Well-posedness and dynamics of impulsive fractional stochastic evolution equations with unbounded delay

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Abstract

This paper is concerned with the well-posedness and dynamics of (delete) a delay impulsive fractional stochastic evolution equations with time fractional differential operator $\alpha \in (0,1)$. After establishing the well-posedness of the problem, and a result ensuring the existence and uniqueness of mild solutions globally defined in future, the existence of a minimal global attracting set is investigated in the mean-square topology, under general assumptions not ensuing the uniqueness of solutions. Furthermore, in the case of uniqueness, it is possible to provide more information about the geometrical structure of such global attracting set. In particular, it is proved that the minimal compact globally attracting set for the solutions of the problem becomes a singleton. It is remarkable that the attraction property is proved in the usual forward sense, unlike the pullback concept used in the context of random dynamical systems, but the main point is that the model under study has not been proved to generate a random dynamical system.

Keywords: Impulsive fractional stochastic evolution equations; Infinite delay; Mild solutions; Global forward attracting set; Singleton.

1 Introduction

There are numerous examples [5, 14, 15, 16, 24] of evolutionary systems that are subjected to rapid changes at certain instants in time. The interest in describing such processes by appropriated mathematical models, which are so-called differential equations with impulsive effects, mainly arose in recent years. In the simulations of such systems it is often convenient to neglect the durations of the rapid changes and to assume that the changes are represented by state jumps, see, e.g., [2, 7, 9, 13].

The study of fractional evolution equations involving impulses has been investigated to a large extent recently. In [25] the authors established the existence of mild solutions for impulsive

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fractional partial differential equations by means of a fixed point theorem. In [7, 9] it is studied the local and global existence of mild solutions for impulsive fractional integral-differential equation (without) with infinite delay by using fractional solution operator theory. In [27] the authors analyzed the impulsive fractional differential equations with weakly continuous nonlinearity by using the Schauder fixed point theorem. Most of the previous research only concerns the well-posedness and existence of (mild) solutions to different kinds of impulsive differential(-integral) equations, but there has been little research regarding the asymptotic behavior of (mild) solutions to these models.

Xu and Caraballo analyzed in [28] these previously mentioned issues for the following impulsive fractional stochastic delay differential equation (see [28] for the details on the operators and noise)

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + f(t, x_t) + g(t, x_t)\frac{dB(t)}{dt} + h(t)\frac{dB_Q^H(t)}{dt}, \quad t \geq 0, \\ \qquad t \neq t_k, \quad \frac{1}{2} < \alpha < 1, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad t = t_k, \quad k = 1, 2, \cdots, \\ x(t) = \phi(t), \quad t \in (-\infty, 0]. \end{cases} \tag{1.1}$$

The local and global existence and uniqueness of mild solutions to problem (1.1) were studied by means of a fixed point theorem, and the properties of α-order fractional solution operator $T_\alpha(t)$ and resolvent operator $S_\alpha(t)$. Moreover, the exponential decay to zero of the mild solutions to problem (1.1) was also proved. However, the lack of compactness of the α-order resolvent operator $S_\alpha(t)$ did not allow to establish the existence and structure of attracting sets, which is a key concept for the understanding of the dynamical properties of the model. To this respect, in [5] the authors already studied the existence of attractors for impulsive non-autonomous dynamical systems when $\alpha = 1$, since the operator that generates the infinitesimal generator A is a semigroup (see [18, 20] for more details). Fortunately, to overcome this difficulty in our fractional situation, we can take advantage of the compactness of α-order fractional solution operator $T_\alpha(t)$ which has been proven in [23, 26], and this is one of our motivations to heuristically propose, in this paper, the analysis of existence (and eventual uniqueness) of mild solutions and the global forward attracting set of the following fractional stochastic impulsive differential equations with infinite delay

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + I_t^{1-\alpha} f(t, x_t) + [I_t^{1-\alpha} g(t, x_t)]\frac{dB(t)}{dt} + [I_t^{1-\alpha} h(t)]\frac{dB_Q^H(t)}{dt}, \quad t \geq 0, \\ \qquad t \neq t_k, \quad 0 < \alpha < 1, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad t = t_k, \quad k = 1, 2, \cdots, \\ x(t) = \phi(t), \quad t \in (-\infty, 0], \end{cases} \tag{1.2}$$

where D_t^α is the Caputo fractional derivative of order $0 < \alpha < 1$, $I_t^{1-\alpha}$ is the $(1-\alpha)$-order fractional integral operator, $x(\cdot)$ takes the value in the separable Hilbert sapce $\mathbb{H}$. $A : D(A) \subseteq \mathbb{H} \to \mathbb{H}$ is the infinitesimal generator of an α-order fractional compact and analytic operator $\mathcal{T}_\alpha(t)(t \geq 0)$ (the same as the operator $T_\alpha(t)$ in [28]). As usual, $B(t)$ and $B_Q^H(t)$ denote, respectively, a $\mathbb{K}$-valued Q-cylindrical Brownian motion and fractional Brownian motion defined on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. The nonlinear maps $f : \mathbb{R}^+ \times \mathcal{PC} \to \mathbb{H}$, $g : \mathbb{R}^+ \times \mathcal{PC} \to \mathcal{L}(\mathbb{K}, \mathbb{H})$ and $h : \mathbb{R}^+ \to \mathcal{L}(\mathbb{K}, \mathbb{H})$ that satisfies $hQ^{\frac{1}{2}}$ is a Hilbert-Schmidt operator, are appropriate functions which will be specified later. Moreover, the initial data ϕ belongs to $\mathcal{PC}$ (which is the abstract phase space defined in Section 3), also for any x defined on $(-\infty, \infty)$ and any $t \in [0, \infty)$, we denote by x_t the segment of solution which is a function defined on $(-\infty, 0]$ via the relation

$$x_t(\theta) = x(t + \theta), \qquad \theta \in (-\infty, 0].$$

Additionally, $I_k \in C(\mathbb{H}, \mathbb{H})$ for each $k \in \mathbb{N}^+$, $x(t_k^+) = \lim_{h\to 0} x(t_k + h)$ and $x(t_k^-) = \lim_{h\to 0} x(t_k - h)$, represent the right and left-hand limits of $x(t)$ at $t = t_k$ respectively, and the fixed times t_k, where the impulses take place, satisfy $0 = t_0 < t_1 < \cdots < t_k \to +\infty$ as $k \to \infty$.

It is worth noticing that, the nonlinear terms of the right hand side of problem (1.2) have higher regularity because of the integral operator $I_t^{1-\alpha}$. For this model, by a fractional variation of constants formula (see Definition 3.1), the mild solution to problem (1.2) is only involving the α-order fractional solution operator $\mathcal{T}_\alpha(t)$ $(t \geq 0)$ which is compact, so that we are able to answer the open problem given in [28] but for our new model (1.2). We emphasize that the main advantage of our model is that we can extend the results obtained in [28] for the model (1.1) when $\alpha \in (\frac{1}{2}, 1)$ to our model (1.2) for $\alpha \in (0, 1)$. Also, thanks to the good properties of the α-order fractional solution operator (see Lemma 2.11) we are able to prove the existence of attracting sets and provide interesting information about the dynamics of the problem.

The content of our paper is as follows. Section 2 is devoted to introduce and recall some basic notations, preliminaries and lemmas which will be helpful throughout this paper. Next, in Section 3 we state a set of assumptions and conditions to prove the well-posedness and the existence and uniqueness of mild solutions to problem (1.1) by exploiting the properties of α-order fractional solution operator $\mathcal{T}_\alpha(t)$ $(t \geq 0)$. Finally, in the last section, we first present a general case in which one can ensure the existence of a compact global attracting set to problem (1.2) under continuity and linear growth conditions on the operators of the problem, which is not sufficient to ensure uniqueness of solutions. However, when we impose some stronger assumptions, ensuring in particular uniqueness of solutions, we can prove that the compact global attracting set becomes a singleton which is determining the dynamics in mean square of the problem. In particular, our results extend previous ones in the literature (see, e.g. [17] for a case without impulses and without fractional noise).

2 Preliminaries

In this section, we present the basic definitions, notations and lemmas which will be used further in this paper. Although the content of this section can be found in several published works (see, e.g. [28]), we prefer to include it in our paper to make it more readable and as much as self-contained as possible.

2.1 (Fractional) Brownian motion

In this subsection, we introduce the Brownian motion as well as fractional Brownian motion, and some lemmas are established that will be used the whole paper.

Throughout this paper, let $\mathbb{H}$ and $\mathbb{K}$ be two separable Hilbert spaces and $\mathcal{L}(\mathbb{K},\mathbb{H})$ denote the space of all bounded linear operators from $\mathbb{K}$ to $\mathbb{H}$. When no confusion is possible we will use the same notation $\|\cdot\|$ for the norm in different spaces (e.g. $\mathbb{H}$, $\mathbb{K}$, $\mathcal{L}(\mathbb{K},\mathbb{H})$), and use $(\cdot,\cdot)$ to denote the inner product of $\mathbb{H}$ and $\mathbb{K}$. Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ be a complete filtered probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$).

Let $B=(B(t))_{t\geq 0}$ and $B_Q^H=(B_Q^H(t))_{t\geq 0}$ be a $\mathbb{K}$-valued Q-cylindrical Brownian motion and a fractional Brownian motion respectively, defined on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ with $TrQ<\infty$, where Q is a symmetric nonnegative trace class operator from $\mathbb{K}$ into itself. We assume that there exists a complete orthonormal basis $\{e_k\}_{k\geq 1}$ in $\mathbb{K}$, a bounded sequence of nonnegative real numbers λ_k such that $Qe_k=\lambda_k e_k$, $k=1,2,\cdots$. Then $B(\cdot)$ and $B_Q^H(\cdot)$ admit the following expansions

$$B(t)=\sum_{k=1}^{\infty}\sqrt{\lambda_k}\beta_k(t)e_k,\qquad B_Q^H(t)=\sum_{k=1}^{\infty}\sqrt{\lambda_k}\beta_k^H(t)e_k,\qquad t\geq 0,$$

where $\{\beta_k\}_{k\geq 1}$ and $\{\beta_k^H\}_{k\geq 1}$ are, respectively, a sequence of two-sided one-dimensional real valued standard Brownian motions and a sequence of fractional Brownian motions mutually independent on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$.

For φ, $\psi\in\mathcal{L}(\mathbb{K},\mathbb{H})$, we define $(\varphi,\psi)=Tr[\varphi Q\psi^*]$, where ψ^* is the adjoint operator of ψ. Thus, for each $\varphi\in\mathcal{L}(\mathbb{K},\mathbb{H})$,

$$\|\varphi\|_Q^2=Tr[\varphi Q\varphi^*]=\sum_{k=1}^{\infty}\|\sqrt{\lambda_k}\varphi e_k\|^2.$$

An element $\varphi\in\mathcal{L}(\mathbb{K},\mathbb{H})$ is said to be a Q-Hilbert-Schmidt operator if $\|\varphi\|_Q^2<\infty$. For a more detailed description we refer the readers to [12, 19] and the references therein.

The following important results will be helpful throughout the paper.

Lemma 2.1 *([8]) Let $T>0$ and denote*

$$\mathrm{M}(\mathbb{K},\mathbb{H})=\Big\{\Phi(\cdot,\cdot):\Phi \text{ is an } \mathcal{L}(\mathbb{K},\mathbb{H})\text{-valued stochastic process on } [0,T]\times\Omega \text{ such that } \Phi(t) \text{ is measurable relative to } \mathcal{F}_t \text{ for all } t\in[0,T],\ \int_0^T E\|\Phi(t)\|^2dt<\infty\Big\}.$$

If Φ is an element of $\mathrm{M}(\mathbb{K},\mathbb{H})$, then

$$E\Big\|\int_0^T\Phi(s)dB(s)\Big\|^2\leq Tr(Q)\int_0^T E\|\Phi(s)\|^2ds. \tag{2.1}$$

Lemma 2.2 *([6, Lemma 2]) Let $\Phi : [0,T] \to \mathcal{L}(\mathbb{K},\mathbb{H})$ be such that $\Phi Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator, and let $H \in (\frac{1}{2},1)$ be such that*

$$\sum_{n=1}^{\infty} \|\Phi Q^{\frac{1}{2}} e_n\|_{L^{1/H}([0,T];\mathbb{H})} < \infty.$$

Then, for any α, $\beta \in [0,T]$ with $\alpha > \beta$,

$$E\left|\int_{\beta}^{\alpha} \Phi(s)dB_Q^H(s)\right|_{\mathbb{H}}^2 \leq cH(2H-1)(\alpha-\beta)^{2H-1}\sum_{n=1}^{\infty}\int_{\beta}^{\alpha} |\Phi(s)Q^{1/2}e_n|_{\mathbb{H}}^2 ds,$$

where $c = c(H)$.

If, in addition, $\sum\limits_{n=1}^{\infty} |\Phi(t)Q^{1/2}e_n|_{\mathbb{H}}$ is uniformly convergent for $t \in [0,T]$, then

$$E\left|\int_{\beta}^{\alpha} \Phi(s)dB_Q^H(s)\right|_{\mathbb{H}}^2 \leq cH(2H-1)(\alpha-\beta)^{2H-1}\int_{\beta}^{\alpha} \|\Phi(s)\|_Q^2 ds. \tag{2.2}$$

For more details about fractional Brownian motions, the reader is referred to [6] and the references therein.

2.2 Fractional setting

We now recall some facts about the theory of delete(the) fractional calculus.

For $\alpha > 0$, we consider the function $g_\alpha : \mathbb{R} \to \mathbb{R}$ defined by

$$g_\alpha(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} t^{\alpha-1}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

where $\Gamma(\alpha)$ is the Euler Gamma function. Now, assume that $T > 0$.

Definition 2.3 *([11],[21]) The fractional integral of order $\alpha > 0$ with the lower limit 0 for a function f is defined as*

$$I_t^\alpha f(t) := g_\alpha(t) * f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds, \quad t \in [0,T].$$

Thus, based on the definition of Riemann-Liouville fractional integral operator, we present the Caputo fractional differential operator.

Definition 2.4 *([11],[21]) Let $f : [0,T] \to \mathbb{R}$ be a function which possesses absolutely continuous derivatives up to order $n-1$ ($n \in \mathbb{N}$) on $[0,T]$. The Caputo fractional derivative of f of order $\alpha > 0$ with the lower limit 0 is defined as*

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds = I^{n-\alpha} f^n(t), \quad \textit{for} \;\; n-1 < \alpha < n, \quad n \in \mathbb{N}.$$

Notice that the previous definitions can be extended to functions f from $[0,T]$ into an abstract space Banach or Hilbert space $\mathbb{H}$ by considering the integrals in definitions 2.3 and 2.4 in the Bochner sense. Recall that a measurable function $u:[0,\infty)\to\mathbb{H}$ is Bochner integrable whenever $\|u\|$ is Lebesgue integrable.

Now we introduce some properties of a kind of special functions. Denote by $E_{\alpha,\beta}$ the generalized Mittag-Leffler special function defined by

$$E_{\alpha,\beta}(z):=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}=\frac{1}{2\pi i}\int_C\frac{t^{\alpha-\beta}e^t}{t^\alpha-z}dt,\quad \alpha,\beta>0,\ z\in\mathbb{C},$$

where C is a contour which starts and ends at $-\infty$ and encircles the disc $|t|\le|z|^{\frac{1}{\alpha}}$ counterclockwise. For short, we denote $E_\alpha(z)=E_{\alpha,1}(z)$ which is an entire function generalizing, in a simple way, the exponential function $E_1(z)=e^z$. Moreover, this function plays a crucial role in the field of fractional differential equations. It is worth highlighting that some interesting properties of the Mittag-Leffler functions are related to their Laplace integral,

$$\int_0^\infty e^{-\lambda t}t^{\beta-1}E_{\alpha,\beta}(\omega t^\alpha)dt=\frac{\lambda^{\alpha-\beta}}{\lambda^\alpha-\omega},\quad Re\lambda>\omega^{\frac{1}{\alpha}},\quad \omega>0.$$

The reader can find more properties in the references [1, 17, 21].

2.3 Fractional solution operator

Let us recall the following definitions of sectorial operator and α-order fractional solution operator which will be essential to solve our problems.

Definition 2.5 *([25]) A linear closed densely defined operator A is said to be sectorial if there are constants $\omega\in\mathbb{R}$, $\theta\in[\frac{\pi}{2},\pi]$, $M>0$, such that the following two conditions are satisfied:*
(1) $\sigma(A)\subset\sum_{\omega,\theta}=\{\lambda\in\mathbb{C},\lambda\ne\omega,|\arg(\lambda-\omega)|<\theta\}$, where $\sigma(A)$ denotes the spectrum of operator A.

(2) $\|\mathbb{R}(\lambda;A)\|\le\frac{M}{|\lambda-\omega|}$, $\lambda\in\sum_{\omega,\theta}$, where $\mathbb{R}(\lambda;A)$ denotes the resolvent operator associated to A.

Definition 2.6 *([3]) A function $\mathcal{T}_\alpha:\mathbb{R}^+\to\mathcal{L}(\mathbb{H})$ is said to be an α-order fractional solution operator generated by operator A if:*
(1) $\mathcal{T}_\alpha(t)$ is strongly continuous for $t\ge0$ and $\mathcal{T}_\alpha(0)=I$;

(2) $\mathcal{T}_\alpha(t)D(A)\subseteq D(A)$ and $A\mathcal{T}_\alpha(t)x=\mathcal{T}_\alpha(t)Ax$ for all $x\in D(A)$ and $t\ge0$;

(3) For all $x\in D(A)$ and $t\ge0$, $\mathcal{T}_\alpha(t)x$ is a solution of (delete)the following equation

$$z(t)=x+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}Az(s)ds.$$

Definition 2.7 *([10], [22]) An α-order fractional solution operator $\mathcal{T}_\alpha(t)$ $(t \geq 0)$ is called analytic if $\mathcal{T}_\alpha(t)$ possesses an analytic extension to a sector $\sum_{\theta_0} := \{\lambda \in \mathbb{C}\backslash\{0\} : |\arg\lambda| < \theta_0\}$ for some $\theta_0 \in (0, \frac{\pi}{2}]$. An analytic α-order fractional solution operator $\mathcal{T}_\alpha(t)$, generated by A, is said to be of analyticity type (ω_0, θ_0) if for each $\theta < \theta_0$ and $\omega > \omega_0$ $(\omega_0 \in \mathbb{R})$, there exists $M = M(\omega, \theta) > 0$ such that $\|\mathcal{T}_\alpha(\zeta)\| \leq Me^{\omega Re(\zeta)}$, $\zeta \in \sum_\theta$.*

In particular, we will denote by $\mathbb{A}^\alpha(\omega_0, \theta_0)$ the set of operators A which generate analytic α-order fractional solution operators $\mathcal{T}_\alpha$ of analyticity type (ω_0, θ_0).

Definition 2.8 *([26]) An α-order fractional solution operator $\mathcal{T}_\alpha(t)$ $(t \geq 0)$ is said to be compact, if $\mathcal{T}_\alpha(t)$ is a compact operator for all $t > 0$.*

Lemma 2.9 *([26, Lemma 3.1]), If the resolvent $\mathbb{R}(\lambda; A)$ is compact for every $\lambda > 0$, then $\mathcal{T}_\alpha(t)$ is compact for every $t > 0$, and therefore $\mathcal{T}_\alpha(\cdot)$ is compact.*

Thanks to the arguments in the proof of Lemma 3.8 in [10], one can prove the continuity of the α-order fractional solution operator $\mathcal{T}_\alpha(t)$ in the uniform operator topology for $t > 0$.

Lemma 2.10 *Assume $A \in \mathbb{A}^\alpha(\omega_0,\theta_0)$, and the α-order fractional solution operator $\mathcal{T}_\alpha(t)$ $(t > 0)$ is compact. Then the following properties are fulfilled:*
(i) $\lim_{h\to 0} \|\mathcal{T}_\alpha(t+h) - \mathcal{T}_\alpha(t)\| = 0$ *and* *(ii)* $\lim_{h\to 0^+} \|\mathcal{T}_\alpha(t) - \mathcal{T}_\alpha(h)\mathcal{T}_\alpha(t-h)\| = 0$ *for* $t > 0$.

Lemma 2.11 *([25],[10]) Let $\alpha \in (0, 1)$ and $A \in \mathbb{A}^\alpha(\omega_0,\theta_0)$. Then $\|\mathcal{T}_\alpha(t)\| \leq Me^{\omega t}$ for all $t > 0$ and $\omega > \omega_0$ $(\omega_0 \in \mathbb{R}^+)$. Furthermore,*

$$\|\mathcal{T}_\alpha(t)\| \leq M_T, \quad \textit{where} \quad M_T := \sup_{0\leq t\leq T} \|\mathcal{T}_\alpha(t)\|. \tag{2.3}$$

3 Well-posedness of the problem and existence of mild solutions

We discuss in this section the well-posedness of problem (1.2) and the existence and uniqueness of mild solutions.

To start off, we first present the abstract phase space $\mathcal{PC}$. Let $L^2(\Omega; \mathbb{H})$ denote the Hilbert space of all strongly-measurable, square-integrable $\mathbb{H}$-valued random variable equipped with the norm $\|u(\cdot)\|^2_{L^2} = E\|u(\cdot)\|^2$, where the expectation E is defined by $Eu = \int_\Omega u(\cdot)d\mathbb{P}$. The abstract phase space $\mathcal{PC}$ is defined by

$$\mathcal{PC} = \Big\{\xi : (-\infty, 0] \to L^2(\Omega; \mathbb{H}) \text{ is } \mathcal{F}_0\text{-adapted and continuous except in at most a countable number of points } \{\theta_k\}, \text{ at which there exist } \xi(\theta_k^+) \text{ and } \xi(\theta_k^-) \text{ with } \xi(\theta_k) = \xi(\theta_k^-), \text{ and } \sup_{\theta\in(-\infty,0]} e^{\gamma\theta} E\|\xi(\theta)\|^2 < \infty\Big\},$$

for some fixed parameter $\gamma > 0$. If $\mathcal{PC}$ is endowed with the norm

$$\|\xi\|_{\mathcal{PC}} = \left(\sup_{\theta \in (-\infty, 0]} e^{\gamma\theta} E\|\xi(\theta)\|^2 \right)^{\frac{1}{2}}, \qquad \xi \in \mathcal{PC},$$

then, $(\mathcal{PC}, \|\cdot\|_{\mathcal{PC}})$ is a Banach space.

Let us now enumerate the following conditions which will be assumed throughout the paper.

(H_1) $f : [0, \infty) \times \mathcal{PC} \to \mathbb{H}$, $g : [0, \infty) \times \mathcal{PC} \to \mathcal{L}(\mathbb{K}, \mathbb{H})$ are continuous and there exist two positive constants l_1, l_2 such that

$$E\|f(t, x) - f(t, y)\|^2 \leq l_1 \|x - y\|^2_{\mathcal{PC}}, \quad E\|g(t, x) - g(t, y)\|^2 \leq l_2 \|x - y\|^2_{\mathcal{PC}},$$

for every x, $y \in \mathcal{PC}$, and almost every $t > 0$. Moreover, $g(t, \cdot)$ is measurable relative to $\mathcal{F}_t$ for all $t \in [0, \infty)$ satisfying $\int_0^\infty E\|g(t, x)\|^2 dt < \infty$.

(H_2) $h : [0, \infty) \to \mathcal{L}(\mathbb{K}, \mathbb{H})$ satisfying $hQ^{\frac{1}{2}}$ is a Hilbert-Schmidt operator, and there exists a constant $\Lambda > 0$, such that for every $t \in [0, \infty)$, there holds

$$\int_0^t \|h(s)\|^2_Q ds < \Lambda.$$

(H_3) The functions $I_k : L^2(\Omega; \mathbb{H}) \to L^2(\Omega; \mathbb{H})$ are linear and continuous for each $k \in \mathbb{N}^+$, and there exists $N_k > 0$ with $\sum_{k=1}^\infty N_k < +\infty$, such that

$$E\|I_k(x)\|^2 \leq N_k E\|x\|^2, \text{ for all } x \in L^2(\Omega; \mathbb{H}).$$

Notice that these assumptions imply that $N_k \to 0$ as $k \to +\infty$, and there exists a positive constant N such that

$$E\|I_k(x)\|^2 \leq N E\|x\|^2, \text{ for all } x \in L^2(\Omega; \mathbb{H}).$$

(H_4) $\beta = \inf\limits_{k \in \mathbb{N}^+} \{t_k - t_{k-1}\} > 0$, $\eta = \sup\limits_{k \in \mathbb{N}^+} \{t_k - t_{k-1}\} < \infty$.

Next we state the definition of mild solution to problem (1.2). Namely it is a fractional variation of constants formula which involves the Mittag-Leffler families. More details can be found in [17, 25] and the references therein.

Definition 3.1 *Let $\mathcal{F}_t = \mathcal{F}_0$ for all $t \in (-\infty, 0]$, and let $\phi \in \mathcal{PC}$ be an initial value. An $\mathcal{F}_t$-adapted stochastic process $x : (-\infty, T] \to \mathbb{H}$ is said to be a mild solution of the equation (1.2) if*

$x(t)=\phi(t)$ for $t\in(-\infty,0]$, and for $t\in[0,T]$, $x(t)$ satisfies the integral equation

$$x(t)=\begin{cases}\mathcal{T}_\alpha(t)\phi(0)+\int_0^t\mathcal{T}_\alpha(t-s)f(s,x_s)ds+\int_0^t\mathcal{T}_\alpha(t-s)g(s,x_s)dB(s)\\ +\int_0^t\mathcal{T}_\alpha(t-s)h(s)dB_Q^H(s),\quad t\in[0,t_1],\\ \mathcal{T}_\alpha(t-t_1)(x(t_1^-)+I_1(x(t_1^-)))+\int_{t_1}^t\mathcal{T}_\alpha(t-s)f(s,x_s)ds\\ +\int_{t_1}^t\mathcal{T}_\alpha(t-s)g(s,x_s)dB(s)+\int_{t_1}^t\mathcal{T}_\alpha(t-s)h(s)dB_Q^H(s),\quad t\in(t_1,t_2],\\ \cdots,\\ \mathcal{T}_\alpha(t-t_m)(x(t_m^-)+I_m(x(t_m^-)))+\int_{t_m}^t\mathcal{T}_\alpha(t-s)f(s,x_s)ds\\ +\int_{t_m}^t\mathcal{T}_\alpha(t-s)g(s,x_s)dB(s)+\int_{t_m}^t\mathcal{T}_\alpha(t-s)h(s)dB_Q^H(s),\quad t\in(t_m,T],\end{cases}\tag{3.1}$$

where $t_m=\max\{t_k,\ t_k<T,\ k=0,1,2,\cdots\}$, and we recall that

$$\mathcal{T}_\alpha(t)=E_{\alpha,1}(At^\alpha)=\frac{1}{2\pi i}\int_C e^{\lambda t}\frac{\lambda^{\alpha-1}}{\lambda^\alpha-A}d\lambda.$$

After having presented the required estimates and properties of the Wiener integral and α-order fractional solution operator, we can now state and prove the main results of this section.

Theorem 3.2 *Let $\alpha\in(0,1)$, $A\in\mathbb{A}^\alpha(\omega_0,\theta_0)$ with $\theta_0\in(0,\frac{\pi}{2}]$ and $\omega_0\in\mathbb{R}^+$. Assume (H_1)-(H_4) hold, and the α-order fractional solution operator $\mathcal{T}_\alpha(t)$ $(t\geq 0)$ is compact. Then, for every initial data $\phi\in\mathcal{PC}$ and every $T>0$, the problem (1.2) has a unique mild solution defined on $(-\infty,T]$.*

Proof. We only include a sketch of this proof since the similar result is proved in ([28, Theorem 12]) but for a different kernels. The main idea is to use a Picard iteration scheme. To do this, we first define two proper abstract phase spaces $\mathcal{PC}^T$ and $\mathcal{PC}^T_\phi$ which are the same as in the proof of Theorem 12 of [28]. Then, we construct a sequence $\{x^n(t)\}_{n=0}^\infty$ as follows, for a fixed but arbitrary $\psi\in\mathcal{PC}^T_\phi$,

$$\begin{cases}x^0(t)=\psi(t),\\ x^n(t)=\chi_{(-\infty,0]}(t)\psi(t)+\sum_k\chi_{(t_k,t_{k+1}]}(t)\Big\{\mathcal{T}_\alpha(t-t_k)(x^{n-1}(t_k^-)+I_k(x^{n-1}(t_k^-)))\\ \quad+\int_{t_k}^t\mathcal{T}_\alpha(t-s)f(s,x_s^{n-1})ds+\int_{t_k}^t\mathcal{T}_\alpha(t-s)g(s,x_s^{n-1})dB(s)\\ \quad+\int_{t_k}^t\mathcal{T}_\alpha(t-s)h(s)dB_Q^H(s)\Big\},\qquad t\in(-\infty,T],\quad k=0,1,2\cdots,\end{cases}\tag{3.2}$$

where $I_0=0$, $t_k<T$ and χ denotes the characteristic function.

By slightly modifying the proof of Theorem 12 in [28], we can first prove that $x^n(\cdot)\in\mathcal{PC}^T_\phi$ for all $n\geq 1$. Then, one can prove that it is a Cauchy sequence in $\mathcal{PC}^T_\phi$ and its limit is a solution to our problem. The uniqueness follows from the next result below. □

The next result ensures the continuous dependence of mild solutions with respect the initial data, and can be proved by using similar arguments to those used in [28, Theorem 14]. We omit the detailed proof here.

Theorem 3.3 *Assume the hypotheses of Theorem 3.2. Then, the mild solution to problem (1.2) is continuous with respect to the initial value* ϕ*. That is, if* $x(t)$*,* $y(t)$ *are the corresponding mild solutions to the initial data* ϕ *and* φ *on* $[0,T]$*, we have*

$$\|x_t - y_t\|^2_{\mathcal{PC}} \leq (3M_T^2+1)\|\phi-\varphi\|^2_{\mathcal{PC}} e^{\left(A_1+\frac{\ln A_2}{\beta}\right)t}, \qquad \forall t\in[0,T], \tag{3.3}$$

where $A_1 = 3M_T^2(l_1\eta + l_2 Tr(Q))$ *and* $A_2 = 6M_T^2(N+1)+e^{A_1\eta}$*.*

4 Asymptotic behavior of global mild solutions

Now we can study the long time behavior of the global mild solutions to our problem. First we enumerate some assumptions which will be imposed in our further analysis.

(C_1) $A : D(A)\subset \mathbb{H}\to\mathbb{H}$ is the infinitesimal generator of an α-order fractional compact and analytic solution operator $\mathcal{T}_\alpha(t)$ $(t\geq 0)$ on a separable Hilbert space $\mathbb{H}$ with

$$\|\mathcal{T}_\alpha(t)\| \leq Me^{-\mu t}, \qquad \forall t\geq 0, \quad M\geq 1, \quad \mu\in\mathbb{R}^+.$$

(C_2) There exist two nonnegative continuous functions $k_1(t)$, $k_2(t)\in L^1(\mathbb{R}^+)$, such that the continuous function $f:\mathbb{R}^+\times\mathcal{PC}\to\mathbb{H}$ satisfies

$$E\|f(t,x)\|^2 \leq k_1(t)+k_2(t)\|x\|^2_{\mathcal{PC}},$$

for $t\in[0,\infty)$ and every $x\in\mathcal{PC}$.

(C_3) There exist two nonnegative continuous functions $k_3(t)$, $k_4(t)\in L^1(\mathbb{R}^+)$, such that the continuous function $g:\mathbb{R}^+\times\mathcal{PC}\to\mathcal{L}(\mathbb{K},\mathbb{H})$ satisfies

$$E\|g(t,x)\|^2 \leq k_3(t)+k_4(t)\|x\|^2_{\mathcal{PC}},$$

for $t\in[0,\infty)$ and every $x\in\mathcal{PC}$.

At this point some remarks are in order.

Remark 4.1 *i) Notice that, under the Lipschitz condition* (H_1)*, we can obtain the local existence and uniqueness of mild solution to problem (1.2) (see Theorem 3.2). However, in this section we are interested in analyzing the asymptotic behavior of mild solutions to equation (1.2) no matter how many solutions the problem may have for each initial condition. Therefore, our analysis can be carried out without imposing* (H_1)*. Instead, in order to guarantee that we have mild solutions* *globally defined in time, it is enough to assume conditions* (C_2) *and* (C_3) *as above. A well-known conclusion is that conditions* (C_2) *and* (C_3) *hold automatically once we assume condition* (H_1) *holds true. Henceforth, throughout this paper, we will assume either condition* (H_1) *(when we need uniqueness of solution) or* (C_2)*-*(C_3)*.*

ii) Due to the fact that the continuous functions $k_i(s) \in L^1(\mathbb{R}^+)$ *appearing in conditions* (C_2) *and* (C_3) *are nonnegative, we will denote in the sequel*

$$\int_0^\infty k_i(s)ds := K_i < \infty, \quad i = 1, 2, 3, 4,$$

where K_i *are positive constants.*

Throughout the paper, the symbol C will denote a generic constant whose value may change from one line to another and even in the same line.

4.1 Global existence and estimates of mild solutions

This subsection is first concerned with a theorem ensuring the global existence and uniqueness of mild solution to problem (1.2).

Theorem 4.2 *Assume hypotheses of Theorem 3.2 and* (C_1) *hold. Then for every initial value* $\phi \in \mathcal{PC}$*, the initial value problem (1.2) has a unique solution defined on* $[0, \infty)$ *in the sense of Definition 3.1.*

Proof. Thanks to assumption (C_1), the estimates which are necessary to prove Theorem 3.2 are independent of T. This implies that the solution is defined in $(-\infty, T]$ for all $T > 0$. More details for a similar problem can be found in [28, Theorem 15]. □

In what follows, we shall obtain the estimate of solutions which will imply that the solutions are bounded uniformly with respect to bounded sets of initial conditions and positive values of time. This also implies the existence of an absorbing set for the solutions which is also a property on the ultimate boundedness of solutions.

Theorem 4.3 *Assume* (H_2)*-*(H_4)*,* (C_1)*-*(C_3)*, and*

$$\gamma > 2\mu, \tag{4.1}$$

also, let

$$2\mu - \frac{\ln \mathcal{K}_1}{\beta} > 0, \tag{4.2}$$

where

$$\mathcal{K}_1 = 8M^2(N+1)(1+L_3), \qquad L_3 = 4M^2(\eta K_2 + Tr(Q)K_4)\exp\left(4M^2(\eta K_2 + Tr(Q)K_4)\right).$$

Then every solution $x(\cdot)$ *of problem* (1.2) *with* $x_0 = \phi \in \mathcal{PC}$*, defined globally in time, verifies*

$$\|x_t\|^2_{\mathcal{PC}} \leq C\|\phi\|^2_{\mathcal{PC}} e^{-\left(2\mu - \frac{\ln \mathcal{K}_1}{\beta}\right)t} + C, \quad \forall t \geq 0.$$

Proof. For the sake of convenience, we split the proof into three steps.

Step 1. By (2.1), (2.2), (2.3), Definition 3.1, the Cauchy-Schwarz inequality, (H_2)-(H_4) and (C_1)-(C_3), we obtain that for $t \in [0, t_1]$,

$$\begin{aligned} E\|x(t)\|^2 &\leq 4E\|\mathcal{T}_\alpha(t)\phi(0)\|^2 + 4\eta\int_0^t \|\mathcal{T}_\alpha(t-s)\|^2 E\|f(s,x_s)\|^2 ds \\ &\quad + 4Tr(Q)\int_0^t \|\mathcal{T}_\alpha(t-s)\|^2 E\|g(s,x_s)\|^2 ds \\ &\quad + 4cH(2H-1)t^{2H-1}\int_0^t \|\mathcal{T}_\alpha(t-s)\|^2 \|h(s)\|_Q^2 ds \\ &\leq 4M^2 e^{-2\mu t} E\|\phi(0)\|^2 + 4M^2\eta\int_0^t e^{-2\mu(t-s)}\left(k_1(s) + k_2(s)\|x_s\|_{\mathcal{PC}}^2\right) ds \\ &\quad + 4M^2 Tr(Q)\int_0^t e^{-2\mu(t-s)}\left(k_3(s) + k_4(s)\|x_s\|_{\mathcal{PC}}^2\right) ds \\ &\quad + 4cH(2H-1)t^{2H-1}\int_0^t \|\mathcal{T}_\alpha(t-s)\|^2 \|h(s)\|_Q^2 ds \\ &\leq 4M^2 e^{-2\mu t}\|\phi\|_{\mathcal{PC}}^2 + 4M^2(\eta K_1 + Tr(Q)K_3) + L_1 \\ &\quad + 4M^2\int_0^t (\eta k_2(s) + Tr(Q)k_4(s))\, e^{-2\mu(t-s)}\|x_s\|_{\mathcal{PC}}^2 ds, \end{aligned} \tag{4.3}$$

where we have used the notation

$$L_1 = 4M^2cH(2H-1)\eta^{2H-1}\int_0^\eta \|h(s)\|_Q^2 ds \leq 4M^2cH(2H-1)\eta^{2H-1}\Lambda.$$

By assumption (4.1), we have $2\mu < \gamma$, then $e^{-(2\mu-\gamma)\theta} \leq 1$ holds immediately for any $\theta \leq 0$. Multiplying (4.3) by $e^{\gamma\theta}$ and replacing t by $t+\theta$, it follows

$$\begin{aligned} \sup_{\theta\in(-t,0]} e^{\gamma\theta}E\|x(t+\theta)\|^2 &\leq 4M^2 e^{-2\mu(t+\theta)}e^{\gamma\theta}\|\phi\|_{\mathcal{PC}}^2 + 4M^2(\eta K_1 + Tr(Q)K_3)e^{\gamma\theta} + L_1 e^{\gamma\theta} \\ &\quad + 4M^2\int_0^{t+\theta} (\eta k_2(s) + Tr(Q)k_4(s))\, e^{-2\mu(t+\theta-s)}e^{\gamma\theta}\|x_s\|_{\mathcal{PC}}^2 ds \\ &\leq 4M^2 e^{-2\mu t}\|\phi\|_{\mathcal{PC}}^2 + 4M^2(\eta K_1 + Tr(Q)K_3) + L_1 \\ &\quad + 4M^2\int_0^t (\eta k_2(s) + Tr(Q)k_4(s))\, e^{-2\mu(t-s)}\|x_s\|_{\mathcal{PC}}^2 ds. \end{aligned}$$

Note that

$$e^{\gamma\theta}E\|x(t+\theta)\|^2 = e^{-\gamma t}e^{\gamma(t+\theta)}E\|x(t+\theta)\|^2 \leq e^{-\gamma t}\|\phi\|_{\mathcal{PC}}^2 \leq e^{-2\mu t}\|\phi\|_{\mathcal{PC}}^2, \qquad \forall\theta \in (-\infty, -t].$$

Therefore,

$$\begin{aligned} e^{2\mu t}\|x_t\|_{\mathcal{PC}}^2 &\leq 4M^2\|\phi\|_{\mathcal{PC}}^2 + 4M^2\left(\eta K_1 + Tr(Q)K_3\right)e^{2\mu t} + L_1 e^{2\mu t} \\ &\quad + 4M^2\int_0^t (\eta k_2(s) + Tr(Q)k_4(s))\, e^{2\mu s}\|x_s\|_{\mathcal{PC}}^2 ds. \end{aligned}$$

Applying the Gronwall inequality, we have for $t \in [0, t_1]$ that

$$\|x_t\|_{\mathcal{PC}}^2 \le 4M^2\|\phi\|_{\mathcal{PC}}^2(1+L_3)e^{-2\mu t} + L_2 + L_2L_3, \tag{4.4}$$

where we used the notation

$$L_2 = 4M^2(\eta K_1 + Tr(Q)K_3) + L_1, \quad L_3 = 4M^2(\eta K_2 + Tr(Q)K_4)\exp\left(4M^2(\eta K_2 + Tr(Q)K_4)\right).$$

In particular,

$$E\|x(t_1)\|^2 \le 4M^2\|\phi\|_{\mathcal{PC}}^2(1+L_3)e^{-2\mu t_1} + L_2 + L_3L_3 := D_1^*. \tag{4.5}$$

Step 2: Similar to (4.3) and in view of (H_3), we obtain for $t \in (t_1, t_2]$,

$$\begin{aligned}
E\|x(t)\|^2 \le\ & 4E\|\mathcal{T}_\alpha(t-t_1)(x(t_1^-) + I_1(x(t_1^-)))\|^2 + 4\eta\int_{t_1}^t \|\mathcal{T}_\alpha(t-s)\|^2 E\|f(s,x_s)\|^2 ds \\
& + 4Tr(Q)\int_{t_1}^t \|\mathcal{T}_\alpha(t-s)\|^2 E\|g(s,x_s)\|^2 ds \\
& + 4cH(2H-1)(t-t_1)^{2H-1}\int_{t_1}^t \|\mathcal{T}_\alpha(t-s)\|^2\|h(s)\|_Q^2 ds \\
\le\ & 8M^2e^{-2\mu(t-t_1)}(N+1)E\|x(t_1^-)\|^2 + 4M^2(\eta K_1 + Tr(Q)K_3) \\
& + 4M^2\int_{t_1}^t (\eta k_2(s) + Tr(Q)k_4(s))\, e^{-2\mu(t-s)}\|x_s\|_{\mathcal{PC}}^2 ds + L_1.
\end{aligned}$$

Arguing as in Step 1, we derive for $t+\theta > t_1$ (where $\theta \in (-\infty, 0]$) that

$$\begin{aligned}
& \sup_{\theta\in(t_1-t,0]} e^{\gamma\theta}E\|x(t+\theta)\|^2 \\
& \le 8M^2e^{-2\mu(t+\theta-t_1)}e^{\gamma\theta}(N+1)E\|x(t_1^-)\|^2 + 4M^2(\eta K_1 + Tr(Q)K_3)e^{\gamma\theta} + L_1e^{\gamma\theta} \\
& \quad + 4M^2\int_{t_1}^{t+\theta} (\eta k_2(s) + Tr(Q)k_4(s))\, e^{-2\mu(t+\theta-s)}e^{\gamma\theta}\|x_s\|_{\mathcal{PC}}^2 ds \\
& \le 8M^2e^{-2\mu(t-t_1)}(N+1)E\|x(t_1^-)\|^2 + 4M^2(\eta K_1 + Tr(Q)K_3) + L_1 \\
& \quad + 4M^2\int_{t_1}^t (\eta k_2(s) + Tr(Q)k_4(s))\, e^{-2\mu(t-s)}\|x_s\|_{\mathcal{PC}}^2 ds.
\end{aligned}$$

It follows from (4.4) and (4.5) that for $t \in (t_1, t_2]$ such that $t+\theta < t_1$,

$$\begin{aligned}
e^{\gamma\theta}E\|x(t+\theta)\|^2 & \le 4M^2\|\phi\|_{\mathcal{PC}}^2(1+L_3)e^{-2\mu t_1}e^{-2\mu(t+\theta-t_1)} + L_2 + L_2L_3 \\
& \le (4M^2\|\phi\|_{\mathcal{PC}}^2(1+L_3)e^{-2\mu t_1} + L_2 + L_2L_3)e^{-2\mu(t+\theta-t_1)} \\
& = D_1^*e^{-2\mu(t+\theta-t_1)}.
\end{aligned}$$

Hence,

$$e^{2\mu(t-t_1)}\|x_t\|^2_{\mathcal{PC}} \le 8M^2(N+1)D_1^* + 4M^2(\eta K_1 + Tr(Q)K_3)e^{2\mu(t-t_1)} + L_1 e^{2\mu(t-t_1)} + 4M^2\int_{t_1}^{t}(\eta k_2(s) + Tr(Q)k_4(s))\, e^{2\mu(s-t_1)}\|x_s\|^2_{\mathcal{PC}}ds.$$

Thanks to the Gronwall inequality, we deduce for $t \in (t_1, t_2]$

$$\|x_t\|^2_{\mathcal{PC}} \le 8M^2(N+1)D_1^*(1+L_3)e^{-2\mu(t-t_1)} + L_2 + L_2L_3, \tag{4.6}$$

and, consequently,

$$E\|x(t_2)\|^2 \le 8M^2(N+1)D_1^*(1+L_3)e^{-2\mu(t_2-t_1)} + L_2 + L_2L_3 := D_2^*. \tag{4.7}$$

Step 3: In a similar way, for $t \in (t_k, t_{k+1}]$ with $k \ge 2$, we have

$$\|x_t\|^2_{\mathcal{PC}} \le 8M^2(N+1)D_k^*(1+L_3)e^{-2\mu(t-t_k)} + L_2 + L_2L_3, \tag{4.8}$$

and

$$E\|x(t_{k+1})\|^2 \le 8M^2(N+1)D_k^*(1+L_3)e^{-2\mu(t_{k+1}-t_k)} + L_2 + L_2L_3 := D_{k+1}^*. \tag{4.9}$$

For convenience, let $\mathcal{K}_1 = 8M^2(N+1)(1+L_3)$, $\mathcal{K}_2 = L_2+L_2L_3$. Then, by using the mathematical induction method, we obtain for $k \ge 2$ that

$$\begin{aligned} D_k^* &= \mathcal{K}_1 D_{k-1}^* e^{-2\mu(t_k-t_{k-1})} + \mathcal{K}_2 \\ &\le \mathcal{K}_1^{k-1} D_1^* e^{-2\mu(t_k-t_1)} + \mathcal{K}_2 \sum_{j=0}^{k-2} \mathcal{K}_1^{k} e^{-2\mu(t_k-t_{k-j})}. \end{aligned} \tag{4.10}$$

Noticing that (H_4) implies that $k-1 \le \frac{t_k-t_1}{\beta}$ and $k \le \frac{t_k-t_{k-j}}{\beta}$, it then follows from (4.2) and (4.10) that

$$\begin{aligned} D_k^* &\le \mathcal{K}_1^{\frac{t_k-t_1}{\beta}} e^{-2\mu(t_k-t_1)} D_1^* + \mathcal{K}_2 \sum_{j=0}^{k-2} C_1^{\frac{t_k-t_{k-j}}{\beta}} e^{-2\mu(t_k-t_{k-j})} \\ &= e^{\frac{t_k-t_1}{\beta}\ln \mathcal{K}_1} e^{-2\mu(t_k-t_1)} D_1^* + \mathcal{K}_2 \sum_{j=0}^{k-2} e^{\frac{t_k-t_{k-j}}{\beta}\ln \mathcal{K}_1} e^{-2\mu(t_k-t_{k-j})} \\ &\le e^{-\left(2\mu-\frac{\ln \mathcal{K}_1}{\beta}\right)(t_k-t_1)} D_1^* + \mathcal{K}_2 \sum_{j=0}^{k-2} e^{-\left(2\mu-\frac{\ln \mathcal{K}_1}{\beta}\right)(t_k-t_{k-j})} \\ &\le e^{-\left(2\mu-\frac{\ln \mathcal{K}_1}{\beta}\right)(t_k-t_1)} D_1^* + \mathcal{K}_2 \frac{e^{\left(2\mu-\frac{\ln \mathcal{K}_1}{\beta}\right)\beta}}{e^{\left(2\mu-\frac{\ln \mathcal{K}_1}{\beta}\right)\beta}-1}. \end{aligned} \tag{4.11}$$

Therefore, by (4.5), (4.8) and (4.11), we deduce that for all $t \in (t_k, t_{k+1}]$ with $k \geq 2$,

$$\begin{aligned}
\|x_t\|_{\mathcal{PC}}^2 &\leq \mathcal{K}_1 e^{-2\mu(t-t_k)} D_k^* + \mathcal{K}_2 \\
&\leq \mathcal{K}_1 e^{-\left(2\mu - \frac{\ln \mathcal{K}_1}{\beta}\right)(t-t_1)} D_1^* + \mathcal{K}_1 \mathcal{K}_2 \frac{e^{(2\mu\beta - \ln \mathcal{K}_1)}}{e^{(2\mu\beta - \ln \mathcal{K}_1)} - 1} + \mathcal{K}_2 \\
&\leq \mathcal{K}_1^2 e^{-\left(2\mu - \frac{\ln \mathcal{K}_1}{\beta}\right)(t-t_1)} e^{-2\mu t_1} \|\phi\|_{\mathcal{PC}}^2 + C \qquad (4.12) \\
&\leq \mathcal{K}_1^2 e^{-\left(2\mu - \frac{\ln \mathcal{K}_1}{\beta}\right)t} \|\phi\|_{\mathcal{PC}}^2 + C \\
&:= C\|\phi\|_{\mathcal{PC}}^2 e^{-\left(2\mu - \frac{\ln \mathcal{K}_1}{\beta}\right)t} + C,
\end{aligned}$$

which, on account of (4.4) and (4.6), implies

$$\|x_t\|_{\mathcal{PC}}^2 \leq C\|\phi\|_{\mathcal{PC}}^2 e^{-\left(2\mu - \frac{\ln \mathcal{K}_1}{\beta}\right)t} + C, \quad \text{for all } t \geq 0.$$

The proof is finished. □

Remark 4.4 *We emphasize that under assumptions (C_2)-(C_3) (instead of Lipschitz condition (H_1)), we may only have a general result ensuring that there exists at least one mild solution to problem (1.2), i.e., as Theorem 3.3 may not hold, the uniqueness of solutions in Theorem 3.2 is not ensured. Hence, in Theorem 4.3 we have proved that for any solution corresponding to the initial value $\phi \in \mathcal{PC}$, this a priori estimate holds true.*

4.2 Existence of global attracting sets: General case

A general result concerning the existence of a minimal compact set in $\mathcal{PC}$ which is globally attracting for the solutions of our problem will be proved in this subsection. To that end we first need the following compactness conclusion.

Lemma 4.5 *Assume the conditions of Theorem 4.3. Then for any bounded subset D of $\mathcal{PC}$, any sequence $\{\tau_n\}$ with $\tau_n \to \infty$ $(n \to \infty)$, $\{\phi_n\}$ with $\phi_n \in D$, and any sequence of solutions $\{x^n(\cdot)\}$ of problem (1.2) with $x_0^n = \phi_n \in D$, the sequence $\{x_{\tau_n}^n\}$ is relatively compact in $\mathcal{PC}$.*

Proof. Without loss of generality, we assume that $\|\phi\|_{\mathcal{PC}} \leq d$ for all $\phi \in D$. For any $\phi_n \in D$, we define $u_{\tau_n}^n(\cdot) : (-\infty, 0] \to \mathbb{H}$ by

$$u^n(\tau_n + \theta) = \begin{cases} \phi_n(\tau_n + \theta), & \tau_n + \theta \in (-\infty, 0], \\ \mathcal{T}_\alpha(\tau_n + \theta)\phi(0), & \tau_n + \theta \in [0, t_1], \\ \mathcal{T}_\alpha(\tau_n + \theta - t_k)(u^n(t_k^-) + I_k(u^n(t_k^-))), & \tau_n + \theta \in (t_k, t_{k+1}], \quad k = 1, 2, \cdots. \end{cases}$$

Let us do estimates likewise as in the proof of Theorem 16 in [28], by (C_1) and (H_3) we find that

$$\|u_{\tau_n}^n\|_{\mathcal{PC}} \leq Ce^{-\mu\tau_n}\|\phi\|_{\mathcal{PC}}. \qquad (4.13)$$

Next we define the function $z^n_{\tau_n}(\cdot):(-\infty,0]\to\mathbb{H}$ by

$$z^n(\tau_n+\theta)=\begin{cases}0, & \tau_n+\theta\in(-\infty,0],\\ \displaystyle\int_0^{\tau_n+\theta}\mathcal{T}_\alpha(\tau_n+\theta-s)f(s,x^n_s)ds+\int_0^{\tau_n+\theta}\mathcal{T}_\alpha(\tau_n+\theta-s)g(s,x^n_s)dB(s)\\ \quad+\displaystyle\int_0^{\tau_n+\theta}\mathcal{T}_\alpha(\tau_n+\theta-s)h(s)dB^H_Q(s), \quad \tau_n+\theta\in[0,t_1],\\ \mathcal{T}_\alpha(\tau_n+\theta-t_k)(z^n(t_k^-)+I_k(z^n(t_k^-)))+\displaystyle\int_{t_k}^{\tau_n+\theta}\mathcal{T}_\alpha(\tau_n+\theta-s)f(s,x^n_s)ds\\ \quad+\displaystyle\int_{t_k}^{\tau_n+\theta}\mathcal{T}_\alpha(\tau_n+\theta-s)g(s,x^n_s)dB(s)\\ \quad+\displaystyle\int_{t_k}^{\tau_n+\theta}\mathcal{T}_\alpha(\tau_n+\theta-s)h(s)dB^H_Q(s), \ \ \tau_n+\theta\in(t_k,t_{k+1}], \ \ k=1,2,\cdots.\end{cases} \tag{4.14}$$

It is important to observe that if $x^n_{\tau_n}(\cdot)$ satisfies (3.1), then $x^n_{\tau_n}=u^n_{\tau_n}+z^n_{\tau_n}$ for $\tau_n\in[0,\infty)$ since the impulse functions I_k $(k\in\mathbb{N}^+)$ are linear. In order to prove that $\{x^n_{\tau_n}\}$ is relatively compact in $\mathcal{PC}$, by the decomposition of $x^n_{\tau_n}$ and (4.13), it is enough to state $\{z^n_{\tau_n}\}$ is compact in $\mathcal{PC}$ as $\tau_n\to\infty$.

Initially, we show that $\{z^n(\tau_n+\cdot)\}_{n=0}^{\infty}$ is equicontinuous on $[-T^*,0]$ for any fixed $T^*>0$. For such T^* fixed, there exists $n_0\in\mathbb{N}$ such that for all $n\geq n_0$ the points $\tau_n+\theta>r$ for all $\theta\in[-T^*,0]$, for some $r>0$. Then, to prove the equicontinuity in the interval $[-T^*,0]$, it is sufficient to assume that for each $n\geq n_0$, and $\theta_1,\theta_2\in[-T^*,0]$ with $\theta_1<\theta_2$ (with $\theta_2-\theta_1$ sufficiently small), we have that $\tau_n+\theta_1$, $\tau_n+\theta_2\in(t_k,t_{k+1}]\cap[r,+\infty)$, for some $k\in\mathbb{N}$ and $r>0$. Once we have proved the equicontinuity for this case, the possibility that the points $\tau_n+\theta$,$\tau_n+\theta_2$ may belong to different intervals can be handled by comparing with the values of the solution in the impulse time t_k, by using Theorem 4.3, estimate (4.13) and the properties of the impulsive linear function I_k, in particular the fact that $N_k\to0$ as $k\to\infty$. Consequently, given $\varepsilon>0$, we assume that $\tau_n+\theta_2\in(t_k,t_{k+1}]\cap[r,+\infty)$ for all $\theta_1<\theta_2$ in the interval $[-T^*,0]$, with $\theta_2-\theta_1$

sufficiently small as we will determine below. From (4.14) we can deduce, for all $n \geq n_0$,

$$
\begin{aligned}
&E\|z^n(\tau_n+\theta_2)-z^n(\tau_n+\theta_1)\|^2 \\
&\leq 7\|\mathcal{T}_\alpha(\tau_n+\theta_2-t_k)-\mathcal{T}_\alpha(\tau_n+\theta_1-t_k)\|^2 E\|z^n(t_k^-)+I_k(z^n(t_k^-))\|^2 \\
&\quad +7E\left\|\int_{t_k}^{\tau_n+\theta_1}(\mathcal{T}_\alpha(\tau_n+\theta_2-s)-\mathcal{T}_\alpha(\tau_n+\theta_1-s))f(s,x_s^n)ds\right\|^2 \\
&\quad +7E\left\|\int_{\tau_n+\theta_1}^{\tau_n+\theta_2}\mathcal{T}_\alpha(\tau_n+\theta_2-s)f(s,x_s^n)ds\right\|^2 \\
&\quad +7E\left\|\int_{t_k}^{\tau_n+\theta_1}(\mathcal{T}_\alpha(\tau_n+\theta_2-s)-\mathcal{T}_\alpha(\tau_n+\theta_1-s))g(s,x_s^n)dB(s)\right\|^2 \\
&\quad +7E\left\|\int_{\tau_n+\theta_1}^{\tau_n+\theta_2}\mathcal{T}_\alpha(\tau_n+\theta_2-s)g(s,x_s^n)dB(s)\right\|^2 \\
&\quad +7E\left\|\int_{t_k}^{\tau_n+\theta_1}(\mathcal{T}_\alpha(\tau_n+\theta_2-s)-\mathcal{T}_\alpha(\tau_n+\theta_1-s))h(s)dB_Q^H(s)\right\|^2 \\
&\quad +7E\left\|\int_{\tau_n+\theta_1}^{\tau_n+\theta_2}\mathcal{T}_\alpha(\tau_n+\theta_2-s)h(s)dB_Q^H(s)\right\|^2 \\
&:= \mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4+\mathcal{I}_5+\mathcal{I}_6+\mathcal{I}_7.
\end{aligned}
\tag{4.15}
$$

Hereafter, we assume that $k \geq 1$, since the proof of the case $k = 0$ is similar.

Since $\phi_n \in D$, by Theorem 4.3 we find that for all $s \geq 0$ and $n \in \mathbb{N}$,

$$\|x_s^n\|_{\mathcal{PC}}^2 \leq C\|\phi\|_{\mathcal{PC}}^2 e^{-\left(2\mu-\frac{\ln \mathcal{K}_1}{\beta}\right)s} + C \leq C\left(1+e^{-\left(2\mu-\frac{\ln \mathcal{K}_1}{\beta}\right)s}\right). \tag{4.16}$$

For every $0 < \varepsilon < 1$, in view of (H_2) and Remark 4.1 *ii*), we obtain from the absolute continuity of the integral that there exists $0 < \sigma < \varepsilon$ such that

$$\sup_{t\geq 0}\int_t^{t+\sigma} k_i(s)ds < \varepsilon, \quad i=1,2,3,4, \qquad \text{and} \qquad \sup_{t\geq 0}\int_t^{t+\sigma}\|h(s)\|_Q^2 ds < \varepsilon. \tag{4.17}$$

Moreover, by Lemma 2.10 (i) and (C_1), we deduce that $\mathcal{T}_\alpha(s)$ is uniformly continuous for $s \in [\sigma, \infty)$, i.e., there exists $0 < \delta < \sigma$ such that for all s_1, $s_2 \in [\sigma,\infty)$ with $|s_1 - s_2| < \delta$, we have

$$\|\mathcal{T}_\alpha(s_2)-\mathcal{T}_\alpha(s_1)\| < \varepsilon. \tag{4.18}$$

Hence, by (4.15)-(4.17) and the fact that $x_{\tau_n}^n = u_{\tau_n}^n + z_{\tau_n}^n$, for all $|\theta_1-\theta_2| < \delta$ and $n > n_0$ such that $\tau_n+\theta_1$, $\tau_n+\theta_2 \in (t_k, t_{k+1}] \cap [\sigma,\infty)$ (we choose $r = \sigma$ here), it follows that

$$\mathcal{I}_1 \leq C\|\mathcal{T}_\alpha(\tau_n+\theta_2-t_k)-\mathcal{T}_\alpha(\tau_n+\theta_1-t_k)\|^2 e^{-\left(2\mu-\frac{\ln \mathcal{K}_1}{\beta}\right)t_k} \leq C\varepsilon. \tag{4.19}$$

For the term $\mathcal{I}_2$, by (H_4), (C_1)-(C_2), (4.16)-(4.18), and the Cauchy-Schwarz inequality, we have for all $n \geq n_0$ and $|\theta_2 - \theta_1| < \delta$,

$$\begin{aligned}
\mathcal{I}_2 &\leq C\eta \int_{t_k}^{\tau_n+\theta_1-\sigma} \|\mathcal{T}_\alpha(\tau_n+\theta_2-s) - \mathcal{T}_\alpha(\tau_n+\theta_1-s)\|^2 \left(k_1(s) + k_2(s)\|x_s^n\|^2_{\mathcal{PC}}\right) ds \\
&\quad + C\sigma \int_{\tau_n+\theta_1-\sigma}^{\tau_n+\theta_1} \left(e^{-2\mu(\tau_n+\theta_2-s)} + e^{-2\mu(\tau_n+\theta_1-s)}\right) \left(k_1(s) + k_2(s)\|x_s^n\|^2_{\mathcal{PC}}\right) ds \\
&\leq C \int_{t_k}^{\tau_n+\theta_1-\sigma} \|\mathcal{T}_\alpha(\tau_n+\theta_2-s) - \mathcal{T}_\alpha(\tau_n+\theta_1-s)\|^2 \left(k_1(s) + Ck_2(s)\left(1 + e^{-\left(2\mu - \frac{\ln \mathcal{K}_1}{\beta}\right)s}\right)\right) ds \\
&\quad + C\sigma \int_{\tau_n+\theta_1-\sigma}^{\tau_n+\theta_1} (k_1(s) + k_2(s)) ds \leq C\varepsilon.
\end{aligned} \tag{4.20}$$

Now, for $\mathcal{I}_3$, thanks to (C_1)-(C_2), (4.16), and the Cauchy-Schwarz inequality, we find that for all $n \geq n_0$ and $|\theta_1 - \theta_2| < \delta$,

$$\begin{aligned}
\mathcal{I}_3 &\leq C(\theta_2-\theta_1) \int_{\tau_n+\theta_1}^{\tau_n+\theta_2} \|\mathcal{T}_\alpha(\tau_n+\theta_2-s)\|^2 E\|f(s, x_s^n)\|^2 ds \\
&\leq C(\theta_2-\theta_1) \int_{\tau_n+\theta_1}^{\tau_n+\theta_2} e^{-2\mu(\tau_n+\theta_2-s)} \left(k_1(s) + Ck_2(s)\left(1 + e^{-\left(2\mu - \frac{\ln \mathcal{K}_1}{\beta}\right)s}\right)\right) ds \leq C\varepsilon.
\end{aligned} \tag{4.21}$$

Analogously, (C_1), (C_3), (2.1), (4.16)-(4.18) and the Cauchy-Schwarz inequality imply, for all $n \geq n_0$ and $|\theta_2 - \theta_1| < \delta$,

$$\begin{aligned}
\mathcal{I}_4 &\leq CTr(Q) \int_{t_k}^{\tau_n+\theta_1-\sigma} \|\mathcal{T}_\alpha(\tau_n+\theta_2-s) - \mathcal{T}_\alpha(\tau_n+\theta_1-s)\|^2 (k_3(s) + k_4(s)\|x_s^n\|^2_{\mathcal{PC}}) ds \\
&\quad + CTr(Q) \int_{\tau_n+\theta_1-\sigma}^{\tau_n+\theta_1} \left(e^{-2\mu(\tau_n+\theta_1-s)} + e^{-2\mu(\tau_n+\theta_2-s)}\right) \left(k_3(s) + k_4(s)\|x_s^n\|^2_{\mathcal{PC}}\right) ds \\
&\leq C \int_{t_k}^{\tau_n+\theta_1-\sigma} \|\mathcal{T}_\alpha(\tau_n+\theta_2-s) - \mathcal{T}_\alpha(\tau_n+\theta_1-s)\|^2 \left(k_3(s) + Ck_4(s)\left(1 + e^{-\left(2\mu - \frac{\ln \mathcal{K}_1}{\beta}\right)s}\right)\right) ds \\
&\quad + C \int_{\tau_n+\theta_1-\sigma}^{\tau_n+\theta_1} (k_3(s) + k_4(s)) \leq C\varepsilon.
\end{aligned} \tag{4.22}$$

Using the same arguments as for $\mathcal{I}_3$, by (2.1), (C_1), (C_3) and (4.15) we find that, for all $n \geq n_0$ and $|\theta_2 - \theta_1| < \delta$,

$$\begin{aligned}
\mathcal{I}_5 &\leq CTr(Q) \int_{\tau_n+\theta_1}^{\tau_n+\theta_2} \|\mathcal{T}_\alpha(\tau_n+\theta_2-s)\|^2 \left(k_3(s) + Ck_4(s)\left(1 + e^{-\left(2\mu - \frac{\ln \mathcal{K}_1}{\beta}\right)s}\right)\right) ds \\
&\leq C \int_{\tau_n+\theta_1}^{\tau_n+\theta_2} e^{-2\mu(\tau_n+\theta_2-s)} (k_3(s) + k_4(s)) ds \leq C\varepsilon.
\end{aligned} \tag{4.23}$$

To estimate $\mathcal{I}_6$, we can see that (2.2), (H_4), (C_1), (4.17)-(4.18) imply that, for all $n \geq n_0$ and

$|\theta_2-\theta_1|<\delta$,

$$\begin{aligned}
\mathcal{I}_6 &\le C\eta\int_{t_k}^{\tau_n+\theta_1-\sigma}\|\mathcal{T}_\alpha(\tau_n+\theta_2-s)-\mathcal{T}_\alpha(\tau_n+\theta_1-s)\|^2\|h(s)\|_Q^2ds\\
&\quad+C\sigma^{2H-1}\int_{\tau_n+\theta_1-\sigma}^{\tau_n+\theta_1}\|\mathcal{T}_\alpha(\tau_n+\theta_2-s)-\mathcal{T}_\alpha(\tau_n+\theta_1-s)\|^2\|h(s)\|_Q^2ds\\
&\le C\int_{t_k}^{\tau_n+\theta_1-\sigma}\|\mathcal{T}_\alpha(\tau_n+\theta_2-s)-\mathcal{T}_\alpha(\tau_n+\theta_1-s)\|^2\|h(s)\|_Q^2ds\\
&\quad+C\sigma^{2H-1}\int_{\tau_n+\theta_1-\sigma}^{\tau_n+\theta_1}\left(e^{-2\mu(\tau_n+\theta_1-s)}+e^{-2\mu(\tau_n+\theta_2-s)}\right)\|h(s)\|_Q^2ds\\
&\le C\int_{t_k}^{\tau_n+\theta_1-\sigma}\|\mathcal{T}_\alpha(\tau_n+\theta_2-s)-\mathcal{T}_\alpha(\tau_n+\theta_1-s)\|^2\|h(s)\|_Q^2ds\\
&\quad+C\sigma^{2H-1}\int_{\tau_n+\theta_1-\sigma}^{\tau_n+\theta_1}\|h(s)\|_Q^2ds\le C\varepsilon.
\end{aligned}\tag{4.24}$$

As for the last term $\mathcal{I}_7$, by (2.2), (4.17) and (C_1), we obtain that, for all $n\ge n_0$ and $|\theta_2-\theta_1|<\delta$,

$$\begin{aligned}
\mathcal{I}_7 &\le C(\theta_2-\theta_1)^{2H-1}\int_{\tau_n+\theta_1}^{\tau_n+\theta_2}\|\mathcal{T}_\alpha(\tau_n+\theta_2-s)\|^2\|h(s)\|_Q^2ds\\
&\le C(\theta_2-\theta_1)^{2H-1}\int_{\tau_n+\theta_1}^{\tau_n+\theta_2}e^{-2\mu(\tau_n+\theta_2-s)}\|h(s)\|_Q^2ds\le C\varepsilon.
\end{aligned}\tag{4.25}$$

Therefore, (4.19)-(4.25) show the sequence $\{z^n(\tau_n+\cdot):n\in\mathbb{N}\}$ is equicontinous on $[-T^*,0]$.

Next, we state that the sequence $\{z^n(\tau_n+\theta)\}_{n=1}^\infty$ is relatively compact in $L^2(\Omega;\mathbb{H})$ for each fixed $\theta\in[-T^*,0]$. Then, for such fixed $\theta\in[-T^*,0]$, there exists $n_0\in\mathbb{N}$ such that for each $n\ge n_0$ $0<\lambda<\tau_n+\theta$. Now, for a fixed $n\ge n_0$ there exists $k\in\mathbb{N}$ such taht $\tau_n+\theta-\lambda\in(t_k,t_{k+1}]\cap[\sigma,\infty)$, and we can define

$$\begin{aligned}
z_\lambda^n(\tau_n+\theta)&=\mathcal{T}_\alpha(\lambda)\left(\mathcal{T}_\alpha(\tau_n+\theta-\lambda-t_k)(z^n(t_k^-)+I_k(z^n(t_k^-)))\right)\\
&\quad+\mathcal{T}_\alpha(\lambda)\bigg(\int_{t_k}^{\tau_n+\theta-\lambda}\mathcal{T}_\alpha(\tau_n+\theta-\lambda-s)f(s,x_s^n)ds\\
&\quad+\int_{t_k}^{\tau_n+\theta-\lambda}\mathcal{T}_\alpha(\tau_n+\theta-\lambda-s)g(s,x_s^n)dB(s)\\
&\quad+\int_{t_k}^{\tau_n+\theta-\lambda}\mathcal{T}_\alpha(\tau_n+\theta-\lambda-s)h(s)dB_Q^H(s)\bigg)\\
&:=\mathcal{T}_\alpha(\lambda)\mathcal{Z}_1^n(\tau_n+\theta-\lambda)+\mathcal{T}_\alpha(\lambda)\mathcal{Z}_2^n(\tau_n+\theta-\lambda).
\end{aligned}$$

On the one hand, if $t_k=0$, then this implies $z^n(0)=0$ and $\mathcal{Z}_1^n=0$. On the other hand, if $t_k>0$, then by (H_4), (C_1) and (4.16), we deduce

$$\begin{aligned}
E\|\mathcal{Z}_1^n(\tau_n+\theta-\lambda)\|^2&\le\|\mathcal{T}_\alpha(\tau_n+\theta-\lambda-t_k)\|^2E\|z^n(t_k^-)+I_k(z^n(t_k^-))\|^2\\
&\le CM^2e^{-2\mu(\tau_n+\theta-\lambda-t_k)}\left(1+e^{-\left(2\mu-\frac{\ln\mathcal{K}_1}{\beta}\right)t_k}\right)\le C,
\end{aligned}$$

and by (2.1)-(2.2), (H_2), (H_4), (C_1)-(C_3), (4.16) and the Cauchy-Schwarz inequality,

$$\begin{aligned}
&E\|\mathcal{Z}_2^n(\tau_n+\theta-\lambda)\|^2 \le 3\eta\int_{t_k}^{\tau_n+\theta-\lambda}\|\mathcal{T}_\alpha(\tau_n+\theta-\lambda-s)\|^2E\|f(s,x_s^n)\|^2ds\\
&\quad+3Tr(Q)\int_{t_k}^{\tau_n+\theta-\lambda}\|\mathcal{T}_\alpha(\tau_n+\theta-\lambda-s)\|^2E\|g(s,x_s^n)\|^2ds\\
&\quad+3cH(2H-1)\eta^{2H-1}\int_{t_k}^{\tau_n+\theta-\lambda}\|\mathcal{T}_\alpha(\tau_n+\theta-\lambda-s)\|^2\|h(s)\|_Q^2ds\\
&\le CM^2\int_{t_k}^{\tau_n+\theta-\lambda}e^{-2\mu(\tau_n+\theta-\lambda-s)}\left(k_1(s)+Ck_2(s)\left(1+e^{-\left(2\mu-\frac{\ln\mathcal{K}_1}{\beta}\right)s}\right)\right)ds\\
&\quad+CM^2\int_{t_k}^{\tau_n+\theta-\lambda}e^{-2\mu(\tau_n+\theta-\lambda-s)}\left(k_3(s)+Ck_4(s)\left(1+e^{-\left(2\mu-\frac{\ln\mathcal{K}_1}{\beta}\right)s}\right)\right)ds\\
&\quad+CM^2\int_{t_k}^{\tau_n+\theta-\lambda}e^{-2\mu(\tau_n+\theta-\lambda-s)}\|h(s)\|_Q^2ds\\
&\le C\int_{t_k}^{\tau_n+\theta-\lambda}\left(k_1(s)+k_2(s)+k_3(s)+k_4(s)+\|h(s)\|_Q^2\right)ds\le C.
\end{aligned}$$

By assumption, $\mathcal{T}_\alpha(t)$ is compact for every $t>0$, the set $\{z_\lambda^n(\tau_n+\theta)\}_{n=1}^\infty$ is relatively compact in $L^2(\Omega;\mathbb{H})$ for every $0<\lambda<\tau_n+\theta$. Moreover, for all $n\ge n_0$, there exists a constant $\lambda>0$ such that $\tau_n+\theta$, $\tau_n+\theta-\lambda\in(t_k,t_{k+1}]\cap[\sigma,\infty)$ $(k\in\mathbb{N})$, thus we have

$$\begin{aligned}
E\|z_\lambda^n(\tau_n+\theta)-z^n(\tau_n+\theta)\|^2 &\le 2E\|\mathcal{T}_\alpha(\lambda)\mathcal{Z}_1^n(\tau_n+\theta-\lambda)-\mathcal{Z}_1^n(\tau_n+\theta)\|^2\\
&\quad+2E\|\mathcal{T}_\alpha(\lambda)\mathcal{Z}_2^n(\tau_n+\theta-\lambda)-\mathcal{Z}_2^n(\tau_n+\theta)\|^2:=\mathcal{G}_1+\mathcal{G}_2.
\end{aligned}\tag{4.26}$$

Using the same argument as for (4.17), by Lemma 2.10 (ii), (4.18) and (C_1), we find that there exists $0<\delta^*<\delta$ such that for all $0\le\lambda<\delta^*$, we have

$$\sup_{t\in[\sigma,\infty)}\|\mathcal{T}_\alpha(\lambda)\mathcal{T}_\alpha(t-\lambda)-\mathcal{T}_\alpha(t)\|<C\varepsilon,\tag{4.27}$$

where σ and δ are given in (4.17) and (4.18). When $\tau_n+\theta\in[0,t_1]$, it is obvious that $\mathcal{Z}_1^n(\tau_n+\theta-\lambda)=\mathcal{Z}_1^n(\tau_n+\theta)=0$ and $\mathcal{G}_1=0$. When $\tau_n+\theta$, $\tau_n+\theta-\lambda\in(t_k,t_{k+1}]\cap[\sigma,\infty)$ $(k\ge1)$, by (4.16) and (4.27), for all $n\ge n_0$ and $0<\lambda<\delta^*$, we have

$$\begin{aligned}
\mathcal{G}_1&\le C\|\mathcal{T}_\alpha(\lambda)\mathcal{T}_\alpha(\tau_n+\theta-\lambda-t_k)-\mathcal{T}_\alpha(\tau_n+\theta-t_k)\|^2E\|z^n(t_k^-)+I_k(z^n(t_k^-))\|^2\\
&\le C\|\mathcal{T}_\alpha(\lambda)\mathcal{T}_\alpha(\tau_n+\theta-\lambda-t_k)-T_\alpha(\tau_n+\theta-t_k)\|^2\left(1+e^{-\left(2\mu-\frac{\ln\mathcal{K}_1}{\beta}\right)t_k}\right)\le C\varepsilon,
\end{aligned}\tag{4.28}$$

and

$$
\begin{aligned}
\mathcal{G}_2 &\leq 6E\bigg\|\mathcal{T}_\alpha(\lambda)\int_{t_k}^{\tau_n+\theta-\lambda}\mathcal{T}_\alpha(\tau_n+\theta-\lambda-s)f(s,x_s^n)ds-\int_{t_k}^{\tau_n+\theta}\mathcal{T}_\alpha(\tau_n+\theta-s)f(s,x_s^n)ds\bigg\|^2\\
&+6E\bigg\|\mathcal{T}_\alpha(\lambda)\int_{t_k}^{\tau_n+\theta-\lambda}\mathcal{T}_\alpha(\tau_n+\theta-\lambda-s)g(s,x_s^n)dB(s)-\int_{t_k}^{\tau_n+\theta}\mathcal{T}_\alpha(\tau_n+\theta-s)g(s,x_s^n)dB(s)\bigg\|^2\\
&+6E\bigg\|\mathcal{T}_\alpha(\lambda)\int_{t_k}^{\tau_n+\theta-\lambda}\mathcal{T}_\alpha(\tau_n+\theta-\lambda-s)h(s)dB_Q^H(s)-\int_{t_k}^{\tau_n+\theta}\mathcal{T}_\alpha(\tau_n+\theta-s)h(s)dB_Q^H(s)\bigg\|^2\\
&:=\mathcal{G}_{21}+\mathcal{G}_{22}+\mathcal{G}_{23}.
\end{aligned}
\tag{4.29}
$$

In what follows, we will do estimates for (4.29) one by one. By (4.16), (4.27), (H_4), (C_1)-(C_2) and the Cauchy-Schwarz inequality, we obtain for all $n \geq n_0$ and $0<\lambda<\delta^*$,

$$
\begin{aligned}
\mathcal{G}_{21} &\leq C\eta\int_{t_k}^{\tau_n+\theta-\sigma}\|\mathcal{T}_\alpha(\lambda)\mathcal{T}_\alpha(\tau_n+\theta-\lambda-s)-\mathcal{T}_\alpha(\tau_n+\theta-s)\|^2E\|f(s,x_s^n)\|^2ds\\
&+C(\sigma-\lambda)\int_{\tau_n+\theta-\sigma}^{\tau_n+\theta-\lambda}\|\mathcal{T}_\alpha(\lambda)\mathcal{T}_\alpha(\tau_n+\theta-\lambda-s)-\mathcal{T}_\alpha(\tau_n+\theta-s)\|^2E\|f(s,x_s^n)\|^2ds\\
&+C\lambda\int_{\tau_n+\theta-\lambda}^{\tau_n+\theta}\|\mathcal{T}_\alpha(\tau_n+\theta-s)\|^2E\|f(s,x_s^n)\|^2ds\\
&\leq C\bigg(\int_{t_k}^{\tau_n+\theta-\sigma}\|\mathcal{T}_\alpha(\lambda)\mathcal{T}_\alpha(\tau_n+\theta-\lambda-s)-\mathcal{T}_\alpha(\tau_n+\theta-s)\|^2+(\sigma-\lambda)\int_{\tau_n+\theta-\sigma}^{\tau_n+\theta-\lambda}\\
&\left(e^{-2\mu\lambda}e^{-2\mu(\tau_n+\theta-\lambda-s)}+e^{-2\mu(\tau_n+\theta-s)}\right)\bigg)\times\left(k_1(s)+Ck_2(s)\left(1+e^{-\left(2\mu-\frac{\ln\mathcal{K}_1}{\beta}\right)s}\right)\right)ds\\
&+C\lambda\int_{\tau_n+\theta-\lambda}^{\tau_n+\theta}e^{-2\mu(\tau_n+\theta-s)}\left(k_1(s)+Ck_2(s)\left(1+e^{-\left(2\mu-\frac{\ln\mathcal{K}_1}{\beta}\right)s}\right)\right)ds\\
&\leq C\varepsilon+C(\sigma-\lambda)\int_{\tau_n+\theta-\sigma}^{\tau_n+\theta-\lambda}(k_1(s)+k_2(s))ds+C\lambda\int_{\tau_n+\theta-\lambda}^{\tau_n+\theta}(k_1(s)+k_2(s))ds\leq C\varepsilon.
\end{aligned}
\tag{4.30}
$$

For $\mathcal{G}_{22}$, in a similar way, by (2.1), (4.16), (4.27), (C_1) and (C_3), we find that, for all $n \geq n_0$ and

$0 < \lambda < \delta^*$,

$$
\begin{aligned}
\mathcal{G}_{22} \leq & CTr(Q) \int_{t_k}^{\tau_n+\theta-\sigma} \|\mathcal{T}_\alpha(\lambda)\mathcal{T}_\alpha(\tau_n+\theta-\lambda-s) - \mathcal{T}_\alpha(\tau_n+\theta-s)\|^2 \\
& \times \left(k_3(s) + Ck_4(s)\left(1 + e^{-\left(2\mu - \frac{\ln \mathcal{K}_1}{\beta}\right)s}\right)\right) ds \\
& + CTr(Q) \int_{\tau_n+\theta-\sigma}^{\tau_n+\theta-\lambda} \left(e^{-2\mu\lambda} e^{-2\mu(\tau_n+\theta-\lambda-s)} + e^{-2\mu(\tau_n+\theta-s)}\right) \\
& \times \left(k_3(s) + Ck_4(s)\left(1 + e^{-\left(2\mu - \frac{\ln \mathcal{K}_1}{\beta}\right)s}\right)\right) ds \\
& + CTr(Q) \int_{\tau_n+\theta-\lambda}^{\tau_n+\theta} e^{-2\mu(\tau_n+\theta-s)} \left(k_3(s) + Ck_4(s)\left(1 + e^{-\left(2\mu - \frac{\ln \mathcal{K}_1}{\beta}\right)s}\right)\right) ds \\
\leq & C\varepsilon + C\int_{\tau_n+\theta-\sigma}^{\tau_n+\theta-\lambda} (k_3(s) + k_4(s)) ds + C\int_{\tau_n+\theta-\lambda}^{\tau_n+\theta} (k_3(s) + k_4(s)) ds \leq C\varepsilon.
\end{aligned}
\tag{4.31}
$$

For the last term $\mathcal{G}_{23}$, it follows from (2.2), (4.27), (H_2), (H_4) and (C_1) that, for all $n \geq n_0$ and $0 < \lambda < \delta^*$,

$$
\begin{aligned}
\mathcal{G}_{23} \leq & C\eta^{2H-1} \int_{t_k}^{\tau_n+\theta-\sigma} \|\mathcal{T}_\alpha(\lambda)\mathcal{T}_\alpha(\tau_n+\theta-\lambda-s) - \mathcal{T}_\alpha(\tau_n+\theta-s)\|^2 \|h(s)\|_Q^2 ds \\
& + C(\sigma-\lambda)^{2H-1} \int_{\tau_n+\theta-\sigma}^{\tau_n+\theta-\lambda} \left(e^{-2\mu\lambda} e^{-2\mu(\tau_n+\theta-\lambda-s)} + e^{-2\mu(\tau_n+\theta-s)}\right) \|h(s)\|_Q^2 ds \\
& + C\lambda^{2H-1} \int_{\tau_n+\theta-\lambda}^{\tau_n+\theta} e^{-2\mu(\tau_n+\theta-s)} \|h(s)\|_Q^2 ds \\
\leq & C\varepsilon + C(\sigma-\lambda)^{2H-1} \int_{\tau_n+\theta-\sigma}^{\tau_n+\theta-\lambda} \|h(s)\|_Q^2 ds + C\lambda^{2H-1} \int_{\tau_n+\theta-\lambda}^{\tau_n+\theta} \|h(s)\|_Q^2 ds \leq C\varepsilon.
\end{aligned}
\tag{4.32}
$$

Thus, (4.28)-(4.32) imply

$$
E\|z_\lambda^n(\tau_n+\theta) - z^n(\tau_n+\theta)\|^2 \to 0
$$

as $\lambda \to 0$, uniformly for n. Hence, $\{z_{\tau_n}^n(\theta)\}_{n=1}^\infty$ is precompact in $L^2(\Omega;\mathbb{H})$ for any $\theta \in [-T^*, 0]$. By the Arzelà-Ascoli theorem there exists a subsequence $\{z_{\tau_{n'}}^{n'}(\theta)\}_{n=1}^\infty$ and a function $\xi : \mathbb{R}^- \to L^2(\Omega;\mathbb{H})$ which is the uniform limit of $\{z_{\tau_{n'}}^{n'}(\cdot)\}$ on every interval $[-T^*, 0]$.

Eventually, let us show that $x_{\tau_{n'}}^{n'}(\cdot)$ converges to ξ in $L^2(\Omega;\mathbb{H})$. To do so, we choose some $n \geq n_0$ such that $\tau_n + \theta \in (t_k, t_{k+1}] \cap [\sigma, \infty)$. In view of (2.1)-(2.2), (H_2)-(H_4), (C_1)-(C_3), (4.16) and the Cauchy-Schwarz inequality, together with the fact that $x_{\tau_n}^n = u_{\tau_n}^n + z_{\tau_n}^n$, the following a priori estimate holds,

$$\begin{aligned}
E\|z^n(\tau_n+\theta)\|^2 \le\, & 4E\|\mathcal{T}_\alpha(\tau_n+\theta-t_k)(z^n(t_k^-)+I_k(z^n(t_k^-)))\|^2 \\
& + 4E\Big\|\int_{t_k}^{\tau_n+\theta}\mathcal{T}_\alpha(\tau_n+\theta-s)f(s,x_s^n)ds\Big\|^2 \\
& + 4E\Big\|\int_{t_k}^{\tau_n+\theta}\mathcal{T}_\alpha(\tau_n+\theta-s)g(s,x_s^n)dB(s)\Big\|^2 \\
& + 4E\Big\|\int_{t_k}^{\tau_n+\theta}\mathcal{T}_\alpha(\tau_n+\theta-s)h(s)dB_Q^H(s)\Big\|^2 \\
\le\, & CM^2(N+1)\left(1+e^{-\left(2\mu-\frac{\ln\mathcal{K}_1}{\beta}\right)t_k}\right) \\
& + C\eta M^2\int_{t_k}^{\tau_n+\theta}e^{-2\mu(\tau_n+\theta-s)}\left(k_1(s)+Ck_2(s)\left(1+e^{-\left(2\mu-\frac{\ln\mathcal{K}_1}{\beta}\right)s}\right)\right)ds \\
& + CTr(Q)M^2\int_{t_k}^{\tau_n+\theta}e^{-2\mu(\tau_n+\theta-s)}\left(k_3(s)+Ck_4(s)\left(1+e^{-\left(2\mu-\frac{\ln\mathcal{K}_1}{\beta}\right)s}\right)\right)ds \\
& + C\eta^{2H-1}M^2\int_{t_k}^{\tau_n+\theta}e^{-2\mu(\tau_n+\theta-s)}\|h(s)\|_Q^2ds \le C, \qquad \theta\le 0,
\end{aligned} \tag{4.33}$$

where we assumed that $k\ge 1$, since the proof of the case $k=1$ is similar. From (4.33), we know that

$$\sup_{\theta\in[-T^*,0]} e^{\gamma\theta}E\|z_{\tau_{n'}}^{n'}(\theta)\|^2 = \sup_{\theta\in[-T^*,0]} e^{\gamma\theta}E\|z^{n'}(\tau_{n'}+\theta)\|^2 \le C,$$

and thus for every $T^*>0$,

$$\sup_{\theta\in[-T^*,0]} e^{\gamma\theta}E\|\xi(\theta)\|^2 = \lim_{n'\to\infty}\sup_{\theta\in[-T^*,0]} e^{\gamma\theta}E\|z_{\tau_{n'}}^{n'}(\theta)\|^2 \le C, \tag{4.34}$$

which implies that $\xi\in\mathcal{PC}$ on $[-T^*,0]$ and $\|\xi\|_{\mathcal{PC}}\le C$.

What we want to prove is $z_{\tau_{n'}}^{n'}(\cdot)$ converges to ξ on $(-\infty,0]$. That is, we need to check that for every $\varepsilon>0$, there exists $N(\varepsilon)$ such that

$$\|z_{\tau_{n'}}^{n'}-\xi\|_{\mathcal{PC}}^2 = \sup_{\theta\in(-\infty,0]} e^{\gamma\theta}E\|z_{\tau_{n'}}^{n'}(\theta)-\xi(\theta)\|^2 \le \varepsilon, \qquad n'>N(\varepsilon). \tag{4.35}$$

Thanks to (4.34), we have that $z_{\tau_{n'}}^{n'}(\cdot)$ converges to ξ on $[-T^*,0]$ for arbitrarily fixed $T^*>0$. Therefore, we choose $T^*\ge\tau_{n'}$, where $n'>N(\varepsilon)$ defined in (4.35). Obviously in order to prove (4.35), it only remains to show that

$$\sup_{\theta\in(-\infty,-T^*]} e^{\gamma\theta}E\|z_{\tau_{n'}}^{n'}(\theta)-\xi(\theta)\|^2 \le \varepsilon, \qquad n'>N(\varepsilon), \quad T^*\ge\tau_{n'}. \tag{4.36}$$

Observe that for $n' > N(\varepsilon)$, $T^* \geq \tau_{n'}$, combining with (4.33)-(4.34), we have

$$\begin{aligned}\sup_{\theta\in(-\infty,-T^*]} e^{\gamma\theta}E\|z^{n'}_{\tau_{n'}}(\theta)-\xi(\theta)\|^2 &\leq \sup_{\theta\in(-\infty,-T^*]} e^{\gamma\theta}E\|z^{n'}_{\tau_{n'}}(\theta)\|^2 + \sup_{\theta\in(-\infty,-T^*]} e^{\gamma\theta}E\|\xi(\theta)\|^2 \\ &\leq Ce^{-\gamma T^*} + \lim_{n'\to\infty}\sup_{\theta\in(-\infty,-T^*]} e^{\gamma\theta}E\|z^{n'}_{\tau_{n'}}(\theta)\|^2 \leq C\varepsilon.\end{aligned} \tag{4.37}$$

On account of (4.35) and (4.37), the convergence of $\{z^{n'}_{\tau_{n'}}(\cdot)\}$ to ξ in $\mathcal{PC}$ follows immediately. Recall that the previous decomposition shows $x^n_{\tau_n} = u^n_{\tau_n} + z^n_{\tau_n}$. Moreover, (4.13) implies that

$$\lim_{\tau_n\to\infty}\|u^n_{\tau_n}\|_{\mathcal{PC}} = 0,$$

since $\phi_n \in D$. Thus we have found the convergence of $x^{n'}_{\tau_{n'}}$ to ξ in $\mathcal{PC}$, and the proof of this lemma is finished. □

Now we analyze the properties of the omega limit sets for our problem.

Theorem 4.6 *Assume the conditions of Theorem 4.3. Then for any bounded subset D of $\mathcal{PC}$, the set*

$$\begin{aligned}\omega(D) =&\{x : \exists\tau_n\to\infty, \phi_n\in D \text{ and a sequence of solutions } x^n(\cdot) \text{ of problem } (1.2) \\ &\text{with } x^n_0 = \phi_n \in D \text{ such that } x^n_{\tau_n}\to x \text{ in } \mathcal{PC}\}\end{aligned}$$

is nonempty, compact and attracts D. (Note that we use the same letter ω because it is a standard notation but no confusion is possible with the events of the probability space).

Proof. The definition of omega limit set and Lemma 4.5 imply that $\omega(D)$ is nonempty and compact immediately. Now we show that $\omega(D)$ attracts D. We argue by contradiction, then it the result were not true, there would exist $\varepsilon > 0$ and sequences $\{\tau_n\}$ with $\tau_n \to \infty$ $(n\to\infty)$, $\{\phi_n\}$ with $\phi_n \in D$ and solutions $\{x^n(\cdot)\}$ of (1.2) with initial values $x^n_0 = \phi_n$ such that

$$dist(x^n_{\tau_n}, \omega(D)) > \varepsilon, \quad \forall n\in\mathbb{N}, \tag{4.38}$$

where $\text{dist}(\cdot,\cdot)$ is the metric for the topology of $\mathcal{PC}$. By Lemma 4.5, we can ensure that $x^n_{\tau_n}$ is relatively compact and possesses at least one cluster point $z\in\mathcal{PC}$. Obviously $z\in\omega(D)$, and this contradicts (4.38). Thus this theorem is completed. □

We are now ready to state the following key result.

Theorem 4.7 *Assume the conditions in Theorem 4.3. Then the set*

$$A = \overline{\bigcup\{\omega(D):\ D\subset\mathcal{PC},\ D \text{ bounded}\}}$$

is compact in $\mathcal{PC}$, and, moreover, is the minimal closed set that attracts all bounded subsets of $\mathcal{PC}$ in the topology of $\mathcal{PC}$. In other words, for any bounded set $D\subset\mathcal{PC}$ and any $\varepsilon>0$, there exists $t(D,\varepsilon)>0$ such that for any $\phi\in D$ and any solution $x(\cdot)$ of (1.2) with initial value ϕ, it holds

$$dist(x_t, A) \leq \varepsilon, \quad \text{for all } t\geq t(D,\varepsilon).$$

Proof. Let us denote

$$\widetilde{A} = \{\omega(D) : D \subset \mathcal{PC},\ D \text{ bounded}\},$$

and let us first prove that $\widetilde{A}$ is relatively compact, which will imply the compactness of A.

Indeed, let $\{\xi^n\}_{n=1}^{\infty}$ be a sequence in $\widetilde{A}$ with $\xi^n \in \omega(D_n)$, and $\|D_n\|_{\mathcal{PC}} = \sup_{\phi_n \in D_n} \|\phi_n\|_{\mathcal{PC}} \leq d_n$. Thanks to the definition of $\omega(D)$, there exist sequences $\{\tau_n\}$ with $\tau_n \to +\infty$ and

$$\max\left\{\frac{d_n}{e^{2\mu\tau_n}},\quad \frac{d_n^2}{e^{(2\mu - \ln \mathcal{K}_1/\beta)\tau_n}}\right\} \to 0 \qquad \text{as} \qquad n \to +\infty, \tag{4.39}$$

and $\{\phi_n\}$ with $\phi_n \in D_n$ whose solutions $\{x^n(\cdot)\}$ of problem (1.2) corresponding to the initial datum $x_0^n = \phi_n \in D_n$ satisfy, for all $n \in \mathbb{N}$,

$$\|x_{\tau_n}^n - \xi^n\|_{\mathcal{PC}} < \frac{1}{n}. \tag{4.40}$$

Arguing as in the proof of Lemma 4.5, taking into account (4.39), one can prove that $\{x_{\tau_n}^n\}_{n=0}^{\infty}$ is relatively compact in $\mathcal{PC}$. Therefore, this result and (4.40) imply that $\{\xi^n\}_{n=1}^{\infty}$ is relatively compact in $\mathcal{PC}$, and thus A is compact in $\mathcal{PC}$.

Finally we show that A is the minimal closed set attracting any bounded set $D \subset \mathcal{PC}$. To prove this, notice that if A' is another closed subset which attracts any bounded set $D \subset \mathcal{PC}$, then by the definition of $\omega(D)$, we have that $\omega(D) \subset A'$, and thus $\bigcup\{\omega(D) : D \subset \mathcal{PC}, D \text{ bounded}\}$ belongs to A'. Since A' is closed,

$$A = \overline{\bigcup\{\omega(B) : B \subset \mathcal{PC},\ B \text{ bounded}\}} \subseteq A',$$

and the proof is complete. □

4.3 Existence of the singleton global attracting set in the case of uniqueness of solutions

In general, we cannot obtain much more information about the attracting set just proved to exist in the previous section. In fact, such attracting sets may have a complex structure, even of fractal nature as the vast literature on the theory of global attractors has shown over the last decades. However, in the case of uniqueness of solutions, we can provide more details of the geometrical structure of this set. In fact, we will be able to prove in this subsection that it becomes a singleton, which means the solutions are attracted by a single point in $\mathcal{PC}$, which is not in general an equilibrium point of the problem.

We start this section with the following a priori estimate.

Theorem 4.8 *Let $A \in \mathbb{A}^{\alpha}(\omega_0, \theta_0)$ with $\theta_0 \in (0, \frac{\pi}{2}]$. Assume that (H_1), (H_3), (H_4) and (C_1)hold and, in addition,*

$$\gamma > 2\mu > \mathcal{R} + \frac{\ln w_1}{\beta}, \tag{4.41}$$

(C_4)

$$\int_0^\infty e^{2\mu s} E\|f(s,0)\|^2 ds < \infty, \quad \int_0^\infty e^{2\mu s} E\|g(s,0)\|^2 ds < \infty, \quad \int_0^\infty e^{2\mu s} \|h(s)\|_Q^2 ds < \infty.$$

Then, for every $\phi \in \mathcal{PC}$, there exists a unique mild solution to problem (1.2) fulfilling

$$\|x_t\|_{\mathcal{PC}}^2 \leq C(1 + \|\phi\|_{\mathcal{PC}}^2) e^{-\left(2\mu - \mathcal{R} - \frac{\ln w_1}{\beta}\right)t},$$

where

$$\mathcal{R} = 8M^2(\eta l_1 + Tr(Q) l_2), \qquad w_1 = 8M^2(1+N) + e^{\mathcal{R}\eta}.$$

Proof. The proof of this theorem follows the lines of the corresponding one of Theorem 16 in [28]. Thus we only sketch it. We split the proof into three steps.

Step 1: From Definition 3.1, (2.1), (2.2), (H_1), (C_1) and the Cauchy-Schwarz inequality, we obtain for $t \in [0, t_1]$,

$$\begin{aligned} E\|x(t)\|^2 \leq\ & 4M^2 e^{-2\mu t} \|\phi\|_{\mathcal{PC}}^2 + 8M^2 e^{-2\mu t}(\eta l_1 + Tr(Q) l_2) \int_0^t e^{2\mu s} \|x_s\|_{\mathcal{PC}}^2 ds \\ & + 8M^2 e^{-2\mu t} \int_0^t e^{2\mu s} \left(\eta E\|f(s,0)\|^2 + Tr(Q) E\|g(s,0)\|^2\right) ds \\ & + 4cH(2H-1)\eta^{2H-1} M^2 e^{-2\mu t} \int_0^t e^{2\mu s} \|h(s)\|_Q^2 ds, \end{aligned} \tag{4.42}$$

and condition (C_4) ensures that there exist three constants $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ such that

$$\begin{gathered} cH(2H-1)\eta^{2H-1} \int_0^t e^{2\mu s} \|h(s)\|_Q^2 ds \leq \mathcal{R}_1, \\ \int_0^t e^{2\mu s} E\|f(s,0)\|^2 ds \leq \mathcal{R}_2, \qquad \int_0^t e^{2\mu s} E\|g(s,0)\|^2 ds \leq \mathcal{R}_3. \end{gathered} \tag{4.43}$$

Replacing (4.43) into (4.42) implies, for $t \in [0, t_1]$,

$$E\|x(t)\|^2 \leq 4M^2 e^{-2\mu t} \|\phi\|_{\mathcal{PC}}^2 + 8M^2 e^{-2\mu t} \mathcal{R}_4 + 8M^2 e^{-2\mu t} \mathcal{R}_5 \int_0^t e^{2\mu s} \|x_s\|_{\mathcal{PC}}^2 ds,$$

where we have used the notation

$$\mathcal{R}_4 = (\eta \mathcal{R}_2 + Tr(Q)\mathcal{R}_3) + \frac{\mathcal{R}_1}{2}, \qquad \mathcal{R}_5 = \eta l_1 + Tr(Q) l_2.$$

An analogous argument to the one in Theorem 4.3 to deal with the infinite delay, together with the Gronwall inequality, show that

$$\|x_t\|_{\mathcal{PC}}^2 \leq (4M^2 \|\phi\|_{\mathcal{PC}}^2 + 8M^2 \mathcal{R}_4) e^{-(2\mu - \mathcal{R})t},$$

and, consequently,

$$E\|x(t_1)\|^2 \leq (4M^2 \|\phi\|_{\mathcal{PC}}^2 + 8M^2 \mathcal{R}_4) e^{-(2\mu - \mathcal{R})t_1} := B_1^*. \tag{4.44}$$

Step 2: Similar to (4.32), in view of (H_3), we find for $t \in (t_1, t_2]$ that

$$E\|x(t)\|^2 \leq 8M^2 e^{-2\mu(t-t_1)}(1+N)E\|x(t_1^-)\|^2 + 8M^2 e^{-2\mu(t-t_1)}\mathcal{R}_4 e^{-(2\mu-\mathcal{R})t_1} + 8M^2 e^{-2\mu(t-t_1)}\mathcal{R}_5 \int_{t_1}^{t} e^{2\mu(s-t_1)}\|x_s\|^2_{\mathcal{PC}} ds.$$

We proceed now as in the proof of Step 2 of Theorem 16 in [28], combined the Gronwall inequality. It then follows, for $t \in (t_1, t_2]$,

$$\|x_t\|^2_{\mathcal{PC}} \leq (8M^2(1+N) + e^{\mathcal{R}\eta})B_1^* + 8M^2\mathcal{R}_4 e^{-(2\mu-\mathcal{R})t_1} e^{-(2\mu-\mathcal{R})(t-t_1)}, \tag{4.45}$$

and, therefore,

$$E\|x(t_2)\|^2 \leq (8M^2(1+N) + e^{\mathcal{R}\eta})B_1^* + 8M^2\mathcal{R}_4 e^{-(2\mu-\mathcal{R})t_1} e^{-(2\mu-\mathcal{R})(t_2-t_1)} := B_2^*. \tag{4.46}$$

Step 3: The same reasoning as above implies that for $t \in (t_k, t_{k+1}]$ with $k \geq 2$,

$$\|x_t\|^2_{\mathcal{PC}} \leq (8M^2(1+N) + e^{\mathcal{R}\eta})B_k^* + 8M^2\mathcal{R}_4 e^{-(2\mu-\mathcal{R})t_k} e^{-(2\mu-\mathcal{R})(t-t_k)}, \tag{4.47}$$

and

$$E\|x(t_k)\|^2 \leq (8M^2(1+N) + e^{\mathcal{R}\eta})B_k^* + 8M^2\mathcal{R}_4 e^{-(2\mu-\mathcal{R})t_k} e^{-(2\mu-\mathcal{R})(t_{k+1}-t_k)} := B_{k+1}^*. \tag{4.48}$$

For convenience, let $w_1 = 8M^2(1+N) + e^{\mathcal{R}\eta}$. It is obvious that $w_1 > 2$ so that $\sum_{k=0}^{k-2} w_1^j \leq \frac{w_1^{k-1}}{w_1 - \frac{w_1}{2}} \leq 2w_1^{k-2}$. In addition, condition (H_4) implies that $k-1 \leq \frac{t_k - t_1}{\beta}$ and $k\beta < t_k$. Then for $k \geq 2$, the mathematical induction method furnishes that

$$B_k^* \leq B_1^* e^{-\left(2\mu-\mathcal{R}-\frac{\ln w_1}{\beta}\right)}(t_k - t_1) + 16M^2\mathcal{R}_4 e^{-\left(2\mu-\mathcal{R}-\frac{\ln w_1}{\beta}\right)t_k}. \tag{4.49}$$

Therefore, by (4.44), (4.47) and (4.49), we deduce that, for $\in (t_k, t_{k+1}]$ with $k \geq 2$,

$$\|x_t\|^2_{\mathcal{PC}} \leq C(1 + \|\phi\|^2_{\mathcal{PC}})e^{-\left(2\mu-\mathcal{R}-\frac{\ln w_1}{\beta}\right)t},$$

which, thanks to (4.46) and (4.49), implies that, for all $t > 0$,

$$\|x_t\|^2_{\mathcal{PC}} \leq C(1 + \|\phi\|^2_{\mathcal{PC}})e^{-\left(2\mu-\mathcal{R}-\frac{\ln w_1}{\beta}\right)t}.$$

This completes the proof. □

In order to show that the global attracting set is a singleton set, we first establish the second moment exponential stability of solutions to problem (1.2).

Lemma 4.9 *Assume the conditions of Theorem 4.8. Then, for any two solutions $x(t)$ and $y(t)$ of problem (1.2) corresponding to initial values ψ and ϕ in $\mathcal{PC}$, we have*

$$\|x_t - y_t\|^2_{\mathcal{PC}} \leq 3M^2\|\psi - \phi\|^2_{\mathcal{PC}} e^{-\left(2\mu-\mathcal{R}-\frac{\ln w_1}{\beta}\right)t}, \qquad \forall t \geq 0,$$

where $\mathcal{R}$ and w_1 are defined in Theorem 4.8.

Proof. It is straightforward that we are able to obtain global existence (without uniqueness) of mild solutions to problem (1.2) by the conditions of this lemma. An analogous argument to that already applied in the proof of Theorem 14 in [28] proves the exponential asymptotic behavior with condition (C_4), so we omit the details here. □

Now we state and prove the main results of this subsection.

Theorem 4.10 *Assume the conditions of Theorem 4.8. Then*

- (i) *For any bounded subset D of $\mathcal{PC}$, any sequence $\{\tau_n\}$ with $\tau_n \to +\infty$ $(n \to +\infty)$, $\{\phi_n\}$ with $\phi_n \in D$, and any sequence of solution $\{x^n(\cdot)\}$ of problem (1.2) with $x_0^n = \phi_n \in D$, this last sequence $\{x_{\tau_n}^n\}$ is relatively compact in $\mathcal{PC}$.*

- (ii) *For any bounded subset D of $\mathcal{PC}$, the set*

$$\begin{aligned}\omega(D) \ &= \ \{x : \exists \tau_n \to \infty, \phi_n \in D \textit{ and a sequence of solutions } x^n(\cdot) \textit{of problem } (1.2)\\ &\qquad \textit{with } \ x_0^n = \phi_n \in D \ \ \textit{such that } \ x_{\tau_n}^n \to x \textit{ in } \mathcal{PC}\}\end{aligned}$$

  is a singleton set and attracts D.

- (iii) *The set*

$$\mathcal{A} = \bigcup \{\omega(D) : \ D \subset \mathcal{PC}, \ D \textit{ bounded } \}$$

  is a singleton set, and the minimal set that attracts all bounded subsets of $\mathcal{PC}$.

Proof. (i) To prove $\{x_{\tau_n}^n\}_{n=1}^{\infty}$ is precompact in $\mathcal{PC}$, we only need to state that $\{x_{\tau_n}^n\}$ is a Cauchy sequence in $\mathcal{PC}$. Thanks to Theorem 4.8 and Lemma 4.9, we deduce that

$$\begin{aligned}\|x_{\tau_n}^n - x_{\tau_m}^m\|_{\mathcal{PC}}^2 &\le 3\|x_{\tau_n}^n - x_{\tau_n}^m\|_{\mathcal{PC}}^2 + 3\|x_{\tau_n}^m\|_{\mathcal{PC}}^2 + 3\|x_{\tau_m}^m\|_{\mathcal{PC}}^2\\ &\le C\|\phi_n - \phi_m\|_{\mathcal{PC}}^2 e^{-\left(2\mu - \mathcal{R} - \frac{\ln w_1}{\beta}\right)\tau_n}\\ &\quad + C(1 + \|\phi_m\|_{\mathcal{PC}}^2)\left(e^{-\left(2\mu - \mathcal{R} - \frac{\ln w_1}{\beta}\right)\tau_n} + e^{-\left(2\mu - \mathcal{R} - \frac{\ln w_1}{\beta}\right)\tau_m}\right).\end{aligned} \tag{4.50}$$

Furthermore, as D is a bounded subset of $\mathcal{PC}$ then

$$\|D\|_{\mathcal{PC}} := \sup_{\phi \in D} \|\phi\|_{\mathcal{PC}} \le d, \tag{4.51}$$

and hence (4.50) and (4.51) imply that $\{x_{\tau_n}^n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{PC}$ as $n, m \to \infty$.

(ii) Now we need to prove that $\omega(D)$ is a singleton set. If this were not the case, then there would exist $x, y \in \omega(D)$ such that $x \ne y$. By the definition of $\omega(D)$, we see there exist sequences $\{\tau_n\}$ and $\{s_m\}$ with $\tau_n \to (n \to \infty)$ and $s_m \to \infty$ $(m \to \infty)$, $\{\psi_n\}$ and $\{\phi_m\}$ with $\psi_n, \phi_m \in D$, the solutions $\{x^n(\cdot)\}$ and $\{y^m(\cdot)\}$ of problem (1.2) with $x_0^n = \psi_n$ and $y_0^m = \phi_m$ such that

$$x_{\tau_n}^n \to x \ (n \to \infty) \quad \text{and} \quad y_{s_m}^m \to y \ (n \to \infty).$$

Taking into account Theorem 4.8 and Lemma 4.9, we derive that

$$\begin{aligned}\|x^n_{\tau_n} - y^m_{s_m}\|^2_{\mathcal{PC}} &\leq 3\|x^n_{\tau_n} - y^m_{\tau_n}\|^2_{\mathcal{PC}} + 3\|y^m_{\tau_n}\|^2_{\mathcal{PC}} + 3\|y^m_{s_m}\|^2_{\mathcal{PC}} \\ &\leq C\|\psi_n - \phi_m\|^2_{\mathcal{PC}} e^{-\left(2\mu - \mathcal{R} - \frac{\ln w_1}{\beta}\right)\tau_n} \\ &\quad + C(1 + \|\phi_m\|^2_{\mathcal{PC}}) \left(e^{-\left(2\mu - \mathcal{R} - \frac{\ln w_1}{\beta}\right)\tau_n} + e^{-\left(2\mu - \mathcal{R} - \frac{\ln w_1}{\beta}\right)s_m} \right),\end{aligned}$$

which implies that

$$\|x^n_{\tau_n} - y^m_{s_m}\|^2_{\mathcal{PC}} \to 0 \qquad \text{as} \quad n,\ m \to \infty.$$

Hence $\|x - y\|_{\mathcal{PC}} = 0$, and this is a contradiction since $x \neq y$.

(iii) The set $\mathcal{A}$ is a bounded subset of $\mathcal{PC}$ thanks to Theorem 4.8. The assertion (ii) implies that $\omega(\mathcal{B}(0, \rho))$ is a singleton for each $\rho \in \mathbb{R}^+$, where $\mathcal{B}(0, \rho) = \{x \in \mathcal{PC} : \|x\|_{\mathcal{PC}} \leq \rho\}$. From the definition of omega limit set we have that $\omega(\mathcal{B}(0, 1)) \subset \omega(\mathcal{B}(0, 2)) \subset \cdots \subset \omega(\mathcal{B}(0, n)) \cdots$, and as all of them are singleton sets, all of them must coincide, i.e., $\omega(\mathcal{B}(0, 1)) = \omega(\mathcal{B}(0, 2)) = \cdots = \omega(\mathcal{B}(0, n)) = \cdots$. Consequently,

$$\mathcal{A} = \bigcup \{\omega(D) : \ D \subset \mathcal{PC}, D \text{ bounded}\} = \bigcup_{\rho \in \mathbb{N}} \left\{ \omega(\mathcal{B}(0, \rho)) \right\}$$

is a singleton set. Therefore, $\mathcal{A}$ is the minimal set attracting any bounded set $D \subset \mathcal{PC}$, and we have completed the proof of Theorem 4.10. □

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