beta_{j+1}^{(l)} - E_j\left[\beta_{j+1}^{(l)}\right] = \frac{\left[\Delta W_{j+1}^{(n)}\right]_c}{\Delta}$$

for $l = 1, \dots, 5$. In order to compute the $\mathbb{R}$-valued coefficient $a^*$, and, hence, the constant approximation $\tilde{y}_j(x, b) = a^*$ to $y_j$, we implement the "training and testing" approach of Remark 3.4.1 with $\Lambda^{mini} = \Lambda^{test} = 1000$ paths and $\{\gamma_1, \dots, \gamma_{21}\} = \{0,\ 0.025, \dots, 0.5\}$. We find that $a^*$, as an approximation of $Y_0$, ranges between 16 and 17.5 for our different choices of $J$ and $\rho$, and as $a^* > 0$, the input subsolution $Y^{low,0}$ is constructed from the constant control $u(R^L)$. For the computation of upper and lower bounds with up to two iterative improvements, we take $\Lambda^{out} = 1000$ outer paths, $\Lambda^{mid} = 200$ middle paths and $\Lambda^{in} = 50$ inner paths. The resulting estimators for the upper and lower bounds from the $k$-th improvement are denoted by $\hat{Y}_0^{up,k,a}$ and $\hat{Y}_0^{low,k,a}$. For comparison, we also state the upper bound estimator $\hat{Y}_0^{up,0,0}$ which is computed by choosing $a = 0$, i.e., by setting all martingale increments to zero.

Table 3.1 presents upper and lower bounds for two different choices of $\rho$, namely $\rho = 0.3$ and $\rho = -0.3$.

We first observe that the upper bound is very sensitive with respect to the input martingale. Even optimizing a very crude constant approximation for $Y$ has a huge impact, and, e.g., leads to a half as large upper bound for $J = 40$ time steps in the negative correlation case compared to the upper bound $\hat{Y}_0^{up,0,0}$ computed from the zero martingale. Nonetheless, the relative width of the 95% confidence interval based on the optimal constant approximation is still more than 16% for 40 time steps in the positive correlation case and even larger in the negative correlation case. Improving upper and lower confidence bound once, shrinks the 95% confidence interval to a quite acceptable relative width of less than 3.5% in the positive correlation case, while a second iterative improvement of the upper bound leads to a relative width of less than 1.5%. The negative correlation apparently

| $\rho$ | 0.3 | | | −0.3 | | |
|---|---|---|---|---|---|---|
| $J$ | 20 | 30 | 40 | 20 | 30 | 40 |
| $\hat{Y}_0^{up,0,0}$ | 18.7084 (0.2193) | 20.9437 (0.2369) | 21.9248 (0.2746) | 26.0757 (0.2624) | 30.3401 (0.2996) | 33.9942 (0.3659) |
| $\hat{Y}_0^{up,0,a^*}$ | 14.1415 (0.1366) | 14.7452 (0.1316) | 14.8168 (0.1361) | 15.8132 (0.1001) | 16.2220 (0.0979) | 16.6361 (0.0986) |
| $\hat{Y}_0^{up,1,a^*}$ | 13.1298 (0.0126) | 13.2443 (0.0139) | 13.3392 (0.0151) | 14.5053 (0.0120) | 14.7067 (0.0129) | 14.9055 (0.0152) |
| $\hat{Y}_0^{up,2,a^*}$ | 13.0608 (0.0132) | 13.0564 (0.0140) | 13.1203 (0.0142) | 14.2127 (0.0096) | 14.2593 (0.0101) | 14.3247 (0.0104) |
| $\hat{Y}_0^{low,0,a^*}$ | 12.5648 (0.0228) | 12.6002 (0.0273) | 12.5813 (0.0303) | 13.7964 (0.0271) | 13.7688 (0.0324) | 13.7555 (0.0387) |
| $\hat{Y}_0^{low,1,a^*}$ | 12.9757 (0.0133) | 12.9827 (0.0159) | 12.9545 (0.0185) | 14.0569 (0.0162) | 14.0400 (0.0190) | 13.9903 (0.0268) |

Table 3.1: Upper and lower bounds based on the generic minimization algorithm for different time discretizations. Standard deviations are given in brackets.

makes the problem harder to solve numerically. But, still, after two iteration steps for the upper bound and one iteration step for the lower bound, we end up with a 95% confidence interval of a relative width of less than 3%. We also observe a significant decrease in the empirical standard deviations of the upper bound estimators through the improvement steps, as expected since the martingales approach the pathwise optimal Doob martingale of $\beta Y$.

Taking into account that no problem-specific information was used to construct the above confidence intervals in a five-dimensional problem with non-smooth coefficients and non-trivial correlation structure, the numerical results are convincing. We note, however, that the second iteration step increases the computational costs by a factor of $\Lambda^{in} \cdot (J/3)$ (e.g., a factor of 667 in our setting for $J = 40$ time steps) compared to a single improvement step. Thus, we next explore to what extent the results can be improved by putting more effort into the construction of the input approximation.

#### Non-generic minimization and LSMC algorithms

Following ideas of Andersen and Broadie (2004) for the pricing of Bermudan options on the maximum of several assets, we now incorporate information about option prices on the largest and second-largest asset into the function basis. To this end, we define the two adapted processes $l^{(1)}$ and $l^{(2)}$ by

$$l_j^{(1)} := \inf\left\{ l_0 \in \{1,\dots,5\} \,\middle|\, X_j^{(l_0)} \geq X_j^{(l)} \;\forall l = 1,\dots,5 \right\}$$
$$l_j^{(2)} := \inf\left\{ l_0 \in \{1,\dots,5\} \setminus \left\{l_j^{(1)}\right\} \,\middle|\, X_j^{(l_0)} \geq X_j^{(l)} \;\forall l \in \{1,\dots,5\} \setminus \left\{l_j^{(1)}\right\} \right\}$$

for $j = 0,\dots,J$. Hence, $l_j^{(1)}$ and $l_j^{(2)}$ indicate the largest respectively second-largest asset at time $t_j$. In particular, they can be viewed as functions of $X_j$. Based on this, we define the following functions which serve as a basis for our approximations of $Y$:

$$\eta_{j,1}(X_{j-1}, X_j) := 1, \quad \eta_{j,\iota+1}(X_{j-1}, X_j) := \sum_{l=1}^{5} X_j^{(l)} \mathbb{1}_{\{l_{j-1}^{(\iota)} = l\}}, \; \iota = 1, 2,$$

$$\eta_{j,\iota+3}(X_{j-1}, X_j) := \sum_{l=1}^{5} E\left[ \left(X_J^{(l)} - K_1\right)_+ - 2\left(X_J^{(l)} - K_2\right)_+ \middle| X_j^{(l)} \right] \mathbb{1}_{\{l_{j-1}^{(\iota)} = l\}}, \; \iota = 1, 2,$$

$$\eta_{j,6}(X_{j-1},X_j) := \sum_{l=1}^{5} E\left[\left(X_J^{(l)}-K_2\right)_+\middle| X_j^{(l)}\right] \mathbb{1}_{\{l_{j-1}^{(1)}=l\}}.$$

Here, we write, for simplicity and in slight abuse of notation, the basis functions as functions of $(X_{j-1}, X_j)$ instead of $(X_{j-1}, B_j)$, cp. Example 1.7.1. Note that, e.g., the fourth basis function represents the price of the corresponding call spread option at time $t_j$ on the asset which is the largest one at time $t_{j-1}$. Shifting the time index in the indicator by one time step (compared to the more intuitive function basis in Andersen and Broadie (2004) which is based on the largest asset at time $t_j$) turned out to be inessential in this numerical example, but ensures that the one-step conditional expectations $R_{j-1,k}(X_{j-1})$ in (1.53) are available in closed form (when neglecting the truncations of the Brownian increments for the closed form computations). Indeed, we have for $R^{(0)}_{j-1,k}(X_{j-1}) := E_{j-1}[\eta_{j,k}(X_{j-1},X_j)]$ that

$$R^{(0)}_{j-1,1}(X_{j-1}) = 1, \quad R^{(0)}_{j-1,\iota+1}(X_{j-1}) = \sum_{l=1}^{5} e^{\mu\Delta} X_{j-1}^{(l)} \mathbb{1}_{\{l_{j-1}^{(\iota)}=l\}}, \; \iota = 1,2,$$

$$R^{(0)}_{j-1,\iota+3}(X_{j-1}) = \sum_{l=1}^{5} E\left[\left(X_J^{(l)}-K_1\right)_+ - 2\left(X_J^{(l)}-K_2\right)_+\middle| X_{j-1}^{(l)}\right] \mathbb{1}_{\{l_{j-1}^{(\iota)}=l\}}, \; \iota = 1,2,$$

$$R^{(0)}_{j-1,6}(X_{j-1}) = \sum_{l=1}^{5} E\left[\left(X_J^{(l)}-K_2\right)_+\middle| X_{j-1}^{(l)}\right] \mathbb{1}_{\{l_{j-1}^{(1)}=l\}}.$$

For the conditional expectations $R^{(l)}_{j-1,k}(X_{j-1}) := E_{j-1}[(\Delta W_j^{(l)}/\Delta)\eta_{j,k}(X_{j-1},X_j)]$, we first note that for each $n, l \in \{1,\ldots,d\}$

$$E\left[\frac{\Delta W_j^{(l)}}{\Delta} h\left(X_J^{(n)}\right)\middle| X_{j-1}^{(n)} = x^{(n)}\right] = x^{(n)}\sigma_{n,l}\frac{\partial}{\partial x^{(d)}} E\left[h\left(X_J^{(n)}\right)\middle| X_{j-1}^{(n)} = x^{(n)}\right] \tag{3.39}$$

for functions $h : \mathbb{R} \to \mathbb{R}$ satisfying suitable growth conditions. This identity is straightforward and we provide the details in Appendix A.4. Then, we obtain that

$$\begin{aligned}
R^{(l)}_{j-1,1}(X_{j-1}) &= 0, \quad R^{(l)}_{j-1,\iota+1}(X_{j-1}) = \sum_{n=1}^{5} \sigma_{n,k} X_{j-1}^{(n)} e^{\mu\Delta} \mathbb{1}_{\{l_{j-1}^{(\iota)}=n\}}, \; \iota = 1,2, \\
R^{(l)}_{j-1,\iota+3}(X_{j-1}) &= \sum_{n=1}^{5} \sigma_{n,k} X_{j-1}^{(n)} \cdot \Big(\mathcal{N}(d_+(t_J - t_j, X_{j-1}^{(n)}, K_1)) \\
&\quad -2\mathcal{N}(d_+(t_J - t_j, X_{j-1}^{(n)}, K_2))\Big) \mathbb{1}_{\{l_{j-1}^{(\iota)}=n\}}, \; \iota = 1,2, \\
R^{(l)}_{j-1,6}(X_{j-1}) &= \sum_{n=1}^{5} \sigma_{n,k} X_{j-1}^{(n)} \mathcal{N}(d_+(t_J - t_j, X_{j-1}^{(n)}, K_2)) \mathbb{1}_{\{l_{j-1}^{(1)}=n\}}.
\end{aligned}$$

Hence we rely essentially on Black-Scholes prices and Black-Scholes deltas of European options at time $t_{j-1}$ on the asset which is the (second) largest at time $t_{j-1}$. Note that we, again, dropped the truncation of the Brownian increments $\Delta W_j$ in the computation of $R^{(l)}_{j-1,k}$ as the truncation error is negligible for this choice of basis functions $\eta_{j,k}$, cp. Appendix C.1.

With these basis functions, we construct input approximations from both, the regression-later and the minimization approach. For the martingale minimization algorithm, we run as before

$\Lambda^{mini} = \Lambda^{test} = 1000$ paths and take the penalization parameter from the set $\{\gamma_1, \dots, \gamma_{21}\} = \{0,\ 0.025, \dots, 0.5\}$. The regression-later approach is applied with $\Lambda^{reg} = 100.000$ regression paths. Tables 3.2 and 3.3 below display the corresponding upper and lower bound estimators as well as iterative improvements up to the second order, based on these two input approximations. As before, we denote by $\hat{Y}_0^{up,k}$ and $\hat{Y}_0^{low,k}$ the upper respectively lower bound resulting from the $k$-th improvement.

| $\rho$ | 0.3 | | | $-0.3$ | | |
|---|---|---|---|---|---|---|
| $J$ | 20 | 30 | 40 | 20 | 30 | 40 |
| $\hat{Y}_0^{up,0,mini}$ | 13.2790 (0.0676) | 13.6548 (0.0721) | 13.6082 (0.0748) | 14.7490 (0.0654) | 14.9249 (0.0678) | 14.9845 (0.0585) |
| $\hat{Y}_0^{up,1,mini}$ | 13.0343 (0.0067) | 13.0548 (0.0062) | 13.0736 (0.0074) | 14.2254 (0.0066) | 14.2828 (0.0065) | 14.3574 (0.0069) |
| $\hat{Y}_0^{up,2,mini}$ | 13.0455 (0.0067) | 13.0635 (0.0071) | 13.0646 (0.0072) | 14.1659 (0.0059) | 14.2023 (0.0059) | 14.2234 (0.0055) |
| $\hat{Y}_0^{low,0,mini}$ | 12.9829 (0.0084) | 12.9750 (0.0080) | 12.9871 (0.0093) | 14.0692 (0.0098) | 14.0616 (0.0100) | 14.0820 (0.0120) |
| $\hat{Y}_0^{low,1,mini}$ | 13.0078 (0.0071) | 13.0136 (0.0065) | 13.0052 (0.0081) | 14.1118 (0.0074) | 14.1005 (0.0080) | 14.1022 (0.0093) |

Table 3.2: Upper and lower bounds based on the non-generic minimization algorithm for different time discretizations and $R^B = 0.06$. Standard deviations are given in brackets.

| $\rho$ | 0.3 | | | $-0.3$ | | |
|---|---|---|---|---|---|---|
| $J$ | 20 | 30 | 40 | 20 | 30 | 40 |
| $\hat{Y}_0^{up,0,reg}$ | 13.1765 (0.0673) | 13.5839 (0.0701) | 13.5552 (0.0757) | 14.7160 (0.0714) | 14.8928 (0.0768) | 15.0624 (0.0740) |
| $\hat{Y}_0^{up,1,reg}$ | 13.0271 (0.0058) | 13.0503 (0.0057) | 13.0675 (0.0057) | 14.2127 (0.0060) | 14.2732 (0.0061) | 14.3315 (0.0066) |
| $\hat{Y}_0^{up,2,reg}$ | 13.0510 (0.0064) | 13.0714 (0.0065) | 13.0874 (0.0070) | 14.1817 (0.0060) | 14.2157 (0.0058) | 14.2501 (0.0063) |
| $\hat{Y}_0^{low,0,reg}$ | 12.9873 (0.0073) | 13.0009 (0.0067) | 12.9945 (0.0084) | 14.0566 (0.0100) | 14.0367 (0.0108) | 14.0658 (0.0128) |
| $\hat{Y}_0^{low,1,reg}$ | 13.0087 (0.0070) | 13.0149 (0.0065) | 13.0089 (0.0080) | 14.1119 (0.0073) | 14.1009 (0.0079) | 14.1140 (0.0091) |

Table 3.3: Upper and lower bounds based on the regression-later approach for different time discretizations and $R^B = 0.06$. Standard deviations are given in brackets.

By and large, we find that the quality of the upper bound estimators $\hat{Y}_0^{up,0,mini}$ and $\hat{Y}_0^{up,0,reg}$, computed from the two different methods to obtain the coefficients for the input approximation, is almost identical. They typically vary by less than two empirical standard deviations. The same holds true for the lower bounds $\hat{Y}_0^{low,0,mini}$ and $\hat{Y}_0^{low,0,reg}$. We also observe that, compared to the generic implementation, the input lower bounds $\hat{Y}_0^{low,0,mini}$ and $\hat{Y}_0^{low,0,reg}$ are of the same quality as the generic lower bounds in Table 3.1 $\hat{Y}_0^{low,1,a^*}$ after one iterative improvement. Similarly, one improvement step of the upper bound in both non-generic cases $\hat{Y}_0^{up,1,mini}$ and $\hat{Y}_0^{up,1,reg}$ is comparable with two improvement steps in the generic setting $\hat{Y}_0^{up,2,a^*}$. Recalling the large computational costs for the second improvement step, we observe that incorporating soft problem information into the function basis (here, the indicator function on the largest and second-largest asset one time step before) can significantly help to pin down the non-linear option price $Y_0$ into a rather tight confidence interval after one iteration step only (and, hence, at moderate costs). For the sake of completeness, we also report the numerical results after performing a second iteration step for the upper bounds in the non-generic case. While in the case of negative correlation, we obtain a further improvement and end up with a confidence interval of a relative width of less than 1.2 % for $J = 40$ time steps, the situation for the positive correlation case is different. Here, the theoretical improvement of the upper bound is offset by the additional upward bias due to the small number of inner paths. In this case, however, the relative width of the 95% confidence interval is about 0.75% already after one iteration step, and, thus, any further improvement seems to be unnecessary for the option pricing problem under consideration.

We finally check the performance of our algorithm when the influence of the non-linearity is further increased. To this end, we change the borrowing rate from $R^B = 0.06$ to $R^B = 0.21$, resulting in an increase of the Lipschitz constant by a factor of 4. While an interest rate of 21% may be viewed as unrealistic, we note that a large value of $R^B$ penalizes borrowing and the superhedging price under the no-borrowing constraint is known to arise in the limit $R^B \to \infty$.

| $\rho$ | 0.3 | | | $-0.3$ | | |
|---|---|---|---|---|---|---|
| $J$ | 20 | 30 | 40 | 20 | 30 | 40 |
| $\hat{Y}_0^{up,0,mini}$ | 15.1756 (0.0629) | 15.4021 (0.0659) | 15.6797 (0.0827) | 16.6270 (0.0661) | 17.0378 (0.0873) | 17.2585 (0.0888) |
| $\hat{Y}_0^{up,1,mini}$ | 14.3803 (0.0066) | 14.6102 (0.0066) | 14.8575 (0.0073) | 15.9797 (0.0071) | 16.3250 (0.0067) | 16.6146 (0.0076) |
| $\hat{Y}_0^{up,2,mini}$ | 14.0527 (0.0049) | 14.2220 (0.0047) | 14.3928 (0.0050) | 15.5550 (0.0066) | 16.0712 (0.0097) | 16.6958 (0.0159) |
| $\hat{Y}_0^{low,0,mini}$ | 13.7119 (0.0273) | 13.7081 (0.0288) | 13.6582 (0.0415) | 14.3663 (0.0508) | 14.4026 (0.0596) | 14.2957 (0.0706) |
| $\hat{Y}_0^{low,1,mini}$ | 13.8733 (0.0139) | 13.8560 (0.0160) | 13.8620 (0.0205) | 14.6236 (0.0394) | 14.5642 (0.0464) | 14.4738 (0.0593) |

Table 3.4: Upper and lower bounds based on the non-generic minimization algorithm for different time discretizations and $R^B = 0.21$. Standard deviations are given in brackets.

| $\rho$ | 0.3 | | | $-0.3$ | | |
|---|---|---|---|---|---|---|
| $J$ | 20 | 30 | 40 | 20 | 30 | 40 |
| $\hat{Y}_0^{up,0,reg}$ | 15.3014 (0.1212) | 16.3805 (0.1762) | 17.0170 (0.2460) | 19.6349 (0.3717) | 23.4699 (0.6203) | 26.8002 (0.7822) |
| $\hat{Y}_0^{up,1,reg}$ | 14.3939 (0.0113) | 14.7545 (0.0135) | 15.1279 (0.0175) | 16.7786 (0.0289) | 18.5106 (0.0462) | 20.7646 (0.0700) |
| $\hat{Y}_0^{up,2,reg}$ | 14.1012 (0.0069) | 14.2952 (0.0072) | 14.4788 (0.0088) | 15.7018 (0.0115) | 16.6096 (0.0177) | 17.9005 (0.0282) |
| $\hat{Y}_0^{low,0,reg}$ | 13.8339 (0.0166) | 13.8350 (0.0179) | 13.8730 (0.0214) | 14.6735 (0.0409) | 14.6399 (0.0505) | 14.7351 (0.0502) |
| $\hat{Y}_0^{low,1,reg}$ | 13.8745 (0.0142) | 13.8756 (0.0168) | 13.8892 (0.0195) | 14.6257 (0.0386) | 14.6864 (0.0433) | 14.6676 (0.0509) |

Table 3.5: Upper and lower bounds based on the regression-later approach for different time discretizations and $R^B = 0.21$. Standard deviations are given in brackets.

Tables 3.4 and 3.5 illustrate the numerical results for this parameter choice. Except for the borrowing rate, all other parameters and the choice of basis functions remain unchanged. We observe that in this more challenging test case, the input upper bounds of the minimization algorithm are superior to those computed from the regression approach, and vice versa for the lower bounds for both choices of $\rho$. However, after one improvement step, the lower bounds based on the minimization approach are within two empirical standard deviations compared to the one step improvements of the regression lower bounds, while the upper bounds of the minimization approach are still significantly below the regression upper bounds after two improvement steps for $\rho = 0.3$. The overall performance of the improvement algorithm is (in spite of the larger Lipschitz constant) still very acceptable for $\rho = 0.3$ and both input types. Indeed, the relative width of the 95% confidence interval is about 4% for 40 time steps in the minimization approach and of about 4.5% in the regression approach.

In the case of negative correlation, the results are however not fully satisfactory. Although the effect of the improvement algorithm is clearly visible, the relative width of the corresponding confidence intervals is significantly larger, even after two improvements of the upper bounds. For $J = 30$ time steps, the relative widths are about 10% for the minimization approach and even 12% for the regression approach. Combining the once improved regression lower bound and the twice improved minimization upper bound, the relative width of the 95% confidence interval $[14.60, 16.09]$ can be reduced to about 9%. This clearly indicates that a better input approximation is required for this problem.

# Appendix A

# Appendix to Chapter 1

## A.1 Derivation of the Malliavin Monte Carlo weights in Example 1.1.3

In this appendix, we provide a detailed derivation of the discretized Malliavin Monte Carlo weights proposed in the context of the uncertain volatility model. In contrast to Fournié et al. (1999), we rely on re-writing the conditional expectation as integrals on $\mathbb{R}$ with respect to the Gaussian density. Then, a straightforward application of the integration by parts formula leads to the asserted representation.

To this end, we briefly recall the setting of Example 1.1.3. Let $0 = t_0 < t_1 < \ldots < t_J = T$ be a partition of $[0, T]$ and $W$ be a Brownian motion. Further, the price of the risky asset $X^{\hat{\rho}}$ under risk-neutral dynamics and in discounted units at time $t_j$ is given by

$$X_j^{\hat{\rho}} = X_{j-1}^{\hat{\rho}} \exp\left\{\hat{\rho}\Delta W_{j+1} - \frac{1}{2}\hat{\rho}^2\Delta\right\}, \quad x_0 \in \mathbb{R},$$

for a given constant volatility $\hat{\rho} > 0$ and $\Delta W_{j+1} := W_{t_{j+1}} - W_{t_j}$. Then, the value process $(Y_j)_{j=0,\ldots,J}$ of a European option with maturity $T$ and payoff $g(X_J^{\hat{\rho}})$ is given by $Y_j = y^j(W_{t_j})$, where the deterministic function $y^j$ is given by the recursive scheme

$$\begin{aligned}
y^J(x) &= g\left(x_0 e^{\hat{\rho}x - \frac{1}{2}\hat{\rho}^2 T}\right), \quad x \in \mathbb{R}, \\
\bar{y}_t^j(t,x) &= -\frac{1}{2}\bar{y}_{xx}^j(t,x), \quad (t,x) \in [t_j, t_{j+1}) \times \mathbb{R}, \\
\bar{y}^j(t_{j+1},x) &= y^{j+1}(x), \quad x \in \mathbb{R}, \\
y^j(x) &= \bar{y}^j(t_j,x) + \Delta \max_{\sigma \in \{\sigma_{low}, \sigma_{up}\}} \left\{\frac{1}{2}\left(\frac{\sigma^2}{\hat{\rho}^2} - 1\right)\left(\bar{y}_{xx}^j(t_j,x) - \hat{\rho}\bar{y}_x^j(t_j,x)\right)\right\}, \quad x \in \mathbb{R}.
\end{aligned}$$

As stated in Example 1.1.3, the partial derivatives $y_x^j(t_j, \cdot)$ and $y_{xx}^j(t_j, \cdot)$ may be represented via the Malliavin Monte Carlo weights

$$\bar{y}_x^j(t_j, W_{t_j}) = E_j\left[\frac{\Delta W_{j+1}}{\Delta} Y_{j+1}\right], \quad \bar{y}_{xx}^j(t_j, W_{t_j}) = E_j\left[\left(\frac{\Delta W_{j+1}^2}{\Delta^2} - \frac{1}{\Delta}\right) Y_{j+1}\right].$$

To see this, we first note that under the given assumptions, differentiation and integration can be interchanged:

$$\begin{aligned}
\bar{y}^j_x(t_j,x) &= \frac{d}{dx}E\left[Y_{j+1}|W_{t_j}=x\right] \\
&= \frac{d}{dx}E\left[y^{j+1}(W_{t_{j+1}})|W_{t_j}=x\right] \\
&= \frac{d}{dx}E\left[y^{j+1}(W_{t_j}+(W_{t_{j+1}}-W_{t_j}))|W_{t_j}=x\right] \\
&= \frac{d}{dx}\int_{-\infty}^{\infty} y^{j+1}(x+\sqrt{\Delta}u)\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}\,du \\
&= \int_{-\infty}^{\infty}\frac{d}{dx}y^{j+1}(x+\sqrt{\Delta}u)\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}\,du.
\end{aligned}$$

Replacing the derivative of $y^{j+1}$ with respect to $x$ by the derivative with respect to $u$ and integrating by parts, shows the first assertion:

$$\begin{aligned}
\bar{y}^j_x(t_j,x) &= \int_{-\infty}^{\infty}\frac{1}{\sqrt{\Delta}}\frac{d}{du}\left(y^{j+1}(x+\sqrt{\Delta}u)\right)\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}\,du \\
&= \frac{1}{\sqrt{2\pi\Delta}}\left(\left[y^{j+1}(x+\sqrt{\Delta}u)\cdot e^{-\frac{1}{2}u^2}\right]_{-\infty}^{\infty}+\int_{-\infty}^{\infty}uy^{j+1}(x+\sqrt{\Delta}u)\cdot e^{-\frac{1}{2}u^2}\,du\right) \\
&= \int_{-\infty}^{\infty}\frac{u}{\sqrt{\Delta}}y^{j+1}(x+\sqrt{\Delta}u)\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}\,du \\
&= E\left[\frac{\Delta W_{j+1}}{\Delta}y^{j+1}(W_{t_{j+1}})\,\middle|\,W_{t_j}=x\right] \\
&= E\left[\frac{\Delta W_{j+1}}{\Delta}Y_{j+1}\,\middle|\,W_{t_j}=x\right].
\end{aligned}$$

Following the same line of reasoning and integrating by parts twice, yields the second claim:

$$\begin{aligned}
\bar{y}^j_{xx}(t_j,x) &= \frac{d^2}{dx^2}E\left[Y_{j+1}|W_{t_j}=x\right] \\
&= \frac{d^2}{dx^2}E\left[y^{j+1}(W_{t_{j+1}})|W_{t_j}=x\right] \\
&= \frac{d^2}{dx^2}E\left[y^{j+1}(W_{t_j}+(W_{t_{j+1}}-W_{t_j}))|W_{t_j}=x\right] \\
&= \frac{d^2}{dx^2}\int_{-\infty}^{\infty}y^{j+1}(x+\sqrt{\Delta}u)\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}\,du \\
&= \int_{-\infty}^{\infty}\frac{d^2}{dx^2}y^{j+1}(x+\sqrt{\Delta}u)\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}\,du \\
&= \frac{1}{\sqrt{2\pi}\Delta}\int_{-\infty}^{\infty}\left(\frac{d^2}{du^2}y^{j+1}(x+\sqrt{\Delta}u)\right)\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}\,du \\
&= \frac{1}{\sqrt{2\pi}\Delta}\left(\left[\left(\frac{d}{du}y^{j+1}(x+\sqrt{\Delta}u)\right)\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}\right]_{-\infty}^{\infty}\right. \\
&\quad\left.+\int_{-\infty}^{\infty}\left(\frac{d}{du}y^{j+1}(x+\sqrt{\Delta}u)\right)\cdot ue^{-\frac{1}{2}u^2}\,du\right) \\
&= \frac{1}{\sqrt{2\pi}\Delta}\int_{-\infty}^{\infty}\left(\frac{d}{du}y^{j+1}(x+\sqrt{\Delta}u)\right)\cdot ue^{-\frac{1}{2}u^2}\,du
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}\Delta}\left(\left[y^{j+1}(x+\sqrt{\Delta}u)\cdot ue^{-\frac{1}{2}u^2}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} y^{j+1}(x+\sqrt{\Delta}u)\cdot(1-u^2)e^{-\frac{1}{2}u^2}\,du\right)\\
&= \int_{-\infty}^{\infty} \frac{u^2-1}{\Delta} y^{j+1}(x+\sqrt{\Delta}u)\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}\,du\\
&= E\left[\left(\frac{\Delta W_{j+1}^2}{\Delta^2}-\frac{1}{\Delta}\right) y^{j+1}(W_{t_{j+1}})\middle| W_{t_j}=x\right]\\
&= E\left[\left(\frac{\Delta W_{j+1}^2}{\Delta^2}-\frac{1}{\Delta}\right) Y_{j+1}\middle| W_{t_j}=x\right].
\end{aligned}
$$

## A.2 Convex conjugate for a class of piecewise-linear functions

In this appendix, we derive the convex conjugate for functions $f:\mathbb{R}^d\to\mathbb{R}$ given by

$$
f(x)=\sum_{k=1}^{d} a^{(k)}x^{(k)}+c_1+\left(\sum_{k=1}^{d} b^{(k)}x^{(k)}+c_2\right)_+
$$

for vectors $a,b\in\mathbb{R}^d$ with $b^{(d)}\neq 0$ and coefficients $c_1,c_2\in\mathbb{R}$. Note that we assume $b^{(d)}\neq 0$ in order to exclude the trivial case where $f$ is linear. Considering this general type of piecewise linear functions allows us to capture the numerical examples presented in this thesis. From straightforward manipulations and the definition of $(\cdot)_+$, it follows that

$$
\begin{aligned}
f^{\#}(u) &= \sup_{x\in\mathbb{R}^d}\sum_{k=1}^{d} u^{(k)}x^{(k)} - f(x)\\
&= \sup_{x\in\mathbb{R}^d}\sum_{k=1}^{d}\left(u^{(k)}-a^{(k)}\right)x^{(k)} - c_1 - \left(\sum_{k=1}^{d} b^{(k)}x^{(k)}+c_2\right)_+\\
&= -c_1 + \sup_{x\in\mathbb{R}^d}\left(\min\left\{\sum_{k=1}^{d}\left(u^{(k)}-a^{(k)}\right)x^{(k)}, \sum_{k=1}^{d}\left(u^{(k)}-\left(a^{(k)}+b^{(k)}\right)\right)x^{(k)}-c_2\right\}\right)\\
&= -c_1 - \frac{u^{(d)}-a^{(d)}}{b^{(d)}}c_2\\
&\quad + \sup_{x\in\mathbb{R}^d}\left(\min\left\{\sum_{k=1}^{d}\left(u^{(k)}-a^{(k)}\right)x^{(k)} + \frac{u^{(d)}-a^{(d)}}{b^{(d)}}c_2,\right.\right.\\
&\quad \left.\left.\sum_{k=1}^{d}\left(u^{(k)}-\left(a^{(k)}+b^{(k)}\right)\right)x^{(k)} - c_2 + \frac{u^{(d)}-a^{(d)}}{b^{(d)}}c_2\right\}\right)\\
&= -c_1 - \frac{u^{(d)}-a^{(d)}}{b^{(d)}}c_2\\
&\quad + \sup_{x\in\mathbb{R}^d}\left(\min\left\{\sum_{k=1}^{d-1}\left(u^{(k)}-a^{(k)}\right)x^{(k)} + \left(u^{(d)}-a^{(d)}\right)\left(x^{(d)}+\frac{c_2}{b^{(d)}}\right),\right.\right.\\
&\quad \left.\left.\sum_{k=1}^{d-1}\left(u^{(k)}-\left(a^{(k)}+b^{(k)}\right)\right)x^{(k)} + \left(u^{(d)}-\left(a^{(d)}+b^{(d)}\right)\right)\left(x^{(d)}+\frac{c_2}{b^{(d)}}\right)\right\}\right)
\end{aligned}
$$

$$= \begin{cases} -c_1 - \frac{u^{(d)}-a^{(d)}}{b^{(d)}} c_2, & u \in \operatorname{conv}\{a, a+b\} \\ +\infty, & \text{else,} \end{cases}$$

where conv denotes the convex hull. Hence, the effective domain of $f^{\#}$ is given by

$$D_{f^{\#}} = \operatorname{conv}\{a, a+b\}.$$

## A.3 Conditional expectations for basis functions in Section 1.7.2.1

In this appendix, we provide the details for the computation of the conditional expectations involving the payoff function in the numerical example of Section 1.7.2.1.

To this end, let $0 = t_0 < t_1 < \ldots < t_J = T$ be an equidistant partition of the interval $[0, T]$ with increments $\Delta$. Furthermore, let $W = (W^{(1)}, \ldots, W^{(d)})$ be a $d$-dimensional Brownian motion and denote by $\mathcal{F}_j$ the $\sigma$-algebra generated by $W$ up to $t_j$. We denote by $X = (X^{(1)}, \ldots, X^{(d)})$ $d$ independent identically distributed geometric Brownian motions, whose dynamics on the grid $\{t_0, \ldots, t_J\}$ are given by

$$X_j^{(k)} = x_0 \exp\left\{\left(R^L - \frac{1}{2}\sigma^2\right) t_j + \sigma W_{t_j}^{(k)}\right\}, \quad k = 1, \ldots, d,$$

with drift $R^L \geq 0$ and volatility $\sigma > 0$. Here, we again use the shorthand notation $X_j^{(k)} := X_{t_j}^{(k)}$. Moreover, we define the process $(B_j)_{j=1,\ldots,J}$ by

$$B_j = \left(1, \frac{\Delta W_j}{\Delta}\right)^\top, \quad j = 1, \ldots, J.$$

In the following, we compute the conditional expectations

$$E_j\left[f_l(X_j, B_{j+1})\right] \quad \text{and} \quad E_j\left[\frac{\Delta W_{j+1}}{\Delta} f_l(X_j, B_{j+1})\right], \tag{A.1}$$

where

$$\begin{aligned} f_l(X_j, B_{j+1}) \;=\; & \sqrt{\frac{T-t_j}{T-t_{j+1}}} \int_{\mathbb{R}} \bar{h}\left(X_j^{(l)} e^{\left(R^L - \frac{1}{2}\sigma^2\right)(T-t_j) + \sigma z\sqrt{T-t_j}}\right) e^{\frac{z^2}{2} - \frac{(\sqrt{T-t_j}z - \Delta W_{j+1}^{(l)})^2}{2(T-t_{j+1})}} \\ & \times \prod_{l' \in \{1,\ldots,d\} \setminus \{l\}} \mathcal{N}\left(\sqrt{\frac{T-t_j}{T-t_{j+1}}} z + \frac{\ln(X_j^{(l)}) - \ln(X_j^{(l')})}{\sigma\sqrt{T-t_{j+1}}} - \frac{\Delta W_{j+1}^{(l')}}{\sqrt{T-t_{j+1}}}\right) \\ & \times \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz. \end{aligned} \tag{A.2}$$

Here, $\bar{h} : \mathbb{R} \to \mathbb{R}$ is a function which is of polynomial growth and $\mathcal{N}$ denotes the cumulative distribution function of the standard normal distribution.

In order to simplify the computation of (A.1), we first prove the following lemma.

**Lemma A.3.1.** *Let $\gamma \in \mathbb{R}$ and let $U$ be standard normally distributed random variable. Furthermore, denote by $\mathcal{N}$ the cumulative distribution function of a standard normal distribution. Then, it holds:*

*(i)* $E\left[\exp\left\{-\frac{1}{2}\left(\gamma-\sqrt{\frac{\Delta}{T-t_{j+1}}}U\right)^2\right\}\right]=\sqrt{\frac{T-t_{j+1}}{T-t_j}}\exp\left\{-\frac{1}{2}\frac{T-t_{j+1}}{T-t_j}\gamma^2\right\}$.

*(ii)* $E\left[\mathcal{N}\left(\gamma-\sqrt{\frac{\Delta}{T-t_{j+1}}}U\right)\right]=\mathcal{N}\left(\sqrt{\frac{T-t_{j+1}}{T-t_j}}\gamma\right)$.

*Proof.* We first show $(i)$. By straightforward calculations, we obtain that:

$$\begin{aligned}
&E\left[\exp\left\{-\frac{1}{2}\left(\gamma-\sqrt{\frac{\Delta}{T-t_{j+1}}}U\right)^2\right\}\right]\\
&= \int_{\mathbb{R}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}\exp\left\{-\frac{1}{2}\left(\gamma-\sqrt{\frac{\Delta}{T-t_{j+1}}}u\right)^2\right\}du\\
&= \int_{\mathbb{R}}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\gamma^2-2\gamma\sqrt{\frac{\Delta}{T-t_{j+1}}}u+\left(\frac{\Delta}{T-t_{j+1}}+1\right)u^2\right)\right\}du\\
&= \int_{\mathbb{R}}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\gamma^2-2\gamma\sqrt{\frac{\Delta}{T-t_{j+1}}}u+\frac{T-t_j}{T-t_{j+1}}u^2\right)\right\}du\\
&= \int_{\mathbb{R}}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\gamma^2-2\gamma\sqrt{\frac{\Delta}{T-t_j}}\sqrt{\frac{T-t_j}{T-t_{j+1}}}u+\frac{T-t_j}{T-t_{j+1}}u^2\right)\right\}du\\
&= \int_{\mathbb{R}}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\sqrt{\frac{\Delta}{T-t_j}}\gamma-\sqrt{\frac{T-t_j}{T-t_{j+1}}}u\right)^2-\frac{1}{2}\left(1-\frac{\Delta}{T-t_j}\right)\gamma^2\right\}du\\
&= \int_{\mathbb{R}}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\frac{T-t_j}{T-t_{j+1}}\left(\sqrt{\frac{\Delta(T-t_{j+1})}{(T-t_j)^2}}\gamma-u\right)^2\right\}\exp\left\{-\frac{1}{2}\frac{T-t_{j+1}}{T-t_j}\gamma^2\right\}du\\
&= \sqrt{\frac{T-t_{j+1}}{T-t_j}}\exp\left\{-\frac{1}{2}\frac{T-t_{j+1}}{T-t_j}\gamma^2\right\}\\
&\quad\cdot\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi\frac{T-t_{j+1}}{T-t_j}}}\exp\left\{-\frac{1}{2}\frac{T-t_j}{T-t_{j+1}}\left(\sqrt{\frac{\Delta(T-t_{j+1})}{(T-t_j)^2}}\gamma-u\right)^2\right\}du\\
&= \sqrt{\frac{T-t_{j+1}}{T-t_j}}\exp\left\{-\frac{1}{2}\frac{T-t_{j+1}}{T-t_j}\gamma^2\right\}.
\end{aligned}$$

For the second claim, we first note that

$$\begin{aligned}
&E\left[\mathcal{N}\left(\gamma-\sqrt{\frac{\Delta}{T-t_{j+1}}}U\right)\right]\\
&= \int_{\mathbb{R}}\left(\int_{-\infty}^{\gamma-\sqrt{\frac{\Delta}{T-t_{j+1}}}u}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}v^2}dv\right)\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}du\\
&= \int_{\mathbb{R}}\left(\int_{-\infty}^{\gamma}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(y-\sqrt{\frac{\Delta}{T-t_{j+1}}}u\right)^2\right\}dy\right)\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}du
\end{aligned}$$

$$= \int_{-\infty}^{\gamma} \frac{1}{\sqrt{2\pi}} \left( \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2}\left( y - \sqrt{\frac{\Delta}{T-t_{j+1}}}u \right)^2 - \frac{1}{2}u^2 \right\} du \right) dy,$$

due to the substitution $v = y - \sqrt{\frac{\Delta}{T-t_{j+1}}}u$ and Fubini's theorem. By $(i)$ and the substitution $z = \sqrt{\frac{T-t_{j+1}}{T-t_j}}y$, we obtain

$$\begin{aligned} E\left[\mathcal{N}\left(\gamma - \sqrt{\frac{\Delta}{T-t_{j+1}}}U\right)\right] &= \int_{-\infty}^{\gamma} \frac{1}{\sqrt{2\pi}}\sqrt{\frac{T-t_{j+1}}{T-t_j}} \exp\left\{-\frac{1}{2}\frac{T-t_{j+1}}{T-t_j}y^2\right\} dy, \\ &= \int_{-\infty}^{\sqrt{\frac{T-t_{j+1}}{T-t_j}}\gamma} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}z^2\right\} dz, \\ &= \mathcal{N}\left(\sqrt{\frac{T-t_{j+1}}{T-t_j}}\gamma\right). \end{aligned}$$

□

With this lemma at hand, we now turn to the calculation of the conditional expectations in (A.1). First note that by Fubini's theorem and the independence of the components of the Brownian motion $W$

$$\begin{aligned} & E\left[f_l(X_j, B_{j+1}) \middle| X_j = x\right] \\ &= E\left[f_l(x, B_{j+1})\right] \\ &= E\Bigg[\sqrt{\frac{T-t_j}{T-t_{j+1}}} \int_{\mathbb{R}} \bar{h}\left(x^{(l)} e^{\left(R^L - \frac{1}{2}\sigma^2\right)(T-t_j) + \sigma z\sqrt{T-t_j}}\right) e^{\frac{z^2}{2} - \frac{(\sqrt{T-t_j}z - \Delta W_{j+1}^{(l)})^2}{2(T-t_{j+1})}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \\ &\quad \times \prod_{l' \in \{1,\dots,d\}\setminus\{l\}} \mathcal{N}\left(\sqrt{\frac{T-t_j}{T-t_{j+1}}}z + \frac{\ln(X_j^{(l)}) - \ln(X_j^{(l')})}{\sigma\sqrt{T-t_{j+1}}} - \frac{\Delta W_{j+1}^{(l')}}{\sqrt{T-t_{j+1}}}\right) dz\Bigg] \\ &= \sqrt{\frac{T-t_j}{T-t_{j+1}}} \int_{\mathbb{R}} \bar{h}\left(x^{(l)} e^{\left(R^L - \frac{1}{2}\sigma^2\right)(T-t_j) + \sigma z\sqrt{T-t_j}}\right) e^{\frac{z^2}{2}} E\left[e^{-\frac{(\sqrt{T-t_j}z - \Delta W_{j+1}^{(l)})^2}{2(T-t_{j+1})}}\right] \\ &\quad \times \prod_{l' \in \{1,\dots,d\}\setminus\{l\}} E\left[\mathcal{N}\left(\sqrt{\frac{T-t_j}{T-t_{j+1}}}z + \frac{\ln(x^{(l)}) - \ln(x^{(l')})}{\sigma\sqrt{T-t_{j+1}}} - \frac{\Delta W_{j+1}^{(l')}}{\sqrt{T-t_{j+1}}}\right)\right] \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\ &= \sqrt{\frac{T-t_j}{T-t_{j+1}}} \int_{\mathbb{R}} \bar{h}\left(x^{(l)} e^{\left(R^L - \frac{1}{2}\sigma^2\right)(T-t_j) + \sigma z\sqrt{T-t_j}}\right) e^{\frac{z^2}{2}} \\ &\quad \times E\left[\exp\left(-\frac{1}{2}\left(\sqrt{\frac{T-t_j}{T-t_{j+1}}}z - \frac{\Delta W_{j+1}^{(l)}}{T-t_{j+1}}\right)^2\right)\right] \\ &\quad \times \prod_{l' \in \{1,\dots,d\}\setminus\{l\}} E\left[\mathcal{N}\left(\sqrt{\frac{T-t_j}{T-t_{j+1}}}z + \frac{\ln(x^{(l)}) - \ln(x^{(l')})}{\sigma\sqrt{T-t_{j+1}}} - \frac{\Delta W_{j+1}^{(l')}}{\sqrt{T-t_{j+1}}}\right)\right] \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \end{aligned}$$

for every $x \in \mathbb{R}^d$. Applying Lemma A.3.1, we end up with

$$
\begin{aligned}
&E\left[f_l(X_j, B_{j+1}) \middle| X_j = x\right] \\
= &\sqrt{\frac{T-t_j}{T-t_{j+1}}} \int_{\mathbb{R}} \bar{h}\left(x^{(l)} e^{\left(R^L - \frac{1}{2}\sigma^2\right)(T-t_j) + \sigma z \sqrt{T-t_j}}\right) e^{\frac{z^2}{2}} \\
&\times \left(\sqrt{\frac{T-t_{j+1}}{T-t_j}} \exp\left\{-\frac{1}{2}\frac{T-t_{j+1}}{T-t_j}\left(\sqrt{\frac{T-t_j}{T-t_{j+1}}} z\right)^2\right\}\right) \\
&\times \prod_{l' \in \{1,\ldots,d\}\setminus\{l\}} \mathcal{N}\left(\sqrt{\frac{T-t_{j+1}}{T-t_j}}\left(\sqrt{\frac{T-t_j}{T-t_{j+1}}} z + \frac{\ln(x^{(l)}) - \ln(x^{(l')})}{\sigma\sqrt{T-t_{j+1}}}\right)\right) \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz \\
= &\int_{\mathbb{R}} \bar{h}\left(x^{(l)} e^{\left(R^L - \frac{1}{2}\sigma^2\right)(T-t_j) + \sigma z \sqrt{T-t_j}}\right) \prod_{l' \in \{1,\ldots,d\}\setminus\{l\}} \mathcal{N}\left(z + \frac{\ln(x^{(l)}) - \ln(x^{(l')})}{\sigma\sqrt{T-t_j}}\right) \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz.
\end{aligned}
$$

Finally, we compute $E_j[\frac{\Delta W_{j+1}^{(k)}}{\Delta} f_l(X_j, B_{j+1})]$. To do this, we distinguish two different cases, namely $k \neq l$ and $k = l$ and apply the identity

$$
E_j\left[\frac{\Delta W_{j+1}^{(k)}}{\Delta} f_l(X_j, B_{j+1})\right] = \frac{d}{dh} E_j\left[f_l(X_j, B_{j+1} + h e_{d+1+k})\right]\Big|_{h=0}, \tag{A.3}
$$

where $e_n$ denotes the $n$-th canonical vector in $\mathbb{R}^{2d+1}$. Note that (A.3) follows by a similar computation than the one in Appendix A.1.

We first consider the case $k \neq l$. Then, we get by (A.3) and the same arguments as above, that

$$
\begin{aligned}
&E\left[\frac{\Delta W_{j+1}^{(k)}}{\Delta} f_l(X_j, B_{j+1}) \middle| X_j = x\right] \\
= &\frac{d}{dh} E\left[f_l(X_j, B_{j+1} + h e_{d+1+k}) | X_j = x\right]\Big|_{h=0} \\
= &\frac{d}{dh} E\left[f_l(x, B_{j+1} + h e_{d+1+k})\right]\Big|_{h=0} \\
= &\frac{d}{dh} E\left[\sqrt{\frac{T-t_j}{T-t_{j+1}}} \int_{\mathbb{R}} \bar{h}\left(x^{(l)} e^{\left(R^L - \frac{1}{2}\sigma^2\right)(T-t_j) + \sigma z\sqrt{T-t_j}}\right) e^{\frac{z^2}{2} - \frac{(\sqrt{T-t_j} z - \Delta W_{j+1}^{(l)})^2}{2(T-t_{j+1})}}\right. \\
&\times \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \cdot \mathcal{N}\left(\sqrt{\frac{T-t_j}{T-t_{j+1}}} z + \frac{\ln(x^{(l)}) - \ln(x^{(k)})}{\sigma\sqrt{T-t_{j+1}}} - \frac{(\Delta W_{j+1}^{(k)} + h)}{\sqrt{T-t_{j+1}}}\right) \\
&\left.\times \prod_{l' \in \{1,\ldots,d\}\setminus\{l,k\}} \mathcal{N}\left(\sqrt{\frac{T-t_j}{T-t_{j+1}}} z + \frac{\ln(x^{(l)}) - \ln(x^{(l')})}{\sigma\sqrt{T-t_{j+1}}} - \frac{\Delta W_{j+1}^{(l')}}{\sqrt{T-t_{j+1}}}\right) dz\right]\Bigg|_{h=0} \\
= &\frac{d}{dh}\left(\sqrt{\frac{T-t_j}{T-t_{j+1}}} \int_{\mathbb{R}} \bar{h}\left(x^{(l)} e^{\left(R^L - \frac{1}{2}\sigma^2\right)(T-t_j) + \sigma z\sqrt{T-t_j}}\right) e^{\frac{z^2}{2}} E\left[e^{-\frac{(\sqrt{T-t_j} z - \Delta W_{j+1}^{(l)})^2}{2(T-t_{j+1})}}\right]\right. \\
&\times \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \cdot E\left[\mathcal{N}\left(\sqrt{\frac{T-t_j}{T-t_{j+1}}} z + \frac{\ln(x^{(l)}) - \ln(x^{(k)})}{\sigma\sqrt{T-t_{j+1}}} - \frac{(\Delta W_{j+1}^{(k)} + h)}{\sqrt{T-t_{j+1}}}\right)\right]
\end{aligned}
$$

$$\times \prod_{l'\in\{1,\dots,d\}\setminus\{l,k\}} E\left[\mathcal{N}\left(\sqrt{\frac{T-t_j}{T-t_{j+1}}}z+\frac{\ln(x^{(l)})-\ln(x^{(l')})}{\sigma\sqrt{T-t_{j+1}}}-\frac{\Delta W_{j+1}^{(l')}}{\sqrt{T-t_{j+1}}}\right)\right] dz\right)\Bigg|_{h=0}$$

$$= \frac{d}{dh}\left(\sqrt{\frac{T-t_j}{T-t_{j+1}}}\int_{\mathbb{R}} \bar{h}\left(x^{(l)}e^{\left(R^L-\frac{1}{2}\sigma^2\right)(T-t_j)+\sigma z\sqrt{T-t_j}}\right)e^{\frac{z^2}{2}}\right.$$

$$\times E\left[\exp\left(-\frac{1}{2}\left(\sqrt{\frac{T-t_j}{T-t_{j+1}}}z-\frac{\Delta W_{j+1}^{(l)}}{T-t_{j+1}}\right)^2\right)\right]$$

$$\times\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\cdot E\left[\mathcal{N}\left(\sqrt{\frac{T-t_j}{T-t_{j+1}}}z+\frac{\ln(x^{(l)})-\ln(x^{(k)})}{\sigma\sqrt{T-t_{j+1}}}-\frac{(\Delta W_{j+1}^{(k)}+h)}{\sqrt{T-t_{j+1}}}\right)\right]$$

$$\times \prod_{l'\in\{1,\dots,d\}\setminus\{l,k\}} E\left[\mathcal{N}\left(\sqrt{\frac{T-t_j}{T-t_{j+1}}}z+\frac{\ln(x^{(l)})-\ln(x^{(l')})}{\sigma\sqrt{T-t_{j+1}}}-\frac{\Delta W_{j+1}^{(l')}}{\sqrt{T-t_{j+1}}}\right)\right] dz\right)\Bigg|_{h=0}.$$

From Lemma A.3.1 we conclude, as before, that

$$E\left[\frac{\Delta W_{j+1}^{(k)}}{\Delta}f_l(X_j,B_{j+1})\middle| X_j=x\right]$$

$$= \frac{d}{dh}\left(\sqrt{\frac{T-t_j}{T-t_{j+1}}}\int_{\mathbb{R}} \bar{h}\left(x^{(l)}e^{\left(R^L-\frac{1}{2}\sigma^2\right)(T-t_j)+\sigma z\sqrt{T-t_j}}\right)e^{\frac{z^2}{2}}\right.$$

$$\times\left(\sqrt{\frac{T-t_{j+1}}{T-t_j}}\exp\left\{-\frac{1}{2}\frac{T-t_{j+1}}{T-t_j}\left(\sqrt{\frac{T-t_j}{T-t_{j+1}}}z\right)^2\right\}\right)$$

$$\times\mathcal{N}\left(\sqrt{\frac{T-t_{j+1}}{T-t_j}}\left(\sqrt{\frac{T-t_j}{T-t_{j+1}}}z+\frac{\ln(x^{(l)})-\ln(x^{(k)})}{\sigma\sqrt{T-t_{j+1}}}-\frac{h}{\sqrt{T-t_{j+1}}}\right)\right)$$

$$\times \prod_{l'\in\{1,\dots,d\}\setminus\{l,k\}} \mathcal{N}\left(\sqrt{\frac{T-t_{j+1}}{T-t_j}}\left(\sqrt{\frac{T-t_j}{T-t_{j+1}}}z+\frac{\ln(x^{(l)})-\ln(x^{(l')})}{\sigma\sqrt{T-t_{j+1}}}\right)\right)\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\, dz\right)\Bigg|_{h=0}$$

$$= \frac{d}{dh}\left(\int_{\mathbb{R}} \bar{h}\left(x^{(l)}e^{\left(R^L-\frac{1}{2}\sigma^2\right)(T-t_j)+\sigma z\sqrt{T-t_j}}\right)\mathcal{N}\left(z+\frac{\ln(x^{(l)})-\ln(x^{(k)})}{\sigma\sqrt{T-t_j}}-\frac{h}{\sqrt{T-t_j}}\right)\right.$$

$$\times \prod_{l'\in\{1,\dots,d\}\setminus\{l,k\}} \mathcal{N}\left(z+\frac{\ln(x^{(l)})-\ln(x^{(l')})}{\sigma\sqrt{T-t_j}}\right)\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\, dz\right)\Bigg|_{h=0}$$

$$= -\frac{1}{\sqrt{T-t_j}}\int_{\mathbb{R}} \bar{h}\left(x^{(l)}e^{\left(R^L-\frac{1}{2}\sigma^2\right)(T-t_j)+\sigma z\sqrt{T-t_j}}\right)\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(z+\frac{\ln(x^{(l)})-\ln(x^{(k)})}{\sigma\sqrt{T-t_j}}\right)^2\right\}$$

$$\times \prod_{l'\in\{1,\dots,d\}\setminus\{l,k\}} \mathcal{N}\left(z+\frac{\ln(x^{(l)})-\ln(x^{(l')})}{\sigma\sqrt{T-t_j}}\right)\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\, dz.$$

For $k=l$, we follow the same argumentation and end up with

$$E\left[\frac{\Delta W_{j+1}^{(l)}}{\Delta}f_l(X_j,B_{j+1})\middle| X_j=x\right]$$

$$
\begin{aligned}
&= \frac{d}{dh} E\left[f_l(X_j, B_{j+1} + he_{d+1+l}) | X_j = x\right]\Big|_{h=0} \\
&= \frac{d}{dh} E\left[\sqrt{\frac{T-t_j}{T-t_{j+1}}} \int_{\mathbb{R}} \bar{h}\left(x^{(l)} e^{\left(R^L - \frac{1}{2}\sigma^2\right)(T-t_j) + \sigma z \sqrt{T-t_j}}\right) e^{\frac{z^2}{2} - \frac{(\sqrt{T-t_j} z - (\Delta W_{j+1}^{(l)} + h))^2}{2(T-t_{j+1})}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\right. \\
&\quad \left.\times \prod_{l' \in \{1,\dots,d\} \setminus \{l\}} \mathcal{N}\left(\sqrt{\frac{T-t_j}{T-t_{j+1}}} z + \frac{\ln(x^{(l)}) - \ln(x^{(l')})}{\sigma\sqrt{T-t_{j+1}}} - \frac{\Delta W_{j+1}^{(l')}}{\sqrt{T-t_{j+1}}}\right) dz\right]\Bigg|_{h=0} \\
&= \frac{d}{dh}\left(\sqrt{\frac{T-t_j}{T-t_{j+1}}} \int_{\mathbb{R}} \bar{h}\left(x^{(l)} e^{\left(R^L - \frac{1}{2}\sigma^2\right)(T-t_j) + \sigma z \sqrt{T-t_j}}\right) e^{\frac{z^2}{2}}\right. \\
&\quad \times E\left[e^{-\frac{(\sqrt{T-t_j} z - (\Delta W_{j+1}^{(l)} + h))^2}{2(T-t_{j+1})}}\right] \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \\
&\quad \left.\times \prod_{l' \in \{1,\dots,d\} \setminus \{l\}} E\left[\mathcal{N}\left(\sqrt{\frac{T-t_j}{T-t_{j+1}}} z + \frac{\ln(x^{(l)}) - \ln(x^{(l')})}{\sigma\sqrt{T-t_{j+1}}} - \frac{\Delta W_{j+1}^{(l')}}{\sqrt{T-t_{j+1}}}\right)\right] dz\right)\Bigg|_{h=0} \\
&= \frac{d}{dh}\left(\sqrt{\frac{T-t_j}{T-t_{j+1}}} \int_{\mathbb{R}} \bar{h}\left(x^{(l)} e^{\left(R^L - \frac{1}{2}\sigma^2\right)(T-t_j) + \sigma z \sqrt{T-t_j}}\right) e^{\frac{z^2}{2}}\right. \\
&\quad \times \left(\sqrt{\frac{T-t_{j+1}}{T-t_j}} \exp\left\{-\frac{1}{2}\frac{T-t_{j+1}}{T-t_j}\left(\sqrt{\frac{T-t_j}{T-t_{j+1}}} z - \frac{h}{T-t_{j+1}}\right)^2\right\}\right) \\
&\quad \left.\times \prod_{l' \in \{1,\dots,d\} \setminus \{l\}} \mathcal{N}\left(\sqrt{\frac{T-t_{j+1}}{T-t_j}}\left(\sqrt{\frac{T-t_j}{T-t_{j+1}}} z + \frac{\ln(x^{(l)}) - \ln(x^{(l')})}{\sigma(\sqrt{T-t_{j+1}})}\right)\right) \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz\right)\Bigg|_{h=0} \\
&= \frac{d}{dh}\left(\int_{\mathbb{R}} \bar{h}\left(x^{(l)} e^{\left(R^L - \frac{1}{2}\sigma^2\right)(T-t_j) + \sigma z \sqrt{T-t_j}}\right) e^{\frac{z^2}{2}}\right. \\
&\quad \times \exp\left\{-\frac{1}{2}\frac{T-t_{j+1}}{T-t_j}\left(\sqrt{\frac{T-t_j}{T-t_{j+1}}} z - \frac{h}{T-t_{j+1}}\right)^2\right\} \\
&\quad \left.\times \prod_{l' \in \{1,\dots,d\} \setminus \{l\}} \mathcal{N}\left(z + \frac{\ln(x^{(l)}) - \ln(x^{(l')})}{\sigma(\sqrt{T-t_j})}\right) \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz\right)\Bigg|_{h=0} \\
&= \int_{\mathbb{R}} \bar{h}\left(x^{(l)} e^{\left(R^L - \frac{1}{2}\sigma^2\right)(T-t_j) + \sigma z \sqrt{T-t_j}}\right) \frac{z}{\sqrt{T-t_j}} \prod_{l' \in \{1,\dots,d\} \setminus \{l\}} \mathcal{N}\left(z + \frac{\ln(x^{(l)}) - \ln(x^{(l')})}{\sigma(\sqrt{T-t_j})}\right) \\
&\quad \times \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz
\end{aligned}
$$

## A.4 Closed-form representations for conditional expectations

In the numerical examples presented in this thesis, we require the computation of conditional expectations of basis functions which depend on geometric Brownian motions. In this appendix, we provide a straightforward generalization of Example 3.1 in Bender and Steiner (2012) and show that these conditional expectations can essentially be expressed by the first- and second-order derivatives of the basis functions.

Let $0 = t_0 < t_1 < \ldots < t_J = T$ be a partition of $[0,T]$ and define $\Delta_{j+1} := t_{j+1} - t_j$ for $j = 0, \ldots, J-1$. Further, let $W$ be a $d$-dimensional Brownian motion and define the $d$-dimensional process $X$ on the grid $\{t_0, \ldots t_J\}$ by

$$X_j^{(n)} = x_0^{(n)} \exp\left\{\left(\mu - \frac{1}{2}\sum_{l=1}^{d} \sigma_{n,l}^2\right) t_j + \sum_{l=1}^{d} \sigma_{n,l} W_{t_j}^{(l)}\right\}, \quad n = 1, \ldots, d,$$

where $x_0^{(1)}, \ldots, x_0^{(d)}, \mu \in \mathbb{R}_+$ and $\sigma$ is an invertible $d \times d$-matrix with entries in $\mathbb{R}$. Then, we show that

$$E\left[\left.\frac{\Delta W_{j+1}^{(k)}}{\Delta_{j+1}} h\left(X_J^{(n)}\right)\right| X_j = x\right] = x^{(n)} \sigma_{n,k} \frac{\partial}{\partial x^{(n)}} E\left[\left. h\left(X_J^{(n)}\right)\right| X_j = x\right]$$

and

$$E\left[\left.\left(\left(\frac{\Delta W_{j+1}^{(k)}}{\Delta_{j+1}}\right)^2 - \sigma_{n,k}\frac{\Delta W_{j+1}^{(k)}}{\Delta_{j+1}} - \frac{1}{\Delta_{j+1}}\right) h\left(X_J^{(n)}\right)\right| X_j = x\right]$$
$$= \left(x^{(n)}\right)^2 \sigma_{n,k}^2 \frac{\partial^2}{\partial (x^{(n)})^2} E\left[\left. h\left(X_J^{(n)}\right)\right| X_j = x\right]$$

holds for all $n, k = 1, \ldots, d$ and all functions $h : \mathbb{R} \to \mathbb{R}$ satisfying appropriate growth and differentiability conditions.

In order to simplify the following computations, we introduce some further notation. We denote by $u^{(-l)}$ the $\mathbb{R}^{d-1}$-dimensional vector $(u^{(1)}, \ldots, u^{l-1}, u^{l+1}, \ldots, u^{(d)})$. Additionally, we define the function $\tilde{h}_n : \mathbb{R} \to \mathbb{R}$ for every $n = 1, \ldots, d$ by

$$\tilde{h}_n(x) = h\left(x \exp\left\{\left(\mu - \frac{1}{2}\sum_{l=1}^{d}\sigma_{n,l}^2\right)(t_J - t_{j+1}) + \sum_{l=1}^{d}\sigma_{n,l}\left(\tilde{W}_{t_J}^{(l)} - \tilde{W}_{t_{j+1}}^{(l)}\right)\right\}\right).$$

Moreover, the function $f : \mathbb{R}^d \to \mathbb{R}$ is given by

$$f(u) = \exp\left\{\left(\mu - \frac{1}{2}\sum_{l=1}^{d}\sigma_{n,l}^2\right)\Delta_{j+1} + \sqrt{\Delta_{j+1}}\sum_{l=1}^{d}\sigma_{n,l} u^{(l)}\right\}.$$

Then, a straightforward computation, involving Fubini's Theorem and integration by parts, yields

$$\begin{aligned}
& x^{(n)}\sigma_{n,k}\frac{\partial}{\partial x^{(n)}} E\left[\left. h\left(X_J^{(n)}\right)\right| X_j = x\right] \\
= \;& x^{(n)}\sigma_{n,k}\frac{\partial}{\partial x^{(n)}} E\left[\left. \tilde{h}_n\left(X_{j+1}^{(n)}\right)\right| X_j = x\right] \\
= \;& x^{(n)}\sigma_{n,k}\frac{\partial}{\partial x^{(n)}} \int_{\mathbb{R}^d} \tilde{h}_n\left(x^{(n)} f(u)\right) \prod_{l=1}^{d}\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u^{(l)})^2}\right) d(u^{(1)}, \ldots, u^{(d)}) \\
= \;& x^{(n)}\sigma_{n,k} \int_{\mathbb{R}^d} \frac{\partial}{\partial x^{(n)}} \tilde{h}_n\left(x^{(n)} f(u)\right) \prod_{l=1}^{d}\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u^{(l)})^2}\right) d(u^{(1)}, \ldots, u^{(d)})
\end{aligned}$$

$$
\begin{aligned}
&= x^{(n)}\sigma_{n,k}\int_{\mathbb{R}^d} f(u)\tilde{h}_n'\left(x^{(n)}f(u)\right)\prod_{l=1}^{d}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(u^{(l)})^2}\right)\,d(u^{(1)},\ldots,u^{(d)})\\
&= \frac{1}{\sqrt{\Delta_{j+1}}}\int_{\mathbb{R}^d}\frac{\partial}{\partial u^{(k)}}\tilde{h}_n\left(x^{(n)}f(u)\right)\prod_{l=1}^{d}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(u^{(l)})^2}\right)\,d(u^{(1)},\ldots,u^{(d)})\\
&= \frac{1}{\sqrt{\Delta_{j+1}}}\int_{\mathbb{R}^{d-1}}\prod_{l=1,l\neq k}^{d}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(u^{(l)})^2}\right)\\
&\quad\times\left(\int_{\mathbb{R}}\left(\frac{\partial}{\partial u^{(k)}}\tilde{h}_n\left(x^{(n)}f(u)\right)\right)\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(u^{(k)})^2}\,du^{(k)}\right)du^{(-k)}\\
&= \frac{1}{\sqrt{\Delta_{j+1}}}\int_{\mathbb{R}^{d-1}}\prod_{l=1,l\neq k}^{d}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(u^{(l)})^2}\right)\left(\int_{\mathbb{R}}\tilde{h}_n\left(x^{(n)}f(u)\right)\cdot\frac{u^{(k)}}{\sqrt{2\pi}}e^{-\frac{1}{2}(u^{(k)})^2}\,du^{(k)}\right)du^{(-k)}\\
&= \int_{\mathbb{R}^d}\frac{u^{(k)}}{\sqrt{\Delta_{j+1}}}\tilde{h}_n\left(x^{(n)}f(u)\right)\prod_{l=1}^{d}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(u^{(l)})^2}\right)\,d(u^{(1)},\ldots,u^{(d)})\\
&= E\left[\frac{\Delta W_{j+1}^{(k)}}{\Delta_{j+1}}\tilde{h}_n\left(X_{j+1}^{(n)}\right)\middle|\,X_j^{(n)}=x^{(n)}\right]\\
&= E\left[\frac{\Delta W_{j+1}^{(k)}}{\Delta_{j+1}}h\left(X_J^{(n)}\right)\middle|\,X_j^{(n)}=x^{(n)}\right].
\end{aligned}
$$

For the proof of the second claim, we first observe that

$$
\begin{aligned}
&\left(x^{(n)}\right)^2\sigma_{n,k}^2\frac{\partial^2}{\partial(x^{(n)})^2}E\left[h\left(X_J^{(n)}\right)\middle|\,X_j=x\right]\\
&= \left(x^{(n)}\right)^2\sigma_{n,k}^2\frac{\partial^2}{\partial(x^{(n)})^2}E\left[\tilde{h}_n\left(X_{j+1}^{(n)}\right)\middle|\,X_j=x\right]\\
&= \left(x^{(n)}\right)^2\sigma_{n,k}^2\frac{\partial^2}{\partial(x^{(n)})^2}\int_{\mathbb{R}^d}\tilde{h}_n\left(x^{(n)}f(u)\right)\prod_{l=1}^{d}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(u^{(l)})^2}\right)\,d(u^{(1)},\ldots,u^{(d)})\\
&= \left(x^{(n)}\right)^2\sigma_{n,k}^2\int_{\mathbb{R}^d}\frac{\partial^2}{\partial(x^{(n)})^2}\tilde{h}_n\left(x^{(n)}f(u)\right)\prod_{l=1}^{d}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(u^{(l)})^2}\right)\,d(u^{(1)},\ldots,u^{(d)})\\
&= \left(x^{(n)}\right)^2\sigma_{n,k}^2\int_{\mathbb{R}^d}\tilde{h}_n''\left(x^{(n)}f(u)\right)f(u)^2\prod_{l=1}^{d}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(u^{(l)})^2}\right)\,d(u^{(1)},\ldots,u^{(d)}).
\end{aligned}
$$

From the definition of $\tilde{h}_n$ and $f$, we obtain that

$$
\begin{aligned}
&\left(x^{(n)}\right)^2\sigma_{n,k}^2\Delta_{j+1}f(u)^2\tilde{h}_n''\left(x^{(n)}f(u)\right)\\
&\qquad= \frac{\partial^2}{\partial(u^{(k)})^2}\tilde{h}_n\left(x^{(n)}f(u)\right)-\sigma_{n,k}\sqrt{\Delta_{j+1}}\frac{\partial}{\partial(u^{(k)})}\tilde{h}_n\left(x^{(n)}f(u)\right).
\end{aligned}
$$

By exploiting this identity, Fubini's theorem, and integration by parts, we conclude that

$$
\left(x^{(n)}\right)^2\sigma_{n,k}^2\frac{\partial^2}{\partial(x^{(n)})^2}E\left[h\left(X_J^{(n)}\right)\middle|\,X_j=x\right]
$$

$$
\begin{aligned}
= \quad & \frac{1}{\Delta_{j+1}} \int_{\mathbb{R}^d} \left( \frac{\partial^2}{\partial (u^{(k)})^2} \tilde{h}_n \left( x^{(n)} f(u) \right) - \sigma_{n,k} \sqrt{\Delta_{j+1}} \frac{\partial}{\partial (u^{(k)})} \tilde{h}_n \left( x^{(n)} f(u) \right) \right) \\
& \times \prod_{l=1}^{d} \left( \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u^{(l)})^2} \right) d(u^{(1)}, \ldots, u^{(d)}) \\
= \quad & \frac{1}{\Delta_{j+1}} \int_{\mathbb{R}^{d-1}} \prod_{l=1, l \neq k}^{d} \left( \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u^{(l)})^2} \right) \left( \int_{\mathbb{R}} \left( \frac{\partial^2}{\partial (u^{(k)})^2} \tilde{h}_n \left( x^{(n)} f(u) \right) \right.\right. \\
& \left.\left. - \sigma_{n,k} \sqrt{\Delta_{j+1}} \frac{\partial}{\partial (u^{(k)})} \tilde{h}_n \left( x^{(n)} f(u) \right) \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u^{(k)})^2} \, du^{(k)} \right) du^{(-k)} \\
= \quad & \frac{1}{\Delta_{j+1}} \int_{\mathbb{R}^{d-1}} \prod_{l=1, l \neq k}^{d} \left( \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u^{(l)})^2} \right) \\
& \times \left( \int_{\mathbb{R}} \left( \left( u^{(k)} \right)^2 - \sigma_{n,k} \sqrt{\Delta_{j+1}} u^{(k)} - 1 \right) \tilde{h}_n \left( x^{(n)} f(u) \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u^{(k)})^2} \, du^{(k)} \right) du^{(-k)} \\
= \quad & \int_{\mathbb{R}^d} \left( \frac{\left( u^{(k)} \right)^2}{\Delta_{j+1}} - \sigma_{n,k} \frac{u^{(k)}}{\sqrt{\Delta_{j+1}}} - \frac{1}{\Delta_{j+1}} \right) \tilde{h}_n \left( x^{(n)} f(u) \right) \\
& \times \prod_{l=1}^{d} \left( \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u^{(l)})^2} \right) d(u^{(1)}, \ldots, u^{(d)}) \\
= \quad & E\left[ \left( \left( \frac{\Delta W_{j+1}^{(k)}}{\Delta_{j+1}} \right)^2 - \sigma_{n,k} \frac{\Delta W_{j+1}^{(k)}}{\Delta_{j+1}} - \frac{1}{\Delta_{j+1}} \right) \tilde{h}_n \left( X_{j+1}^{(n)} \right) \middle| X_j = x \right] \\
= \quad & E\left[ \left( \left( \frac{\Delta W_{j+1}^{(k)}}{\Delta_{j+1}} \right)^2 - \sigma_{n,k} \frac{\Delta W_{j+1}^{(k)}}{\Delta_{j+1}} - \frac{1}{\Delta_{j+1}} \right) h \left( X_J^{(n)} \right) \middle| X_j = x \right].
\end{aligned}
$$

# Appendix B

# Appendix to Chapter 2

## B.1 Conditional expectations for basis functions in Section 2.5

In this appendix, we derive the conditional expectations of the basis functions used in Section 2.5. To this end, recall that the stochastic processes $(x_t, y_t, \gamma_t, \tilde{p}_t)_{t\in[0,T]}$ are given by the stochastic differential equations

$$\begin{aligned} dx_t &= -\kappa_x x_t dt + \sigma_x dW_t^x, \\ dy_t &= -\kappa_y y_t dt + \sigma_y dW_t^y \\ d\gamma_t &= \kappa_\gamma(\mu_\gamma - \gamma_t)dt + \sigma_\gamma\sqrt{\gamma_t}dW_t^\gamma, \\ d\tilde{p}_t &= \kappa_p(\mu_p - \tilde{p}_t)dt + \sigma_p dW_t^p \end{aligned}$$

for real constants $\kappa_x$, $\kappa_y$, $\sigma_x$, $\sigma_y$, $\kappa_\gamma$, $\mu_\gamma$, $\sigma_\gamma$, $\kappa_p$, $\mu_p$, $\sigma_p$. Here, $W^x$, $W^y$ and $W^\gamma$ are Brownian motions with instantaneous correlations $\rho_{xy}$, $\rho_{x\gamma}$ and $\rho_{y\gamma}$. Moreover, we have that $W_t^p = \rho_{\gamma p}W_t^\gamma + \sqrt{1-\rho_{\gamma p}^2}W_t$ where the Brownian motion $W$ is independent of $(W^x, W^y, W^\gamma)$. Further, let $0 = t_0 < t_1 < \ldots < t_J = T$ be an equidistant partition with time increments $\Delta$ and denote by $\mathcal{T} = \{T_1, \ldots, T_K\} \subseteq \{t_0, \ldots, t_J\}$ the set of tenor dates. The tenor dates all have the same distance which is denoted by $\delta$. For the processes $\gamma$ and $\tilde{p}$, we apply the following discretization scheme

$$\begin{aligned} \tilde{\gamma}_j &= \tilde{\gamma}_{j-1} - \kappa_\gamma\Delta((\tilde{\gamma}_{j-1})_+ - \mu_\gamma) + \sigma_\gamma\sqrt{(\tilde{\gamma}_{j-1})_+}\Delta W_j^\gamma, \\ \tilde{p}_j &= \tilde{p}_{j-1}e^{-\kappa_p\Delta} + \mu_p(1-e^{-\kappa_p\Delta}) + \sigma_p\sqrt{\frac{1-e^{-2\kappa_p\Delta}}{2\kappa_p\Delta}}\Delta W_j^p, \end{aligned}$$

where we use the shorthand notation $U_j := U_{t_j}$ for $U \in \{\tilde{\gamma}, \tilde{p}\}$. Then, we observe by a straightforward computation that the conditional expectation $E_{j-1}[\tilde{\gamma}_j]$ is given by

$$\begin{aligned} E_{j-1}[\tilde{\gamma}_j] &= E_{j-1}\left[\tilde{\gamma}_{j-1} - \kappa_\gamma\Delta((\tilde{\gamma}_{j-1})_+ - \mu_\gamma) + \sigma_\gamma\sqrt{(\tilde{\gamma}_{j-1})_+}\Delta W_j^\gamma\right] \\ &= \tilde{\gamma}_{j-1} - \kappa_\gamma\Delta((\tilde{\gamma}_{j-1})_+ - \mu_\gamma) + \sigma_\gamma\sqrt{(\tilde{\gamma}_{j-1})_+}E_{j-1}\left[\Delta W_j^\gamma\right] \\ &= \tilde{\gamma}_{j-1} - \kappa_\gamma\Delta((\tilde{\gamma}_{j-1})_+ - \mu_\gamma) \end{aligned}$$

for every $j = 1, \dots, J$. In order to compute $E_{j-1}[\tilde{\gamma}_j \tilde{p}_j]$, we exploit the definition of $W^p$ and obtain that

$$
\begin{aligned}
E_{j-1}[\tilde{\gamma}_j \tilde{p}_j] &= E_{j-1}\Bigg[\left(\tilde{\gamma}_{j-1} - \kappa_\gamma \Delta((\tilde{\gamma}_{j-1})_+ - \mu_\gamma) + \sigma_\gamma \sqrt{(\tilde{\gamma}_{j-1})_+} \Delta W_j^\gamma\right) \\
&\quad \left(\tilde{p}_{j-1} e^{-\kappa_p \Delta} + \mu_p (1 - e^{-\kappa_p \Delta}) + \sigma_p \sqrt{\frac{1 - e^{-2\kappa_p \Delta}}{2\kappa_p \Delta}} \Delta W_j^p\right)\Bigg] \\
&= \left(\tilde{\gamma}_{j-1} - \kappa_\gamma \Delta((\tilde{\gamma}_{j-1})_+ - \mu_\gamma)\right)\left(\tilde{p}_{j-1} e^{-\kappa_p \Delta} + \mu_p (1 - e^{-\kappa_p \Delta})\right) \\
&\quad + \left(\tilde{\gamma}_{j-1} - \kappa_\gamma \Delta((\tilde{\gamma}_{j-1})_+ - \mu_\gamma)\right) \sigma_p \sqrt{\frac{1 - e^{-2\kappa_p \Delta}}{2\kappa_p \Delta}} E_{j-1}\left[\Delta W_j^p\right] \\
&\quad + \left(\tilde{p}_{j-1} e^{-\kappa_p \Delta} + \mu_p (1 - e^{-\kappa_p \Delta})\right) \sigma_\gamma \sqrt{(\tilde{\gamma}_{j-1})_+} E_{j-1}\left[\Delta W_j^\gamma\right] \\
&\quad + \sigma_\gamma \sqrt{(\tilde{\gamma}_{j-1})_+} \sigma_p \sqrt{\frac{1 - e^{-2\kappa_p \Delta}}{2\kappa_p \Delta}} E_{j-1}\left[\Delta W_j^\gamma \Delta W_j^p\right] \\
&= \left(\tilde{\gamma}_{j-1} - \kappa_\gamma \Delta((\tilde{\gamma}_{j-1})_+ - \mu_\gamma)\right)\left(\tilde{p}_{j-1} e^{-\kappa_p \Delta} + \mu_p (1 - e^{-\kappa_p \Delta})\right) \\
&\quad + \sigma_\gamma \sigma_p \rho_{\gamma p} \sqrt{(\tilde{\gamma}_{j-1})_+ \Delta} \sqrt{\frac{1 - e^{-2\kappa_p \Delta}}{2\kappa_p}}.
\end{aligned}
$$

It thus remains to compute the expected value of the clean swap price, which is given by

$$
S_{t_j} = P(t_j, T_{\tau(j)}) C_{T_{\tau(j)}} + N \cdot \sum_{i=\tau(j)+1}^{K} \left(P(t_j, T_{i-1}) - (1 + R\delta) P(t_j, T_i)\right),
$$

where $\tau(j)$ denotes the index of the first tenor date weakly after $t_j$ and $P(t, s)$ is for $t, s \in [0, T]$, $t < s$, given by

$$
P(t, s) = \exp\left\{-r_0 (s - t) - \frac{1 - e^{-\kappa_x (s-t)}}{\kappa_x} x_t - \frac{1 - e^{-\kappa_y (s-t)}}{\kappa_y} y_t + \frac{1}{2} V(t, s)\right\}.
$$

Here, the deterministic function $V$ is defined by

$$
\begin{aligned}
V(t, s) &= \frac{\sigma_x^2}{\kappa_x^2}\left(s - t + \frac{2}{\kappa_x} e^{-\kappa_x (s-t)} - \frac{1}{2\kappa_x} e^{-2\kappa_x (s-t)} - \frac{3}{2\kappa_x}\right) \\
&\quad + \frac{\sigma_y^2}{\kappa_y^2}\left(s - t + \frac{2}{\kappa_y} e^{-\kappa_y (s-t)} - \frac{1}{2\kappa_y} e^{-2\kappa_y (s-t)} - \frac{3}{2\kappa_y}\right) \\
&\quad + 2\rho_{xy} \frac{\sigma_x \sigma_y}{\kappa_x \kappa_y}\left(s - t + \frac{e^{-\kappa_x (s-t)} - 1}{\kappa_x} + \frac{e^{-\kappa_y (s-t)} - 1}{\kappa_y} - \frac{e^{-(\kappa_x + \kappa_y)(s-t)} - 1}{\kappa_x + \kappa_y}\right),
\end{aligned}
$$

see Chapter 4.2 in Brigo and Mercurio (2006). Hence, in order to derive a closed form expression for $E_{j-1}[S_{t_j}]$ it is sufficient to compute $E_{j-1}[P(t_j, T_i)]$ for tenor dates $T_i \geq t_j$, since

$$
E_{j-1}[S_{t_j}] = E_{j-1}\left[P(t_j, T_{\tau(j)})\right] C_{T_{\tau(j)}} + N \cdot \sum_{i=\tau(j)+1}^{K} \left(E_{j-1}\left[P(t_j, T_{i-1})\right] - (1 + R\delta) E_{j-1}\left[P(t_j, T_i)\right]\right)
$$

by the $\mathcal{F}_{T_{\tau(j)-1}}$-measurability of $C_{T_{\tau(j)}}$. By definition of $P(t,s)$, the computation of this conditional expectation boils down to the computation of

$$E[\exp\{\alpha x_s + \beta y_s\}|\mathcal{F}_t]$$

for any $t, s \in [0,T]$ with $t < s$ and arbitrary $\alpha, \beta \in \mathbb{R}$.

Hence, let $t, s \in [0,T]$, $t < s$, and $\alpha, \beta \in \mathbb{R}$ be fixed from now on. Then, it is well-known, that, given $\mathcal{F}_t$, the random variable $\alpha x_s + \beta y_s$ is normally distributed, see e.g. Section 3.3 in Glasserman (2004). Applying the martingale property of the stochastic integrals, we observe that the mean is given by

$$\begin{aligned}
E[\alpha x_s + \beta y_s|\mathcal{F}_t] &= E\left[\alpha\left(x_t e^{-\kappa_x(s-t)} + \sigma_x \int_t^s e^{-\kappa_x(s-u)}\, dW_u^x\right)\middle|\mathcal{F}_t\right] \\
&\quad + E\left[\beta\left(y_t e^{-\kappa_y(s-t)} + \sigma_y \int_t^s e^{-\kappa_y(s-u)}\, dW_u^y\right)\middle|\mathcal{F}_t\right] \\
&= \alpha x_t e^{-\kappa_x(s-t)} + \sigma_x \alpha E\left[\int_t^s e^{-\kappa_x(s-u)}\, dW_u^x\middle|\mathcal{F}_t\right] \\
&\quad + \beta y_t e^{-\kappa_y(s-t)} + \sigma_y \beta E\left[\int_t^s e^{-\kappa_y(s-u)}\, dW_u^y\middle|\mathcal{F}_t\right] \\
&= \alpha x_t e^{-\kappa_x(s-t)} + \beta y_t e^{-\kappa_y(s-t)}.
\end{aligned}$$

For the variance, a straightforward application of the Itô-isometry shows that

$$\begin{aligned}
&\mathrm{Var}(\alpha x_s + \beta y_s|\mathcal{F}_t) \\
&= \alpha^2 E\left[(x_s - E[x_s|\mathcal{F}_t])^2\middle|\mathcal{F}_t\right] + \beta^2 E\left[(y_s - E[y_s|\mathcal{F}_t])^2\middle|\mathcal{F}_t\right] \\
&\quad + 2\alpha\beta E[(x_s - E[x_s|\mathcal{F}_t])(y_s - E[y_s|\mathcal{F}_t])|\mathcal{F}_t] \\
&= \alpha^2 E\left[\left(x_t e^{-\kappa_x(s-t)} + \sigma_x \int_t^s e^{-\kappa_x(s-u)}\, dW_u^x - x_t e^{-\kappa_x(s-t)}\right)^2\middle|\mathcal{F}_t\right] \\
&\quad + \beta^2 E\left[\left(y_t e^{-\kappa_y(s-t)} + \sigma_y \int_t^s e^{-\kappa_y(s-u)}\, dW_u^y - y_t e^{-\kappa_y(s-t)}\right)^2\middle|\mathcal{F}_t\right] \\
&\quad + 2\alpha\beta E\left[\left(x_t e^{-\kappa_x(s-t)} + \sigma_x \int_t^s e^{-\kappa_x(s-u)}\, dW_u^x - x_t e^{-\kappa_x(s-t)}\right)\right. \\
&\qquad \left.\left(y_t e^{-\kappa_y(s-t)} + \sigma_y \int_t^s e^{-\kappa_y(s-u)}\, dW_u^y - y_t e^{-\kappa_y(s-t)}\right)\middle|\mathcal{F}_t\right] \\
&= \alpha^2 E\left[\left(\sigma_x \int_t^s e^{-\kappa_x(s-u)}\, dW_u^x\right)^2\middle|\mathcal{F}_t\right] + \beta^2 E\left[\left(\sigma_y \int_t^s e^{-\kappa_y(s-u)}\, dW_u^y\right)^2\middle|\mathcal{F}_t\right] \\
&\quad + 2\alpha\beta E\left[\left(\sigma_x \int_t^s e^{-\kappa_x(s-u)}\, dW_u^x\right)\left(\sigma_y \int_t^s e^{-\kappa_y(s-u)}\, dW_u^y\right)\middle|\mathcal{F}_t\right] \\
&= \alpha^2\sigma_x^2 \int_t^s e^{-2\kappa_x(s-u)}\, du + \beta^2\sigma_y^2 \int_t^s e^{-2\kappa_y(s-u)}\, du + 2\alpha\beta\sigma_x\sigma_y\rho_{xy} \int_t^s e^{-(\kappa_x+\kappa_y)(s-u)}\, du \\
&= \frac{\alpha^2\sigma_x^2}{2\kappa_x}\left(1 - e^{-2\kappa_x(s-t)}\right) + \frac{\beta^2\sigma_y^2}{2\kappa_y}\left(1 - e^{-2\kappa_y(s-t)}\right) + \frac{2\alpha\beta\sigma_x\sigma_y\rho_{xy}}{\kappa_x+\kappa_y}\left(1 - e^{-(\kappa_x+\kappa_y)(s-t)}\right).
\end{aligned}$$

Hence, we conclude that

$$E\left[\exp\left\{\alpha x_s + \beta y_s\right\}\middle|\mathcal{F}_t\right] = \exp\left\{m_{xy}(t,s) + \frac{1}{2}\sigma_{xy}^2(t,s)\right\},$$

where $m_{xy}(t,s) := E[\alpha x_s + \beta y_s|\mathcal{F}_t]$ and $\sigma_{xy}(t,s) := \text{Var}(\alpha x_s + \beta y_s|\mathcal{F}_t)$. From this identity, we finally obtain that

$$E_{j-1}[P(t_j,T_i)] \quad = \quad \exp\left\{-r_0(T_i - t_j) - \left(m_{xy}(t_{j-1},t_j) + \frac{1}{2}\sigma_{xy}^2(t_{j-1},t_j)\right) + \frac{1}{2}V(t_j,T_i)\right\},$$

for every $t_j \in \{t_0, \ldots, t_{J-1}\}$ and $T_i \in \mathcal{T}$ with $T_i \geq t_j$, where $m_{xy}(t_{j-1},t_j)$ and $\sigma_{xy}^2(t_{j-1},t_j)$ are given as above with

$$\alpha = \frac{1 - e^{-\kappa_x(t_j - t_{j-1})}}{\kappa_x} \quad \text{and} \quad \beta = \frac{1 - e^{-\kappa_y(t_j - t_{j-1})}}{\kappa_y}.$$

# Appendix C

# Appendix to Chapter 3

## C.1 Estimation of the truncation error

In the following, we discuss the truncation error which arises in Section 3.4.3 when computing the conditional expectations of the basis functions. More precisely, we show that this error is negligible if the functions satisfy certain growth conditions and the mesh of a given partition is small enough.

To this end, let $W$ be a $d'$-dimensional Brownian motion and $h : \mathbb{R}^d \times \mathbb{R}^{d'} \to \mathbb{R}$ be a function which grows at most exponentially in its second argument. Moreover, let $0 = t_0 < t_1 < \ldots < t_J = T$ be a partition of $[0, T]$ and denote by $\Delta_{j+1}$ the distance between the discretization points $t_j$ and $t_{j+1}$. Further, we denote by $\Delta W_{j+1} := W_{t_{j+1}} - W_{t_j}$ the increment of the Brownian motion between the time points $t_j$ and $t_{j+1}$ and by $[\Delta W_{j+1}]_c$ its componentwise truncation at $\pm c$ for $c \geq 0$. Then, we show that

$$\left\| E\left[\frac{\Delta W_{j+1}}{\Delta_{j+1}} h(x, \Delta W_{j+1})\right] - E\left[\frac{[\Delta W_{j+1}]_c}{\Delta_{j+1}} h(x, \Delta W_{j+1})\right] \right\|^2 \leq \left(\frac{18}{\pi c^2}\right)^{\frac{1}{4}} \Delta_{j+1}^{-\frac{3}{4}} d' \cdot e^{-\frac{c^2}{4\Delta_{j+1}}} E\left[h(x, \Delta W_{j+1})^2\right] \quad \text{(C.1)}$$

for every $x \in \mathbb{R}^d$, where $\| \cdot \|$ denotes the Euclidean norm on $\mathbb{R}^{d'}$. To see this, we require the following tail estimate for standard normally distributed random variables $U$:

$$P(\{U \geq c\}) \leq \frac{1}{c\sqrt{2\pi}} e^{-\frac{1}{2}c^2}. \quad \text{(C.2)}$$

This follows simply by exploiting that $\frac{u}{c} \geq 1$ for all $u \in [c, \infty)$:

$$P(\{U \geq c\}) = \int_c^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} \, du \leq \int_c^\infty \frac{u}{c} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} \, du = \frac{1}{c\sqrt{2\pi}} e^{-\frac{1}{2}c^2}.$$

For the proof of (C.1), we first note that by Jensen's inequality (applied to the convex function $x \mapsto \|x\|^2$) as well as Hölder's inequality

$$\begin{aligned} &\left\| E\left[\frac{\Delta W_{j+1}}{\Delta_{j+1}} h(x, \Delta W_{j+1})\right] - E\left[\frac{[\Delta W_{j+1}]_c}{\Delta_{j+1}} h(x, \Delta W_{j+1})\right] \right\|^2 \\ &= \frac{1}{\Delta_{j+1}^2} \left\| E\left[\left(\Delta W_{j+1} - [\Delta W_{j+1}]_c\right) h(x, \Delta W_{j+1})\right] \right\|^2 \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Delta_{j+1}^2} E\left[\left\| \left(\Delta W_{j+1} - [\Delta W_{j+1}]_c\right) h(x, \Delta W_{j+1}) \right\|^2\right] \\
&\leq \frac{1}{\Delta_{j+1}^2} \left( E\left[\left\| \Delta W_{j+1} - [\Delta W_{j+1}]_c \right\|^2\right]^{\frac{1}{2}} E\left[h(x, \Delta W_{j+1})^2\right]^{\frac{1}{2}} \right)^2
\end{aligned}
$$

holds. In the next step, the definition of the Euclidean norm on $\mathbb{R}^{d'}$ and the fact that the components of $W$ are identically distributed yield

$$
\begin{aligned}
&\left\| E\left[\frac{\Delta W_{j+1}}{\Delta_{j+1}} h(x, \Delta W_{j+1})\right] - E\left[\frac{[\Delta W_{j+1}]_c}{\Delta_{j+1}} h(x, \Delta W_{j+1})\right] \right\|^2 \\
&\leq \frac{1}{\Delta_{j+1}^2} E\left[h(x, \Delta W_{j+1})^2\right] E\left[\left\| \Delta W_{j+1} - [\Delta W_{j+1}]_c \right\|^2\right] \\
&= \frac{1}{\Delta_{j+1}^2} E\left[h(x, \Delta W_{j+1})^2\right] \sum_{n=1}^{d'} E\left[\left(\Delta W_{j+1}^{(n)} - \left[\Delta W_{j+1}^{(n)}\right]_c\right)^2\right] \\
&= \frac{1}{\Delta_{j+1}^2} E\left[h(x, \Delta W_{j+1})^2\right] d' \cdot E\left[\left(\Delta W_{j+1}^{(1)} - \left[\Delta W_{j+1}^{(1)}\right]_c\right)^2\right].
\end{aligned}
$$

Since $1 - \mathbb{1}_A = \mathbb{1}_{A^C}$ for every $A \in \mathcal{F}$, we observe by Hölder's inequality that

$$
\begin{aligned}
&\left\| E\left[\frac{\Delta W_{j+1}}{\Delta_{j+1}} h(x, \Delta W_{j+1})\right] - E\left[\frac{[\Delta W_{j+1}]_c}{\Delta_{j+1}} h(x, \Delta W_{j+1})\right] \right\|^2 \\
&\leq \frac{1}{\Delta_{j+1}^2} E\left[h(x, \Delta W_{j+1})^2\right] d' \cdot E\left[\left(\Delta W_{j+1}^{(1)} - \Delta W_{j+1}^{(1)} \mathbb{1}_{\{\Delta W_{j+1}^{(1)} \in [-c,c]\}}\right)^2\right] \\
&= \frac{1}{\Delta_{j+1}^2} E\left[h(x, \Delta W_{j+1})^2\right] d' \cdot E\left[\left(\Delta W_{j+1}^{(1)} \mathbb{1}_{\{\Delta W_{j+1}^{(1)} \notin [-c,c]\}}\right)^2\right] \\
&= \frac{1}{\Delta_{j+1}^2} E\left[h(x, \Delta W_{j+1})^2\right] d' \cdot \left( E\left[\left(\Delta W_{j+1}^{(1)} \mathbb{1}_{\{\Delta W_{j+1}^{(1)} \notin [-c,c]\}}\right)^2\right]^{\frac{1}{2}} \right)^2 \\
&\leq \frac{1}{\Delta_{j+1}^2} E\left[h(x, \Delta W_{j+1})^2\right] d' \cdot \left( E\left[\left(\Delta W_{j+1}^{(1)}\right)^4\right]^{\frac{1}{4}} \right)^2 \left( E\left[\left(\mathbb{1}_{\{\Delta W_{j+1}^{(1)} \notin [-c,c]\}}\right)^4\right]^{\frac{1}{4}} \right)^2 \\
&= \frac{1}{\Delta_{j+1}^2} E\left[h(x, \Delta W_{j+1})^2\right] d' \cdot E\left[\left(\Delta W_{j+1}^{(1)}\right)^4\right]^{\frac{1}{2}} E\left[\mathbb{1}_{\{\Delta W_{j+1}^{(1)} \notin [-c,c]\}}\right]^{\frac{1}{2}}.
\end{aligned}
$$

Exploiting the symmetry of the density function of the normal distribution, we conclude that

$$
\begin{aligned}
&\left\| E\left[\frac{\Delta W_{j+1}}{\Delta_{j+1}} h(x, \Delta W_{j+1})\right] - E\left[\frac{[\Delta W_{j+1}]_c}{\Delta_{j+1}} h(x, \Delta W_{j+1})\right] \right\|^2 \\
&\leq \frac{1}{\Delta_{j+1}^2} E\left[h(x, \Delta W_{j+1})^2\right] d' \cdot \sqrt{3\Delta_{j+1}^2} P\left(\left\{\Delta W_{j+1}^{(1)} \notin [-c,c]\right\}\right)^{\frac{1}{2}} \\
&= \frac{\sqrt{3}}{\Delta_{j+1}} E\left[h(x, \Delta W_{j+1})^2\right] d' \cdot \left( P\left(\left\{\Delta W_{j+1}^{(1)} < -c\right\}\right) + P\left(\left\{\Delta W_{j+1}^{(1)} > c\right\}\right) \right)^{\frac{1}{2}} \\
&= \frac{\sqrt{3}}{\Delta_{j+1}} E\left[h(x, \Delta W_{j+1})^2\right] d' \cdot \left( 2P\left(\left\{\Delta W_{j+1}^{(1)} > c\right\}\right) \right)^{\frac{1}{2}}
\end{aligned}
$$

$$= \frac{\sqrt{6}}{\Delta_{j+1}} E\left[h(x, \Delta W_{j+1})^2\right] d' \cdot P\left(\left\{\Delta W_{j+1}^{(1)} > c\right\}\right)^{\frac{1}{2}}.$$

Finally, the assertion (C.1) follows from (C.2) and standard calculations:

$$\begin{aligned}
&\left\| E\left[\frac{\Delta W_{j+1}}{\Delta_{j+1}} h(x, \Delta W_{j+1})\right] - E\left[\frac{[\Delta W_{j+1}]_c}{\Delta_{j+1}} h(x, \Delta W_{j+1})\right]\right\|^2 \\
\leq\ & \frac{\sqrt{6}}{\Delta_{j+1}} E\left[h(x, \Delta W_{j+1})^2\right] d' \cdot \left(\frac{1}{\frac{c}{\sqrt{\Delta_{j+1}}} \cdot \sqrt{2\pi}} e^{-\frac{1}{2}\cdot\left(\frac{c}{\sqrt{\Delta_{j+1}}}\right)^2}\right)^{\frac{1}{2}} \\
=\ & \frac{\sqrt{6}}{\Delta_{j+1}} E\left[h(x, \Delta W_{j+1})^2\right] d' \cdot \left(\frac{\sqrt{\Delta_{j+1}}}{\sqrt{2\pi c^2}} e^{-\frac{c^2}{2\Delta_{j+1}}}\right)^{\frac{1}{2}} \\
=\ & \frac{\sqrt{6}}{\Delta_{j+1}} \cdot \left(\frac{\Delta_{j+1}}{2\pi c^2}\right)^{\frac{1}{4}} E\left[h(x, \Delta W_{j+1})^2\right] d' \cdot e^{-\frac{c^2}{4\Delta_{j+1}}} \\
=\ & \left(\frac{18}{\pi c^2}\right)^{\frac{1}{4}} \Delta_{j+1}^{-\frac{3}{4}} d' \cdot e^{-\frac{c^2}{4\Delta_{j+1}}} E\left[h(x, \Delta W_{j+1})^2\right].
\end{aligned}$$

# List of Figures

# List of Tables

# Bibliography


S. Alanko and M. Avellaneda. Reducing variance in the numerical solution of BSDEs. *Comptes Rendus Mathematique*, 351(3):135–138, 2013.

L. Andersen and M. Broadie. Primal-dual simulation algorithm for pricing multidimensional American options. *Management Science*, 50(9):1222–1234, 2004.

M. Avellaneda, A. Levy, and A. Parás. Pricing and hedging derivative securities in markets with uncertain volatilities. *Applied Mathematical Finance*, 2(2):73–88, 1995.

S. Balder, A. Mahayni, and J. Schoenmakers. Primal-dual linear Monte Carlo algorithm for multiple stopping—an application to flexible caps. *Quantitative Finance*, 13(7):1003–1013, 2013.

V. Bally and G. Pagès. A quantization algorithm for solving multi-dimensional discrete-time optimal stopping problems. *Bernoulli*, 9(6):1003–1049, 2003.

D. Belomestny. Solving optimal stopping problems via empirical dual optimization. *Annals of Applied Probability*, 23(5):1988–2019, 2013.

C. Bender and R. Denk. A forward scheme for backward SDEs. *Stochastic Processes and their Applications*, 117(12):1793–1812, 2007.

C. Bender and J. Steiner. Least-squares Monte Carlo for Backward SDEs. In R. Carmona, P. Del Moral, P. Hu, and N. Oudjane, editors, *Numerical Methods in Finance*, pages 257–289. Springer, 2012.

C. Bender, J. Schoenmakers, and J. Zhang. Dual representations for general multiple stopping problems. *Mathematical Finance. An International Journal of Mathematics, Statistics and Financial Economics*, 25(2):339–370, 2015.

C. Bender, N. Schweizer, and J. Zhuo. A primal-dual algorithm for BSDEs. *Mathematical Finance*, 27(3):866–901, 2017.

Y. Z. Bergman. Option pricing with differential interest rates. *Review of Financial Studies*, 8(2): 475–500, 1995.

D. P. Bertsekas. *Dynamic Programming and Optimal Control*, volume I. Athena Scientific, 3rd edition, 2005.

C. Beveridge and M. Joshi. Monte Carlo bounds for game options including convertible bonds. *Management Science*, 57(5):960–974, 2011.

B. Bouchard and N. Touzi. Discrete-time approximation and Monte-Carlo simulation of backward stochastic differential equations. *Stochastic Processes and their Applications*, 111(2):175–206, 2004.

D. Brigo and F. Mercurio. *Interest Rate Models – Theory and Practice: With Smile, Inflation and Credit*. Springer, 2nd edition, 2006.

D. Brigo and A. Pallavicini. Counterparty risk pricing under correlation between default and interest rates. In J. H. Miller, D. C. Edelman, and J. A. D. Appleby, editors, *Numerical Methods for Finance*, pages 63–81. Chapman and Hall/CRC, 2007.

D. Brigo, M. Morini, and A. Pallavicini. *Counterparty Credit Risk, Collateral and Funding: With Pricing Cases for All Asset Classes*. John Wiley & Sons, 2013.

D. B. Brown and M. B. Haugh. Information relaxation bounds for infinite horizon markov decision processes. *Preprint*, 2016.

D. B. Brown and J. E. Smith. Dynamic portfolio optimization with transaction costs: Heuristics and dual bounds. *Management Science*, 57(10):1752–1770, 2011.

D. B. Brown, J. E. Smith, and P. Sun. Information relaxations and duality in stochastic dynamic programs. *Operations Research*, 58(4):785–801, 2010.

R. Carmona and N. Touzi. Optimal multiple stopping and valuation of swing options. *Mathematical Finance. An International Journal of Mathematics, Statistics and Financial Economics*, 18(2): 239–268, 2008.

S. S. Chandramouli and M. B. Haugh. A unified approach to multiple stopping and duality. *Operations Research Letters*, 40(4):258–264, 2012.

N. Chen and P. Glasserman. Additive and multiplicative duals for American option pricing. *Finance and Stochastics*, 11(2):153–179, 2007.

P. Cheridito, H. M. Soner, N. Touzi, and N. Victoir. Second-order backward stochastic differential equations and fully nonlinear parabolic PDEs. *Communications on Pure and Applied Mathematics*, 60(7):1081–1110, 2007.

P. Cheridito, M. Kupper, and N. Vogelpoth. Conditional analysis on $\mathbb{R}^d$. In A. H. Hamel, F. Heyde, A. Löhne, B. Rudloff, and C. Schrage, editors, *Set Optimization and Applications in Finance - The State of the Art*, pages 179–211. Springer, 2015.

S. Crépey, R. Gerboud, Z. Grbac, and N. Ngor. Counterparty risk and funding: The four wings of the TVA. *International Journal of Theoretical and Applied Finance*, 16(2):1350006, 2013.

S. Crépey, T. R. Bielecki, and D. Brigo. *Counterparty Risk and Funding: A Tale of Two Puzzles*. Chapman and Hall/CRC, 2014.

D. Crisan and K. Manolarakis. Solving backward stochastic differential equations using the cubature method: application to nonlinear pricing. *SIAM Journal on Financial Mathematics*, 3(1):534–571, 2012.

V. V. Desai, V. F. Farias, and C. C. Moallemi. Pathwise optimization for optimal stopping problems. *Management Science*, 58(12):2292–2308, 2012.

N. El Karoui, S. Peng, and M. C. Quenez. Backward stochastic differential equations in finance. *Mathematical Finance*, 7(1):1–71, 1997.

A. Fahim, N. Touzi, and X. Warin. A probabilistic numerical method for fully nonlinear parabolic PDEs. *Annals of Applied Probability*, 21(4):1322–1364, 2011.

E. Fournié, J.-M. Lasry, J. Lebuchoux, P.-L. Lions, and N. Touzi. Applications of Malliavin calculus to Monte Carlo methods in finance. *Finance and Stochastics*, 3(4):391–412, 1999.

A. Friedman. *Partial differential equations of parabolic type.* Prentice-Hall, 1964.

P. Glasserman. *Monte Carlo Methods in Financial Engineering.* Springer, 2004.

P. Glasserman and B. Yu. Simulation for American options: Regression now or regression later? In H. Niederreiter, editor, *Monte Carlo and Quasi-Monte Carlo Methods 2002*, pages 213–226. Springer, 2004.

E. Gobet and C. Labart. Error expansion for the discretization of backward stochastic differential equations. *Stochastic Processes and their Applications*, 117(7):803–829, 2007.

E. Gobet and A. Makhlouf. $\mathbf{L}_2$-time regularity of BSDEs with irregular terminal functions. *Stochastic Processes and their Applications*, 120(7):1105–1132, 2010.

E. Gobet, J.-P. Lemor, and X. Warin. A regression-based Monte Carlo method to solve backward stochastic differential equations. *Annals of Applied Probability*, 15(3):2172–2202, 2005.

E. Gobet, J. G. López-Salas, P. Turkedjiev, and C. Vásquez. Stratified regression Monte-Carlo scheme for semilinear PDEs and BSDEs with large scale parallelization on GPUs. *SIAM Journal on Scientific Computing*, 38(6):C652–C677, 2016.

J. Guyon and P. Henry-Labordère. The uncertain volatility model: A Monte Carlo approach. *Journal of Computational Finance*, 14(3):37–71, 2011.

M. Haugh and A. E. B. Lim. Linear–quadratic control and information relaxations. *Operations Research Letters*, 40(6):521–528, 2012.

M. B. Haugh and L. Kogan. Pricing American options: A duality approach. *Operations Research*, 52(2):258–270, 2004.

M. B. Haugh and C. Wang. Information relaxations and dynamic zero-sum games. *arXiv preprint 1405.4347*, 2015.

R. A. Howard. *Dynamic Programming and Markov Processes.* Technology Press and Wiley, 1960.

I. Karatzas and S. Shreve. *Brownian Motion and Stochastic Calculus.* Springer, 2nd edition, 1991.

I. Kharroubi, N. Langrené, and H. Pham. A numerical algorithm for fully nonlinear HJB equations: an approach by control randomization. *Monte Carlo Methods and Applications*, 20(2):145–165, 2014.

A. Kolodko and J. Schoenmakers. Iterative construction of the optimal Bermudan stopping time. *Finance and Stochastics*, 10(1):27–49, 2006.

H. Kraft and F. T. Seifried. Stochastic differential utility as the continuous-time limit of recursive utility. *Journal of Economic Theory*, 151:528–550, 2014.

J.-P. Laurent, P. Amzelek, and J. Bonnaud. An overview of the valuation of collateralized derivative contracts. *Review of Derivatives Research*, 17(3):261–286, 2014.

J.-P. Lemor, E. Gobet, and X. Warin. Rate of convergence of an empirical regression method for solving generalized backward stochastic differential equations. *Bernoulli*, 12(5):889–916, 2006.

F. A. Longstaff and E. S. Schwartz. Valuing American options by simulation: A simple least-squares approach. *The Review of Financial Studies*, 14(1):113–147, 2001.

R. Lord, R. Koekkoek, and D. van Dijk. A comparison of biased simulation schemes for stochastic volatility models. *Quantitative Finance*, 10(2):177–194, 2010.

T. J. Lyons. Uncertain volatility and the risk-free synthesis of derivatives. *Applied Mathematical Finance*, 2(2):117–133, 1995.

J. Neveu. *Discrete-parameter martingales.* North-Holland Publishing Co.; American Elsevier Publishing Co., revised edition, 1975.

T. Nie and M. Rutkowski. A BSDE approach to fair bilateral pricing under endogenous collateralization. *Finance and Stochastics*, 20(4):855–900, 2016.

É. Pardoux. Backward stochastic differential equations and viscosity solutions of systems of semilinear parabolic and elliptic PDEs of second order. In *Stochastic Analysis and Related Topics VI (Geilo 1996)*, pages 79–127. Birkhäuser, 1998.

E. Pardoux and S. Peng. Backward stochastic differential equations and quasilinear parabolic partial differential equations. In B. L. Rozovskii and R. B. Sowers, editors, *Stochastic partial differential equations and their applications*, pages 200–217. Springer, 1992.

H. Pham. *Continuous-time Stochastic Control and Optimization with Financial Applications.* Springer, 2009.

W. B. Powell. *Approximate Dynamic Programming: Solving the curses of dimensionality.* John Wiley & Sons, 2nd edition, 2011.

M. L. Puterman. *Markov Decision Processes: Discrete Stochastic Dynamic Programming.* Wiley, 1994.

N. S. Rasmussen. Control variates for Monte Carlo valuation of American options. *Journal of Computational Finance*, 9(1):84–102, 2005.

R. T. Rockafellar. *Convex Analysis.* Princeton University Press, 1970.

L. C. G. Rogers. Monte Carlo valuation of American options. *Mathematical Finance*, 12(3):271–286, 2002.

L. C. G. Rogers. Pathwise stochastic optimal control. *SIAM Journal on Control and Optimization*, 46(3):1116–1132, 2007.

J. Schoenmakers. A pure martingale dual for multiple stopping. *Finance and Stochastics*, 16(2): 319–334, 2012.

L. S. Shapley. Stochastic games. *Proceedings of the National Academy of Sciences*, 39(10):1095–1100, 1953.

J. N. Tsitsiklis and B. Van Roy. Regression methods for pricing complex American-style options. *IEEE Transactions on Neural Networks*, 12(4):694–703, 2001.

J. M. Vanden. Exact superreplication strategies for a class of derivative assets. *Applied Mathematical Finance*, 13(01):61–87, 2006.

G. Zhang, M. Gunzburger, and W. Zhao. A sparse-grid method for multi-dimensional backward stochastic differential equations. *Journal of Computational Mathematics*, 31(3):221–248, 2013.

J. Zhang. A numerical scheme for BSDEs. *Annals of Applied Probability*, 14(1):459–488, 2004.